COMBINATORIAL ANALYSIS

Practice Midterm 1 (MIT, Fall 2021)

This practice midterm will give you an idea about the format, length, and difficulty level of the first midterm. As the actual midterm, this practice midterm should be taken as a closed-book exam with a duration of 50 minutes.

Problem 1. *Prove that there exists* $n \in \mathbb{N}$ *such that* 211 *divides* $18^n - 1$.

Proof. Consider the positive integers $18 - 1, 18^2 - 1, \dots, 18^{212} - 1$. Since there are only 211 remainders modulo 211, the PHP guarantees the existence of $i, j \in [212]$ with $i \neq j$ such that $18^i - 1 \equiv 18^j - 1 \pmod{211}$ (i.e., $18^i - 1$ and $18^j - 1$ leave the same remainder modulo 211). Assume, without loss of generality, that $i < j$. Because 211 divides $18^j - 18^i = 18^i(18^{j-i} - 1)$, the fact that 18^i and 211 are relatively prime (note that neither 2 nor 3 divide 211) ensures that 211 divides $18^{j-i} - 1$, which concludes our argument. $\square$

Problem 2. *Using a combinatorial argument, prove that*

$$\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$$

for every $m, n \in \mathbb{N}$ *with* $m \leq n$.

Proof. Suppose that in a class of n students we want to select a group of m students and then make some of them leaders of the group. We can create the group by choosing m students out of the whole class in $\binom{n}{m}$ ways, and then we can select a subset of leaders in the already-chosen group of m students in 2^m ways. Hence we can perform our task in $2^m\binom{n}{m}$ ways, which is the right-hand side of the desired identity. We can also do the same as follows. First we choose, for each $k \in [\![0, m]\!]$, a group of k leaders in $\binom{n}{k}$ ways, and then we enlarge the size of the chosen group of leaders to a group of m students by choosing the remaining $m - k$ non-leader out of the remaining part of the class, which can be done in $\binom{n-k}{m-k}$ ways. Hence we have performed the same task in $\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k}$ different ways, which is precisely the left-hand side of the desired identity. $\square$

Problem 3. *For every* $n \in \mathbb{N}$ *with* $n \geq 2$, *prove that the number of compositions of* n *into parts greater than* 1 *is* F_{n-1}, *the* $(n-1)$*-th Fibonacci term.*

Proof. For each $n \in \mathbb{N}$ with $n \geq 2$, let T_n be the set of compositions of n into parts greater than 1, and set $t_n := |T_n|$. Observe that the subset A_n of T_n consisting of partitions whose first part is at least 3 is in bijection with T_{n-1}, where the bijection consists in subtracting 1 to the first part. So $|A_n| = |T_{n-1}|$. On the other hand, the subset B_n of T_n consisting of partitions whose first part is 2 is in bijection with T_{n-2}, where the bijection consists in dropping the first part. Thus, $|B_n| = |T_{n-2}|$. Since $\{A_n, B_n\}$ is a partition of T_n, it follows that

$$t_n = |T_n| = |T_{n-1}| + |T_{n-2}| = t_{n-1} + t_{n-2}.$$

It is clear that $t_2 = 1 = F_1$ and $t_3 = 1 = F_2$. Finally, note that if $t_k = F_{k-1}$ for every $k \in [\![2, n]\!]$, then

$$t_{n+1} = t_n + t_{n-1} = F_{n-1} + F_{n-2} = F_n.$$

Hence it follows by induction that $|T_n| = F_{n-1}$ for every $n \geq 2$, as desired. □

Problem 4. *Using only the combinatorial definition of Stirling numbers of the second kind, find a formula for $S(n, n-2)$ for $n \geq 3$.*

Proof. Let π be a partition of $[n]$ into $n-2$ blocks. We first note that π must have at least one block of size larger than 1. Observe, in addition, that π cannot have any block of size s larger than 3 as otherwise $n \geq s + n - 3 > 3 + (n-3) = n$. Therefore π must have either a block of size 2 or a block of size 3.

If π has a block of size 2, then the remaining blocks of the partition form a partition π' of $n-2$ into $n-3$ blocks, and we have seen before that π' must consist of a block of size 2 and $n-4$ blocks of size 1. Hence if π has a block of size 2, then π must consist of two blocks of size 2 and $n-4$ blocks of size 1. As we can choose the first block of size 2 in $\binom{n}{2}$ ways and the second block of size 2 in $\binom{n-2}{2}$ ways, there must be $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$ partitions of $[n]$ into $n-2$ blocks having a block of size 2 (we have divided by 2 to compensate for double counting).

If π has no block of size 2, then π must have a block of size 3, in which case the remaining blocks of π are completely determined: they are $n-3$ blocks of size 1. Hence there are $\binom{n}{3}$ partitions of $[n]$ into $n-2$ having no block of size 2. Thus, the formula we are looking for is $S(n, n-2) = \frac{1}{2}\binom{n}{2}\binom{n-2}{2} + \binom{n}{3}$. □

Problem 5. *Prove that the number of partitions of n into at most k parts equals the number of partitions of $n+k$ into k parts.*

Proof. Let $P_{\leq k}(n)$ denote the set of partitions of n into at most k parts, and let $P_k(n+k)$ denote the set of partitions of $n+k$ into k parts. Define $f \colon P_{\leq k}(n) \to P_k(n+k)$ as follows. For each $p \in P_{\leq k}(n)$ with Ferrer diagram F, let $f(p)$ be the partition corresponding to the Ferrer diagram we obtain by attaching a first column of size k to F. As the number of parts of $f(p)$ is given by the first column of its Ferrer diagram, $f(p)$ has exactly k parts, and so $f(p) \in P_k(n+k)$. Although it is not hard to verify that f is both injective and surjective, we can show that f is a bijection by explicitly verifying that it has an inverse function. Observe that the function $g\colon P_k(n+k) \to P_{\leq k}(n)$ consisting in removing the first column of the corresponding Ferrer diagram is the inverse function of f. Hence f is a bijection, which implies that $|P_{\leq k}(n)| = |P_k(n+k)| = p_k(n+k)$. $\square$