Problem Set 4 (Solutions by Daniil Kliuev)

Problem 1 *Let G be a connected simple graph with $V = \{v_1, \dots, v_n\}$ and $d_i = \deg v_i$ for every $i \in [n]$. In terms of $(d_1, \dots, d_n)$, find the minimum number of edges that must be added to G to obtain a graph with an Eulerian circuit.*

Solution. Let k be the number of vertices in G that have odd degree. This is the same as the number of odd integers in $(d_1, d_2, \dots, d_n)$. Using handshaking lemma we see that k is even. We claim that the answer is $m = \frac{k}{2}$.

First we prove that we should add at least m edges. Suppose that we added l edges to G and obtained a graph G' with Eulerian circuit. Then all degrees of vertices in G' are even. Hence there are at least k vertices that have different degree in G and G'. Adding an edge changes the degree of two vertices. Hence $2l \geq k$, so $l \geq m$.

Now we prove that m edges is enough. Suppose that $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ are all vertices with odd degree. Consider edges connecting v_{i_s} to $v_{i_{s+m}}$ for $s = 1, \dots, m$. Adding these edges to G increases the degree of $v_{i_1}, \dots, v_{i_k}$ by one and does not change other degrees. Hence we obtain a graph G' where each vertex has an even degree. Since G was connected, G' is also connected. Therefore G' contains an Eulerian circuit.

□

Problem 2 *Let G be a simple graph on n vertices. Suppose that $\deg v + \deg w \geq n$ for each pair of distinct non-adjacent vertices v and w of G.*

1. *Prove that G is connected.*

2. *Prove that G has a Hamiltonian cycle.*

Solution. It is enough to prove the second statement. Consider the longest path in G. We can assume that this path is $v_1 v_2 \cdots v_l$, where $2 \leq l \leq n$. There are two cases: $l < n$ and $l = n$. We start with the first one.

Suppose that $l < n$. Since we chose the longest path v_l is not adjacent to $v_{l+1}, \cdots, v_n$. In particular, $\deg v_l + \deg v_n \geq n$. Let A be the subset of $[l]$ consisting of indices i such that v_i is adjacent to v_l. Let B be the subset of $[l]$ consisting of indices i such that v_i is adjacent to v_n. Since v_l is not adjacent to $v_{l+1}, \dots, v_n$ we have $\deg v_l = |A|$. Since there are $n-l-1$ vertices distinct from v_n in $\{v_{l+1}, \dots, v_n\}$ we have $\deg v_n \leq |B| + n - l - 1$. Comparing this with $\deg v_l + \deg v_n \geq n$ we get $|A| + |B| \geq l + 1$.

Since v_l is not adjacent to itself we have $A \subset [l-1]$. Consider the set $A + 1 = \{a+1 \mid a \in A\} \subset [l]$. We have $|A+1| + |B| = |A| + |B| \geq l+1 > l = |[l]|$. It follows that $A + 1$ and B intersect. Let a be an element of the intersection. This means that v_{a-1} is adjacent to v_l and v is adjacent to v_n. So there is a path $v_1 \cdots v_{a-1} v_n v_{n-1} \cdots v_a v_n$ that has length $l + 1$. Since we chose the longest path we get a contradiction.

Suppose that $l = n$. If v_1 and v_n are adjacent we already got a Hamiltonian cycle. We assume that v_1 and v_n are not adjacent. In this case $\deg v_1 + \deg v_n \geq n$. Define A, B as above: A consists of indices i such that v_i and v_1 are adjacent, B consists of indices i such that v_i and v_n are adjacent. We have $|A| + |B| \geq n$.

Since v_1 and v_n are not adjacent to themselves and to each other we have $A, B \subset \{2, 3, \ldots, n-1\}$. Hence $A - 1$ and B are subsets of $[n-1]$. Since $|A-1| + |B| = |A| + |B| \geq n > |[n-1]|$ subsets $A - 1$ and B intersect. Let a be an element of the intersection. This means that v_a is adjacent to v_n and v_{a+1} is adjacent to v_1. We get a Hamiltonian cycle $v_1 \cdots v_a v_n v_{n-1} \cdots v_{a+1}$.

$\square$

Problem 3 *For what values of n, can we decompose K_n into the union of edge-disjoint Hamiltonian cycles?*

Solution. The graph K_n contains $\frac{n(n-1)}{2}$ edges. Each Hamiltonian cycle contains n edges. If the set of edges of K_n can be decomposed into a disjoint union of Hamiltonian cycles then n divides $\frac{n(n-1)}{2}$. Hence $\frac{n-1}{2}$ is an integer, so n is odd.

We claim that for odd n the graph K_n can be decomposed into a union of edge-disjoint Hamiltonian cycles. Let $n = 2k+1$. Denote vertices by $u, v_0, \ldots, v_{2k-1}$. For an integer i consider the cycle $C_i = u v_i v_{i+1} v_{i-1} v_{i+2} v_{i-2} \cdots v_{i+k-1} v_{i-k+1} v_{i+k}$. Here we use cyclic numeration of vertices, so $v_{2k} = v_0$, $v_{2k+1} = v_1$ and so on.

Each C_i is a Hamiltonian cycle and $C_i = C_{i+k}$ for all i. So we have k Hamiltonian cycles $C_0, \ldots, C_{k-1}$. They contain $nk = \frac{n(n-1)}{2}$ edges. So it is enough to prove that each edge of K_n belongs to one of C_i, it will follows that the edges of $C_0, \ldots, C_{k-1}$ are disjoint.

Consider any edge $v_a v_b$, where $0 \leq a, b < 2k$. If a, b have the same parity let $i = \frac{a+b}{2}$, $j = \frac{a-b}{2}$. Since $j < k$ the cycle C_i contains edge $v_{i-j} v_{i+j} = v_a v_b$. If a, b have different parity let $i = \frac{a+b-1}{2}$. Then exactly one of $a - i$, $b - i$ is nonnegative, let j be this value. We have $j < \frac{1}{2}|b - a| < k$. Then the cycle C_i contains the edge $v_{i-j} v_{i+j+1} = v_a v_b$. Here we use that $i - j$ equals to a or b and $a + b = 2i + 1$.

$\square$

Problem 4 *Let T be a tournament that is not strongly connected. Prove that the set of vertices of T can be partitioned into nonempty subsets A and B such that all edges between A and B go from A to B.*

Solution. We say that two vertices v and w are equivalent if $v = w$ or there is a path from v to w and from w to v. This relation is symmetric and transitive by definition. Let $E_1, \ldots, E_l \subset V(T)$ be equivalence classes of this relation. Each of E_i is strongly connected by definition, they are called strongly connected components of T. Since T is not strongly connected we have $l \geq 2$.

Let G be the following directed graph: $V(G) = [l]$ and for $i \neq j$ it has edge $i \to j$ if there exists an edge $v \to w$ in T with $v \in E_i$, $w \in E_j$. We claim that G has no cycles. Suppose that we have cycle $i_1 \cdots i_k$. This means we have edges $v_{i_s} \to w_{i_{s+1}}$, where $v_i, w_i \in E_i$ for all i. Using paths P_i from w_i to v_i we obtain a cycle $v_{i_1} w_{i_2} P_{i_2} v_{i_2} w_{i_3} P_{i_3} \cdots w_{i_k} P_{i_k} v_{i_k} w_{i_1} P_{i_1}$. We see that v_{i_1} is equivalent to w_{i_2}. Since $i_1 \neq i_2$ we get a contradiction.

So we proved that G has no cycles. Consider the longest path $v_{i_1} \cdots v_{i_k}$ in G. Since this path is the longest there are no edges from v_{i_k} to other vertices. Since G has no cycles there are no edges from v_{i_k} to other vertices of this path. Hence v_{i_k} is a sink. Now we take $B = E_{i_k}$ and define A to be the union of all other strongly connected components. Any edge from B to A gives an edge from v_{i_k} to another vertex of G, so there are no edges from B to A. □

Problem 5 *A plane rooted tree is a tree with a distinguished vertex, usually called* ***root****, with a total ordering on the children of each vertex. For each $n \in \mathbb{N}$, find the number of plane rooted trees with n edges.*

Solution. Denote the answer by t_n. We will write recursion for t_n. Let T be a rooted tree with root r. Let v be the first child of r. Let T_v be a set of descendants of v, including v. This is a rooted tree with the root v. Let $T_2 = T \setminus T_v$. This is also a rooted tree: we left descendants of other children of r.

We claim for any $0 \leq m \leq n-1$ there is a bijection between the set of rooted trees T such that T_v has m edges and set of pairs (T_1, T_2), where T_1 is a rooted tree with m edges and T_2 is a rooted tree with $n-1-m$ edges.

The map in one direction is given by $T \mapsto (T_v, T_2)$. The map in other direction takes T_1, T_2 and attaches the root v of T_1 to the root r of T_2. The order on the children of r is that v is first, the rest are ordered as in T_2. We see that these two maps are inverse two each other. Therefore we can write $t_n = \sum_{i=0}^{n-1} t_i t_{n-1-i}$. We also have $t_0 = 1$. It follows that $t_n = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number. □

Problem 6 *Let k be a positive integer, and let T be a tree with precisely one vertex of degree j for every $j \in [\![2, k]\!]$. Find the number of vertices of T if the rest of the vertices of T are leaves (i.e., have degree one).*

Solution. Denote the number of leaves in T by l. Suppose that T has a vertices and b edges. It follows that $a = k - 1 + l$: one vertex of degree j for all $j \in \{2, \dots, k\}$ and l leaves. The sum of degrees of vertices is $2 + 3 + \cdots + k + l = \frac{k(k+1)}{2} - 1 + l$. By handshake lemma this equals to $2b$. We also have $b = a - 1$. Hence $2b = 2a - 2$. Using equations above we get

$$\frac{k(k+1)}{2} - 1 + l = 2(k - 1 + l) - 2 = 2k + 2l - 4.$$

We deduce that $l = \frac{k(k-3)}{2} + 3$. Hence $a = \frac{k(k-1)}{2} + 2$.

□

Problem 7 *We say that a tree T is **trivalent** if it satisfies that $\deg v \in \{1, 3\}$ for all $v \in V(T)$. Let T be a trivalent tree with ℓ leaves.*

1. *Find the number of three-degree vertices of T.*
2. *Prove that if $\ell > 3$, then there is a vertex of T that is adjacent to two leaves.*

Solution.

1. Denote the number of three-degree vertices by k. Then T has $k + l$ vertices with the sum of degrees $3k+l$. Using the handshake lemma we see that T has $\frac{1}{2}(3k+l)$ edges. The number of edges is the number of vertices minus one, hence

$$\frac{1}{2}(3k + l) = k + l - 1.$$

   It follows that $k = l - 2$.

2. Suppose that a vertex v is adjacent to three leaves w_1, w_2, w_3. Since the degree of v is three and the degree of leaf is one there are no more edges going from v, w_1, w_2, w_3. Since a tree is connected we deduce that v, w_1, w_2, w_3 are all vertices of T, so $l = 3$. We get a contradiction with $l > 3$.

   Hence all three-degree vertices are adjacent to zero, one or two leaves. If there is no vertex that is adjacent to two leaves all three-degree vertices are adjacent to zero or one leaf. Let m be the number of edges between three-degree vertices and leaves. We see that $m \leq k$.

   If two leaves are adjacent to each other then T consists of two vertices and $l = 2$. Hence each leaf if adjacent to a three-degree vertex. It follows that $m \geq l$. It follows that $k \geq l$, contradiction with $k = l-2$. Hence there exists a three-degree vertex that is adjacent to two leaves.

□

Problem 8 *Let G be the graph obtained from the complete graph K_n by removing an edge. Find the number of spanning trees of G.*

Solution.

Let k be the number of trees that contain an edge, this number is the same for all edges. Then the number of spanning trees of G equals to the number of spanning trees of K_n minus k, so the answer is $n^{n-2} - k$. It remains to find k. Let N be the number of pairs (T, e), where T is a spanning tree of K_n and e is an edge of T. On one hand, there are $\frac{n(n-1)}{2}$ choices for e and k choices of T for fixed e, so $N = \frac{n(n-1)}{2} \cdot k$. On the other hand there are n^{n-2} choices of T and for each T there are $n-1$ choices of e, so $N = (n-1)n^{n-2}$. It follows that $k = 2n^{n-3}$, so the answer is $n^{n-3}(n-2)$. $\square$