Practice Midterm 3

Problem 1 *Consider the simple graph G with*

$$V(G) = \{1, 2, \dots, 30\} \quad \textit{and} \quad E(G) = \{ij \mid i \le 10 < j\}.$$

Find the number of Hamiltonian cycles of G.

Solution. The number of Hamiltonian cycles of G is 0. To argue this, set $A = [10]$ and $B = [30] \setminus [10]$. Since not two vertices in A (or in B) are adjacent in G, any cycle of G must alternate vertices of A and B and, therefore, must be a cycle of even length passing through the same number of vertices of A and B. Thus, the fact that $|A| < |B|$ implies that no cycle of G can pass through all the vertices of B and, as a result, G cannot contain any Hamiltonian cycle. □

Problem 2 *Prove the following statements.*

(1) In any tree, any two longest paths cross each other (i.e., share at least a vertex).

(2) A tree with no vertex of degree 2 has more leaves than non-leaf vertices.

Solution. (1) Let T be a tree. Suppose, towards a contradiction, that P_1 and P_2 are two disjoint paths in T with maximum length possible. For a path P, we let $\ell(P)$ denote the length of P. Since T is connected, there must be a minimum-length path P_0 in T starting at a vertex v_1 of P_1 and ending at a vertex v_2 of P_2. The minimality of P_0 ensures that it does not share any edges with neither P_1 nor P_2. Among the two paths we obtain by cutting P_1 (resp., P_2) at v_1 (resp., v_2), let P_1' (resp., P_2') be one with bigger length (note that they may have the same length). Then $P_1'P_0P_2'$ (i.e., the concatenation of P_1', P_0, and P_2') is a path in T satisfying

$$\ell(P_1'P_0P_2') = \ell(P_1') + \ell(P_0) + \ell(P_2') \ge \frac{1}{2}\ell(P_1) + 1 + \frac{1}{2}\ell(P_2) = \frac{1}{2}\ell(P_1) + 1 + \frac{1}{2}\ell(P_1) > \ell(P_1),$$

which contradicts the fact that P_1 is one of the longest paths of T.

(2) Let T be a tree. Set $n := |V(T)|$ and assume that T has no vertices of degree 2. Let L and N denote the sets of leaves and non-leaves of G, respectively, and set $m = |L|$. Since T has no vertices of degree 2, we see that

$$2(n-1) = 2|E(T)| = \sum_{v \in L} \deg v + \sum_{v \in N} \deg v \ge m + 3(n-m) = 3n - 2m,$$

that is, $2n - 2 \geq 3n - 2m$. This inequality implies that

$$|L| = m \geq (n - m) + 2 > n - m = |N|,$$

which concludes our proof. $\square$

Problem 3 *Let G be a simple connected graph with weight function $\omega\colon E(G) \to \mathbb{R}_+$, and assume that ω is injective. If C is a cycle in G and e is the heaviest edge in C, prove that no minimum-weight spanning tree of G contains e.*

Solution. Suppose, towards a contradiction, that T is a minimum-weight spanning tree of G containing the heaviest edge $e = uv$ of a cycle C in G. Let P denote the path from u to v that we obtain after removing e from C. The tree T' obtained from T by removing the edge e has two connected components, namely, T_u and T_v. Assume, without loss of generality, that $u \in V(T_u)$ and $v \in V(T_v)$. As P connects u to v, there is an edge e' in P connecting T_u and T_v. Since ω is injective, the maximality of e ensures that $\omega(e') < \omega(e)$. However, in this case, the spanning tree of G we obtain from T by replacing e by e' would have strictly less weight than T does, which contradicts the fact that T is a minimum-weight spanning tree. $\square$

Problem 4 *Explain how to count the number of 3-cycles of a simple graph using its adjacency matrix.*

Solution. Let G be a single graph with $|V(G)| = n$, and label the vertices of G by $1, 2, \dots, n$. Let A be the adjacency matrix of G. Observe that every closed walk of length 3 yields a 3-cycle (note that this is not true, in general, if we replace 3 by some $\ell \in \mathbb{N}$). Therefore there is a bijection between closed walks of length 3 and 3-cycles with a distinguished vertex (the starting vertex of the walk). Thus, we can start by counting the closed walks of length 3. We know that $A^3_{i,j}$ equals the number of walks of length 3 from the vertex i to the vertex j (for any $i, j \in [n]$) and, in particular, $A^3_{i,i}$ equals the number of walks of length 3from i to itself (for each $i \in [n]$). As a result, the number of 3-cycles of G is

$$\frac{A^3_{1,1} + A^3_{2,2} + \cdots + A^3_{n,n}}{3},$$

where we have divided by 3 because we want to count cycles (instead of cycles with distinguished vertices). $\square$