Problem Set 1 (Due on Wednesday, 09/22)

Problem 1 *Show that at any given moment of this semester, we can choose two students in our class having the same number of friends inside our class.*

Problem 2 *Show that* $\big(n/3\big)^n < n! < (n/2)^n$ *for every* $n \in \mathbb{Z}$ *with* $n \geq 6$.

Problem 3 *Consider the sequence* $(F_n)_{n\geq 0}$ *obtained by setting* $F_0 = 0, F_1 = 1$, *and* $F_n = F_{n-1} + F_{n-2}$ *for every* $n \geq 2$. *Prove that* 18211 *divides* F_n *for some* $n \in \mathbb{N}$. *[This is called the* Fibonacci sequence *and we will learn more about it throughout the course].*

Problem 4 *Let* T *be a triangle with two angles of* $30°$. *Prove that* T *can be subdivided into* n *smaller triangles similar to it for all* $n > 3$.

Problem 5 *For* $n \in \mathbb{N}$ *and* $k \in \mathbb{Z}$ *with* $0 \leq k \leq n$, *let* $N(n,k)$ *be the number of* k*-subsets of* $[n]$ *that do not contain a pair of consecutive integers.*

1. *Prove that* $N(n,k) = \binom{n-k+1}{k}$.
2. *Prove that* $\sum_{k=0}^{n} N(n,k) = F_{n+2}$, *where* F_{n+2} *is the* $(n+2)$*-th term of the Fibonacci sequence.*

Problem 6 *Prove that*

$$\sum_{k\in\mathbb{N}} \binom{2r}{2k-1}\binom{k-1}{s-1} = 2^{2r-2s+1}\binom{2r-s}{s-1}$$

for all $r, s \in \mathbb{N}_0$ *by using a combinatorial argument.*

Problem 7 *What is the number of northeastern lattice paths from* $(0,0)$ *to* (n,n) *that only touch the segment between* $(0,0)$ *and* (n,n) *at its endpoints?*

Problem 8 *In the decimal representation of* $(\sqrt{2}+\sqrt{3})^{2020}$, *what digit is immediately on the right of the decimal point?*