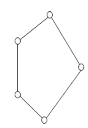
Problem Set 7 (Bonus)

Problem 1 *Le P be a finite poset with no isolated elements (an isolated element of a poset is an element that is only comparable with itself). Suppose that the longest chain of P has length ℓ, and assume that for all $s, t \in P$ with $s \lessdot t$ there is a chain of length ℓ containing both s and t. Prove that every maximal chain of P has length ℓ.*

Problem 2 *Prove that a lattice is modular if and only if it does not contain the pentagonal lattice below as a sublattice (i.e., a subposet that is a lattice).*



Problem 3 *Let L be a finite semimodular lattice, and let S be a subposet of L consisting of elements that are joins of atoms of L with $\hat{0}$ as the empty join ($\hat{0}$ and $\hat{1}$ denote the minimum and maximum of L, respectively). Prove that S is a geometric lattice.*

Problem 4 *Let W be a subspace of $\mathbb{R}^n$. The **support** of $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ is the set $\operatorname{supp}(v) := \{k \in [n] \mid v_k \neq 0\}$. Set $L := \{\operatorname{supp}(v) \mid v \in W\}$. Consider the following order relation on L: for $S_2, S_2 \in L$ the relation $S_1 \le S_2$ holds if and only if $S_2 \subseteq S_1$. Prove that L is a geometric lattice.*

Problem 5 *Let $f(n)$ be the number of sublattices of rank n of B_n, and let $g(n)$ be the number of sublattices of B_n that contain the empty set and $[n]$ (i.e., the $\hat{0}$ and $\hat{1}$ of B_n).*

(1) Prove that $f(n)$ is also the number of poset on $[n]$.

(2) Prove that if $F(x)$ and $G(x)$ are the exponential generating functions of $(f(n))_{n\ge 0}$ and $(g(n))_{n\ge 0}$, respectively, then $G(x) = F(e^x - 1)$.

Problem 6 *Let E_n denote the poset of all subsets of $[n]$ whose elements have even sum (ordered by inclusion).*

(1) Find $|E_n|$.

(2) Compute $|\mu(S,T)|$ for all $S \le T$ in E_n, where μ is the Möbius function of E_n.

Problem 7 *Prove that, for a finite poset P, the following conditions are equivalent.*

(a) For all $s < t$, the interval $[s, t]$ has an odd number of atoms.

(b) For all $s < t$, the interval $[s, t]$ has an odd number of coatoms.

*Recall that a **coatom** of a bounded poset is an element covered by $\hat{1}$. [Hint: Consider $\mu(s, t)$ modulo 2.]*

Problem 8 *Let L be a finite lattice with n atoms. Prove that $|\mu(\hat{0}, \hat{1})| \le \binom{n-1}{\lfloor (n-1)/2 \rfloor}$, and find a finite lattice with n atoms such that the equality holds.*