Problem Set 6 (Due on Wednesday, 12/08)

Problem 1 Recall that two graphs $G_1$ and $G_2$ are called isomorphic if there exists a bijective function $f : V(G_1) \to V(G_2)$, called an isomorphism, such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. Which of the following graphs are isomorphic? Justify your answer.

![Graphs](image1.png)

Problem 2 Let $G$ be a graph. If $f : V(G) \to V(G)$ is an isomorphism of graphs, then we call $f$ an **automorphism** of $G$. How many automorphisms does the following graph have? Justify your answer.

![Graph](image2.png)

Problem 3 Let $G$ be a simple graph. We say that $e \in E(G)$ is a bridge if the graph $(V(G), E(G) \setminus \{e\})$ has more connected components than $G$. Let $G$ be a bipartite $k$-regular graph for $k \geq 2$. Prove that $G$ has no bridge.

Problem 4 For every $n \in \mathbb{N}$ with $n \geq 3$, find the chromatic polynomial of $C_n$, the cycle graph on $[n]$.

Problem 5 For $n \in \mathbb{N}$, prove that the chromatic polynomial of the complete bipartite graph $K_{n,2}$ is $x(x-1)^n + x(x-1)(x-2)^n$.

Problem 6 Let $G$ be a simple connected graph that is neither an odd cycle nor a complete graph, and assume that $G$ has no cut-vertices. Prove that if the subgraph $G \setminus \{v\}$ of $G$ contains a cut-vertex for some $v \in V(G)$, then $\chi(G) \leq \Delta(G)$.

Problem 7 Is it possible to subdivide a square into finitely many concave quadrilateral?

Problem 8 Let $P$ be a convex polyhedron with triangular faces. Suppose that the edges of $P$ are oriented. A **singularity** of $P$ is a face whose edges form an oriented cycle or a vertex $v$ with $\text{indeg}(v) \cdot \text{outdeg}(v) = 0$. Prove that $P$ has at least two singularities.