Problem Set 4 (Solutions by Daniil Kliuev)

Problem 1 Let $G$ be a connected simple graph with $V = \{v_1, \ldots, v_n\}$ and $d_i = \deg v_i$ for every $i \in [n]$. In terms of $(d_1, \ldots, d_n)$, find the minimum number of edges that must be added to $G$ to obtain a graph with an Eulerian circuit.

Solution. Let $k$ be the number of vertices in $G$ that have odd degree. This is the same as the number of odd integers in $(d_1, d_2, \ldots, d_n)$. Using handshaking lemma we see that $k$ is even. We claim that the answer is $m = \frac{k}{2}$.

First we prove that we should add at least $m$ edges. Suppose that we added $l$ edges to $G$ and obtained a graph $G'$ with Eulerian circuit. Then all degrees of vertices in $G'$ are even. Hence there are at least $k$ vertices that have different degree in $G$ and $G'$. Adding an edge changes the degree of two vertices. Hence $2l \geq k$, so $l \geq m$.

Now we prove that $m$ edges is enough. Suppose that $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are all vertices with odd degree. Consider edges connecting $v_{i_s}$ to $v_{i_{s+1}}$ for $s = 1, \ldots, m$. Adding these edges to $G$ increases the degree of $v_{i_1}, \ldots, v_{i_k}$ by one and does not change other degrees. Hence we obtain a graph $G'$ where each vertex has an even degree. Since $G$ was connected, $G'$ is also connected. Therefore $G'$ contains an Eulerian circuit.

\[ \square \]

Problem 2 Let $G$ be a simple graph on $n$ vertices. Suppose that $\deg v + \deg w \geq n$ for each pair of distinct non-adjacent vertices $v$ and $w$ of $G$.

1. Prove that $G$ is connected.

2. Prove that $G$ has a Hamiltonian cycle.

Solution. It is enough to prove the second statement. Consider the longest path in $G$. We can assume that this path is $v_1v_2\cdots v_l$, where $2 \leq l \leq n$. There are two cases: $l < n$ and $l = n$. We start with the first one.

Suppose that $l < n$. Since we chose the longest path $v_l$ is not adjacent to $v_{l+1}, \ldots, v_n$. In particular, $\deg v_l + \deg v_n \geq n$. Let $A$ be the subset of $[l]$ consisting of indices $i$ such that $v_i$ is adjacent to $v_l$. Let $B$ be the subset of $[l]$ consisting of indices $i$ such that $v_i$ is adjacent to $v_n$. Since $v_l$ is not adjacent to $v_{l+1}, \ldots, v_n$ we have $\deg v_l = |A|$. Since there are $n - l - 1$ vertices distinct from $v_n$ in $\{v_{l+1}, \ldots, v_n\}$ we have $\deg v_n \leq |B| + n - l - 1$. Comparing this with $\deg v_l + \deg v_n \geq n$ we get $|A| + |B| \geq l + 1$.

Since $v_l$ is not adjacent to itself we have $A \subset [l - 1]$. Consider the set $A + 1 = \{a + 1 \mid a \in A\} \subset [l]$. We have $|A + 1| + |B| = |A| + |B| \geq l + 1 > l = |[l]|$. It follows that $A + 1$ and $B$ intersect. Let $a$ be an element of the intersection. This means that $v_{a-1}$ is adjacent to $v_l$ and $v$ is adjacent to $v_n$. So there is a path $v_1 \cdots v_{a-1}v_nv_{n-1} \cdots v_av_n$ that has length $l + 1$. Since we chose the longest path we get a contradiction.
Suppose that \( l = n \). If \( v_1 \) and \( v_n \) are adjacent we already got a Hamiltonian cycle. We assume that \( v_1 \) and \( v_n \) are not adjacent. In this case \( \deg v_1 + \deg v_n \geq n \). Define \( A, B \) as above: \( A \) consists of indices \( i \) such that \( v_i \) and \( v_1 \) are adjacent, \( B \) consists of indices \( i \) such that \( v_i \) and \( v_n \) are adjacent. We have \( |A| + |B| \geq n \).

Since \( v_1 \) and \( v_n \) are not adjacent to themselves and to each other we have \( A, B \subset \{2, 3, \ldots, n - 1\} \). Hence \( A - 1 \) and \( B \) are subsets of \([n - 1]\). Since \( |A - 1| + |B| = |A| + |B| \geq n > |[n - 1]| \) subsets \( A - 1 \) and \( B \) intersect. Let \( a \) be an element of the intersection. This means that \( v_a \) is adjacent to \( v_n \) and \( v_{a+1} \) is adjacent to \( v_1 \). We get a Hamiltonian cycle \( v_1 \cdots v_a v_n v_{a-1} \cdots v_{a+1} \).

\[ \square \]

**Problem 3** For what values of \( n \), can we decompose \( K_n \) into the union of edge-disjoint Hamiltonian cycles?

**Solution.** The graph \( K_n \) contains \( \frac{n(n-1)}{2} \) edges. Each Hamiltonian cycle contains \( n \) edges. If the set of edges of \( K_n \) can be decomposed into a disjoint union of Hamiltonian cycles then \( n \) divides \( \frac{n(n-1)}{2} \). Hence \( \frac{n-1}{2} \) is an integer, so \( n \) is odd.

We claim that for odd \( n \) the graph \( K_n \) can be decomposed into a union of edge-disjoint Hamiltonian cycles. Let \( n = 2k + 1 \). Denote vertices by \( u, v_0, \ldots, v_{2k-1} \). For an integer \( i \) consider the cycle \( C_i = u v_i v_{i+1} v_{i-1} v_{i+2} \cdots v_{i+k-1} v_{i-k+1} v_{i+k} \). Here we use cyclic numeration of vertices, so \( v_{2k} = v_0, v_{2k+1} = v_1 \) and so on.

Each \( C_i \) is a Hamiltonian cycle and \( C_i = C_{i+k} \) for all \( i \). So we have \( k \) Hamiltonian cycles \( C_0, \ldots, C_{k-1} \). They contain \( nk = \frac{n(n-1)}{2} \) edges. So it is enough to prove that each edge of \( K_n \) belongs to one of \( C_i \), it will follows that the edges of \( C_0, \ldots, C_{k-1} \) are disjoint.

Consider any edge \( v_a v_b \), where \( 0 \leq a, b < 2k \). If \( a, b \) have the same parity let \( i = \frac{a+b}{2}, j = \frac{a-b}{2} \). Since \( j < k \) the cycle \( C_i \) contains edge \( v_{i-j} v_{i+j} = v_a v_b \). If \( a, b \) have different parity let \( i = \frac{a+b-1}{2} \). Then exactly one of \( a - i, b - i \) is nonnegative, let \( j \) be this value. We have \( j < \frac{1}{2} |b-a| < k \). Then the cycle \( C_i \) contains the edge \( v_{i-j} v_{i+j+1} = v_a v_b \). Here we use that \( i - j \) equals to \( a \) or \( b \) and \( a + b = 2i + 1 \).

\[ \square \]

**Problem 4** Let \( T \) be a tournament that is not strongly connected. Prove that the set of vertices of \( T \) can be partitioned into nonempty subsets \( A \) and \( B \) such that all edges between \( A \) and \( B \) go from \( A \) to \( B \).

**Solution.** We say that two vertices \( v \) and \( w \) are equivalent if \( v = w \) or there is a path from \( v \) to \( w \) and from \( w \) to \( v \). This relation is symmetric and transitive by definition. Let \( E_1, \ldots, E_l \subset V(T) \) be equivalence classes of this relation. Each of \( E_i \) is strongly
connected by definition, they are called strongly connected components of $T$. Since $T$ is not strongly connected we have $l \geq 2$.

Let $G$ be the following directed graph: $V(G) = [l]$ and for $i \neq j$ it has edge $i \to j$ if there exists an edge $v \to w$ in $T$ with $v \in E_i$, $w \in E_j$. We claim that $G$ has no cycles. Suppose that we have cycle $i_1 \cdots i_k$. This means we have edges $v_{i_s} \to w_{i_{s+1}}$, where $v_i, w_i \in E_i$ for all $i$. Using paths $P_i$ from $w_i$ to $v_i$ we obtain a cycle $v_{i_1}w_{i_2}v_{i_2}w_{i_3}v_{i_3} \cdots v_{i_k}w_{i_k}v_{i_1}$. We see that $v_{i_1}$ is equivalent to $w_{i_2}$. Since $i_1 \neq i_2$ we get a contradiction.

So we proved that $G$ has no cycles. Consider the longest path $v_{i_1} \cdots v_{i_k}$ in $G$. Since this path is the longest there are no edges from $v_{i_k}$ to other vertices. Since $G$ has no cycles there are no edges from $v_{i_k}$ to other vertices of this path. Hence $v_{i_k}$ is a sink. Now we take $B = E_{i_k}$ and define $A$ to be the union of all other strongly connected components. Any edge from $B$ to $A$ gives an edge from $v_{i_k}$ to another vertex of $G$, so there are no edges from $B$ to $A$. \qed

**Problem 5** A plane rooted tree is a tree with a distinguished vertex, usually called root, with a total ordering on the children of each vertex. For each $n \in \mathbb{N}$, find the number of plane rooted trees with $n$ edges.

**Solution.** Denote the answer by $t_n$. We will write recursion for $t_n$. Let $T$ be a rooted tree with root $r$. Let $v$ be the first child of $r$. Let $T_v$ be a set of descendants of $v$, including $v$. This is a rooted tree with the root $v$. Let $T_2 = T \setminus T_v$. This is also a rooted tree: we left descendants of other children of $r$.

We claim for any $0 \leq m \leq n - 1$ there is a bijection between the set of rooted trees $T$ such that $T_v$ has $m$ edges and set of pairs $(T_1, T_2)$, where $T_1$ is a rooted tree with $m$ edges and $T_2$ is a rooted tree with $n - 1 - m$ edges.

The map in one direction is given by $T \mapsto (T_v, T_2)$. The map in other direction takes $T_1, T_2$ and attaches the root $v$ of $T_1$ to the root $r$ of $T_2$. The order on the children of $r$ is that $v$ is first, the rest are ordered as in $T_2$. We see that these two maps are inverse two each other. Therefore we can write $t_n = \sum_{i=0}^{n-1} t_i t_{n-1-i}$. We also have $t_0 = 1$. It follows that $t_n = C_n$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the Catalan number. \qed

**Problem 6** Let $k$ be a positive integer, and let $T$ be a tree with precisely one vertex of degree $j$ for every $j \in [2, k]$. Find the number of vertices of $T$ if the rest of the vertices of $T$ are leaves (i.e., have degree one).

**Solution.** Denote the number of leaves in $T$ by $l$. Suppose that $T$ has $a$ vertices and $b$ edges. It follows that $a = k - 1 + l$: one vertex of degree $j$ for all $j \in \{2, \ldots, k\}$ and $l$ leaves. The sum of degrees of vertices is $2 + 3 + \cdots + k + l = \frac{k(k+1)}{2} - 1 + l$. By
handshake lemma this equals to $2b$. We also have $b = a - 1$. Hence $2b = 2a - 2$. Using equations above we get

$$\frac{k(k+1)}{2} - 1 + l = 2(k - 1 + l) - 2 = 2k + 2l - 4.$$  

We deduce that $l = \frac{k(k-3)}{2} + 3$. Hence $a = \frac{k(k-1)}{2} + 2$. □

Problem 7 We say that a tree $T$ is trivalent if it satisfies that $\text{deg} v \in \{1, 3\}$ for all $v \in V(T)$. Let $T$ be a trivalent tree with $\ell$ leaves.

1. Find the number of three-degree vertices of $T$.

2. Prove that if $\ell > 3$, then there is a vertex of $T$ that is adjacent to two leaves.

Solution.

1. Denote the number of three-degree vertices by $k$. Then $T$ has $k + l$ vertices with the sum of degrees $3k + l$. Using the handshake lemma we see that $T$ has $\frac{1}{2}(3k + l)$ edges. The number of edges is the number of vertices minus one, hence

$$\frac{1}{2}(3k + l) = k + l - 1.$$  

It follows that $k = l - 2$.

2. Suppose that a vertex $v$ is adjacent to three leaves $w_1, w_2, w_3$. Since the degree of $v$ is three and the degree of leaf is one there are no more edges going from $v, w_1, w_2, w_3$. Since a tree is connected we deduce that $v, w_1, w_2, w_3$ are all vertices of $T$, so $l = 3$. We get a contradiction with $l > 3$.

Hence all three-degree vertices are adjacent to zero, one or two leaves. If there is no vertex that is adjacent to two leaves all three-degree vertices are adjacent to zero or one leaf. Let $m$ be the number of edges between three-degree vertices and leaves. We see that $m \leq k$.

If two leaves are adjacent to each other then $T$ consists of two vertices and $l = 2$. Hence each leaf if adjacent to a three-degree vertex. It follows that $m \geq l$. It follows that $k \geq l$, contradiction with $k = l - 2$. Hence there exists a three-degree vertex that is adjacent to two leaves. □
Problem 8 Let $G$ be the graph obtained from the complete graph $K_n$ by removing an edge. Find the number of spanning trees of $G$.

Solution.

Let $k$ be the number of trees that contain an edge, this number is the same for all edges. Then the number of spanning trees of $G$ equals to the number of spanning trees of $K_n$ minus $k$, so the answer is $n^{n-2} - k$. It remains to find $k$. Let $N$ be the number of pairs $(T, e)$, where $T$ is a spanning tree of $K_n$ and $e$ is an edge of $T$. On one hand, there are $n(n-1)/2$ choices for $e$ and $k$ choices of $T$ for fixed $e$, so $N = \frac{n(n-1)}{2} \cdot k$. On the other hand there are $n^{n-2}$ choices of $T$ and for each $T$ there are $n - 1$ choices of $e$, so $N = (n - 1)n^{n-2}$. It follows that $k = 2n^{n-3}$, so the answer is $n^{n-3}(n - 2)$.

□