Lecture 14: The Sieve Method and Applications

It is well-known that the identity $|A \cup B| = |A| + |B| - |A \cap B|$ holds for any two finite sets $A$ and $B$, and one can visualize such an identity using a Venn diagram. We can also draw a Venn diagram to illustrate the identity

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

for any finite sets $A$, $B$, and $C$. More generally, we have the following result, which is known as the Sieve Method or the Principle of Inclusion and Exclusion.

**Theorem 1.** If $A_1, \ldots, A_n$ are finite sets, then

$$\left| \bigcup_{k=1}^{n} A_k \right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|.$$  

**Proof.** Suppose that $A_1, \ldots, A_n \subseteq U$ for some universe set $U$. For each $A \subseteq U$, consider the characteristic function $f_A: U \to \{\pm 1\}$ defined as $f_A(x) = 1$ if $x \in A$ and $f_A(x) = 0$ if $x \notin A$. In addition, define $F: U \to \{\pm 1\}$ by

$$F(x) = \prod_{k=1}^{n} (1 - f_{A_k}(x)).$$

Observe that $x \in \bigcup_{k=1}^{n} A_k$ if and only if $f_i(x) = 0$ for some $i \in [n]$, which happens precisely when $F(x) = 0$. Hence $F$ is the characteristic function of the set $U \setminus \bigcup_{k=1}^{n} A_k$. Therefore

$$\left| U \setminus \bigcup_{k=1}^{n} A_k \right| = \sum_{x \in U} F(x) = \sum_{x \in U} \prod_{k=1}^{n} (1 - f_{A_k}(x))$$  

$$= \sum_{x \in U} \sum_{I \subseteq [n]} (-1)^{|I|} \prod_{i \in I} f_{A_i}(x)$$  

$$= \sum_{I \subseteq [n]} (-1)^{|I|} \sum_{x \in U} \prod_{i \in I} f_{A_i}(x)$$  

$$= \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|. $$
where the last equality follows from the fact that, for each $I \subseteq [n]$, the function $g_I: U \to \{\pm 1\}$ given by $g_I(x) = \prod_{i \in I} f_{A_i}(x)$ is the characteristic function of $\bigcap_{i \in I} A_i$. Taking into consideration that the intersection of no subsets of $U$ equals $U$, we obtain that

$$|U| - |\bigcup_{k=1}^n A_k| = |U| + \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|.$$

After subtracting $|U|$ and multiplying by $-1$ both sides of the previous equality, we obtain (0.1), which completes our proof. □

The same problem can be more formally stated as follows. A permutation $\pi \in S_n$ is called a derangement if $\pi(i) \neq i$ for any $i \in [n]$. For every $n \in \mathbb{N}$, we let $D(n)$ denote the number of derangement in $S_n$.

**Proposition 2.** For each $n \in \mathbb{N}$, the following identity holds:

$$D(n) = \sum_{k=0}^n (-1)^k \frac{n!}{k!}.$$

**Proof.** For each $k \in [n]$, set $A_k := \{\pi \in S_n | \pi(k) = k\}$. Observe that $|A_k| = (n - 1)!$ for every $k \in [n]$. Instead of counting directly the number of derangements in $S_n$, let us find the size of $\bigcup_{k=1}^n A_k$, which is the set of permutations that are not derangements. In light of the Sieve Method, we see that

$$\left| \bigcup_{k=1}^n A_k \right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| = \sum_{k=1}^n \sum_{I \subseteq [n]: |I| = k} (-1)^{k+1} \left| \bigcap_{i \in I} A_i \right| = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (n - k)!,$$

where the last inequality follows from the fact that, for every $k$ subset $I$ of $[n]$, the set $\bigcap_{i \in I} A_i$ has size $(n - k)!$. Thus, the number of derangements in $S_n$ is

$$D(n) = n! - \left| \bigcup_{k=1}^n A_k \right| = n! - \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (n - k)! = \sum_{k=0}^n (-1)^k \frac{n!}{k!}.$$

□

**Remark 3.** Since $|S_n| = n!$, the quotient $D(n)/n!$ gives us an idea of the proportion of derangements. When $n$ increases, this quotient converges to $e^{-1}$. Indeed,

$$\lim_{n \to \infty} \frac{D(n)}{n!} = \lim_{n \to \infty} \sum_{k=0}^n \frac{(-1)^k}{k!} = \sum_{k=0}^\infty \frac{(-1)^k}{k!} = e^{-1}.$$
We conclude this lecture by establishing a non-recurrent formula for the Stirling numbers of the second kind.

**Proposition 4.** For every \( n \in \mathbb{N} \) and \( k \in [n] \), the following identity holds:

\[
S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n.
\]

**Proof.** Fix \( n \in \mathbb{N} \) and \( k \in [n] \). Recall from previous lectures, that \( S(n, k)k! \) is the number of surjective functions \([n] \to [k]\). In order to establish the desired formula, let us count such surjective functions in a different way. Instead, we will count the functions \([n] \to [k]\) that are not surjective, and to do so we will use the Sieve Method. For each \( j \in [k] \) set

\[
A_j := \{ f : [n] \to [k] \mid j \notin f([n]) \}.
\]

Note that a function \([n] \to [k]\) is not surjective if and only if it belongs to \( \bigcup_{j=1}^{k} A_j \).

Then it follows from the Sieve Method that

\[
\left| \bigcup_{j \in [k]} A_j \right| = \sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} (k - j)^n,
\]

where the last equality follows from the fact that for any \( j\)-subset \( I \) of \([k]\) there are exactly \((k - j)^n\) functions \([n] \to [k]\) that fix the set \( I \). Since there are \( k^n \) functions \([n] \to [k]\), we obtain that

\[
S(n, k)k! = k^n - \left| \bigcup_{j \in [k]} A_j \right| = k^n - \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} (k - j)^n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n.
\]

After dividing the previous equality by \( k! \) we obtain the desired formula. \( \square \)

**Practice Exercises**

**Exercise 1.** In how many ways we can throw 5 dice with different colors and obtain 20 as their total sum?

**Exercise 2.** The Euler totient function \( \phi : \mathbb{N} \to \mathbb{N} \) is defined by letting \( \phi(n) \) be the number of positive integers in \([n]\) that are relatively prime to \( n \). For instance, \( \phi(3) = 2, \phi(4) = 2, \phi(5) = 4, \) and \( \phi(6) = 2 \). For distinct primes \( p_1, \ldots, p_k \), find a formula for \( \phi(p_1 \cdots p_k) \).
References


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