LECTURE 11: PERMUTATIONS I

For every $n \in \mathbb{N}$, let $S_n$ denote the set of all permutations of $[n]$. Recall that, by definition, a permutation in $S_n$ can be represented as a linear arrangement of the elements of $[n]$. This representation is often referred to as a word representation. In addition, we have learned in previous lectures how to represent a permutation based on its set of inversions, and we called this representation the inversion table. The primary purpose of this lecture is to introduce a further representation for permutations. This representation, which consists of certain disjoint cycles, is useful in a wide variety of situations.

Recall that a permutation $\pi = w_1 \ldots w_n$ in $S_n$ can be considered as a function, namely, the function determined by the assignments $\pi(i) = w_i$ for every $i \in [n]$. We have also seen before that, as functions, permutations are indeed bijections on $[n]$. Conversely, every bijection $w : [n] \to [n]$ yields the linear arrangement $w(1) \ldots w(n)$, which is a permutation of $[n]$. Under the operation of composition of functions, $S_n$ is a group, and somehow every finite group is inside $S_n$ for some large $n$. However, we will not delve into the wonderful algebraic structure of $S_n$ as part of this course.

Let $\pi : [n] \to [n]$ be a permutation, and fix $k \in [n]$. By the PHP, there exist $i, j \in [0, n]$ with $i < j$ such that $\pi^i(k) = \pi^j(k)$, where $\pi^m$ denote the bijection we obtain by composing $\pi$ with itself $m$ times (we assume that $\pi^0$ is the identity function on $[n]$, which means that $\pi^0(a) = a$ for every $a \in [n]$). Let $s$ be the smallest element in $[1, n]$ such that there exists $r \in [0, s - 1]$ with $\pi^r(k) = \pi^s(k)$. Note that $r = 0$ as, otherwise, the injectivity of $\pi$ would imply that $\pi^{r-1}(k) = \pi^{s-1}(k)$, contradicting the minimality of $s$. Hence, for every $k \in [n]$, there exists $s \in [n]$ such that $k, \pi(k), \ldots, \pi^{s-1}(k)$ are pairwise distinct and $\pi^s(k) = k$.

**Definition 1.** For a permutation $\pi : [n] \to [n]$ and $k \in [n]$, we call $(k, \pi(k), \ldots, \pi^{s-1}(k))$ a cycle of $\pi$ of length $s$ provided that $k, \pi(k), \ldots, \pi^{s-1}(k)$ are pairwise distinct and $\pi^s(k) = k$.

With notation as in the previous definition, the cycles $(k, \pi(k), \ldots, \pi^{s-1}(k))$ and $(\pi^i(k), \pi^{i+1}(k), \ldots, \pi^{s-1}(k), k, \ldots, \pi^{i-1}(k))$ are considered the same cycle for every $i \in [s - 1]$. Therefore each $k \in [n]$ appears in a unique cycle of $\pi$, and so we can assign a formal product of disjoint cycles $C_1 \cdots C_\ell$ to $\pi$, where every element of $[n]$ belongs to
a unique cycle $C_i$. Two such formal products of cycles are considered the same if one of them can be obtained by permuting the cycles of the other one.

**Definition 2.** The representation of $\pi \in S_n$ as a formal product of disjoint cycles is called the **disjoint cycle decomposition** of $\pi$.

Observe that given a product of disjoint cycles $C_1 \cdots C_\ell$ satisfying that every element of $[n]$ is in one of the cycles $C_i$, there is a permutation $\pi \in S_n$ whose disjoint cycle decomposition is $C_1 \cdots C_\ell$. We can obtain $\pi$ as follows: for each $k$, set $\pi(k) = j$, where $j$ is the first entry in the cycle containing $k$ if $k$ is the last entry and $j$ proceeds $k$ if $k$ is not the last entry.

**Example 3.** Let us find the disjoint cycle decomposition of the permutation $\pi = 783295146 \in S_9$. First, we identify the cycle containing 1. Since $\pi(1) = 7$ and $\pi(7) = 1$, the desired cycle is $C_1 := (1,7)$. Now let us identify the cycle containing the smallest element of $[9]$ that is not an entry of $C_1$, which is 2. Because $\pi(2) = 8$, $\pi(8) = 4$, and $\pi(4) = 2$, the cycle we are looking for is $C_2 := (2,8,4)$. Let us proceed to identify the cycle containing 3, which is the smallest element of $[9]$ that is neither an entry of $C_1$ nor an entry of $C_2$. Since $\pi(3) = 3$, we get that $C_3 := (3)$. Now we identify the cycle containing 5, the smallest element of $[9]$ that is not an entry of any of the cycles already found. As $\pi(5) = 9$, $\pi(9) = 6$, and $\pi(6) = 5$, the desired cycle is $C_4 := (5,9,6)$. Hence the disjoint cycle decomposition of $\pi$ is $(1,7)(2,8,4)(3)(5,9,6)$.

Following standard conventions, we will omit the cycles of length 1 in the disjoint cycle decomposition of any permutation. For instance, if $\pi$ is the permutation in the previous example, we omit the cycle $(3)$ in its disjoint cycle decomposition, simply writing $\pi = (1,7)(2,8,4)(5,9,6)$.

**Definition 4.** Let $\pi \in S_n$. If for every $i \in [n]$ the disjoint cycle decomposition of $\pi$ has precisely $a_i$ cycles of length $i$, then $(a_1, \ldots, a_n)$ is called the **(cycle) type** of $\pi$.

For instance, the cycle type of the permutation $\pi$ in Example 3 is $(1,1,2,0,0,0,0,0,0)$.

**Example 5.** Let us count the set of permutations of $S_9$ whose disjoint cycle decompositions have exactly one cycle, that is, whose cycle type is $(0,0,0,0,0,0,0,0,1)$. Well, we can choose a linear arrangement $w_1 \ldots w_9$ of the elements of $[9]$ in $9!$ ways, and then we can turn such a linear arrangement into the cycle decomposition $(w_1, w_2, \ldots, w_9)$, which consists of precisely one cycle of length 9. However, observe that the 9 rotations of the cycle $(w_1, w_2, \ldots, w_9)$ yield the same permutation. Therefore each permutation with cycle type $(0,0,0,0,0,0,0,0,1)$ has been counted 9 times. Hence there are $9!/9 = 8!$ permutations of $S_9$ consisting of exactly one (disjoint) cycle.

**Example 6.** Now we count the set of permutations of $S_7$ whose disjoint cycle decompositions consist of two cycles, one of them of length 3, that is, whose cycle type is $(0,0,1,1,0,0,0)$. As in the previous example, we choose a linear arrangement $w_1 \ldots w_7$
of [7] in 7! ways, and this time we introduce 2 pairs of parentheses to obtain the cycle decomposition \((w_1, w_2, w_3)(w_4, w_5, w_6, w_7)\). Observe that all the 3 rotations of the cycle \((w_1, w_2, w_3)\) yield the same permutation and also all the 4 rotations of the cycle \((w_4, w_5, w_6, w_7)\) yield the same permutation. Therefore we have to compensate the overcounting caused by these rotations, which amounts to dividing by \(12 = 3 \cdot 4\). Hence there are \(7!/12\) permutations in \(S_7\) whose cycle type is \((0, 0, 1, 1, 0, 0, 0)\).

Keeping the previous two examples in mind, we can establish a formula for the number of permutations of \([n]\) having any prescribed cycle type.

**Theorem 7.** Let \(a_1, \ldots, a_n \in \mathbb{N}_0\) such that \(a_1 + 2a_2 + \cdots + na_n = n\). Then the number of permutations with cycle type \((a_1, \ldots, a_n)\) is

\[
\frac{n!}{a_1!a_2! \cdots a_n! \cdot 1^{a_1} 2^{a_2} \cdots n^{a_n}}.
\]

**Proof.** Suppose that we have \(n\) consecutive blank spaces, and insert \(a_1 + \cdots + a_n\) pairs of parenthesis from left to right in \(n\) steps as follows. In the \(i\)-th step, insert \(a_i\) pairs of parentheses in such a way that

1. the first parenthesis of the first pair is right after the last parenthesis inserted in a previous step (if \(i > 1\)) and right before the first blank (if \(i = 1\)),
2. leave exactly \(i\) blanks between each of the \(a_i\) pairs of parentheses, and
3. leave no blank between consecutive pairs of parentheses.

For instance, when \(n = 9\) the configuration of blanks and parenthesis corresponding to the cycle type \((1, 2, 0, 1, 0, 0, 0, 0, 0)\) is

\[
(\_)(\_\_)(\_\_)(\_\_\_\_\_\_\_).
\]

Now we can fill the \(n\) consecutive blanks by choosing a linear arrangement \(\pi\) of \([n]\) in \(n!\) ways. This gives us a permutation whose cycle type is \((a_1, \ldots, a_n)\). Now for each \(i \in [n]\) there are \(a_i\) cycles of length \(i\) whose \(a_i!\) linear arrangements yield the same permutation; therefore we are overcounting each permutation \(a_1!a_2! \cdots a_n!\) times due to this situation. In addition, for each \(i \in [n]\) all the \(i\) rotations of each of the \(a_i\) cycle of length \(i\) yield the same permutation; therefore we are overcounting each permutation \(1^{a_1} 2^{a_2} \cdots n^{a_n}\) times due to this second situation. Hence we conclude that the number of permutations in \(S_n\) with cycle type \((a_1, \ldots, a_n)\) is

\[
\frac{n!}{a_1!a_2! \cdots a_n! \cdot 1^{a_1} 2^{a_2} \cdots n^{a_n}}.
\]

\(\square\)
Practice Exercises

Exercise 1. [1, Exercise 6.6] Find a recurrence formula for the number of permutations of $S_n$ whose cube is the identity permutation.

Exercise 2. [1, Exercise 6.31] Find the number of permutations of $S_{2n}$ whose largest cycle has length $n$.

References


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