Problem Set 3 (Due on Friday, 10/29)

Problem 1 Let $D(n)$ be the number of derangements in $S_n$.

1. Prove that $D(n) = (n - 1)(D(n - 1) + D(n - 2))$.

2. Deduce that $D(n) = nD(n - 1) + (-1)^n$.

Problem 2 For each $n \in \mathbb{N}_0$, let $C_n$ be the $n$-th Catalan number and set $a_n = nC_n$. Find an explicit formula for the generating function of $(a_n)_{n \geq 0}$.

Problem 3 Find an explicit formula for the number of solutions $(x, y, z) \in \mathbb{N}_0^3$ of the equation $x + y + z = n$ satisfying that $x$ is odd, $y > 2$, and $z < 5$.

Problem 4 Let $a_n$ be the number of compositions of $n$ with an odd number of parts such that every part is at least 3. Find an explicit formula (no summation signs allowed) for the generating function of $(a_n)_{n \geq 0}$.

Problem 5 Let $t_n$ be the number of partitions of $[n]$ into blocks of cardinality two. Find the explicit formula (no summation signs allowed) for the exponential generating function of $(t_n)_{n \geq 0}$.

Problem 6 Find an explicit formula (no summation signs allowed) for the exponential generating function of $(D(n))_{n \geq 0}$, where $D(0) = 1$ and $D(n)$ is the number of derangements of $S_n$.

Problem 7 For each $n \in \mathbb{N}$, let $t_n$ be the number of simple graphs with vertex set $[n]$ with no vertex of degree larger than 2, and assume that $t_0 = 1$. Find an explicit formula for the exponential generating function of $(t_n)_{n \geq 0}$.

Problem 8 Using generating functions, prove that the number of partitions of $n$ into distinct parts equals the number of partitions of $n$ where each part is odd.