Lecture 16: Generating Functions II: Products and Catalan Numbers

In this lecture we continue with ordinary generating functions. We will see that if we have certain combinatorial interpretation for the coefficients of two generating functions, then there is a natural way to interpret the coefficients of the product of such functions. We will also see some applications of this interpretation.

**Proposition 1.** Let \( F(x) \) and \( G(x) \) be the generating functions of the sequences \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\), respectively. Then the following statements hold.

1. \( F'(x) = \sum_{n=0}^{\infty} na_n x^{n-1} \).
2. \( F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n \).
3. \( F(x)G(x) = \sum_{n=0}^{\infty} c_n x^n \), where \( c_n = \sum_{k=0}^{n} a_k b_{n-k} \).

**Proof.**

1. Integrating the formal power series \( \sum_{n=0}^{\infty} a_n x^n \) term by term, we obtain

\[
F'(x) = \left( \sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=0}^{\infty} a_n n x^{n-1}.
\]

2. Adding the for \( F(x) + G(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n \).

3. Observe that the coefficient of \( x^n \) in the product \( F(x)G(x) \) are the sum of the terms \((a_k x^k)(b_\ell x^\ell)\) such that \( k + \ell = n \). Therefore

\[
c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.
\]

**Example 2.** Let \( C_n \) be the number of strings of balanced parentheses of length \( 2n \), where \( C_0 = 1 \). Let us find a explicit formula for \( C_n \) by using generating functions. First, observe that every string of balanced parentheses of length \( 2n \) has the form \((P_k)P_{(n-1)-k}\) for some \( k \in \{0, n-1\} \), where \( P_k \) is a string of balanced parentheses of length \( 2k \), the closing parenthesis \( "\)\)\(^\prime\)), which is in \( 2(k+1)\)-th position, is the match of
the first opening parenthesis, and $P_{(n-1)-k}$ is a string of balanced parenthesis of length $2((n - 1) - k)$. Therefore

\[(0.1) \quad C_n = C_0 C_{n-1} + \cdots + C_{n-1} C_0 = \sum_{k=0}^{n-1} C_k C_{(n-1)-k}.\]

Now let $C(x)$ be the generating function of the sequence $(C_n)_{n \geq 0}$, that is, $C(x) = \sum_{n=0}^{\infty} C_n x^n$. Using the recurrence (0.1), we see that

\[C(x) - 1 = \sum_{n=1}^{\infty} C_n x^n = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} C_k C_{(n-1)-k} \right) x^n = x \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} C_k C_{n-k} \right) x^n = x C(x)^2.\]

Then we see that $xC(x)^2 - C(x) + 1 = 0$, and solving this quadratic equation for $C(x)$ we obtain the solutions

\[C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.\]

By virtue of the Generalized Binomial Theorem,

\[(1-4x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n = \sum_{n=0}^{\infty} \frac{1/2 \cdot 1/2 - 3/2 \cdots -(2n-3)/2}{n!} x^n = 1 - 2x - \sum_{n=2}^{\infty} \frac{2^n (2n - 3)!!}{n!} x^n,\]

where $(2n + 1)!! := 1 \cdot 3 \cdots (2n + 1)$ for every $n \in \mathbb{N}_0$. Thus,

\[\frac{1 - \sqrt{1 - 4x}}{2x} = 1 + \sum_{n=2}^{\infty} \frac{2^{n-1}(2n - 3)!!}{n!} x^n - 1 + \sum_{n=1}^{\infty} \frac{2^n (2n - 1)!!}{(n+1)!} x^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.\]

Hence we conclude that $C_n = \frac{1}{n+1} \binom{2n}{n}$ for every $n \in \mathbb{N}$.

**Practice Exercises**

**Exercise 1.** If $F(x)$, $G(x)$, and $H(x)$ are the generating functions of the sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, and $(c_n)_{n \geq 0}$, respectively. Argue that

\[F(x)G(x)H(x) = \sum_{n=0}^{\infty} \left( \sum_{j+k+\ell = n} a_j b_k c_\ell \right) x^n.\]

Generalize the previous statement for the product of $m$ generating functions $F_1(x), \ldots, F_m(x)$.

**Exercise 2.** Let $a_n$ be the number of ways to provide change for $n$ cents out of pennies, nickels, dimes, and quarters.

1. Find the generating function of the sequence $(a_n)_{n \geq 0}$.
2. Find $a_{211}$. 


Department of Mathematics, MIT, Cambridge, MA 02139
Email address: fgotti@mit.edu