MIT 18.211: COMBINATORIAL ANALYSIS

FELIX GOTTI

Lecture 14: The Sieve Method and Applications

It is well-known that the identity $|A \cup B| = |A| + |B| - |A \cap B|$ holds for any two finite sets A and B, and one can visualize such an identity using a Venn diagram. We can also draw a Venn diagram to illustrate the identity

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

for any finite sets A, B, and C. More generally, we have the following result, which is known as the Sieve Method or the Principle of Inclusion and Exclusion.

Theorem 1. *If $A_1, \dots, A_n$ are finite sets, then*

$$|\bigcup_{k=1}^{n} A_k| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} |\bigcap_{i \in I} A_i|. \tag{0.1}$$

Proof. Suppose that $A_1, \dots, A_n \subseteq U$ for some universe set U. For each $A \subseteq U$, consider the characteristic function $f_A \colon U \to \{\pm 1\}$ defined as $f_A(x) = 1$ if $x \in A$ and $f_A(x) = 0$ if $x \notin A$. In addition, define $F \colon U \to \{\pm 1\}$ by

$$F(x) = \prod_{k=1}^{n} (1 - f_{A_k}(x)).$$

Observe that $x \in \bigcup_{k=1}^{n} A_k$ if and only if $f_i(x) = 0$ for some $i \in [n]$, which happens precisely when $F(x) = 0$. Hence F is the characteristic function of the set $U \setminus \bigcup_{k=1}^{n} A_k$. Therefore

$$\begin{aligned}
\Big|U \setminus \bigcup_{k=1}^{n} A_k\Big| &= \sum_{x \in U} F(x) = \sum_{x \in U} \prod_{k=1}^{n} (1 - f_{A_k}(x)) \\
&= \sum_{x \in U} \sum_{I \subseteq [n]} (-1)^{|I|} \prod_{i \in I} f_{A_i}(x) \\
&= \sum_{I \subseteq [n]} (-1)^{|I|} \sum_{x \in U} \prod_{i \in I} f_{A_i}(x) \\
&= \sum_{I \subseteq [n]} (-1)^{|I|} \Big|\bigcap_{i \in I} A_i\Big|,
\end{aligned}$$

where the last equality follows from the fact that, for each $I \subseteq [n]$, the function $g_I : U \to \{\pm 1\}$ given by $g_I(x) = \prod_{i \in I} f_{A_i}(x)$ is the characteristic function of $\bigcap_{i \in I} A_i$. Taking into consideration that the intersection of no subsets of U equals U, we obtain that

$$|U| - \Big|\bigcup_{k=1}^{n} A_k\Big| = |U| + \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|} \Big|\bigcap_{i \in I} A_i\Big|.$$

After subtracting $|U|$ and multiplying by -1 both sides of the previous equality, we obtain (0.1), which completes our proof. $\square$

The same problem can be more formally stated as follows. A permutation $\pi \in S_n$ is called a *derangement* if $\pi(i) \neq i$ for any $i \in [n]$. For every $n \in \mathbb{N}$, we let $D(n)$ denote the number of derangement in S_n.

Proposition 2. *For each $n \in \mathbb{N}$, the following identity holds:*

$$D(n) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

Proof. For each $k \in [n]$, set $A_k := \{\pi \in S_n \mid \pi(k) = k\}$. Observe that $|A_k| = (n-1)!$ for every $k \in [n]$. Instead of counting directly the number of derangements in S_n, let us find the size of $\bigcup_{k=1}^{n} A_k$, which is the set of permutations that are not derangements. In light of the Sieve Method, we see that

$$\begin{aligned}
\Big|\bigcup_{k=1}^{n} A_k\Big| &= \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \Big|\bigcap_{i \in I} A_i\Big| \\
&= \sum_{k=1}^{n} \sum_{I \subseteq [n] : |I| = k} (-1)^{k+1} \Big|\bigcap_{i \in I} A_i\Big| \\
&= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!,
\end{aligned}$$

where the last inequality follows from the fact that, for every k subset I of $[n]$, the set $\bigcap_{i \in I} A_i$ has size $(n-k)!$. Thus, the number of derangements in S_n is

$$D(n) = n! - \Big|\bigcup_{k=1}^{n} A_k\Big| = n! - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

$\square$

Remark 3. Since $|S_n| = n!$, the quotient $D(n)/n!$ gives us an idea of the proportion of derangements. When n increases, this quotient converges to e^{-1}. Indeed,

$$\lim_{n \in \infty} \frac{D(n)}{n!} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}.$$

We conclude this lecture by establishing a non-recurrent formula for the Stirling numbers of the second kind.

Proposition 4. *For every $n \in \mathbb{N}$ and $k \in [n]$, the following identity holds:*

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n.$$

Proof. Fix $n \in \mathbb{N}$ and $k \in [n]$. Recall from previous lectures, that $S(n,k)k!$ is the number of surjective functions $[n] \to [k]$. In order to establish the desired formula, let us count such surjective functions in a different way. Instead, we will count the functions $[n] \to [k]$ that are not surjective, and to do so we will use the Sieve Method. For each $j \in [k]$ set

$$A_j := \{f \colon [n] \to [k] \mid j \notin f([n])\}.$$

Note that a function $[n] \to [k]$ is not surjective if and only if it belongs to $\bigcup_{j=1}^k A_j$. Then it follows from the Sieve Method that

$$\Big|\bigcup_{j\in[k]} A_j\Big| = \sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|+1}\Big|\bigcap_{i\in I} A_i\Big| = \sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}(k-j)^n,$$

where the last equality follows from the fact that for any j-subset I of $[k]$ there are exactly $(k-j)^n$ functions $[n] \to [k]$ that fix the set I. Since there are k^n functions $[n] \to [k]$, we obtain that

$$S(n,k)k! = k^n - \Big|\bigcup_{j\in[k]} A_j\Big| = k^n - \sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}(k-j)^n = \sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n.$$

After dividing the previous equality by $k!$ we obtain the desired formula. $\square$

Practice Exercises

Exercise 1. *In how many ways we can throw 5 dice with different colors and obtain 20 as their total sum?*

Exercise 2. *The* Euler totient function $\phi \colon \mathbb{N} \to \mathbb{N}$ *is defined by letting $\phi(n)$ be the number of positive integers in $[n]$ that are relatively prime to n. For instance, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(5) = 4$, and $\phi(6) = 2$. For distinct primes $p_1, \dots, p_k$, find a formula for $\phi(p_1 \cdots p_k)$.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

Department of Mathematics, MIT, Cambridge, MA 02139
Email address: fgotti@mit.edu