COMBINATORIAL ANALYSIS

Problem Set 1 Solutions (MIT, Fall 2021)

Problem 1. *Show that at any given moment of this semester, we can choose two students in our class having the same number of friends inside our class.*

Solution. Let S be the set of students in our class, and set $n := |S|$. If one student s_0 does not have any friend in the class, then each of the $n-1$ students in $S \setminus \{s_0\}$ has at most $n-2$ friends in the class, and it follows by the PHP that two of the students in $S \setminus \{s_0\}$ have the same number of friends. On the other hand, if each student has at least one friend, then the number of friends of each of the n students is a number in $[n-1]$, and once again it follows from the PHP that two of them must have the same number of friends inside our class. $\square$

Problem 2. *Show that* $(n/3)^n < n! < (n/2)^n$ *for every* $n \in \mathbb{Z}$ *with* $n \geq 6$.

Solution. First, note that $(6/3)^6 = 64 < 720 = 6!$ and $6! = 720 < 729 = (6/2)^6$. Now assume that $\big(n/3\big)^n < n! < (n/2)^n$ for some $n \in \mathbb{N}$ with $n \geq 6$. Recall from calculus that the sequence $(1+1/n)^n$ increases and $\lim_{n\to\infty} \big(1+\frac{1}{n}\big)^n = e$ (the increasing part can be either taken for granted or proved using Bernoulli's inequality, namely, $(1+x)^n \geq 1+nx$ for every $x > -1$). Therefore

$$2 < \Big(1 + \frac{1}{n}\Big)^n < 3 \tag{0.1}$$

for every $n \geq 2$. From the right inequality of (0.1), we obtain that $(n+1)^n < 3n^n$, and so

$$\Big(\frac{n+1}{3}\Big)^{n+1} = \frac{n+1}{3^{n+1}}(n+1)^n < (n+1)\Big(\frac{n}{3}\Big)^n < (n+1)!,$$

where the last inequality follows from our induction hypothesis. On the other hand, observe that the left inequality of (0.1) ensures that $2n^n < (n+1)^n$. As a consequence, we obtain that

$$(n+1)! < (n+1)\Big(\frac{n}{2}\Big)^n = \frac{n+1}{2^{n+1}}2n^n < \frac{n+1}{2^{n+1}} = \Big(\frac{n+1}{2}\Big)^{n+1},$$

where the first inequality follows from our induction hypothesis. $\square$

Problem 3. *Consider the sequence* $(F_n)_{n\geq 0}$ *obtained by setting* $F_0 = 0, F_1 = 1$, *and* $F_n = F_{n-1} + F_{n-2}$ *for every* $n \geq 2$. *Prove that* 18211 *divides* F_n *for some* $n \in \mathbb{N}$. *[This is called the* Fibonacci sequence *and we will learn more about it throughout the course].*

Solution. Since the set $\{(r_1, r_2) \mid 0 \le r_1, r_2 < 18211\}$ has size 18211^2, it follows from the PHP that there exist $i, j \in [18211^2 + 1]$ with $i < j$ such that $F_i \equiv F_j \pmod{18211}$ and $F_{i+1} \equiv F_{j+1} \pmod{18211}$. Then

$$F_{i-1} = F_{i+1} - F_i \equiv F_{j+1} - F_j = F_{j-1} \pmod{18211}.$$

In a similar way, we can verify that $F_{i-2} \equiv F_{j-2} \pmod{18211}$, and we can continue in this fashion until we obtain that $0 = F_0 \equiv F_{j-i} \pmod{18211}$. Hence F_{j-i} is a Fibonacci number divisible by 18211. □

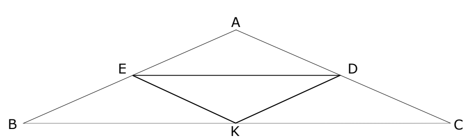
Problem 4. *Let T be a triangle with two angles of $30°$. Prove that T can be subdivided into n smaller triangles similar to it for all $n > 3$.*

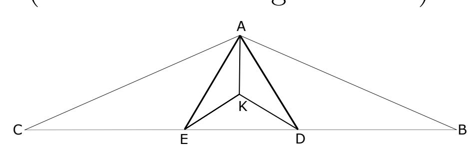
Solution. Let A, B, and C be the vertices of T.

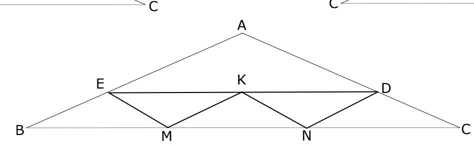
For $n = 4$, consider the subdivision obtained by drawing the triangle $\triangle EDK$, where E, D, and K are the middle points of the segments AB, AC, and BC, respectively (see the top-left figure below).

For $n = 5$, take E and D in the segment CB such that $\angle CAE = \angle DAB = 30°$. Now draw the regular triangle $\triangle AED$, and then draw three segments from the centroid K of $\triangle AED$ to its vertices. This gives us a subdivision of T into five triangles similar to itself (see the top-right figure below).

For $n = 6$, take E and D in the segments AB and AC, respectively, so that $|EB| = \frac{1}{2}|AE|$ and $|DC| = \frac{1}{2}|AD|$. Let K be the middle point of the segment ED. Take M and N in BC satisfying that $|BM| = |MN| = |NC|$. It is easy to check that the triangulation one obtains by drawing the triangles $\triangle EKM$ and $\triangle KDN$ is a subdivision of T into six triangles similar to itself (see the bottom figure below).







Taking the previous cases as base cases, we can proceed by induction. Assume that we can find a desired subdivision of T for every $k \in [\![4, n]\!]$ for some $n \ge 6$. To subdivide T into $n + 1$ triangles similar to itself, we can first subdivide T into $n - 2 \ge 4$ triangles similar to itself, and then we can subdivide one of the triangles of such a subdivision into four triangles similar to T (as in the case when $n = 4$). □

Problem 5. *For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $0 \le k \le n$, let $N(n, k)$ be the number of k-subsets of $[n]$ that do not contain a pair of consecutive integers.*

(1) *Prove that $N(n, k) = \binom{n-k+1}{k}$.*

(2) *Prove that* $\sum_{k=0}^{n} N(n,k) = F_{n+2}$*, where* F_{n+2} *is the* $(n+2)$*-th term of the Fibonacci sequence.*

Solution. (1) Let $T(n,k)$ be the collection of k-subsets of $[n]$ that do not contain a pair of consecutive integers. Define $f\colon T(n,k) \to \binom{[n-k+1]}{k}$ as follows: if $S = \{s_1, \dots, s_k\} \in T(n,k)$ with $s_1 < \cdots < s_k$, then let $f(S) = \{s_1 - 1, s_2 - 2, \dots, s_k - k\}$ and note that the fact that S does not contain two consecutive integers ensures that $|f(S)| = k$. Conversely, we can define $g\colon \binom{[n-k+1]}{k} \to T(n,k)$ as follows: for any $S' = \{s'_1, \dots, s'_k\} \in \binom{[n-k+1]}{k}$ with $s'_1 < \cdots < s'_k$, let $g(S') = \{s'_1 + 1, s'_2 + 2, \dots, s'_k + k\}$ and observe that $1 \le s'_1 < \cdots < s'_k \le n - k + 1$ guarantees that $g(S')$ is a k-subset of $[n]$ that does not contain any two consecutive elements. Finally, one can readily check that f and g are inverses of each other, and so

$$N(n,k) = |T(n,k)| = \binom{n-k+1}{k}.$$

(2) It is clear that $\sum_{k=0}^{n} N(n,k)$ is the size of the set $T(n)$ consisting of all the subsets of $[n]$ that do not contain a pair of consecutive integers. Let us show by induction that $|T(n)| = |F_{n+2}|$. When $n = 1$ none of the two subsets of $[1]$ contains a pair of consecutive elements, and so $|T(1)| = 2 = F_3$. In addition, only one of the four subsets of $[2]$, namely $\{1,2\}$, contains a pair of consecutive integers, and so $|T(2)| = 3 = F_4$. Now suppose that, for some $n \in \mathbb{N}$, the equality $|T(k)| = F_{k+2}$ holds for every $k \le n$. Observe that there are $|T(n)| = F_{n+2}$ subsets in $T(n+1)$ that do not contain $n+1$, those in $T(n)$, and there are $|T(n-1)|$ sets in $T(n+1)$ containing $n+1$, those containing $n+1$ that belong to $T(n-1)$ when $n+1$ is dropped. Hence $|T(n+1)| = |T(n)| + |T(n-1)| = F_{n+1} + F_{n+2} = F_{n+3}$, which completes our inductive argument. $\square$

Problem 6. *Prove that*

$$\sum_{k \in \mathbb{N}} \binom{2r}{2k-1}\binom{k-1}{s-1} = 2^{2r-2s+1}\binom{2r-s}{s-1} \tag{0.2}$$

for all $r, s \in \mathbb{N}_0$ *by using a combinatorial argument.*

Solution. Suppose we have $2r$ delegates labeled $1, 2, \dots, 2r$, from which we choose an odd-size committee $p_1, \dots, p_{2k-1}$, where $p_1 < \cdots < p_{2k-1}$, and then we choose a sub-committee of size $s-1$ consisting ot some of the committee members $p_2, \dots, p_{2k-2}$. We can clearly do this $\sum_{k\in\mathbb{N}} \binom{2r}{2k-1}\binom{k-1}{s-1}$ different ways, which is the left-hand side of (0.2).

Let us argue that the right-hand side of (0.2) also counts the pair of committees and sub-committees we have just described. This time we first choose a size-$(s-1)$ sub-committee $b_1, \dots, b_{s-1}$ with $2 \le b_1 < \cdots < b_{s-1} \le 2r - 1$ such that not two of the labels $b_1, \dots, b_{s-1}$ are consecutive; this can be done in $\binom{(2r-2)-(s-1)+1}{s-1} = \binom{2r-s}{s-1}$ (see Problem 5.1 above). Now we enhance the chosen sub-committee to obtain the desired committee taking into account that the desired committee must satisfy the following conditions:

(1) the committee must have an odd number of delegates,
(2) every member of the sub-committee must occupy an even position in the line we obtain by organizing the members of the committee increasingly by labels.

Notice that achieving this amounts to choosing a subset of odd size from the delegates labeled by $[\![1, b_1 - 1]\!]$ in 2^{b_1-2} different ways, then for every $j \in [s-2]$ a subset of odd size from the delegates labeled by $[\![b_j + 1, b_{j+1} - 1]\!]$ in $2^{b_{j+1}-b_j-2}$ different ways, and finally a subset of odd size from the delegates labeled by $[\![b_{s-1} + 1, 2r]\!]$ in $2^{2r-b_{s-1}-1}$ different ways. Therefore the number of desired pairs of committees and sub-committees is

$$2^{(b_1-2)+\left(\sum_{j=1}^{s-2}(b_{j+1}-b_j-2)\right)+(2r-b_{s-1}-1)}\binom{2r-s}{s-1} = 2^{2r-2s+1}\binom{2r-s}{s-1},$$

which is the right-hand side of (0.2). Hence the identity (0.2) holds. $\square$

Problem 7. *What is the number of northeastern lattice paths from* $(0,0)$ *to* (n,n) *that only touch the segment between* $(0,0)$ *and* (n,n) *at its endpoints?*

Solution. The number C_n of lattice paths from $(0,0)$ to (n,n) below (and possibly repeatedly touching) the line $y = x$ is $C_n = \frac{1}{n+1}\binom{2n}{n}$, which is called the n-th Catalan number (see the solution of Exercise 4.24 in the textbook).

Let E_n be the set of lattice paths from $(0,0)$ to (n,n) whose first unit step is the vector $(1,0)$ and that only touch the line $y = x$ at $(0,0)$ and $(1,1)$. By symmetry, the number we want to determine is $2|E_n|$. Since the last unit step of each lattice path in E_n must be $(0,1)$, the set E_n is in bijection with the set D'_{n-1} consisting of all lattice paths from $(1,0)$ to $(n, n-1)$ that do not go strictly above the line $y = x - 1$: the bijections consists in dropping the first and the last steps. In addition, the set D'_{n-1} is in bijection with the set D_{n-1} consisting of all lattice paths from $(0,0)$ to $(n-1, n-1)$: the bijection consists in translating each lattice path by $(-1,0)$, a unit back. By the remark given in the previous paragraph,

$$2|E_n| = 2|D_{n-1}| = 2\frac{1}{(n-1)+1}\binom{2(n-1)}{n-1} = \frac{2}{n}\binom{2n-2}{n-1}.$$

$\square$

Problem 8. *In the decimal representation of* $(\sqrt{2}+\sqrt{3})^{2020}$*, what digit is immediately on the right of the decimal point?*

Solution. Instead of 2020, we will fix any positive even power $2n$. First, we can use the Binomial Theorem to see that the sum $N := (\sqrt{3}+\sqrt{2})^{2n} + (\sqrt{3}-\sqrt{2})^{2n}$ is an integer:

$$\begin{aligned} N &= \sum_{j=0}^{2n} \binom{2n}{j} \big(\sqrt{3}\big)^{j} \big(\sqrt{2}\big)^{2n-j} + \sum_{j=0}^{2n} \binom{2n}{j} \big(\sqrt{3}\big)^{j} \big(-\sqrt{2}\big)^{2n-k} \\ &= \sum_{j=0}^{n} \binom{2n}{2j} \big(\sqrt{3}\big)^{2j} \big(\sqrt{2}\big)^{2(n-j)} \in \mathbb{N}. \end{aligned}$$

Now observe that $(\sqrt{3}-\sqrt{2})^{2n} = \big(\frac{1}{\sqrt{3}+\sqrt{2}}\big)^{2n} < \frac{1}{2^{2n}} < 0.1$, where the last equality holds as long as $2n > \log_2 10$ (which is clearly the case of $2n = 2020$). Since $(\sqrt{3}+\sqrt{2})^{2n} = N - (\sqrt{3}-\sqrt{2})^{2n} > N - 0.1$, we can conclude that in the decimal expression of $(\sqrt{3}+\sqrt{2})^{2n}$ the digit immediately to the right of the decimal point is 9. $\square$