Problem Set 5 (Due on Monday, 11/22)

Problem 1 *Prove that a connected graph G is a tree if and only if any family of pairwise intersecting (that is, vertex intersecting) paths $P_1, \dots, P_k$ in G have a common vertex.*

Problem 2 *Let A_1, A_2, A_3 be nonempty mutually-disjoint sets with $|A_i| = n_i$ for every $i \in [3]$. The* ***complete tripartite graph*** *on $A_1 \cup A_2 \cup A_3$, denoted by K_{n_1,n_2,n_3}, is the graph G with $V(G) = A_1 \cup A_2 \cup A_3$ having an edge between two vertices if and only if these vertices are not in the same set A_i. For $m, n \in \mathbb{N}$, find a formula for the number of spanning trees of the complete tripartite graph $K_{m,m,n}$.*

Problem 3 *For every $n \in \mathbb{N}$, let t_n be the number of trees on $[n]$. Without using Cayley's theorem (that is, $t_n = n^{n-2}$), prove that*

$$t_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k t_k (n-k) t_{n-k}.$$

Problem 4 *Let T be a tree on the set of vertices $[m]$. For $n \in \mathbb{N}$ with $n > m$, in how many ways can we extend T to a tree on $[n]$?*

Problem 5 *Let G be a simple graph, and let T and T' be two spanning trees of G. Show that for each $e \in E(T)$, we can choose $e' \in E(T')$ such that $(T \setminus \{e\}) \cup \{e'\}$ and $(T' \setminus \{e'\}) \cup \{e\}$ are both spanning trees of G.*

Problem 6 *Let V_1, V_2, and V_3 be three mutually disjoint nonempty sets satisfying $|V_1| = |V_2| = |V_3| = n$, and let G be a simple graph with $V(G) = V_1 \cup V_2 \cup V_3$. Assume that every $v \in V_i$ is adjacent to exactly $n+1$ vertices in $V(G) \setminus V_i$ (v may also be adjacent to some vertices in V_i) for every $i \in [3]$. Prove that there exist $v_1, v_2, v_3 \in V(G)$ with $v_i \in V_i$ for every $i \in [3]$ such that $v_1v_2v_3$ is a cycle in G.*

Problem 7 *Let G be a simple graph in which every vertex has degree 3. Prove that G has a perfect matching if and only if G can be decomposed into paths of length 3 each.*

Problem 8 *Let $k, n \in \mathbb{N}$ such that $k < n/2$. Let G be a bipartite graph with parts V and W satisfying the following condition:*

- *V is the set of k-subsets of $[n]$ and W is the set of $(k+1)$-subsets of $[n]$, and*
- *there is an edge between $S \in V$ and $T \in W$ if and only if $S \subseteq T$.*

Prove that X has a perfect matching into Y

1. *(0.5 pts) by using Hall's theorem, and*
2. *(0.5 pts) by explicitly finding a perfect matching.*