Sato–Tate groups of abelian varieties of dimension up to 3

Francesc Fité (MIT)

VaNTAGe Virtual Seminar.

7th April 2020
1. Sato–Tate conjecture for elliptic curves and equidistribution

2. Sato–Tate axioms for $g \leq 3$

3. Abelian surfaces

4. Abelian threefolds
1 Sato–Tate conjecture for elliptic curves and equidistribution

2 Sato–Tate axioms for $g \leq 3$

3 Abelian surfaces

4 Abelian threefolds
Frobenius traces of elliptic curves

- $k$ a number field.
- $E/k$ an elliptic curve.
- For a prime $p$ of good reduction for $E$, set
  \[ a_p := N(p) + 1 - \#E(F_p). \]
- For $p \nmid \ell$, we have
  \[ a_p = \text{Tr}(\text{Frob}_p | V_\ell(E)). \]
- By the Hasse-Weil bound, the normalized Frobenius trace satisfies
  \[ \bar{a}_p := \frac{a_p}{\sqrt{N(p)}} \in [-2, 2]. \]
- The Sato–Tate conjecture is a prediction for the distribution of the $\bar{a}_p$ on the interval $[-2, 2]$. 
Equidistribution: Basic notions

- Let $X$ be a compact topological space and $C(X)$ be the space of continuous $\mathbb{C}$-valued functions on $X$.
- A measure is a continuous linear form $\mu: C(X) \to \mathbb{C}$.
- Also use the notation $\int_X f \mu := \mu(f)$.
- Assume $\mu(1) = 1$ and $\mu$ positive.
- Assume given a sequence $\{x_n\}_n$ of elements in $X$.
- $\{x_n\}_n$ is said to be $\mu$-equidistributed on $X$ if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \int_X f \mu \quad \text{for every } f \in C(X).$$

- Example: If $X = [0, 1]$ and $\mu$ is the Lebesgue measure, then $\{x_n\}_n$ is $\mu$-equidistributed on $X$ if and only if

$$\lim_{n \to \infty} \frac{1}{n} \#\{i \leq n \mid x_i \in [a, b]\} = b - a \quad \text{for every } [a, b] \subseteq X.$$
The Sato–Tate conjecture for elliptic curves

Sato–Tate conjecture for elliptic curves

Let $E$ be an elliptic curve defined over $k$. The sequence $\{\bar{a}_p\}_p$ is $\mu_I$-equidistributed on $I = [-2, 2]$, where $\mu_I$ is of the form

1) $\frac{1}{2\pi} \sqrt{4 - z^2} dz$ if $E$ does not have CM.

2) $\frac{1}{\pi} \frac{dz}{\sqrt{4 - z^2}}$ if $E$ has CM by $M \subseteq k$.

3) $\frac{1}{2} \delta_0 + \frac{1}{2\pi} \frac{dz}{\sqrt{4 - z^2}}$ if $E$ has CM by $M \nsubseteq k$. 

1)  

2)  

3)
Let $G$ be a compact group and $X = \text{Conj}(G)$.

- $C(X)$ space of $\mathbb{C}$-valued continuous \textit{class functions} on $G$.
- Let $\mu_G$ be the Haar measure of $G$ and let $\mu_X = \pi_* (\mu_G)$, where
  \[ \pi : G \longrightarrow X = \text{Conj}(G). \]

Example $G = \text{SU}(2) := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \text{GL}_2(\mathbb{C}) \mid a\bar{a} + b\bar{b} = 1 \right\}$.

- $\pi : \text{SU}(2) \rightarrow X \simeq [0, \pi]$ sends a matrix with eigenvalues $e^{i\theta}, e^{-i\theta}$ to $\theta$.
- For $f \in C(X)$, we have $\mu_X(f) = \int_X f \mu_X = \frac{2}{\pi} \int_0^\pi f(\theta) \sin^2 \theta d\theta$.

Example $G = \text{U}(1) := \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\} \simeq X$.

- For $f \in C(X)$, we have $\mu_X(f) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$. 
The Sato–Tate conjecture for abelian varieties

- Let $A/k$ be an abelian variety of dimension $g \geq 1$.
- Consider the $\ell$-adic representation attached to $A$
  \[
  \varrho_\ell : G_k \to \text{Aut}(V_\ell(A)).
  \]

Sato–Tate conjecture for abelian varieties (Serre; mid 1990’s)

There exist:

- a compact subgroup $G \subseteq \text{USp}(2g)$;
- For each prime $p$ of good reduction for $A$, an $x_p \in \text{Conj}(G)$ s.t.
  \[
  \text{Charpoly}(x_p) = \text{Charpoly} \left( \frac{\varrho_\ell(\text{Frob}_p)}{\sqrt{N(p)}} \right);
  \]
  such that the sequence $\{x_p\}_p$ is equidistributed on $X = \text{Conj}(G)$ w.r.t. the push forward of the Haar measure of $G$.

Moreover, Serre constructs a candidate $\text{ST}(A)$ for $G$. For $g \leq 3$, Banaszak and Kedlaya define it purely in terms of endomorphisms.
Equidistribution: Moments

- Example: For an elliptic curve $E$, there are three options for $\text{ST}(E)$:
  - $\text{SU}(2)$ if $E$ does not have CM.
  - $\text{U}(1)$ if $E$ has CM by $M \subseteq k$.
  - $N_{\text{SU}(2)}(\text{U}(1)) = \langle \text{U}(1), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$ if $E$ has CM by $M \not\subseteq k$.

- Note that the map
  
  $\text{Conj}(\text{ST}(A)) \rightarrow \text{Conj}(\text{USp}(2g))$, \quad x \mapsto \text{Charpoly}(x)$

  is in general not injective.

- The distribution of the “Charpolys” is captured by the moments.

- Let $G \subseteq \text{USp}(2g)$ be a compact subgroup. $X = \text{Conj}(G)$.

- Let $\chi$ denote the character of the tautological rep. $G \rightarrow \text{GL}_{2g}(\mathbb{C})$.

- For integers $n_1, \ldots, n_g \geq 0$, define the moment

  $M_{n_1, \ldots, n_g}(G) = \int_G \chi^{n_1} \cdot (\wedge^2 \chi)^{n_2} \cdot \cdots \cdot (\wedge^g \chi)^{n_g} \mu_G$. 

Equidistribution: Moments of SU(2)

- Note that:

\[ M_{n_1,\ldots,n_g}(G) = \langle 1, \chi^{n_1} \cdot (\wedge^2 \chi)^{n_2} \cdot \ldots \cdot (\wedge^g \chi)^{n_g} \rangle \in \mathbb{Z}_{\geq 0}. \]

- Example: \( G = SU(2) \).
The irreducible characters of \( SU(2) \) are

\[ \chi_n(\theta) = Sym^n(\chi)(\theta) = e^{-n\theta i} + e^{(2-n)\theta i} \cdot \ldots + e^{(n-2)\theta i} + e^{n\theta i}, \quad n \geq 0. \]

The even moments are:

\[
M_{2n}(SU(2)) = \langle 1, \chi^{2n} \rangle \\
= \langle 1, \chi^{2n} + \left( \binom{2n}{1} - 1 \right)\chi^{2n-1} + \ldots + \left( \binom{2n}{n} - \binom{2n}{n-1} \right)1 \rangle \\
= \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n} = n\text{-th Catalan number}.
\]

- The odd moments are 0.
1. Sato–Tate conjecture for elliptic curves and equidistribution
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3. Abelian surfaces
4. Abelian threefolds
Sato–Tate axioms

- From now on, let $A$ be an abelian variety of dimension $g \leq 3$.
- Banaszak and Kedlaya show that $G = \text{ST}(A)$ satisfies then:

**Hodge condition (ST1)**

There is a homomorphism $\theta : \mathbb{U}(1) \to G^0$ such that $\theta(u)$ has eigenvalues $u$ and $\bar{u}$ each with multiplicity $g$. The image of such a $\theta$ is called a Hodge circle. Moreover, the Hodge circles generate a dense subgroup of $G^0$.

(Expected in general; known if the Mumford–Tate conjecture holds for $A$).

**Rationality condition (ST2)**

For every connected component $H \subseteq G$ and for every irreducible character $\chi : \text{GL}_{2g}(\mathbb{C}) \to \mathbb{C}$:

$$\int_H \chi(h) \mu_G \in \mathbb{Z},$$

where $\mu_G$ is normalized so that $\mu_G(1) = [G : G^0]$.

(Expected in general).
Sato–Tate axioms

Lefschetz condition (ST3)
Write $E := \{ \alpha \in M_{2g}(\mathbb{C}) \mid g \alpha g^{-1} = \alpha \text{ for all } g \in G^0 \}$. Then

$$\{ \gamma \in \text{USp}(2g) \mid \gamma \alpha \gamma^{-1} = \alpha \text{ for all } \alpha \in E \} = G^0.$$ 

Serre condition (ST4)
Let $F/k$ be the minimal extension such that $\text{End}(A_F) \simeq \text{End}(A_{\overline{Q}})$. We call $F$ the endomorphism field of $A$. Then

$$G/G^0 \simeq \text{Gal}(F/k).$$

- None of (ST3) and (ST4) are expected in general. In particular, Mumford has constructed examples of abelian fourthfolds $A$ with

$$\text{End}(A_{\overline{Q}}) = \mathbb{Z} \quad \text{and} \quad G^0 \not\subset \text{USp}(8).$$

- Up to conjugacy, 3 subgroups of USp(2) satisfy the ST axioms.
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Sato–Tate groups of abelian surfaces

- Define the *Galois endomorphism type* of an abelian variety $A/k$ as the isomorphism class of the $\mathbb{R}$-algebra $\text{End}(A_{\mathbb{Q}}) \otimes \mathbb{R}$ equipped with the action of $\text{Gal}(F/k)$.

- Example: There are three Galois types of elliptic curves. They are $\mathbb{R}$, $\mathbb{C}$ (both equipped with the trivial action), and $\mathbb{C}$ equipped with the action of complex conjugation.

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**Theorem (F.-Kedlaya-Rotger-Sutherland; 2012)**

- Up to conjugacy, 52 subgroups of $\text{USp}(4)$ satisfy the ST axioms. All of them occur as ST groups of abelian surfaces over number fields.
- 34 of them occur as ST groups of abelian surfaces over $\mathbb{Q}$.
- The ST group and the GET of an abelian surface determine each other uniquely.
Comments on the classification
(ST1) allows 6 possibilities for $G^0 \subseteq \text{USp}(4)$ ((ST3) is redundant for $g = 2$).

<table>
<thead>
<tr>
<th>$G^0$</th>
<th>$\text{End}(A_{\mathbb{Q}}) \otimes \mathbb{R}$</th>
<th>$N_{\text{USp}(4)}(G^0)/G^0$</th>
<th>#A</th>
</tr>
</thead>
<tbody>
<tr>
<td>USp(4)</td>
<td>$\mathbb{R}$</td>
<td>$C_1$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{SU}(2) \times \text{SU}(2)$</td>
<td>$\mathbb{R} \times \mathbb{R}$</td>
<td>$C_2$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{SU}(2) \times \text{U}(1)$</td>
<td>$\mathbb{R} \times \mathbb{C}$</td>
<td>$C_2$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{U}(1) \times \text{U}(1)$</td>
<td>$\mathbb{C} \times \mathbb{C}$</td>
<td>$D_4$</td>
<td>8</td>
</tr>
<tr>
<td>$\text{SU}(2)_2$</td>
<td>$M_2(\mathbb{R})$</td>
<td>$O(2)$</td>
<td>10</td>
</tr>
<tr>
<td>$\text{U}(1)_2$</td>
<td>$M_2(\mathbb{C})$</td>
<td>$SO(3) \times C_2$</td>
<td>32</td>
</tr>
</tbody>
</table>

- $A$ = set of finite subgroups of $N_{\text{USp}(4)}(G^0)/G^0$ for which (ST2) is satisfied.
- 3 of the groups in the case $G^0 = \text{U}(1) \times \text{U}(1)$ do not satisfy (ST4):
  - $A$ is $\overline{\mathbb{Q}}$-isogenous to a product of abelian varieties $A_i$ with CM by $M_i$.
  - $G/G^0 \cong \text{Gal}(F/k) \cong \prod \text{Gal}(kM_i^*/k) \subseteq C_2 \times C_2, C_4$. 
### Additional facts

#### Remark

As $G$ runs over the 52 groups, $\{M_{i,j}(G)\}_{i,j}$ attains 52 values: Distinct groups yield distinct distributions of charpolys.

#### Corollary

The degree of the endomorphism field of an abelian surface over a number field divides 48. (this refines previous results by Silverberg).

#### Theorem (Johansson, N. Taylor; 2014-19)

For $g = 2$ and $k = \mathbb{Q}$, the ST conjecture holds for 33 of the 34 possible ST groups.
1. Sato–Tate conjecture for elliptic curves and equidistribution

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Sato–Tate groups for $g = 3$

**Theorem (F.-Kedlaya-Sutherland; 2019)**

Up to conjugacy, 410 subgroups of $\text{USp}(6)$ satisfy the ST axioms. All of them occur as Sato–Tate groups of abelian threefolds over number fields.

**Corollary**

The degree of the endomorphism field $[F : \mathbb{Q}]$ of an abelian threefold over a number field divides 192, 336, or 432.

- This refines a previous result of Guralnick and Kedlaya, which asserts $[F : \mathbb{Q}] | 2^6 \cdot 3^3 \cdot 7 = \text{Lcm}(192, 336, 432)$. 
Classification: identity components

(ST1) and (ST3) allow 14 possibilities for $G^0 \subseteq \text{USp}(6)$:

\begin{itemize}
  \item USp(6)
  \item U(3)
  \item SU(2) \times \text{USp}(4)
  \item U(1) \times \text{USp}(4)
  \item U(1) \times SU(2) \times SU(2)
  \item SU(2) \times U(1) \times U(1)
  \item SU(2) \times SU(2)
  \item SU(2) \times U(1)
  \item U(1) \times SU(2)
  \item SU(2) \times U(1)
  \item U(1) \times SU(2)
  \item U(1) \times U(1)
  \item SU(2) \times SU(2) \times SU(2)
  \item U(1) \times U(1) \times U(1)
  \item SU(2)_3
  \item U(1)_3
\end{itemize}

Notations:

- For $d \in \{1, 3\}$:
  \[
  U(d) = \left\{ \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} \mid u \in U(d)^{\text{std}} \right\}
  \]
- For $d \in \{2, 3\}$ and $H \in \{\text{SU}(2), \text{U}(1)\}$:
  \[
  H_d = \{ \text{diag}(u, \ldots, u) \mid u \in H \}
  \]
- Note in particular that
  \[
  \text{SU}(2) \times \text{U}(1)_2 \cong \text{U}(1) \times \text{SU}(2)_2.
  \]
Classification: From $G^0$ to $G$

<table>
<thead>
<tr>
<th>$G^0$</th>
<th>$\text{End}(A_{\mathbb{Q}}) \otimes \mathbb{R}$</th>
<th>$N_{\text{USp}(6)}(G^0)/G^0$</th>
<th>$# \mathcal{A}$</th>
</tr>
</thead>
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<tr>
<td>$\text{USp}(6)$</td>
<td>$\mathbb{R}$</td>
<td>$C_1$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{U}(3)$</td>
<td>$\mathbb{C}$</td>
<td>$C_2$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{SU}(2) \times \text{USp}(4)$</td>
<td>$\mathbb{R} \times \mathbb{R}$</td>
<td>$C_1$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{U}(1) \times \text{USp}(4)$</td>
<td>$\mathbb{C} \times \mathbb{R}$</td>
<td>$C_2$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{U}(1) \times \text{SU}(2) \times \text{SU}(2)$</td>
<td>$\mathbb{C} \times \mathbb{R} \times \mathbb{R}$</td>
<td>$C_2 \times C_2$</td>
<td>5</td>
</tr>
<tr>
<td>$\text{SU}(2) \times \text{U}(1) \times \text{U}(1)$</td>
<td>$\mathbb{R} \times \mathbb{C} \times \mathbb{C}$</td>
<td>$D_4$</td>
<td>8</td>
</tr>
<tr>
<td>$\text{SU}(2) \times \text{SU}(2)_2$</td>
<td>$\mathbb{R} \times \text{M}_2(\mathbb{R})$</td>
<td>$O(2)$</td>
<td>10</td>
</tr>
<tr>
<td>$\text{SU}(2) \times \text{U}(1)_2$</td>
<td>$\mathbb{R} \times \text{M}_2(\mathbb{C})$</td>
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</tr>
<tr>
<td>$\text{U}(1) \times \text{SU}(2)_2$</td>
<td>$\mathbb{C} \times \text{M}_2(\mathbb{R})$</td>
<td>$C_2 \times O(2)$</td>
<td>31</td>
</tr>
<tr>
<td>$\text{U}(1) \times \text{U}(1)_2$</td>
<td>$\mathbb{C} \times \text{M}_2(\mathbb{C})$</td>
<td>$C_2 \times SO(3) \times C_2$</td>
<td>122</td>
</tr>
<tr>
<td>$\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)$</td>
<td>$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$</td>
<td>$S_3$</td>
<td>4</td>
</tr>
<tr>
<td>$\text{U}(1) \times \text{U}(1) \times \text{U}(1)$</td>
<td>$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$</td>
<td>$(C_2 \times C_2 \times C_2) \times S_3$</td>
<td>33</td>
</tr>
<tr>
<td>$\text{SU}(2)_3$</td>
<td>$\text{M}_3(\mathbb{R})$</td>
<td>$SO(3)$</td>
<td>11</td>
</tr>
<tr>
<td>$\text{U}(1)_3$</td>
<td>$\text{M}_3(\mathbb{C})$</td>
<td>$\text{PSU}(3) \times C_2$</td>
<td>171</td>
</tr>
</tbody>
</table>

$\mathcal{A} = \text{set of finite subgroups of } N_{\text{USp}(6)}(G^0)/G^0 \text{ for which (ST2) is satisfied.}
Classification: The 23 spurious groups and invariants

- For $G^0 = SU(2) \times U(1) \times U(1)$:
  As in the case $g = 2$, only 5 of 8 groups in $A$ satisfy (ST4).

- For $G^0 = U(1) \times U(1) \times U(1)$:
  Only 13 of the 33 subgroups in $A$ satisfy (ST4).
  Indeed:
  - $A$ is isogenous to a product of abelian varieties $A_i$ with CM by $M_i$.
  - $G/G^0 \cong \text{Gal}(F/k) \cong \prod \text{Gal}(kM_i^*/k) \subseteq C_2 \times C_2 \times C_2, C_2 \times C_4, C_6$.

- This leaves $433 - 20 - 3 = 410$ groups, of which 33 are maximal
  (w.r.t finite inclusions).

- As $G$ runs over the 410 groups, the sequence $\{M_{i,j,k}(G)\}_{i,j,k}$ attains
  409 values. It only conflates a pair of groups $G_1, G_2$, for which however
  \[ G_1/G_1^0 \cong \langle 54, 5 \rangle \ncong \langle 54, 8 \rangle \cong G_2/G_2^0. \]

- Any possible order of $G/G^0$ divides 192, 336, or 432.
Realization

- It suffices to realize the 33 maximal groups (for prescribed $G^0$). Finite index subgroups are realized by base change.
- For 8 of the 14 possibilities for $G^0$, the maximal groups are of the form $G \cong G_1 \times G_2$ where $G_1$ and $G_2$ are realizable in dimensions 1 and 2. This accounts for 13 maximal groups.
- $\text{USp}(6)$: generic case. Eg.: $y^2 = x^7 - x + 1/\mathbb{Q}$.
- $N(\text{U}(3))$: Picard curves. Eg.: $y^3 = x^4 + x + 1/\mathbb{Q}$.
- $G^0 = \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)$ (1. max. group): $\text{Res}_{\mathbb{Q}}^L(E)$, where $L/\mathbb{Q}$ a non-normal cubic and $E/L$ e.c. which is not a $\mathbb{Q}$-curve.
- $G^0 = \text{U}(1) \times \text{U}(1) \times \text{U}(1)$ (3 max. groups): Products of CM abelian varieties.
- $G^0 = \text{SU}(2)_3$ (2 max. groups): Twists of cubes of non CM e.c.
- $G^0 = \text{U}(1)_3$ (12 max. groups): Twists of cubes of CM elliptic curves.