Lecture 13: Permutations III

In this lecture, we introduce a new statistic of permutations, the number of descents, and we use this statistic to define Eulerian numbers. We will provide two identity involving the Eulerian numbers.

We say that \( i \in [n-1] \) is a descent (resp., an ascent) of \( w = w_1w_2 \ldots w_n \in S_n \) if \( w_i > w_{i+1} \) (resp., \( w_i < w_{i+1} \)). In addition, we set
\[
D(w) := \{ i \in [n-1] \mid w_i > w_{i+1} \} \quad \text{and} \quad \text{des}(w) := |D(w)|.
\]

**Definition 1.** For \( n \in \mathbb{N} \), we let \( A(n, k) \) denote the number of permutations in \( S_n \) having exactly \( k - 1 \) descents, and we call \( A(n, k) \) an Eulerian number.

For \( n \in \mathbb{N} \), observe that \( A(n, k) \neq 0 \) only if \( k \in [n] \). By convention, we set \( A(0, 0) = 1 \) and \( A(0, k) = 0 \) if \( k \neq 0 \).

**Example 2.** In the extremal cases, we see that \( A(n, 1) = 1 \) as the only permutation of \( S_n \) with no descents is the identity permutation, and \( A(n, n) = 1 \) as the only permutation of \( S_n \) with \( n-1 \) descents is \( n(n-1) \ldots 1 \).

As for Stirling numbers, we can obtain a convenient recurrence identity for Eulerian numbers.

**Proposition 3.** For all \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \), the following recurrence identity holds:
\[
A(n, k) = (n - k + 1)A(n - 1, k - 1) + kA(n - 1, k).
\]

**Proof.** By definition of the Eulerian numbers, the left-hand side of (0.1) counts the set of permutations in \( S_n \) having \( k - 1 \) descents. Set \( w' = w_1w_2 \ldots w_{n-1} \in S_{n-1} \), and suppose we want to insert \( n \) in one of the \( n \) blanks of the linear arrangement
\[
\ldots w_1 \ldots w_2 \ldots \ldots w_{n-2} \ldots w_{n-1} \ldots
\]
to obtain a permutation \( w \in S_n \). If we insert \( n \) right after \( w_i \) for some \( i \in [n-2] \), then \( \text{des}(w) = \text{des}(w') \) if and only if \( i \not\in D(w') \) and, otherwise, \( \text{des}(w) = \text{des}(w') + 1 \). In addition, if we insert \( n \) in the last (resp., first) blank of (0.2), then \( \text{des}(w) = \text{des}(w') \) (resp., \( \text{des}(w) = \text{des}(w') + 1 \)). Therefore we can also count the set of permutations in \( S_n \) with \( k - 1 \) descent as follows. Choose the permutation \( w' \) in \( S_{n-1} \) among those having \( k - 1 \) descents in \( A(n - 1, k) \) ways and then insert \( n \) in (0.2) either in the last blank or in the blank right after \( w_i \) for some descent \( i \in D(w') \); this can be done in \( kA(n - 1, k) \).
ways. Now count the rest of the permutations in \( S_n \) with \( k - 1 \) descents by choosing a permutation in \( S_{n-1} \) with \( k - 2 \) descents and inserting \( n \) in (0.2) either in the first blank or after \( w_i \) for some ascent position \( i \in [n - 2] \setminus D(w') \); since the number of ascents of \( w' \) is \((n - 2) - (k - 2)\), this can be done in \((n - k + 1)A(n-1, k-1)\). Hence the number of permutations of \( S_n \) with \( k - 1 \) descents is \((n - k + 1)A(n-1, k-1) + kA(n-1, k)\), which is the right-hand side of (0.1). \( \square \)

Now we can establish the following polynomial identity for the Eulerian numbers.

**Theorem 4.** For each \( n \in \mathbb{N} \), the following polynomial identity holds:

\[
(0.3) \quad x^n = \sum_{k=1}^{n} A(n, k) \left( \frac{x + n - k}{n} \right).
\]

**Proof.** First assume that \( x \in \mathbb{N} \). We proceed by induction on \( n \). If \( n = 1 \), then the left-hand side of (0.3) is \( x \). On the other hand, the right-hand side of (0.3) is

\[
A(1, 1) \left( \frac{x}{1} \right) = x.
\]

Now suppose that (0.3) holds for \( n - 1 \in \mathbb{N} \). After setting \( A_n = \sum_{k=0}^{n} A(n, k) \binom{x+n-k}{n} \), we obtain

\[
A_n = \sum_{k=1}^{n} (n - k + 1)A(n-1, k-1) \binom{x + n - k}{n} + \sum_{k=1}^{n} kA(n-1, k) \binom{x + n - k}{n} \]

\[
= \sum_{k=1}^{n-1} (n - k)A(n-1, k) \binom{x + n - k - 1}{n} + \sum_{k=1}^{n-1} kA(n-1, k) \binom{x + n - k}{n} \]

\[
= \sum_{k=1}^{n-1} A(n-1, k) \left[ (n - k) \binom{x + n - k - 1}{n} + k \binom{x + n - k}{n} \right] \]

\[
= \sum_{k=1}^{n-1} A(n-1, k) \left[ x \binom{(n-1) + x - k}{n-1} \right] = x^n,
\]

where the first equality follows from the recurrence identity in Proposition 3 and the last equality follows from our induction hypothesis. As a result, we have established the identity (0.3) provided that \( x \) is a positive integer. Therefore both sides of the same identity are polynomials having infinitely many common roots, and this guarantees that such polynomials are equal. \( \square \)
Practice Exercises

Exercise 1. How many permutations in $S_n$ has exactly one descent?

Exercise 2. For $n \in \mathbb{N}$ and $k \in [n]$, prove that
\[
A(n, k) = \sum_{j=1}^{k} (-1)^j \binom{n+1}{j} (k-j)^n.
\]

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