Midterm 3 (Solutions by Daniil Kliuev)

Problem 1 *For $m, n \in \mathbb{N}$, let $K_{m,n}$ be the simple graph with $V(K_{m,n}) = [m+n]$ and $E(K_{m,n}) = \{ij \mid i \in [m]$ and $j \in [m+n] \setminus [m]\}$. Find the number of Hamiltonian cycles of $K_{m,n}$.*

Solution. Let $A = [m]$, $B = [m+n] \setminus [m]$ be two components of complete bipartite graph $K_{m,n}$.

Let $C = u_1 \dots u_{m+n}$ be a Hamiltonian cycle in $K_{m,n}$. We can assume that $u_1 \in A$. Since each edge goes from A to B we get that $u_i \in A$ for odd i and $u_i \in B$ for even i. Since $u_1 u_{m+n}$ is an edge it follows that $m+n$ is even. It follows that half of vertices in C belong to A and half of vertices belong to B. Since C goes through all vertices of $K_{m,n}$ we have $|A| = |B|$, so $m = n$.

Hence in the case when $m \neq n$ the answer is zero.

In the case $m = n = 1$ the answer is also zero.

Suppose that $m = n > 1$. We can write each Hamiltonian cycle as $1 = u_1 u_2 \dots u_{2m}$. We count the number of Hamiltonian cycles as follows. Vertex u_2 is any element of B, so there are m choices for u_2. Vertex u_3 is any element of A except u_1, so there are $m-1$ choice for u_3. More generally, u_{2i} is any element of B except $u_2, \dots, u_{2i-2}$, so there are $(m+1-i)$ choices for u_{2i} and u_{2i+1} is any element of A except $u_1, u_3, \dots, u_{2i-1}$, so there are $m - i$ choices for u_{2i+1}.

This gives us $m!(m-1)!$ choices for the sequence $(u_2, \dots, u_{2m})$. Any Hamiltonian cycle can be written in two ways: $1u_2 \dots u_{2m}$ and $1u_{2m} \dots u_2$, so the answer is

$$\frac{m!(m-1)!}{2}.$$

□

Problem 2 *Let G be a simple connected graph. Prove that any two paths in G of maximum length have a vertex in common.*

Solution. Let $P_1 = u_1 u_2 \dots u_k$, $P_2 = v_1 v_2 \dots v_k$ be two paths of maximum length k in G. Suppose they don't have any common vertex. Consider all paths that have start at $V(P_1)$ and end at $V(P_2)$, and choose a path of minimal length $P_0 = u_i w_1 \dots w_l v_j$, where $l \geq 0$.

We claim that $w_1, \dots, w_l$ do not belong to $V(P_1)$. Assume that this is not the case, $w_s \in V(P_1)$ for some $1 \leq s \leq l$. Then $w_s \dots w_l v_j$ is a path that starts at $V(P_1)$, ends at $V(P_2)$ and has smaller length than P_0, contradiction. Similarly we prove that $w_1, \dots, w_l$ do not belong to $V(P_2)$.

It follows that $P_3 = u_1 \dots u_i w_1 \dots w_l v_j \dots v_1$, $P_4 = u_k \dots u_i w_1 \dots w_l v_j \dots v_1$ are simple paths. They have length $l(P_3) = i + l + j$, $l(P_4) = (k - i + 1) + l + (k - j + 1)$. Hence $l(P_3) + l(P_4) = 2k + 2 + 2l \geq 2k + 2$. We deduce that P_3 or P_4 have length at least $k + 1$. Therefore P_1 and P_2 are not paths of maximum length, a contradiction. $\square$

Problem 3 *For $n \in \mathbb{N}$ with $n \geq 3$, let G_n be the directed multi-graph satisfying the following two conditions:*

- *$|V(G_n)| = n$ and $|E(G_n)| = 2n$, and*
- *the vertices of G_n are arranged around a circle and, for every vertex of G_n, there is an arrow to its clockwise neighbor and an arrow to its counterclockwise neighbor.*

1. *(2.5 pts) Compute the number of Eulerian trails of G_n.*
2. *(2.5 pts) Compute the number of rooted spanning trees of G_n (a rooted spanning tree is a spanning tree with a distinguished vertex).*

Solution.

1. Let vertices of G_n be $v_1, \dots, v_n$ with arrows from v_i to $v_{i\pm 1}$ for all i from 1 to n. Here $v_{n+1} = v_1$. We assume that the starting vertex of the trails is v_1, the answer for the other vertices will be the same. We also assume that the first edge of the trail is $v_1 v_2$, the answer for $v_1 v_n$ is the same.

   There are two options: either first $n + 1$ vertices of the trail are $v_1 v_2 \dots v_n v_1$ or there exists $2 \leq k \leq n$ such that the first $k + 1$ vertices of the trail are $v_1 v_2 \dots v_k v_{k-1}$. In the first case we cannot use edge $v_1 \to v_2$ again, so we must go back to v_n. This allows us to write both cases uniformly: for $2 \leq k \leq n + 1$ first $k + 1$ vertices are $v_1 \dots v_k v_{k-1}$.

   The edge $v_{k-1} \to v_k$ is already used, so we must go to v_{k-2}. The edge $v_{k-2} \to v_{k-1}$ is already used, so we must go to v_{k-3} and so on all the way back to v_1: $v_1 \dots v_k v_{k-1} \dots v_1$.

   In the case $k = n + 1$ we are done. In the other cases the only edges left are the path $v_1 \to v_n \to \cdots \to v_k$ and the same path reversed $v_k \to v_{k+1} \to \cdots \to v_1$. We see that the only way to use the edge $v_k \to v_{k+1}$ is to continue as follows: $v_1 \dots v_k v_{k-1} \dots v_1 v_n \dots v_{k+1} v_k v_{k+1} \dots v_n v_1$.

   Hence we obtain one path for each k from 2 to $n + 1$, n paths. There were 2 choices of initial direction and n choices of starting vertex, so there are $2n^2$ Eulerian trails.

2. The underlying simple undirected graph of G_n is a cycle C_n. It has n edges. Any spanning tree has $n-1$ edges, so we have to delete one edge from C_n to obtain a spanning tree. Deleting any of n edges of C_n will give us a spanning tree.

   There are also n ways to choose a root of this tree.

   It remains to choose for any undirected edge which directed edge corresponds to it. There are two ways to do it. Multiplying everything we get that the answer is $n^2 2^{n-1}$.

$\square$

Problem 4 *Let G be a simple connected graph, and let T and T' be two spanning trees of G. Prove that, for every $e \in E(T)$, there exists $e' \in E(T')$ such that the edges $(E(T) \setminus \{e\}) \cup \{e'\}$ form a spanning tree of G.*

Solution. Let n be the number of vertices in G. Since T is a tree $T \setminus \{e\}$ is a forest. It has $n-2$ edges. A forest with k connected components has $n-k$ edges. Hence $T \setminus \{e\}$ consists of two connected components, C_1 and C_2.

Let $u \in V(C_1)$, $v \in V(C_2)$. Consider the path between u and v in T', $u = u_0 u_1 u_2 \cdots u_k = v$. Let i be the minimal positive integer such that u_i belongs to $V(C_2)$. By the definition of i we have $u_{i-1} \in V(C_1)$. It follows that the edge $e = u_{i-1}u_i$ belongs to T' and connects C_1 and C_2.

We deduce that $(T \setminus \{e\}) \cup \{e'\}$ is a connected graph. It has $n-1$ edge, so it is a tree. All edges of this tree belong to $E(G)$, so this is a spanning tree. $\square$