**COMBINATORIAL ANALYSIS**

**PROBLEM SET 2 SOLUTIONS (MIT, FALL 2021)**

**Problem 1.** Find an explicit simple formula for the number of compositions of $2n$ whose largest part is $n$.

*Proof.* First, note that there is exactly 1 composition of $2n$ into exactly 2 parts whose largest part is $n$, namely $(n, n)$. We now count the number of compositions of size greater than 2.

It is known from lecture that the number of $k$-compositions of $n$ is \( \binom{n}{k-1} \). Further, if $k > 2$, then at most 1 element can equal $n$ and the remaining elements sum to $n$. Therefore, a valid $k$-composition of $2n$ whose largest part is $n$ has $k$ spots to place $n$ and \( \binom{n-1}{k-2} \) ways to choose the other elements. Summing this over all values of $k$ from 3 to $n + 1$ gives:

\[
\sum_{k=3}^{n+1} \binom{n-1}{k-2} = \sum_{k=1}^{n-1} \left( \binom{n-1}{k} + 2 \binom{n-1}{k} \right) = (n-1)2^{n-2} + 2(2^{n-1} - 1) = (n+3)2^{n-2} - 2.
\]

Note that the identity \( \sum_{k=1}^{n-1} k(n-1) = (n-1)2^{n-2} \) comes from the argument that both sides count the number of ways to pick a president on a committee of arbitrary size from $n - 1$ people. Adding in the 1 composition of size 2 gives an answer of \( (n+3)2^{n-2} - 1 \). \( \square \)

**Problem 2.** Let $F(n)$ be the number of partitions of $[n]$ that do not contain any block of size 1. Prove combinatorially that $B(n) = F(n) + F(n+1)$, where $B(n)$ is the $n$-th Bell number.

*Proof.* First, it is known that $B(n)$ counts the number of ways to partition a set of exactly $n$ elements. Therefore, $B(n) - F(n)$ counts the number of partitions of $[n]$ that contain at least 1 block of size 1. We now seek to form a bijection between the number of partitions of $[n+1]$ containing no blocks of size 1 and the partitions of $[n]$ that contain at least 1 block of size 1. This will evidently show that $F(n+1) = B(n) - F(n)$.

For each $n \in \mathbb{N}$, let $X_n$ be the set of partitions of $[n]$ that have no blocks of size 1 and $Y_n$ be the set of partitions of $[n]$ that have at least 1 block of size 1. Define a function $f : X_{n+1} \to Y_n$ as follows. Consider an element $x \in X_{n+1}$, and suppose $x$ represents the partition $a_1 \cup a_2 \cup \ldots \cup a_k$. It is known that $|a_i| > 1 \forall i$. Without loss of generality, the element $n+1$ is in set $a_1$. Suppose that $a_1 = \{r_1, r_2, \ldots, r_\ell, n+1\}$, and consider the partition $y = \{r_1\} \cup \{r_2\} \cup \ldots \cup \{r_\ell\} \cup a_2 \cup \ldots \cup a_k$. This is a valid partition of $[n]$ with at least 1 block of size 1 since $\ell \geq 1$. Therefore $y \in Y_n$ and is unique to each $x$.

For the reverse direction, take $f^{-1} : Y_n \to X_{n+1}$. Let $y \in Y_n$ so that the elements in a block by themselves are $r_1, r_2, \ldots, r_\ell$. If $y = \{r_1\} \cup \{r_2\} \cup \ldots \cup \{r_\ell\} \cup a_1 \cup a_2 \cup \ldots \cup a_k$, then the partition $a_1 \cup a_2 \cup \ldots \cup a_k \cup \{r_1, \ldots, r_\ell, n+1\}$ is clearly a valid partition in $X_{n+1}$ and uniquely defined. Therefore $f$ is bijective and we are done. \( \square \)

**Problem 3.** For each $n \in \mathbb{N}$, prove that the number $p_{\text{odd}}$ of partitions of $n$ into odd parts equals the number $q(n)$ of partitions of $n$ into distinct parts.
Proof. We proceed via generating functions. Let $a_n$ be the number of partitions of $n$ into distinct parts and suppose $A(x) = \sum_{i=0}^{\infty} a_ix^i$ is the generating function for $a_n$. As an integer may appear at most once in a partition of $n$, $A(x)$ must be composed of only factors in the form $(1 + x^k)$ for all positive integers $k$. Therefore,

$$A(x) = \prod_{k=1}^{\infty} (1 + x^k) = \prod_{k=1}^{\infty} \frac{1 - x^{2k}}{1 - x^k} = \prod_{k \text{ odd}} \frac{1}{1 - x^k}.$$  

The last equality is due to terms of the form $1 - x^k$ in the denominator for $k$ odd cancelling with the term of the form $1 - x^{2k}$ in the numerator. However, the last product may be represented as

$$\prod_{k \text{ odd}} (1 + x^k + x^{2k} + ...).$$

This representations tells us that we can make partitions out of any number of odd integers, where the term $x^a$ represents using $k$ a total of $a$ times in the partitions. This means that $A(x)$ is the generating function for $q(n)$ in addition to $p_{\text{odd}}$, implying they are equal. \hfill \Box

**Problem 4.** Prove that, for every $n \in \mathbb{N}$, the following identity holds:

$$\prod_{i=1}^{n} (1 + qx^i) = \sum_{k=0}^{n} \binom{n}{k} q^{\binom{k+1}{2}} x^k.$$  

Proof. We claim that both sides count the same event. Consider any term $c_{a,b}x^aq^b$ from the expansion of the left hand side. There must be exactly $a$ terms of the form $qx^i$ to produce the exponent $x^a$ and further, the sum of the exponents of all the $q^i$ terms must be $b$. Additionally, all $q^i$ are distinct. Therefore, $c_{a,b}$ represents the number of partitions of $b$ into $a$ distinct parts, each of which is at most $n$. We wish to show the same is true of the right hand side.

Let $p(i, j, k)$ represent the number of partitions of $k$ into at most $j$ parts, each of which is at most $i$. From lecture, $\sum_{k \geq 0} p(i, j, k)q^k = \binom{i+j}{j}q$. Plugging this into the right hand side gives

$$\sum_{k=0}^{n} x^k q^{\binom{k+1}{2}} \sum_{i \geq 0} p(n-k, k, i)q^i.$$  

Fixing $k$ and $i$, we obtain a term of the form $p(n-k, k, i)x^k q^{i+(\frac{k+1}{2})}$ for $i \geq 0$. Consider some partition $p_1 + p_2 + ... + p_j = i$ with $p_1 \geq p_2 \geq ... \geq p_j$ such that $j \leq n$ and $p_1 \leq n-k$. If $j < k$, then let elements $p_{j+1}, p_{j+2}, ..., p_k$ all be equal to 0. Then, take $p'_i = p_i + i$, and so

$$\sum_{m=1}^{k} p'_m = \sum_{m=1}^{k} (p_m + m) = \binom{k+1}{2} + i,$$

meaning that the $p'_m$ form a partition of $i + \binom{k+1}{2}$ into $k$ distinct elements. Further, the largest that $p'_1$ can be is $n-k + k = n$, meaning that the $p'$ form a unique partition that represents splitting $\binom{k+1}{2} + i$ into $k$ distinct parts of at most $n$, for any $i$ and $k$. This process is reversible, meaning that $c_{k, \binom{k+1}{2} + i}$ is the same as $p(n-k, k, i)$, and so the two sides represent the same function. \hfill \Box

**Problem 5.** For $n \in \mathbb{N}$, what number of cycles do we expect when we take at random a permutation in $S_n$?
Proof. Let $X_k$ denote the random variable equal to the number of cycles of length $k$ in a randomly chosen permutation in $S_n$. The sum $\sum_{k=1}^{n} X_k$ denotes the total number of cycles in the chosen permutation. Then, from linearity of expectation,

$$\mathbb{E} \left[ \sum_{k=1}^{n} X_k \right] = \sum_{k=1}^{n} \mathbb{E}[X_k].$$

Now, we wish to count the number of cycles of length $k$ across all $n!$ permutations. There are $\binom{n}{k}$ ways to pick $k$ elements for a cycle, and $(k-1)!$ distinct ways to arrange the elements in a length $k$ cycle. Therefore, there are a total of $\binom{n}{k}(k-1)!/(n-k)! = \frac{n!}{k}$ cycles of length $k$ across all permutations in $S_n$. Thus, $\mathbb{E}[X_k] = \frac{n!}{k} = \frac{1}{k}$. This gives an answer of

$$\sum_{k=1}^{n} \mathbb{E}[X_k] = \sum_{k=1}^{n} \frac{1}{k} = H_n,$$

where $H_n$ is the $n$th Harmonic number.

□

Problem 6. Let $I(n, j)$ be the number of permutations in $S_n$ with no cycles of length greater than $j$. Prove the following recurrence identity:

$$I(n+1, j) = \sum_{k=n-j+1}^{n} (n)_{n-k} I(k, j),$$

where $(n) := n(n-1)...(n-k+1)$.

Proof. Let $T_{n,j}$ be the set of permutations in $S_n$ with no cycles of length greater than $j$. We claim that the number of permutations $\sigma \in T_{n+1,j}$ where $n+1$ is in a cycle of length $\ell \leq j$ is $(n)_{\ell-1} I(n-\ell+1, j)$. To get this, note that there are $\binom{n}{\ell-1}$ ways to pick the remaining $\ell - 1$ elements in the same cycle as $n+1$, and $(\ell - 1)!$ ways to arrange these elements within the cycle. Further, there are $I((n+1)-\ell, j)$ ways to permute the remaining $n+1 - \ell$ elements into cycles of length at most $j$. Combining this, there are

$$\binom{n}{\ell-1} (\ell - 1)! I(n+1-\ell, j) = n(n-1)(n-2)...(n-\ell+2) I(n+1-\ell, j) = (n)_{\ell-1} I(n+1-\ell, j)$$

total permutations with $n+1$ in a cycle of length $\ell$. Summing this over all possible $\ell$ will give every potential permutation in $T_{n,j}$, thus we obtain

$$I(n+1, j) = \sum_{\ell=1}^{j} (n)_{\ell-1} I(n+1-\ell, j) = \sum_{k=n-j+1}^{n} (n)_{n-k} I(k, j),$$

where the last equality comes from a change of bounds. □

Problem 7. For $n \in \mathbb{N}$ with $n \geq 2$, let $a(n,k)$ be the number of permutations in $S_n$ with $k$ cycles in which the entries 1 and 2 are in the same cycle. Prove the following identity:

$$\sum_{k=1}^{n} a(n, k)x^k = x(x+2)...(x+n-1).$$

Proof. Let $c(n, k)$ denote the number of permutations in $S_n$ that contain exactly $k$ cycles. We claim that $c(n, k) = a(n, k) + a(n, k-1)$. In other words, it is sufficient to find a bijection between the number of permutations in $S_n$ with exactly $k-1$ cycles that contain both 1 and 2 in a single cycle with those permutations in $S_n$ that have $k$ cycles with 1 and 2 in distinct cycles. Let $X_k$ be the set of permutations with $k$ cycles where 1 and 2 are in distinct cycles and $Y_k$ be the set of permutations with $k$ cycles where 1 and 2
are in the same cycle. Define \( f : Y_{k-1} \to X_k \) as follows. Given a \( y \in Y_{k-1} \), let the cycle 1 and 2 be in be \((1, a_1, ..., a_p, 2, b_1, ..., b_q)\). This cycle can have a unique split at \( a_p \), the element just before 2 in the cycle and also after \( b_q \) to give us two distinct cycles \((1, a_1, ..., a_p)\) and \((2, b_1, ..., b_q)\). Further, this new permutation is an element of \( X_k \), thus every element in \( Y_{k-1} \) has a unique mapping to \( Y_k \) under \( f \).

For the inverse, it is not hard to see that we can take the two distinct cycles 1 and 2 would be in, namely \((1, a_1, ..., a_p)\) and \((2, b_1, ..., b_q)\), and merge them in the same way to get the cycle \((1, a_1, ..., a_p, 2, b_1, ..., b_q)\), thus meaning that \( f \) has an inverse map. As such, \( f \) is a bijective function and \( c(n, k) = a(n, k) + a(n, k-1) \).

It was shown in lecture that
\[
\sum_{k=1}^{n} c(n, k)x^k = x(x+1)...(x+n-1).
\]

Using this, we can break apart \( c(n, k) \) to get the following:
\[
\sum_{k=1}^{n} c(n, k)x^k = c(n, 1)x + \sum_{k=2}^{n} c(n, k)x^k = a(n, 1)x + \sum_{k=2}^{n} (a(n, k) + a(n, k-1))x^k =
\]
\[
\sum_{k=1}^{n} a(n, k)x^k + \sum_{k=2}^{n} a(n, k-1)x^k = \left( \sum_{k=1}^{n} a(n, k)x^k \right) (x+1).
\]

With the fact from lecture, we find that
\[
\sum_{k=1}^{n} a(n, k)x^k = x(x+2)(x+3)...(x+n-1).
\]

\(\square\)

**Problem 8.** Each person in a group of \( n \) friends checks a hat and an umbrella when entering a restaurant. When they leave, each of them is given back at random a hat and an umbrella (from the same set of articles they had already checked upon entrance). In how many ways can none of the friends get back her/his own hat or umbrella?

**Proof.** For \( i \in [n] \), let \( A_i \) denote the set of events where person \( i \) gets both their hat and umbrella back. Using Sieve, we can count the total number of ways at least 1 person gets both their own hat and umbrella back. This gives
\[
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \left| \bigcap_{|I|=k, I \subseteq [n]} A_I \right|.
\]

The \( \binom{n}{k} \) coefficient comes from there being \( \binom{n}{k} \) sets of size \( k \) for which we can take the intersection of for Sieve. Further, the intersection represents \( k \) people getting their articles back, and there are \( (n-k)! \) ways to give back the remaining hats and \((n-k)! \) ways to give back the remaining umbrellas. Therefore, this quantity is
\[
\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!^2 = \sum_{k=1}^{n} (-1)^{k+1} \frac{n!(n-k)!}{k!}.
\]

We want the complement of this expression, so we have to subtract this quantity from the total number of ways the hats and umbrellas can be given back, which is \((n!)^2\). Thus, the answer is
\[
(n!)^2 - \sum_{k=1}^{n} (-1)^{k+1} \frac{n!(n-k)!}{k!} = \sum_{k=0}^{n} (-1)^{k} \frac{n!(n-k)!}{k!} = n! \sum_{k=1}^{n} (-1)^{k} \frac{(n-k)!}{k!}. \quad \square
\]