In this lecture, we introduce a new statistic of permutations, the number of descents, and we use this statistic to define Eulerian numbers. We will provide two identity involving the Eulerian numbers.

We say that \( i \in [n-1] \) is a descent (resp., an ascent) of \( w = w_1w_2\ldots w_n \in S_n \) if \( w_i > w_{i+1} \) (resp., \( w_i < w_{i+1} \)). In addition, we set

\[
D(w) := \{ i \in [n-1] \mid w_i > w_{i+1} \} \quad \text{and} \quad \text{des}(w) := |D(w)|.
\]

**Definition 1.** For \( n \in \mathbb{N} \), we let \( A(n, k) \) denote the number of permutations in \( S_n \) having exactly \( k-1 \) descents, and we call \( A(n, k) \) an Eulerian number.

For \( n \in \mathbb{N} \), observe that \( A(n, k) \neq 0 \) only if \( k \in [n] \). By convention, we set \( A(0, 0) = 1 \) and \( A(0, k) = 0 \) if \( k \neq 0 \).

**Example 2.** In the extremal cases, we see that \( A(n, 1) = 1 \) as the only permutation of \( S_n \) with no descents is the identity permutation, and \( A(n, n) = 1 \) as the only permutation of \( S_n \) with \( n-1 \) descents is \( n(n-1)\ldots 1 \).

As for Stirling numbers, we can obtain a convenient recurrence identity for Eulerian numbers.

**Proposition 3.** For all \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \), the following recurrence identity holds:

\[
(0.1) \quad A(n, k) = (n - k + 1)A(n - 1, k - 1) + kA(n - 1, k).
\]

**Proof.** By definition of the Eulerian numbers, the left-hand side of (0.1) counts the set of permutations in \( S_n \) having \( k-1 \) descents. Set \( w' = w_{1}w_{2}\ldots w_{n-1} \in S_{n-1} \), and suppose we want to insert \( n \) in one of the \( n \) blanks of the linear arrangement

\[
(0.2) \quad w_{1}\ldots w_{i}\ldots w_{n-2}\ldots w_{n-1}
\]

to obtain a permutation \( w \in S_{n} \). If we insert \( n \) right after \( w_{i} \) for some \( i \in [n-2] \), then \( \text{des}(w) = \text{des}(w') \) if and only if \( i \in D(w') \) and, otherwise, \( \text{des}(w) = \text{des}(w') + 1 \). In addition, if we insert \( n \) in the last (resp., first) blank of (0.2), then \( \text{des}(w) = \text{des}(w') \) (resp., \( \text{des}(w) = \text{des}(w') + 1 \)). Therefore we can also count the set of permutations in \( S_{n} \) with \( k-1 \) descent as follows. Choose the permutation \( w' \) in \( S_{n-1} \) among those having \( k-1 \) descents in \( A(n - 1, k) \) ways and then insert \( n \) in (0.2) either in the last blank or in the blank right after \( w_{i} \) for some descent \( i \in D(w') \); this can be done in \( kA(n - 1, k) \) ways.
ways. Now count the rest of the permutations in \( S_n \) with \( k - 1 \) descents by choosing a permutation in \( S_{n-1} \) with \( k - 2 \) descents and inserting \( n \) in (0.2) either in the first blank or after \( w_i \) for some ascent position \( i \in [n - 2] \setminus D(w') \); since the number of ascents of \( w' \) is \( (n - 2) - (k - 2) \), this can be done in \((n - k + 1)A(n-1, k-1)\). Hence the number of permutations of \( S_n \) with \( k - 1 \) descents is \((n - k + 1)A(n-1, k-1) + kA(n-1, k)\), which is the right-hand side of (0.1).

□

Now we can establish the following polynomial identity for the Eulerian numbers.

**Theorem 4.** For each \( n \in \mathbb{N} \), the following polynomial identity holds:

\[
(0.3) \quad x^n = \sum_{k=1}^{n} A(n, k) \binom{x + n - k}{n}.
\]

**Proof.** First assume that \( x \in \mathbb{N} \). We proceed by induction on \( n \). If \( n = 1 \), then the left-hand side of (0.3) is \( x \). On the other hand, the right-hand side of (0.3) is

\[
A(1, 1)\left(\frac{x}{1}\right) = x.
\]

Now suppose that (0.3) holds for \( n - 1 \in \mathbb{N} \). After setting \( A_n = \sum_{k=0}^{n} A(n, k)\binom{x + n - k}{n} \), we obtain

\[
A_n = \sum_{k=1}^{n} (n - k + 1)A(n-1, k-1)\binom{x + n - k}{n} + \sum_{k=1}^{n} kA(n-1, k)\binom{x + n - k}{n}
\]

\[
= \sum_{k=1}^{n-1} (n - k)A(n-1, k)\binom{x + n - k - 1}{n} + \sum_{k=1}^{n-1} kA(n-1, k)\binom{x + n - k}{n}
\]

\[
= \sum_{k=1}^{n-1} A(n-1, k) \left[ (n - k)\binom{x + n - k - 1}{n} + k\binom{x + n - k}{n} \right]
\]

\[
= \sum_{k=1}^{n-1} A(n-1, k) \left[ x\binom{(n - 1) + x - k}{n - 1} \right] = x^n,
\]

where the first equality follows from the recurrence identity in Proposition 3 and the last equality follows from our induction hypothesis. As a result, we have established the identity (0.3) provided that \( x \) is a positive integer. Therefore both sides of the same identity are polynomials having infinitely many common roots, and this guarantees that such polynomials are equal.

□
Practice Exercises

Exercise 1. How many permutations in $S_n$ has exactly one descent?

Exercise 2. For $n \in \mathbb{N}$ and $k \in [n]$, prove that

$$A(n, k) = \sum_{j=1}^{k} (-1)^j \binom{n+1}{j} (k - j)^n.$$