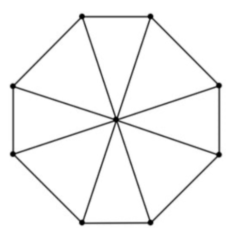
Practice Midterm 4

Problem 1 *Find the chromatic polynomial of a tree with n vertices.*

Solution. We can prove by induction that the chromatic polynomial of a tree on n vertices is $p_n(x) = x(x-1)^{n-1}$. It is clear when $n = 1$ as in this case $p_1(x) = x$ (the tree consists of only one vertex, and so it has exactly k proper k-colorings for every $k \in \mathbb{N}$). Suppose now that $p_n(x) = x(x-1)^{n-1}$ for any tree of n vertices, and let T be a tree on $n+1$ vertices. We can find the number of proper k-colorings of T as follows: remove a leaf v from T to obtain a tree $T \setminus \{v\}$ on n vertices. Then properly k-color $T \setminus \{v\}$ in $p_n(k)$ different ways (here we use the induction hypothesis), and for each of these colorings obtain a proper k-coloring of T by coloring v with any of the $k-1$ colors not used by its adjacent vertex in T. Therefore the number of proper k-colorings of T is $(k-1)p_n(k) = (k-1)k(k-1)^{n-1} = k(k-1)^n$. Thus, the chromatic polynomial of T is $p_{n+1}(x) = x(x-1)^n$. $\square$

Problem 2 *Find the chromatic number and the chromatic polynomial of the 9-vertex wheel graph:*



Solution. Let us use the fact that the chromatic polynomial of the cycle graph C_n is

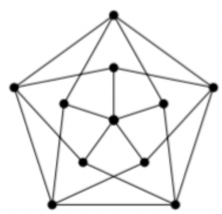
$$q_n(x) = (x-1)^n + (-1)^n(x-1).$$

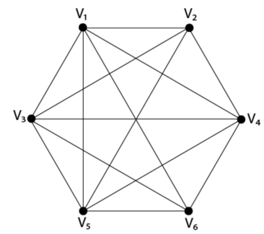
Well, you may have already proved this as part of PS6 (you should feel free tomorrow to use any results in any of our lectures or problem sets (including PS6)). Let W_n denote the wheel graph on n vertices. To obtain a proper k-coloring of W_n, we need to choose one of the k colors for the central vertex v_0 of W_n and, as this vertex is adjacent to the rest of the vertices of W_n, then choose a proper $(k-1)$-coloring of $W_n \setminus \{v_0\}$. Since $W_n \setminus \{v_0\} = C_{n-1}$, the number of proper $(k-1)$-colorings of $W_n \setminus \{v_0\}$ is $q_{n-1}(k-1) = (k-2)^{n-1} + (-1)^{n-1}(k-2)$. Therefore the number of proper k-colorings of W_n is $kq_{n-1}(k-1) = k((k-2)^{n-1} + (-1)^{n-1}(k-2))$, and so the chromatic polynomial of W_n is

$$p(x) = x((x-2)^{(n-1)} + (-1)^{n-1}(x-2)).$$

Problem 3

(1) For each of the two graphs below, determine if it is planar? Justify your answer.

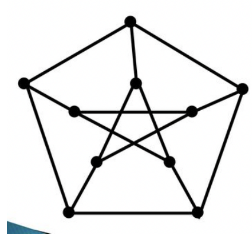


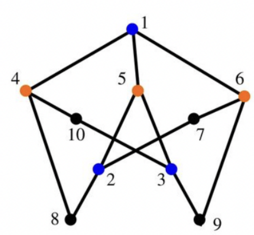


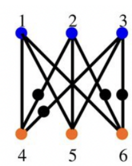
(2) Show that the Petersen graph is not planar.

Solution. (1) None of them is planar by virtue of Kuratwoski's theorem. If we remove the central vertex from the graph on the left, then we obtain a graph that is edge-equivalent to the complete graph K_5. On the other hand, if we remove the edges $v_1v_3, v_3v_5, v_1v_5, v_2v_4$, and v_4v_6 from the graph on the right, then we obtain the complete bipartite graph $K_{3,3}$.

(2) It also follows from Kuratwoski's theorem that the Petersen graph is not planar as it contains a subgraph that is edge equivalent to $K_{3,3}$. This is illustrated in the following pictures.







See also the short video in the link: https://www.youtube.com/watch?v=SbjIdEbd754

□

Problem 4 *Prove that if G is a simple connected planar graph, then the inequality $|E(G)| \leq 3|V(G)| - 6$ holds.*

Solution. Set $E := |E(G)|$ and $V := |V(G)|$. Draw G on the plane such that no two edges of G cross each other. Since G is simple, it has neither loops nor cycles of length 2 and, therefore, the boundary of every face of G consists of at least 3 edges. Hence $3F \leq \sum_{i=1}^{F} E_i = 2E$, where $E_1, \dots, E_F$ are the numbers of edges in the F faces of G (we did this in the lecture on polytopes). Now using Euler's formula, we obtain that

$$E = V + F - 2 \leq V + \frac{2}{3}E - 2,$$

which implies that $\frac{E}{3} \leq V - 2$ or, equivalently, $E \leq 3V - 6$. $\square$

Problem 5 *Prove that in a polytope, there are two vertices having the same number of adjacent vertices.*

Solution. This is Exercise 12.4 in the textbook, which is one of the solved exercises. $\square$