Problem Set 2 (Due on Friday, 10/15)

Problem 1 *Find an explicit simple formula for the number of compositions of $2n$ whose largest part is n.*

Problem 2 *Let $F(n)$ be the number of partitions of $[n]$ that do not contain any block of size 1. Prove combinatorially that $B(n) = F(n) + F(n+1)$, where $B(n)$ is the n-th Bell number.*

Problem 3 *For each $n \in \mathbb{N}$, prove that the number p_{odd} of partitions of n into odd parts equals the number $q(n)$ of partitions of n into distinct parts.*

Problem 4 *Prove that, for every $n \in \mathbb{N}$, the following identity holds:*

$$\prod_{i=1}^{n}(1 + xq^i) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k+1}{2}} x^k.$$

Problem 5 *For $n \in \mathbb{N}$, what number of cycles do we expect when we take at random a permutation in S_n?*

Problem 6 *Let $I(n, j)$ be the number of permutations in S_n with no cycles of length greater than j. Prove the following recurrence identity:*

$$I(n+1, j) = \sum_{k=n-j+1}^{n} (n)_{n-k} I(k, j),$$

where $(n)_k := n(n-1)\cdots(n-k+1)$.

Problem 7 *For $n \in \mathbb{N}$ with $n \geq 2$, let $a(n, k)$ be the number of permutations in S_n with k cycles in which the entries 1 and 2 are in the same cycle. Prove the following identity:*

$$\sum_{k=1}^{n} a(n,k)x^k = x(x+2)(x+3)\cdots(x+n-1).$$

Problem 8 *Each person in a group of n friends checks a hat and an umbrella when entering a restaurant. When they leave, each of them is given back at random a hat and an umbrella (from the same set of articles they had already checked upon entrance). In how many ways none of the friends gets back her/his own hat or umbrella?*