Practice Midterm 2

Problem 1 *For each $n \in \mathbb{N}$, let r_n be the number of permutations $\pi \in S_n$ such that π^2 is the identity permutation (here π^2 means π composed with itself as a function). Prove that $r_{n+1} = r_n + nr_{n-1}$ for every $n \geq 2$.*

Solution. For each $n \in \mathbb{N}$, let T_n be the set of permutations in S_n whose square is the identity. Observe first that if $(a_1\, a_2\, \dots\, a_\ell)$ is a cycle of length greater than 1 in the disjoint cycle decomposition of $\pi \in T_n$, then the fact that $\pi(a_2) = \pi^2(\pi(a_1)) = a_1$ implies that $\ell = 2$. Thus, every cycle in the disjoint cycle decomposition of π has length either 1 or 2. The subset of T_{n+1} whose elements has $(n+1)$ as a cycle is in bijection with T_n, where the bijection consists in dropping the cycle $(n+1)$. On the other hand, we can construct any permutation in T_{n+1} in which the cycle containing n has length greater than one (that is, has length 2) as follows: construct the two-cycle $(j, n+1)$ containing $n+1$ in n different ways, and then make the set $[n] \setminus \{j\}$ into a cycle type decomposition in r_{n-1} different ways. Hence $r_{n+1} = r_n + nr_{n-1}$. □

Problem 2 *In how many ways can we roll a die 8 consecutive times such that all six faces appear at least once?*

Solution. For each $j \in [6]$, let A_j be the set consisting of 8 consecutive rolls of the given die such that the face labeled by j never shows. Observe that if S is a k-subset of [6], then the size of $\bigcap_{j \in S} A_j$ is the number of sequences of 8 consecutive rolls in which none of the elements in S shows, that is $(6-j)^8$. Then it follows from the Sieve Method that

$$\Big|\bigcup_{n=1}^{6} A_n\Big| = \sum_{\emptyset \neq S \subseteq [6]} (-1)^{|S|+1} \Big|\bigcap_{j \in S} A_j\Big| = \sum_{k=1}^{6} \sum_{S \subseteq [6] : |S| = k} (-1)^{|S|+1} \Big|\bigcap_{j \in S} A_j\Big|$$
$$= \sum_{k=1}^{6} \binom{6}{k} (-1)^{k+1} (6-j)^8.$$

Since there are a total of 6^8 sequence of 8 consecutive rolls of the given die, the number we are looking for is

$$6^8 - \Big|\bigcup_{n=1}^{6} A_n\Big| = 6^8 - \sum_{k=1}^{6} \binom{6}{k} (-1)^{k+1} (6-j)^8 = \sum_{k=0}^{6} \binom{6}{k} (-1)^k (6-j)^8.$$

□

Problem 3 *For $n \in \mathbb{N}_0$, let f_n be the number of ways we can have n cents in pennies, nickels, and quarters using at most five nickels. Find the explicit ordinary generating function for $(f_n)_{n\ge 0}$.*

Solution. Let $F(x)$ be the generating function of $(f_n)_{n\ge 0}$. Let a_n and c_n be the number of ways to have n cents in pennies and quarters, respectively, and let b_n be the number of ways to have n cents in at most five nickels. Let $A(x)$, $B(x)$, and $C(x)$ be the generating functions of $(a_n)_{n\ge 0}$, $(b_n)_{\ge 0}$, and $(c_n)_{n\ge 0}$, respectively. Then

$$A(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{and} \quad C(x) = \sum_{n=0}^{\infty} x^{25n} = \frac{1}{1-x^{25}}.$$

In addition, $B(x) = 1 + x^5 + x^{10} + x^{15} + x^{20} + x^{25}$. Then it follows from the product formula that

$$F(x) = A(x)B(x)C(x) = \frac{1 + x^5 + x^{10} + x^{15} + x^{20} + x^{25}}{(1-x)(1-x^{25})}.$$

□

Problem 4 *The sequence $(a_n)_{n\ge 0}$ satisfies $a_0 = 1$ and $a_{n+1} = 3a_n + 2^n$ for every $n \in \mathbb{N}_0$. Find an explicit formula for a_n.*

Solution. Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $(a_n)_{n\ge 0}$. Since $a_0 = 1$,

$$F(x) - 1 = \sum_{n=0}^{\infty} a_{n+1}x^{n+1} = 3x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 2^n x^n = 3xF(x) + \frac{x}{1-2x}.$$

After solving for $F(x)$, we obtain

$$F(x) = \frac{2}{1-3x} - \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2\cdot 3^n x^n - \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty}(2\cdot 3^n - 2^n)x^n.$$

Hence $a_n = 2\cdot 3^n - 2^n$. □

Problem 5 *For each $n \in \mathbb{N}$, let c_n be the number of ways to subdivide a group of n delegates into committees of sizes at least 3, and then select a leader in each committee. Assume that $c_0 = 1$, and find the exponential generating function of $(c_n)_{n\ge 0}$.*

Solution. Let $C(x)$ be the exponential generating function of $(c_n)_{n\ge 0}$. Let a_k be the number of ways to build a committee of size at least 3 using all given k delegates. It is clear that $a_0 = a_1 = a_2 = 0$. On the other hand, $a_n = n$ for every $n \ge 3$. Let $A(x)$ be the exponential generating function of $(a_n)_{n\ge 0}$. Observe that

$$A(x) = \sum_{n=3}^{\infty} n\frac{x^n}{n!} = x\sum_{n=3}^{\infty} \frac{x^{n-1}}{(n-1)!} = x\bigg(\sum_{n=2}^{\infty} \frac{x^n}{n!}\bigg) = x\big(e^x - x - 1\big).$$

Hence, in light of the composition formula for exponential generating functions, after taking $B(x) = \sum_{n=0}^{\infty} x^n/n! = e^x$ we obtain that

$$C(x) = B(A(x)) = e^{x(e^x - x - 1)}.$$