MIT 18.211: COMBINATORIAL ANALYSIS

FELIX GOTTI

Lecture 1: The Pigeonhole Principle

The Pigeonhole Principle, also known as the Dirichlet's (Box) Principle, is a very intuitive statement, which can often be used as a powerful tool in combinatorics (and mathematics in general).

Theorem 1 (Pigeonhole Principle)**.** *Suppose that we place n pigeons into m holes. If $n > m$, then there must be a hole containing at least two pigeons.*

Proof. Suppose, towards a contradiction, that no hole contains more than one pigeon. Label the holes by $1, \dots, m$, and let n_i be the number of pigeons in the i-th hole. Then $n_i \in \{0, 1\}$ for every $i \in [m]$, and so $n = n_1 + \cdots + n_m \le m$, which contradicts that $n > m$. $\square$

To understand how useful can be the Pigeonhole Principle, let us take a look at some examples.

Example 2. Given nine lattice points in the space (i.e., elements of $\mathbb{Z}^3$, we can always choose a pair of points whose midpoint is also a lattice point). Consider the map $f \colon \mathbb{Z}^3 \to \{0,1\}^3$ given by

$$f(a,b,c) = \big(a \pmod 2,\ b \pmod 2,\ c \pmod 2\big).$$

Since $|\{0,1\}^3| = 8$, given a set S consisting of 9 points in $\mathbb{Z}^3$, by the Pigeonhole Principle we can always choose two distinct points $x, y \in S$ such that $f(x) = f(y)$. This implies that $f(x+y) = 0$, which means that each coordinate of the point $x+y$ is even, and so the midpoint $\frac{x+y}{2}$ of x and y belongs to $\mathbb{Z}^3$, whence it is a lattice point.

Example 3. Suppose that each point in the plane is colored either blue or red. Then we claim that we can always choose a rectangle all whose vertices have the same color. Consider the lattice points in $[3] \times [9] \subseteq \mathbb{Z}^2$, which consists of 9 vertically aligned horizontal segments, each consisting of three equidistant lattice points. Since these horizontal segments consist of three lattice points, each of them can be colored in 8 different ways. Therefore the Pigeonhole Principle allows us to choose two of these 9 horizontal segments that are identically colored, namely, H_1 and H_2. Once again we can use the Pigeonhole Principle to assure that two out of the three lattice points of H_1 have the same color, which must coincide with the color of the corresponding two lattice points of H_2 (corresponding here means that they have the same x-coordinate). These four points form the desired rectangle.

Here is a generalized version of the Pigeonhole Principle.

Theorem 4 (Generalized Pigeonhole Principle)**.** *Suppose that we place n pigeons into m holes. If $n > m$, then there must be a hole containing at least $\lceil n/m \rceil$ pigeons.*

Proof. Let $1, 2, \dots, m$ be the labels of the given holes and, for each $i \in [m]$, let n_i denote the number of pigeons in the i-th hole. Suppose, towards a contradiction, that for each $i \in [m]$ the inequality $n_i < \lceil n/m \rceil$ holds, and so $n_i \le \lceil n/m \rceil - 1$. This, along with the fact that $\lceil r \rceil - 1 < r$ for all $r \in \mathbb{R}$, ensures that

$$n = n_1 + \cdots + n_m \le m\Big(\Big\lceil \frac{n}{m} \Big\rceil - 1\Big) < m\frac{n}{m} = n,$$

which is a contradiction. As a result, there must be a hole containing at least $\lceil n/m \rceil$ pigeons, which concludes our proof. $\square$

Let us look at a last example.

Example 5. Suppose that we have 13 points inside a regular hexagon of side 1. Then it turns out that we can always choose a circle of radius $\sqrt{3}/3$ containing 3 of the 13 given points. First subdivide the hexagon into six side-1 equilateral triangles by using its three diagonals (i.e., the three segments joining opposite vertices of the hexagon). Now it follows from the generalized version of the Pigeonhole Principle that one of these equilateral triangles contains $3 = \lceil 13/6 \rceil$ of the 13 given points. Then the circumcircle of this triangle has radius $\sqrt{3}/3$ and contains 3 of the 13 given points.

Practice Exercises

Exercise 1. *For any given set of* 201 *positive integers bounded by* 300*, show that we can always choose two of them whose ratio is a power of* 3*.*

Exercise 2. *Suppose that we are given* 606 *points inside a square with side* 1*. Show that at least* 6 *of them can be covered by a circle of radius* 1/15*.*

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