Lecture 15: Generating Functions I: Generalized Binomial Theorem and Fibonacci Sequence

In this lectures we start our journey through the realm of generating functions. Roughly speaking, a generating function is a formal Taylor series centered at 0, that is, a formal Maclaurin series. In general, if a function \( f(x) \) is smooth enough at \( x = 0 \), then its Maclaurin series can be written as follows:

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,
\]

where \( f^{(n)}(x) \) is the \( n \)-th derivative of \( f(x) \). We know from Calculus that the Maclaurin series of the function \( (1 - x)^{-1} \) is

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n.
\]

The Maclaurin series of every polynomial function is itself. In particular, the Binomial Theorem gives us an explicit formula for the Maclaurin series/polynomial of any nonnegative integer power of the binomial \( 1 + x \):

\[
(1 + x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n.
\]

But what if we want to compute the Maclaurin series of \( (1 + x)^r \) when \( r \) is not a nonnegative integer?

Generalized Binomial Theorem. The Generalized Binomial Theorem allows us to express \( (1 + x)^r \) as a Maclaurin series using a natural generalization of the binomial coefficients. For any \( r \in \mathbb{R} \) and \( n \in \mathbb{N}_0 \), we set

\[
\binom{r}{n} := \frac{r(r - 1) \cdots r - n + 1}{n!}.
\]

Observe that when \( r \in \mathbb{N}_0 \), we recover the standard formula for the binomial coefficients. We are now in a position to generalize the Binomial Theorem.
Theorem 1. For any \( r \in \mathbb{R} \),

\[
(1 + x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n.
\]

Proof. Set \( f(x) = (1 + x)^r \). For each \( n \in \mathbb{N}_0 \), we see that \( f^{(n)}(0)/n! = \binom{r}{n} \). Therefore the Maclaurin formula of \( f(x) \) is that one in the right-hand side of (0.4). \( \square \)

As an application of Theorem 1, we can generalize (0.2).

Example 2. Let us find the Maclaurin series of \( (1 - x)^{-m} \) when \( m \in \mathbb{N} \). First, note that for each \( n \in \mathbb{N}_0 \),

\[
\binom{-m}{n} = \frac{1}{n!} \prod_{i=0}^{n-1} (-m - i) = \frac{(-1)^n}{n!} m(m + 1) \cdots (m + n - 1)
\]

\[
= (-1)^n \frac{(m + n - 1)!}{n!(m - 1)!} = (-1)^n \binom{m + n - 1}{m - 1}.
\]

Now in light of Theorem 1,

\[
(1 + x)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} x^n = \sum_{n=0}^{\infty} (-1)^n \binom{m + n - 1}{m - 1} x^n = \sum_{n=0}^{\infty} \binom{m + n - 1}{m - 1} (-x)^n.
\]

Evaluating the previous identity at \(-x\), we obtain that

\[
(1 - x)^{-m} = \sum_{n=0}^{\infty} \binom{m + n - 1}{m - 1} x^n.
\]

Generating Function of a Sequence. We can associate to any sequence \((a_n)_{n \geq 0}\) of real numbers the formal power series \( \sum_{n=0}^{\infty} a_n x^n \). We call this formal power series the \((\text{ordinary})\ generating\ function\) of the sequence \((a_n)_{n \geq 0}\). When \( \sum_{n=0}^{\infty} a_n \) converges to a function \( F(x) \) in some neighborhood of 0, we also call \( F(x) \) the \((\text{ordinary})\ generating\ function\) of \((a_n)_{n \geq 0}\).

Example 3. The generating function of a sequence \((a_n)_{n \geq 0}\) satisfying that \( a_n = 0 \) for every \( n > d \) is the polynomial \( \sum_{n=0}^{d} a_n x^n \).

Example 4. It follows from (0.2) that \( (1 - x)^{-1} \) is the generating function of the constant sequence all whose terms equal 1.

Example 5. For each \( m \in \mathbb{N} \), we have seen in Example 2 that the generating function of the sequence \( \left( \binom{m+n-1}{m-1} \right)_{n \geq 0} \) is \( (1 - x)^{-m} \).

We can actually use generating functions to find explicit formulas for linear recurrence relations. The following example illustrates how to do this.
Example 6. Consider the sequence \((a_n)_{n \geq 0}\) recurrently defined as follows: \(a_0 = 2\) and \(a_{n+1} = 5a_n\) for every \(n \in \mathbb{N}_0\). Let us find a closed formula for \(a_n\). Let \(F(x) = \sum_{n=0}^{\infty} a_n x^n\) be the generating function of the sequence \((a_n)_{n \geq 0}\). Since \(\sum_{n=0}^{\infty} a_{n+1} x^n = \sum_{n=0}^{\infty} 5a_n x^n\), we see that \(\sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+1} x^{n+1} = 5x \sum_{n=0}^{\infty} a_n x^n\) and, therefore,

\[
F(x) = 2 + \sum_{n=1}^{\infty} a_n x^n = 2 + 5x \sum_{n=0}^{\infty} a_n x^n = 2 + 5xF(x).
\]

Hence \(F(x) = 2(1 - 5x)^{-1}\), and so

\[
\sum_{n=0}^{\infty} a_n x^n = F(x) = \frac{2}{1 - 5x} = 2 \sum_{n=0}^{\infty} (5x)^n = \sum_{n=0}^{\infty} 2 \cdot 5^n x^n,
\]

from which we can obtain the desired explicit formula for \(a_n\), namely, \(a_n = 2 \cdot 5^n\) for every \(n \in \mathbb{N}_0\).

Recall that the Fibonacci sequence is defined by the recurrence \(F_{n+1} = F_n + F_{n-1}\), where \(F_0 = 0\) and \(F_1 = 1\). Let us conclude this lecture providing an explicit formula for the Fibonacci numbers.

Example 7. Let \(F(x)\) be the generating function of the Fibonacci sequence. Then

\[
F(x) - x = \sum_{n=1}^{\infty} F_{n+1} x^{n+1} = x \sum_{n=1}^{\infty} F_n x^n + x^2 \sum_{n=1}^{\infty} F_{n-1} x^{n-1} = xF(x) + x^2F(x).
\]

Solving for \(F(x)\), we obtain that

\[
F(x) = \frac{x}{x^2 + x - 1} = - \left( \frac{A}{x - \alpha} + \frac{B}{x - \beta} \right),
\]

for some \(A, B \in \mathbb{R}\), where \(\alpha\) and \(\beta\) are the real roots of \(x^2 + x - 1\). From \(x = A(x - \beta) + B(x - \alpha)\), we can readily deduce that \(A = \frac{\alpha}{\alpha - \beta}\) and \(B = \frac{\beta}{\beta - \alpha}\). Thus,

\[
F(x) = \frac{A}{\alpha - x} + \frac{B}{\beta - x} = \frac{1}{\alpha - \beta} \left( \frac{x}{\alpha} \right)^{-1} + \frac{1}{\beta - \alpha} \left( \frac{x}{\beta} \right)^{-1} = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left( \frac{x}{\alpha} \right)^n + \frac{1}{\beta - \alpha} \sum_{n=0}^{\infty} \left( \frac{x}{\beta} \right)^n = \sum_{n=0}^{\infty} \left( \frac{\alpha^{-n}}{\alpha - \beta} + \frac{\beta^{-n}}{\beta - \alpha} \right) x^n.
\]

Taking \(\alpha = \frac{-1 + \sqrt{5}}{2}\) and \(\beta = \frac{-1 - \sqrt{5}}{2}\), we obtain the following explicit formula:

\[
F_n = \frac{1}{\sqrt{5}} \left( \frac{2}{-1 + \sqrt{5}} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{2}{-1 - \sqrt{5}} \right)^n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]
Exercise 1. Consider the sequence \((a_n)_{n \geq 0}\) satisfying that \(a_0 = 3\) and \(a_{n+1} = 5a_n + 7^n\) for every \(n \in \mathbb{N}_0\). Find an explicit formula for \(a_n\).

Exercise 2. Find a closed form for the generating function of the sequence \((n^2)_{n \geq 0}\).