COMBINATORIAL ANALYSIS

MIDTERM 1 SOLUTIONS (MIT, FALL 2021)

Problem 1. *Prove that there exist infinitely many positive integers n such that 2021 divides $99^n - 1$.*

Proof. First, we check that $\gcd(99, 2021) = 1$ as neither 3 nor 11 divides 2021. Now consider the positive integers $99^1 - 1, 99^2 - 1, ..., 99^{2022} - 1$. As there are 2021 distinct residues modulo 2021, Pigeonhole Principle guarantees the existence of two distinct integers $i, j \in [2022]$ such that $99^i - 1$ and $99^j - 1$ leave the same remainder when divided by 2021. Assume without loss of generality that $i < j$. This gives that 2021 divides $99^j - 1 - (99^i - 1) = 99^j - 99^i = 99^i(99^{j-i} - 1)$. Since we established that $\gcd(99, 2021) = 1$, it is clear that $99^{j-i} - 1$ is divisible by 2021. However, setting $k = j - i$, our argument concludes that $k > 0$ and $99^k - 1$ is divisible by 2021.

There are several ways to show that the statement is true for an infinite number of positive integers. One such method is to consider integers of the form ck $\forall c \in \mathbb{Z}^+$. We claim all integers of this form work. However, this is not too hard to see as

$$99^{ck} - 1 = (99^k - 1)(99^{k(c-1)} + 99^{k(c-2)} + ... + 1)$$

which is divisible by 2021 since $99^k - 1$ is divisible by 2021. □

Problem 2. *Using a combinatorial argument, prove that*

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$$

for every $n \in \mathbb{N}$ with $n \geq 2$.

Proof. Suppose that there is a class of n students and we wish to select a class council comprised of a committee of at least 2 people, along with a president and a vice president who both serve on the committee and are unique people. We can select a k person committee for any k in $\binom{n}{k}$ different ways. From this committee, we can pick the president in k different ways and the vice president in $k-1$ different ways. That means that there are $\binom{n}{k}k(k-1)$ ways to pick a class council with a committee of size k. Summing this over all sizes from $k = 2$ to n accounts for all possible student council sizes, but this is equivalent to the left hand side of the equation.

Now consider what happens when the president and vice president are chosen before we select the committee. Namely, there are n ways to pick a president from a class of n students and $n-1$ ways to pick a vice president. Both of these students will already be on the committee. From here, the remaining $n-2$ students can either serve on the committee or not serve on the committee, which means there are 2^{n-2} ways to select the rest of the committee. This gives $n(n-1)2^{n-2}$ ways to form the class council which is precisely the right hand side. Thus, these two sides count the same event and are subsequently equal. □

As a remark, this equality can be shown without a combinatorial argument by considering $\frac{d^2}{dx^2}(1+x)^n$, evaluated at $x=1$. It is clear that this is equivalent to $n(n-1)(1+x)^{n-2} = n(n-1)2^{n-2}$ and the left hand side considers taking the second derivative for each $\binom{n}{k}x^k$ term, and summing for all values from $k=2$ to n at $x=1$.

Problem 3. *Prove that the number of compositions of n into odd parts equals to n-th Fibonacci number (assume that the first two Fibonacci numbers are $F_1 = 1$ and $F_2 = 1$).*

Proof. For each $n \in \mathbb{N}$ with $n \geq 2$, let T_n be the set of compositions of n into odd parts, and let $t_n := |T_n|$. Observe that the subset $A_n \subseteq T_n$ consisting of subsets whose first part is 1 is in direct bijection with the odd compositions T_{n-1}. To see why, note that having the first part equal to 1 means the rest of the composition must sum to $n-1$ and will be comprised of odd parts. Therefore, $|A_n| = |T_{n-1}| = t_{n-1}$.

Consider $B_n = T_n \backslash A_n$, the compositions of n into odd parts that do not start with 1. This means that the first part is greater than or equal to 3. Subtracting 2 from the first part gives a composition of $n-2$ into odd parts, and further this process is reversible thus implying that B_n is in direct bijection with T_{n-2}. Therefore $|B_n| = |T_{n-2}| = t_{n-2}$.

As $T_n = A_n \cup B_n$, with A_n and B_n being disjoint, the relation $t_n = t_{n-1} + t_{n-2}$ is made. We finish the proof with a strong induction. We claim that $t_n = F_n$ for all positive integers n. For the base case, it is evident that $t_1 = 1$ and $t_2 = 1$, since $T_1 = \{1\}$ and $T_2 = \{1+1\}$. Assume that $t_n = F_n$ for all positive integers $n \leq k$. It is then given from our work above that $t_{k+1} = t_k + t_{k-1} = F_k + F_{k-1}$ from the induction hypothesis. This means that $t_{k+1} = F_k + F_{k-1} = F_{k+1}$, thus completing the induction, therefore we are done. □

Problem 4. *Let $q_k(n)$ denote the number of partitions of n into k distinct parts. Prove that $q_k(n + \binom{k}{2}) = p_k(n)$.*

Proof. Let $Q_k(n)$ represent the set of partitions of n into k distinct parts and $P_k(n)$ represent the set of partitions of n into k not necessarily distinct parts. We wish to show that $\left|Q_k\left(n + \binom{k}{2}\right)\right| = |P_k(n)|$. Define a function $f : P_k(n) \to Q_k\left(n + \binom{k}{2}\right)$ as follows. Consider an element $p \in P_k(n)$, and let p represent the partition $a_1 + a_2 + ... + a_k$ where $a_i \leq a_{i+1}$ for $i \in [k-1]$. Taking $f(p)$ will produce the partition $b_1 + b_2 + ... + b_k$ where $b_i = a_i + (i-1)$ for all $i \in [k]$. Evaluating this sum, it can be seen that

$$\sum_{i=1}^{k} b_i = \sum_{i=1}^{k}(a_i + i - 1) = -k + \sum_{i=1}^{k}(a_i) + \sum_{i=1}^{k} i = n - k + \frac{k(k+1)}{2} = n + \binom{k}{2}.$$

We further claim that all b_i are distinct. If this were not the case, then there must exist some $b_i = b_j$ for $i \neq j$; however, this would imply $a_i + i - 1 = a_j + j - 1 \iff a_j - a_i = i - j$. Assume without loss of generality that $j > i$, then the quantity $a_j - a_i \geq 0$ but $i - j < 0$, thus implying that these two quantities can not be equal, showing that all b_i are distinct.

Now that it has been shown that f is a valid function, we wish to prove that it is bijective. To do this, it can be shown that f is both injective and surjective, but it also suffices to show that f has a valid inverse. The inverse $f^{-1} : Q_k\left(n + \binom{k}{2}\right) \to P_k(n)$ is defined by taking a valid partition $q \in Q_k\left(n + \binom{k}{2}\right)$, such that q represents $q_1 + q_2 + ... + q_k$ with $q_i < q_{i+1}$ $\forall i \in [k-1]$, and then making a new partition $p_1 + p_2 + ... + p_k$

where $p_i = q_i - (i-1)$ for all $i \in [k]$. It can be seen that this will map to a partition such that $p_i \le p_{i+1}$ for all $i \in [k-1]$ since $q_{i+1} > q_i \implies p_{i+1} + (i+1-1) > p_i + (i-1) \implies p_{i+1} > p_i + 1 \implies p_{i+1} \ge p_i$. Further, it is known that

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i - (i-1) = k + \sum_{i=1}^{k}(q_i) - \sum_{i=1}^{k}(i) = k + n + \binom{k}{2} - \binom{k+1}{2} = n.$$

This means that $p_1 + p_2 + ... + p_k$ is a valid partition of n. Thus, this definition of f^{-1} is well-defined. To see that it is indeed the inverse, consider an element $p \in P_k(n)$ and take $f^{-1}(f(p))$. The partition $p_1+p_2+...+p_k$ gets mapped to $p_1 + (p_2+1) + (p_3+2) + ... + (p_k + k - 1)$ and then f^{-1} maps this to $p_1 + p_2 + ... + p_k = p$. It is evident that this represents a valid inverse to f, and this proves that f is bijective. Therefore, we are done. □

Problem 5. *Find a closed formula for $S(n,3)$.*

Proof. We first present a combinatorial argument. Let π be a partition of $[n]$ into 3 blocks. Note that each element of $[n]$ has 3 choices of where it can go, thus giving 3^n options for where we can put those elements. However, all blocks must be nonempty, meaning that all situations in which there is at least one block without an element have been falsely included.

Counting these, it can be seen that the number of ways to pick exactly one block to not have any elements is 3 and then there are $2^n - 2$ ways to place the elements of $[n]$ into the remaining two blocks so that not all of the elements go entirely into one block or the other. Lastly, there are 3 ways to place all of the elements of $[n]$ into exactly 1 block. That means we have overcounted $3(2^n - 2) + 3 = 3 \cdot 2^n - 3$ cases. Subtracting this from our original count gives a total of $3^n - 3 \cdot 2^n + 3$ valid partitions for π. However, we have to account for the fact that the blocks are indistinguishable, but the elements in them are not. This means we have overcounted by a factor of 3! which is the number of ways to permute the blocks. This gives a final answer of:

$$S(n,3) = \frac{1}{3!}(3^n - 3 \cdot 2^n + 3) = \boxed{\frac{3^{n-1}+1}{2} - 2^{n-1}}.$$

□

A second solution to this question invokes the formula $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ that was shown in lecture. Further note that it was shown that $S(n,2) = 2^{n-1} - 1$, which can be shown again as there are $2^n - 2$ ways to place the items of $[n]$ into one of two boxes so that not all of them items end up in one block, and we overcount by a factor of 2! which is the number of ways to permute the blocks.

Using this, it can be seen that

$$S(n,3) = S(n-1,2) + 3 \cdot S(n-1,3) = S(n-1,2) + 3 \cdot (S(n-2,2) + 3 \cdot S(n-2,3)).$$

Repeating this procedure, it is not hard to see that

$$S(n,3) = \sum_{i=1}^{n-2} 3^{i-1} \cdot S(n-i,2).$$

This can be shown inductively. From here, substituting the value of $S(n-i,2)$ gives:

$$S(n,3)=\sum_{i=1}^{n-2}3^{i-1}(2^{n-i-1}-1)=\sum_{i=1}^{n-2}3^{i-1}\cdot 2^{n-i-1}-\sum_{i=1}^{n-2}3^{i-1}=\sum_{i=0}^{n-3}3^{i}\cdot 2^{n-i-2}-\frac{3^{n-2}-1}{2}.$$

However, this can be simplified to

$$2^{n-2}\sum_{i=0}^{n-3}3^{i}\cdot 2^{-i}-\frac{3^{n-2}-1}{2}=2^{n-2}\left(\frac{\left(\frac{3}{2}\right)^{n-2}-1}{\frac{1}{2}}\right)-\frac{3^{n-2}-1}{2}=2\cdot 3^{n-2}-2^{n-1}-\frac{3^{n-2}-1}{2}=$$

$$\boxed{\frac{3^{n-1}+1}{2}-2^{n-1}}.$$

□

A third and final solution counts the problem directly in another fashion. Namely, if π is a partition of $[n]$ into 3 blocks, we have $\binom{n}{k}$ ways to pick the first block's elements for any k in $[1,n-2]$. Additionally, there are $S(n-k,2)$ ways to place items into the other two blocks. Further, this overcounts by a factor of 3 since we have 3 ways to select which block gets the k elements. This gives the formula

$$S(n,3)=\frac{1}{3}\sum_{k=1}^{n-2}\binom{n}{k}S(n-k,2)=\frac{1}{3}\sum_{k=1}^{n-2}\binom{n}{k}(2^{n-k-1}-1)=\frac{1}{3}\sum_{k=1}^{n-2}\binom{n}{k}2^{n-k-1}-\frac{1}{3}\sum_{k=1}^{n-2}\binom{n}{k}.$$

Using the identity $\sum_{k=0}^{n}\binom{n}{k}=(1+1)^n=2^n$, we obtain

$$S(n,3)=\frac{-2^n+n+2}{3}+\frac{1}{2\cdot 3}\sum_{k=1}^{n-2}\binom{n}{k}2^{n-k}=\frac{-2^n+n+2}{3}-\frac{2^n+2n+1}{6}+\frac{1}{6}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}=$$

$$-2^{n-1}+\frac{1}{2}+\frac{1}{6}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}.$$

Note that the new sum counts the number of ways to select three A,B, and C, where A has k members and the remaining $n-k$ members either go to B or C. However, this also is equal to 3^n as each person has 3 choices for where to go, so this simplifies to

$$S(n,3)=\boxed{-2^{n-1}+\frac{1+3^{n-1}}{2}}.$$

□