MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 12: Permutations II

Recall that for every $n \in \mathbb{N}$ and $k \in [n]$ the Stirling number of the second kind $S(n, k)$ counts the set of partitions of $[n]$ into k blocks. Now that we know how to represent a permutation into disjoint cycles, we introduce the following numbers.

Definition 1. For $n, k \in \mathbb{N}$, we let $c(n, k)$ denote the number of permutations of S_n whose disjoint cycle decompositions consist of k cycles, and we call $c(n, k)$ a *signless Stirling number of the first kind.*

It is clear that $c(n, k) = 0$ when $k \notin [n]$, and it is convenient to extend signless Stirling numbers by setting $c(0, 0) = 1$ and $c(0, k) = 0$ for every $k \in \mathbb{N}$.

Example 2. If the disjoint cycle decomposition of $\pi \in S_n$ consists of n cycles, then each cycle must have length 1, which means that π is the identity permutation. Thus, $c(n, n) = 1$. We have seen before that there are $(n-1)!$ permutations in S_n whose disjoint cycle decomposition consists of only one cycle, so $c(n, 1) = (n-1)!$.

As in the case of Stirling numbers of the second kind, we can express $c(n, k)$ in terms of $c(n-1, k-1)$ and $c(n-1, k)$.

Proposition 3. *For $n \in \mathbb{N}$ and $k \in [n]$, the following recurrence identity holds:*

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k). \tag{0.1}$$

Proof. By the definition of the signless Stirling numbers, the left-hand side of (0.1) equals the number of permutations in S_n whose disjoint cycle decomposition consists of k cycles. Let us count the same set in a different ways. To count the number of permutations in S_n having n as a cycle by itself, choose a permutation of S_{n-1} with $k-1$ cycles in its disjoint cycle decomposition and then include (n) as a cycle; this can be done in $c(n-1, k-1)$ ways. To count the permutations in S_n having n as part of a cycle of length at least 2, choose a permutation of S_{n-1} with k cycles in its disjoint cycle decomposition and then insert n in any cycle after any of the elements in such a cycle (not at the beginning because this would give the same permutation that we have obtained by placing n at the end). This can be done in $(n-1)c(n-1, k)$ ways. Hence the number of permutations in S_n with k cycles in their disjoint cycle decompositions is $c(n-1, k-1) + (n-1)c(n-1, k)$, which is the right-hand side of (0.1). $\square$

We can use the recurrence identity in (0.1) to prove the following result.

Proposition 4. *For each $n \in \mathbb{N}_0$ the following polynomial identity holds:*

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1). \tag{0.2}$$

Proof. Set $F_n(x) := x(x+1)\cdots(x+n-1)$ for every $n \in \mathbb{N}_0$. Observe that, for each $n \in \mathbb{N}$, the function $F_n(x)$ is a polynomial of degree n with nonnegative integer coefficients. Thus, we can write $F_n(x) = \sum_{k=0}^{n} d(n,k)x^k$, where $d(n,k) \in \mathbb{N}_0$ for every $k \in [\![0,n]\!]$. Observe that $F_0(x) = 1$ (the product of no factors is 1 by convention), and so $d(0,0) = 1$ and $d(0,k) = 0$ for any $k \in \mathbb{N}$. In addition,

$$\begin{aligned}
\sum_{k=0}^{n} d(n,k)x^k &= x(x+1)\cdots(x+n-1) = (x+n-1)F_{n-1}(x) \\
&= (x+n-1)\sum_{k=0}^{n-1} d(n-1,k)x^k \\
&= \sum_{k=0}^{n-1} d(n-1,k)x^{k+1} + \sum_{k=0}^{n-1}(n-1)d(n-1,k)x^k \\
&= \sum_{k=1}^{n} d(n-1,k-1)x^k + \sum_{k=0}^{n-1}(n-1)d(n-1,k)x^k.
\end{aligned}$$

As a result, we obtain that $d(n,k) = d(n-1,k-1) + (n-1)d(n-1,k)$ for every $n \in \mathbb{N}$ and $k \in [n]$. Therefore the sequences $c(n,k)$ and $d(n,k)$ satisfy the same recurrence identity. Since they coincide when either $n = 0$ or $k = 0$, they must be equal, whence (0.2) must hold. □

Let us define Stirling numbers of the first kind.

Definition 5. For $n, k \in \mathbb{N}_0$, we call $(-1)^{n-k}c(n,k)$ a *Stirling number of the first kind*, and we denote it by $s(n,k)$.

Recall that Stirling numbers of the second kind satisfy the following recurrence identity

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \tag{0.3}$$

where $(x)_k = x(x-1)\cdots(x-k+1)$. As a consequence of Proposition 4, we obtain the following (somehow) similar polynomial equation for Stirling numbers of the first kind.

Corollary 6. *For each $n \in \mathbb{N}$, the following polynomial identity holds:*

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n, \tag{0.4}$$

Proof. It suffices to see that

$$\sum_{k=0}^{n} s(n,k)x^k = (-1)^n \sum_{k=0}^{n} c(n,k)(-x)^k = (-1)^n \prod_{j=0}^{n-1}(-x+j) = \prod_{j=0}^{n-1}(x-j) = (x)_n,$$

where the second equality follows from Proposition 4. $\square$

We conclude this section with the following linear algebra observation.

Remark 7. Let $\mathbb{Q}[x]$ denote the vector space consisting of all polynomials with coefficients in the field $\mathbb{Q}$ and, for each $n \in \mathbb{N}_0$, let V_n denote the subspace of $\mathbb{Q}[x]$ consisting of all polynomials of $\mathbb{Q}[x]$ of degree at most n. It is clear that the set of monic monomials $\beta := \{x^k \mid k \in [\![0,n]\!]\}$ is a basis of V_n. In addition, it is not hard to argue that the set $\beta' := \{(x)_k \mid k \in [\![0,n]\!]\}$ is also a basis of V_n. By virtue of (0.3), the matrix $B' := (S(n,k))_{n,k \in [\![0,n]\!]}$ is the matrix of change of coordinates from β' to β. Similarly, (0.4) means that the matrix $B := (s(n,k))_{n,k \in [\![0,n]\!]}$ is the matrix of change of coordinates from β to β'. As a result, we obtain that the matrices B and B', whose entries are given by the Stirling numbers of the first and second kind, respectively, are inverses of each other, that is, $BB' = B'B = I_{n+1}$.

Practice Exercises

Exercise 1. [1, Exercise 6.2] *For any positive integer n with $n \geq 2$, find a formula for $c(n, n-2)$.*

Exercise 2. [1, Exercise 6.31] *What is the number of permutations in S_{2n} whose longest cycle has length n?*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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