LECTURES ON CALOGERO-MOSER SYSTEMS

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To my mother Yelena Etingof on her 75-th birthday, with admiration

Introduction

Calogero-Moser systems, which were originally discovered by specialists in integrable systems, are currently at the crossroads of many areas of mathematics and within the scope of interests of many mathematicians. More specifically, these systems and their generalizations turned out to have intrinsic connections with such fields as algebraic geometry (Hilbert schemes of surfaces), representation theory (double affine Hecke algebras, Lie groups, quantum groups), deformation theory (symplectic reflection algebras), homological algebra (Koszul algebras), Poisson geometry, etc. The goal of the present lecture notes is to give an introduction to the theory of Calogero-Moser systems, highlighting their interplay with these fields. Since these lectures are designed for non-experts, we give short introductions to each of the subjects involved, and provide a number of exercises.

We now describe the contents of the lectures in more detail.

In Lecture 1, we give an introduction to Poisson geometry and to the process of classical Hamiltonian reduction. More specifically, we define Poisson manifolds (smooth, analytic, and algebraic), momemt maps and their main properties, and then describe the procedure of (classical) Hamiltonian reduction. We give an example of computation of Hamiltonian reduction in algebraic geometry (the commuting variety). Finally, we define Hamiltonian reduction along a coadjoint orbit, and give the example which plays a central role in these lectures – the Calogero-Moser space of Kazhdan, Kostant, and Sternberg.

In Lecture 2, we give an introduction to classical Hamiltonian mechanics and the theory of integrable systems. Then we explain how integrable systems may sometimes be constructed using Hamiltonian reduction. After this we define the classical Calogero-Moser integrable system using Hamiltonian reduction along a coadjoint orbit (the Kazhdan-Kostant-Sternberg construction), and find its solutions. Then, by introducing coordinates on the Calogero-Moser space, we write both the system and the solutions explicitly, thus recovering the standard results about the Calogero-Moser system. Finally, we generalize these results to construct the trigonometric Calogero-Moser system.

Lecture 3 is an introduction to deformation theory. This lecture is designed, in particular, to enable us to discuss quantum-mechanical versions of the notions and results of Lectures 1 and 2 in a manner parallel to the classical case. Specifically, we develop the theory of formal and algebraic deformations of assosiative algebras, introduce Hochschild cohomology and discuss its role in studying deformations, and define universal deformations. Then we discuss the basics of the theory of deformation quantization of Poisson (in particular, symplectic) manifolds, and state the Kontsevich quantization theorem.

Lecture 4 is dedicated to the quantum-mechanical generalization of the material of Lecture 1. Specifically, we define the notions of quantum moment map and quantum Hamiltonian reduction. Then we give an example of computation of quantum reduction (the Levasseur-Stafford theorem), which is the quantum analog of the example of commuting variety given in Lecture 1. Finally, we define the notion of quantum reduction with respect to an ideal in the enveloping algebra, which is the quantum version of reduction along a coadjoint orbit, and give an example of this reduction – the construction of the spherical subalgebra of the rational Cherednik algebra. Being a quantization of the Calogero-Moser space, this algebra is to play a central role in subsequent lectures.

Lecture 5 contains the quantum-mechanical version of the material of Lecture 1. Namely, after recalling the basics of quantum Hamiltonian mechanics, we introduce the notion of a quantum integrable system. Then we explain how to construct quantum integrable systems by means of quantum reduction (with respect to an ideal), and give an example of this which is central to our exposition – the quantum Calogero-Moser system.

In Lecture 6, we define and study more general classical and quantum Calogero-Moser systems, which are associated to finite Coxeter groups (they were introduced by Olshanetsky and Perelomov). The systems defined in previous lectures correspond to the case of the symmetric group. In general, these integrable systems are not known (or expected) to have a simple construction using reduction; in their construction and study, Dunkl operators are an indispensible tool. We introduce the Dunkl operators (both classical and quantum), and explain how the Olshanetsky-Perelomov Hamiltonians are constructed from them.

Lecture 7 is dedicated to the study of the rational Cherednik algebra, which naturally arises from Dunkl operators (namely, it is generated by Dunkl operators, coordinates, and reflections). Using the Dunkl operator representation, we prove the Poincare-Birkhoff-Witt theorem for this algebra, and study its spherical subalgebra and center.

In Lecture 8, we consider symplectic reflection algebras, associated to a finite group G of automorphisms of a symplectic vector space V. These algebras are natural generalizations of rational Cherednik algebras (although in general they are not related to any integrable system). It turns out that the PBW theorem does generalize to these algebras, but its proof does not, since Dunkl operators don't have a counterpart. Instead, the proof is based on the theory of deformations of Koszul algebras, due to Drinfeld, Braverman-Gaitsgory, Polishchuk-Positselski, and Beilinson-Ginzburg-Soergel. We also study the spherical subalgebra of the symplectic reflection algebra, and show by deformation-theoretic arguments that it is commutative if the Planck constant is equal to zero.

In Lecture 9, we describe the deformation-theoretic interpretation of symplectic reflection algebras. Namely, we show that they are universal deformations of semidirect products of G with the Weyl algebra of V.

In Lecture 10, we study the center of the symplectic reflection algebra, in the case when the Planck constant equals zero. Namely, we consider the spectrum of the center, which is an algebraic variety analogous to the Calogero-Moser space, and show that the smooth locus of this variety is exactly the set of points where the symplectic reflection algebra is an Azumaya algebra; this requires some tools from homological algebra, such as the Cohen-Macaulay property and homological dimension, which we briefly introduce. We also study finite dimensional representations of symplectic reflection algebras with the zero value of the Planck constant. In particular, we show that for G being the symmetric group S_n (i.e. in the case of rational Cherednik algebras of type A), every irreducible representation has dimension $n!$, and irreducible representations are parametrized by the Calogero-Moser space defined in Lecture 1. A similar theorem is valid if $G = S_n \ltimes \Gamma^n$, where Γ is a finite subgroup of $SL_2(\mathbb{C})$.

Lecture 11 is dedicated to representation theory of rational Cherednik algebras with a nonzero Planck constant. Namely, by analogy with semisimple Lie algebras, we develop the theory of category $\mathcal{O}$. In particular, we introduce Verma modules, irreducible highest weight modules, which are labeled by representations of G, and compute the characters of the Verma modules. The main challenge is to compute the characters of irreducible modules, and find out which of them are finite dimensional. We do some of this in the case when $G = S_n \ltimes \Gamma^n$, where Γ is a cyclic group. In particular, we construct and compute the characters of all the finite dimensional simple modules in the case $G = S_n$ (rational

Cherednik algebra of type A). It turns out that a finite dimensional simple module exists if and only if the parameter k of the Cherednik algebra equals r/n, where r is an integer relatively prime to n. For such values of k, such representation is unique, its dimension is $|r|^{n-1}$, and it has no self-extensions.

At the end of each lecture, we provide remarks and references, designed to put the material of the lecture in a broader prospective, and link it with the existing literature. However, due to a limited size and scope of these lectures, we were, unfortunately, unable to give an exhaustive list of references on Calogero-Moser systems; such a list would have been truly enormous.

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1. Poisson manifolds and Hamiltonian reduction

1.1. **Poisson manifolds.** Let A be a commutative algebra over a field k.

Definition 1.1. We say that A is a Poisson algebra if it is equipped with a Lie bracket $\{,\}$ such that $\{a, bc\} = \{a, b\}c + b\{a, c\}$.

Let I be an ideal in A.

Definition 1.2. We say that I is a Poisson ideal if $\{A, I\} \subset I$.

In this case A/I is a Poisson algebra.

Let M be a smooth manifold.

Definition 1.3. We say that M is a Poisson manifold if its structure algebra $C^\infty(M)$ is equipped with a Poisson bracket.

The same definition can be applied to complex analytic and algebraic varieties: a Poisson structure on them is just a Poisson structure on the structure sheaf. Note that this definition may be used even for singular varieties.

Definition 1.4. A morphism of Poisson manifolds (=Poisson map) is a regular map $M \to N$ that induces a homomorphism of Poisson algebras $C^\infty(N) \to C^\infty(M)$, i.e. a map that preserves Poisson structure.

If M is a smooth variety (C^∞, analytic, or algebraic), then a Poisson structure on M is defined by a Poisson bivector $\Pi \in \Gamma(M, \wedge^2 TM)$ such that its Schouten bracket with itself is zero: $[\Pi, \Pi] = 0$. Namely, $\{f, g\} := (df \otimes dg)(\Pi)$ (the condition that $[\Pi, \Pi] = 0$ is equivalent to the Jacobi identity for $\{,\}$. In particular, if M is symplectic (i.e. equipped with a closed nondegenerate 2-form ω) then it is Poisson with $\Pi = \omega^{-1}$, and conversely, a Poisson manifold with nondegenerate Π is symplectic with $\omega = \Pi^{-1}$.

For any Poisson manifold M, we have a homomorphism of Lie algebras $v : C^\infty(M) \to Vect_\Pi(M)$ from the Lie algebra of functions on M to the Lie algebra of vector fields on M preserving the Poisson structure, given by the formula $f \mapsto \{f, ?\}$. In classical mechanics, one says that $v(f)$ is the Hamiltonian vector field corresponding to the Hamiltonian f.

Exercise 1.5. If M is a connected symplectic manifold, then $\mathrm{Ker}(v)$ consists of constant functions. If in addition $H^1(M, \mathbb{C}) = 0$ then the map v is surjective.

Example 1.6. $M = T^*X$, where X is a smooth manifold. Define the Liouville 1-form η on T^*X as follows. Let $\pi : T^*X \to X$ be the projection map. Then given $v \in T_{(x,p)}(T^*X)$, we set $\eta(v) = (d\pi \cdot v, p)$. Thus if x_i are local coordinates on X and p_i are the linear coordinates in the fibers of T^*X with respect to the basis dx_i then $\eta = \sum p_i dx_i$.

Let $\omega = d\eta$. Then ω is a symplectic structure on M. In local coordinates, $\omega = \sum dp_i \wedge dx_i$.

Example 1.7. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Let $\Pi : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$ be the dual map to the Lie bracket. Then Π is a Poisson bivector on $\mathfrak{g}^*$ (whose coefficients are linear). This Poisson structure on $\mathfrak{g}^*$ is called the Lie Poisson structure.

Let $\mathcal{O}$ be an orbit of the coadjoint action in $\mathfrak{g}^*$. Then it is easy to check that the restriction of Π to $\mathcal{O}$ is a section of $\wedge^2 T\mathcal{O}$, which is nondegenerate. Thus $\mathcal{O}$ is a symplectic manifold. The symplectic structure on $\mathcal{O}$ is called the Kirillov-Kostant structure.

1.2. **Moment maps.** Let M be a Poisson manifold and G a Lie group acting on M by Poisson automorphisms. Let $\mathfrak{g}$ be the Lie algebra of G. Then we have a homomorphism of Lie algebras $\phi : \mathfrak{g} \to Vect_\Pi(M)$.

Definition 1.8. A G-equivariant regular map $\mu : M \to \mathfrak{g}^*$ is said to be a moment map for the G-action on M if the pullback map $\mu^* : \mathfrak{g} \to C^\infty(M)$ satisfies the equation $v(\mu^*(a)) = \phi(a)$.

It is easy to see that in this case μ^* is a homomorphism of Lie algebras, so μ is a Poisson map.

A moment map does not always exist, and if it does, it is not always unique. However, if M is a simply connected symplectic manifold, then the homomorphism $\phi : \mathfrak{g} \to Vect_\Pi(M)$ can be lifted to a homomorphism $\hat{\mathfrak{g}} \to C^\infty(M)$, where M is a 1-dimensional central extension of $\mathfrak{g}$. Thus there exists a moment map for the action on M of the simply connected Lie group $\hat{G}$ corresponding to the Lie algebra $\hat{\mathfrak{g}}$. In particular, if in addition the action of G on M is transitive, then M is a coadjoint orbit of $\hat{G}$.

We also see that if M is a connected symplectic manifold then any two moment maps $M \to \mathfrak{g}^*$ differ by shift by a character of $\mathfrak{g}$.

Exercise 1.9. Show that if $M = \mathbb{R}^2$ with symplectic form $dp \wedge dx$ and $G = \mathbb{R}^2$ acting by translations, then there is no moment map $M \to \mathfrak{g}^*$. What is $\hat{G}$ in this case?

Exercise 1.10. Show that if M is simply connected and symplectic and G is compact then there is a moment map $M \to \mathfrak{g}^*$.

Exercise 1.11. Show that if M is symplectic then μ is a submersion near x (i.e., the differential $d\mu_x : T_xM \to \mathfrak{g}^*$ is surjective) if and only if the stabilizer G_x of G is a discrete subgroup of G (i.e. the action is locally free near x).

Example 1.12. Let $M = T^*X$, and let G act on X. Define $\mu : T^*X \to \mathfrak{g}^*$ by $\mu(x,p)(a) = p(\psi(a))$, $a \in \mathfrak{g}$, where $\psi : \mathfrak{g} \to Vect(X)$ is the map defined by the action. Then μ is a moment map.

1.3. **Hamiltonian reduction.** Let M be a Poisson manifold with an action of a Lie group G preserving the Poisson structure. Then the algebra of G-invariants $C^\infty(M)^G$ is a Poisson algebra.

Let J be the ideal in $C^\infty(M)$ generated by $\mu^*(a)$, $a \in \mathfrak{g}$. It is easy to see that J is invariant under Poisson bracket with $C^\infty(M)^G$. Therefore, the ideal J^G in $C^\infty(M)^G$ is a Poisson ideal, and hence the algebra $A := C^\infty(M)^G/J^G$ is a Poisson algebra.

The geometric meaning of the algebra A is as follows. Assume that the action of G on M is proper, i.e. for any two compact sets K_1 and K_2, the set of elements $g \in G$ such that $gK_1 \cap K_2 \neq \emptyset$ is compact. Assume also that the action of G is free. In this case, the quotient M/G is a manifold, and $C^\infty(M)^G = C^\infty(M/G)$. Moreover, as we mentioned in Exercise 1.11, the map μ is a submersion (so $\mu^{-1}(0)$ is a smooth submanifold of M), and the ideal J^G corresponds to the submanifold $M//G := \mu^{-1}(0)/G$ in M. Thus, $A = C^\infty(M//G)$, and so $M//G$ is a Poisson manifold.

Definition 1.13. The manifold $M//G$ is called the Hamiltonian reduction of M with respect to G using the moment map μ.

Exercise 1.14. Show that in this setting, if M is symplectic, so is $M//G$.

This geometric setting can be generalized in various directions. First of all, for $M//G$ to be a manifold, it suffices to require that the action of G be free only near $\mu^{-1}(0)$. Second, one can consider a locally free action which is not necessarily free. In this case, $M//G$ is a Poisson orbifold.

Finally, we can consider a purely algebraic setting which will be most convenient for us: M is a scheme of finite type over $\mathbb{C}$ (for example, a variety), and G is an affine algebraic group. In this case, we do not need to assume that the action of G is locally free (which allows us to consider many more examples). Still, some requirements are needed to ensure the existence of quotients. For example, a sufficient condition that often applies is that M is an affine scheme and G is a reductive group. Then $M//G$ is an affine Poisson scheme (possibly nonreduced and singular even if M was smooth).

Example 1.15. Let G act properly and freely on a manifold X, and $M = T^*X$. Then $M//G$ (for the moment map as in Example 1.12) is isomorphic to $T^*(X/G)$.

On the other hand, the following example shows that when the action of G on an algebraic variety X is not free, the computation of the reduction $T^*X//G$ (as a scheme) may be rather difficult.

Example 1.16. Let $M = T^*\mathrm{Mat}_n(\mathbb{C})$, and $G = PGL_n(\mathbb{C})$ (so $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$). Using the trace form we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and M with $\mathrm{Mat}_n(\mathbb{C}) \oplus \mathrm{Mat}_n(\mathbb{C})$. Then a moment map is given by the formula $\mu(X, Y) = [X, Y]$, for $X, Y \in \mathrm{Mat}_n(\mathbb{C})$. Thus $\mu^{-1}(0)$ is the **commuting scheme** $\mathrm{Comm}(n)$ defined by the equations $[X, Y] = 0$, and the quotient $M//G$ is the quotient $\mathrm{Comm}(n)/G$, whose ring of functions is $A = \mathbb{C}[\mathrm{Comm}(n)]^G$.

It is not known whether the commuting scheme is reduced (i.e. whether the corresponding ideal is a radical ideal); this is a well known open problem. The underlying variety is irreducible (as was shown by Gerstenhaber [Ge1]), but very singular, and has a very complicated structure. However, we have the following result.

Theorem 1.17. *(Gan, Ginzburg,*[GG]*)* $\mathrm{Comm}(n)/G$ *is reduced, and isomorphic to* $\mathbb{C}^{2n}/S_n$. *Thus* $A = \mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]^{S_n}$. *The Poisson bracket on this algebra is induced from the standard symplectic structure on* $\mathbb{C}^{2n}$.

Remark. The hard part of this theorem is to show that $\mathrm{Comm}(n)/G$ is reduced (i.e. A has no nonzero nilpotent elements).

Remark. Let $\mathfrak{g}$ be a simple complex Lie algebra, and G the corresponding group. The commuting scheme $\mathrm{Comm}(\mathfrak{g})$ is the subscheme of $\mathfrak{g} \oplus \mathfrak{g}$ defined by the equation $[X, Y] = 0$. Similarly to the above discussion, $\mathrm{Comm}(\mathfrak{g})/G = T^*\mathfrak{g}//G$. It is conjectured that $\mathrm{Comm}(\mathfrak{g})$ and in particular $\mathrm{Comm}(\mathfrak{g})/G$ is a reduced scheme; the latter is known for $\mathfrak{g} = sl(n)$ thanks to the Gan-Ginzburg theorem. It is also known that the underlying variety $\overline{\mathrm{Comm}(\mathfrak{g})}$ is irreducible (as was shown by Richardson), and $\overline{\mathrm{Comm}(\mathfrak{g})}/G = (\mathfrak{h} \oplus \mathfrak{h})/W$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and W is the Weyl group of $\mathfrak{g}$ (as was shown by Joseph [J]).

1.4. **Hamiltonian reduction along an orbit.** Hamiltonian reduction along an orbit is a generalization of the usual Hamiltonian reduction. For simplicity let us describe it in the situation when M is an affine algebraic variety and G a reductive group. Let $\mathcal{O}$ be a closed coadjoint orbit of G, $I_{\mathcal{O}}$ be the ideal in $S\mathfrak{g}$ corresponding to $\mathcal{O}$, and let $J_{\mathcal{O}}$ be the ideal in $\mathbb{C}[M]$ generated by $\mu^*(I_{\mathcal{O}})$. Then $J_{\mathcal{O}}^G$ is a Poisson ideal in $\mathbb{C}[M]^G$, and $A = \mathbb{C}[M]^G/J_{\mathcal{O}}^G$ is a Poisson algebra.

Geometrically, $\mathrm{Spec}(A) = \mu^{-1}(\mathcal{O})/G$ (categorical quotient).

Definition 1.18. The scheme $\mu^{-1}(\mathcal{O})/G$ is called the Hamiltonian reduction of M with respect to G along $\mathcal{O}$. We will denote by $R(M, G, \mathcal{O})$.

Exercise 1.19. Show that if the action of G on $\mu^{-1}(\mathcal{O})$ is free, and M is a symplectic variety, then $R(M, G, \mathcal{O})$ is a symplectic variety, of dimension $\dim(M) - 2\dim(G) + \dim(\mathcal{O})$.

In a similar way, one can define Hamiltonian reduction along any Zariski closed G-invariant subset of $\mathfrak{g}^*$, for example the closure of a non-closed coadjoint orbit.

1.5. **Calogero-Moser space.** Let M and G be as in Example 1.16, and $\mathcal{O}$ be the orbit of the matrix $\mathrm{diag}(-1, -1, ..., -1, n-1)$, i.e. the set of traceless matrices T such that $T + 1$ has rank 1.

Definition 1.20. (Kazhdan, Kostant, Sternberg, [KKS]) The scheme $\mathcal{C}_n := R(M, G, \mathcal{O})$ is called the Calogero-Moser space.

Thus, $\mathcal{C}_n$ is the space of conjugacy classes of pairs of $n \times n$ matrices (X, Y) such that the matrix $XY - YX + 1$ has rank 1.

Theorem 1.21. *The action of G on $\mu^{-1}(\mathcal{O})$ is free, and thus (by Exercise 1.19) $\mathcal{C}_n$ is a smooth symplectic variety (of dimension $2n$).*

Proof. It suffices to show that if X, Y are such that $XY - YX + 1$ has rank 1, then (X, Y) is an irreducible set of matrices. Indeed, in this case, by Schur's lemma, if $B \in GL(n)$ is such that $BX = XB$ and $BY = YB$ then B is a scalar, so the stabilizer of (X, Y) in PGL_n is trivial.

To show this, assume that $W \neq 0$ is an invariant subspace of X, Y. In this case, the eigenvalues of $[X, Y]$ on W are a subcollection of the collection of $n - 1$ copies of -1 and one copy of $n - 1$. The sum of the elements of this subcollection must be zero, since it is the trace of $[X, Y]$ on W. But then the subcollection must be the entire collection, so $W = \mathbb{C}^n$, as desired. □

In fact, one also has the following more complicated theorem.

Theorem 1.22. *(G. Wilson,* [Wi]*) The Calogero-Moser space is connected.*

1.6. **Notes.** 1. For generalities on Poisson algebras and Poisson manifolds, see e.g. the book [Va]. The basic material on symplectic manifolds can be found in the classical book [Ar]. Moment maps appeared already in the works of S. Lie (1890). Classical Hamiltonian reduction (for symplectic manifolds) was introduced by Marsden and Weinstein [MW]; reduction along an orbit is introduced by Kazhdan-Kostant-Sternberg [KKS]. For basics on moment maps and Hamiltonian reduction see e.g. the book [OR].

2. The commuting variety and the Calogero-Moser space are very special cases of much more general Poisson varieties obtained by reduction, called quiver varieties; they play an important role in geometric representation theory, and were recently studied by many authors, notably W. Crawley-Boevey, G. Lusztig, H. Nakajima. Wilson's connectedness theorem for the Calogero-Moser space can be generalized to quiver varieties; see [CB].

2. Classical mechanics and Integrable systems

2.1. **Classical mechanics.** The basic setting of Hamiltonian classical mechanics is as follows. The phase space of a mechanical system is a Poisson manifold M. The manifold M is usually symplectic and often equals T^*X, where X is another manifold called the configuration space. The dynamics of the system is defined by its Hamiltonian (or energy function) $H \in C^\infty(M)$. Namely, the Hamiltonian flow attached to H is the flow corresponding to the vector field $v(H)$. If y_i are coordinates on M, then the differential equations defining the flow (Hamilton's equations) are written as

$$\frac{dy_i}{dt} = \{H, y_i\},$$

where t is the time. The main mathematical problem in classical mechanics is to find and study the solutions of these equations.

Assume from now on that M is symplectic. Then by Darboux theorem we can locally choose coordinates x_j, p_j on M such that the symplectic form is $\omega = \sum dp_j \wedge dx_j$. Such coordinates are said to be canonical. For them one has $\{p_i, x_j\} = \delta_{ij}, \{p_i, p_j\} = \{x_i, x_j\} = 0$. In canonical coordinates Hamilton's equations are written as

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

Example 2.1. Let $M = T^*X$, where X is a Riemannian manifold. Then we can identify TX with T^*X using the Riemannian metric. Let $H = \frac{p^2}{2} + U(x)$, where $U(x)$ is a smooth function on X called the potential. The flow defined by this Hamiltonian describes the motion of a particle on X in the potential field $U(x)$.

Hamilton's equations for the Hamiltonian H are

$$\dot{x} = p, \ \dot{p} = -\frac{\partial U}{\partial x},$$

where $\dot{p}$ is the covariant time derivative of p with respect to the Levi-Civita connection. (If $X = \mathbb{R}^n$ with the usual metric, it is the usual time derivative). These equations reduce to Newton's equation

$$\ddot{x} = -\frac{\partial U}{\partial x}$$

(again, $\ddot{x}$ is the covariant time derivative of $\dot{x}$). If $U(x)$ is constant, the equation has the form $\ddot{x} = 0$, which defines the so called geodesic flow. Under this flow, the particle moves along geodesics in X (lines if $X = \mathbb{R}^n$) with constant speed.

The law of conservation of energy, which trivially follows from Hamilton's equations, says that the Hamiltonian H is constant along the trajectories of the system. This means that if M is 2-dimensional then the Hamiltonian flow is essentially a 1-dimensional flow along level curves of H, and thus the system can be solved explicitly in quadratures. In higher dimensions this is not the case, and a generic Hamiltonian system on a symplectic manifold of dimension $2n$, $n > 1$, has a very complicated behavior.

2.2. **Symmetries in classical mechanics.** It is well known that if a classical mechanical system has a symmetry, then this symmetry can be used to reduce its order (i.e., the dimension of the phase space). For instance, suppose that $F \in C^\infty(M)$ is a first integral of the system, i.e., F and H are "in involution": $\{F, H\} = 0$. Then F gives rise to a (local) symmetry of the system under the group $\mathbb{R}$ defined by the Hamiltonian flow with Hamiltonian F[1], and if F is functionally independent of H, then we can use F to reduce the order of the system by 2. Another example is motion in a rotationally symmetric field (e.g., motion of planets around the sun), which can be completely solved using the rotational symmetry.

The mathematical mechanism of using symmetry to reduce the order of the system is that of Hamiltonian reduction. Namely, let a Lie group G act on M preserving the Hamiltonian H. Let $\mu : M \to \mathfrak{g}^*$ be a moment map for this action. For simplicity assume that G acts freely on M (this is not an essential assumption). We also assume that generically the Hamiltonian vector field $v(H)$ is transversal to the G-orbits.

It is easy to see that if $y = y(t)$ is a solution of Hamilton's equations, then $\mu(y(t))$ is constant. [2]

Therefore, the flow descends to the symplectic manifolds $R(M, G, \mathcal{O}) = \mu^{-1}(\mathcal{O})/G$, where $\mathcal{O}$ runs over orbits of the coadjoint representation of G, with the same Hamiltonian. These manifolds have a smaller dimensions than that of M.

On the other hand, if we know the image $y_*(t)$ in $M_* := R(M, G, \mathcal{O})$ of a trajectory $y(t) \in \mu^{-1}(\mathcal{O})$ then we can find $y(t)$ explicitly. To do so, let us locally pick a cross-section $i : M_* \to \mu^{-1}(\mathcal{O})$, such that $i(M_*)$ sits inside a level surface $H = \text{const}$ (this can be done since the H-flow is transversal to G-orbits). Then $\mu^{-1}(\mathcal{O})$ gets locally identified with the product $G \times M_*$, and $y(t) = gi(y_*(t))$, where $g \in G$ is such that $y(0) = gi(y_*(0))$.

2.3. **Integrable systems.**

[1]The symmetry is local because the solutions of the system defined by F may not exist for all values of t; this is, however, not essential for our considerations.

[2]This fact is a source of many conservation laws in physics, e.g. conservation of momentum for translational symmetry, or conservation of angular momentum for rotational symmetry; it also explains the origin of the term "moment map".

Definition 2.2. An integrable system on a symplectic manifold M of dimension $2n$ is a collection of smooth functions $H_1, ..., H_n$ on M such that they are in involution (i.e., $\{H_i, H_j\} = 0$), and the differentials dH_i are linearly independent on a dense open set in M.

Let us explain the motivation for this definition. Suppose that we have a Hamiltonian flow on M with Hamiltonian H, and assume that H can be included in an integrable system: $H = H_1, H_2, ..., H_n$. In this case, $H_1, H_2, ..., H_n$ are first integrals of the flow. In particular, we can use H_n to reduce the order of the flow from $2n$ to $2n - 2$ as explained in the previous subsection. Since $\{H_i, H_n\} = 0$, we get a Hamiltonian system on a phase space of dimension $2n - 2$ with Hamiltonian H and first integrals $H = H_1, ..., H_{n-1}$, which form an integrable system. Now we can use H_{n-1} to further reduce the flow to a phase space of dimension $2n - 4$, etc. Continuing in this way, we will reduce the flow to the 2-dimensional phase space, As we mentioned before, such a flow can be integrated in quadratures, hence so can the flow on the original manifold M. This motivates the terminology "integrable system".

Remark. In a similar way one can define integrable systems on complex analytic and algebraic varieties.

Exercise 2.3. Show that the condition that dH_i are linearly independent on a dense open set is equivalent to the requirement that H_i are functionally independent, i.e. there does not exist a nonempty open set $U \subset M$ such that the points $(H_1(u), ..., H_n(u))$, $u \in U$, are contained in a smooth hypersurface in $\mathbb{R}^n$.

Exercise 2.4. Let $H_1, ..., H_n, H_{n+1}$ be a system of functions on a symplectic manifold M, such that $\{H_i, H_j\} = 0$. Let x_0 be a point of M such that dH_i, $i = 1, ..., n$, are linearly independent at x_0. Show that there exists neighborhood U of x_0 and a smooth function F on a neighborhood of $y_0 := (H_1(x_0), ..., H_n(x_0)) \in \mathbb{R}^n$ such that $H_{n+1}(x) = F(H_1(x), ..., H_n(x))$ for $x \in U$.

Theorem 2.5. *(Liouville's theorem) Let M be a symplectic manifold, with an integrable system $H_1, ..., H_n$. Let $c = (c_1, .., c_n) \in \mathbb{R}^n$, and M_c be the set of points $x \in M$ such that $H_i(x) = c_i$. Assume that dH_i are independent at every point of M_c, and that M_c is compact. In this case every connected component N of M_c is a torus T^n, and the n commuting flows corresponding to the Hamiltonians H_i define an action of the group $\mathbb{R}^n$ on N, which identifies N with $\mathbb{R}^n/\Gamma$, where Γ is a lattice in $\mathbb{R}^n$.*

The proof of this theorem can be found, for instance, in [Ar], Chapter 10.

Remark. The compactness assumption of Liouville's theorem is often satisfied. For example, it holds if $M = T^*X$, where X is a compact manifold, and $H = \frac{p^2}{2} + U(x)$.

Remark. One may define a Hamiltonian H on M to be *completely integrable* if it can be included in an integrable system $H = H_1, H_2, ..., H_n$. Unfortunately, this definition is not always satisfactory: for example, it is easy to show that any Hamiltonian is completely integrable in a sufficiently small neighborhood of a point $x \in M$ where $dH(x) \neq 0$. Thus when one says that a Hamiltonian is completely integrable, one usually means not the above definition, but rather that one can explicitly produce $H_2, ..., H_n$ such that $H = H_1, H_2, ..., H_n$ is an integrable system (a property that is hard to define precisely). However, there are situations where the above precise definition is adequate. For example, this is the case in the algebro-geometric situation (M is an algebraic variety, Hamiltonians are polynomial functions), or in the C^∞-situation in the case when the hypersurfaces $H = \text{const}$ are compact. In both of these cases integrability is a strong restriction for $n > 1$. For instance, in the second case integrability implies that the closure of a generic trajectory has dimension n, while it is known that for "generic" H with compact level hypersurfaces this closure is the entire hypersurface $H = \text{const}$, i.e., has dimension $2n - 1$.

2.4. **Action-angle variables of integrable systems.** To express solutions of Hamilton's equations corresponding to H, it is convenient to use the so called "action-angle variables", which are certain corrdinates on M near a point $P \in M$ where dH_i are linearly independent. Namely, the action variables are simply the functions $H_1, ..., H_n$. Note that since H_i are in involution, they define commuting flows on M. To define the angle variables, let us fix (locally near P) a Lagrangian submanifold L of M (i.e., one of dimension n on which the symplectic form vanishes), which is transversal to the joint level surfaces of $(H_1, ..., H_n)$. Then for a point z of M sufficiently close to P one can define the "angle variables" $\phi_1(z), ..., \phi_m(z)$, which are the times one needs to move along the flows defined by $H_1, ..., H_n$ starting from L to reach the point z. By the above explanations, ϕ_i can be found in quadratures if H_i are known.

The reason action-angle variables are useful is that in them both the symplectic form and Hamilton's equations have an extremely simple expression: the symplectic form is $\omega = \sum dH_i \wedge d\phi_i$, and Hamilton's equations are given by the formula $\dot{H}_i = 0$, $\dot{\phi}_i = \delta_{1i}$. Thus finding the action-angle variables is sufficient to solve Hamilton's equations for H. In fact, by producing the action-angle variables, we also solve

Hamilton's equations with Hamiltonians H_i, or, more generally, with Hamiltonian $F(H_1, ..., H_n)$, where F is any given smooth function.

2.5. **Constructing integrable systems by Hamiltonian reduction.** A powerful method of constructing integrable systems is Hamiltonian reduction. Namely, let M be a symplectic manifold, and let $H_1, ..., H_n$ be smooth functions on M such that $\{H_i, H_j\} = 0$ and dH_i are linearly independent everywhere. Assume that M carries a symplectic action of a Lie group G with moment map $\mu : M \to \mathfrak{g}^*$, which preserves the functions H_i, and let $\mathcal{O}$ be a coadjoint orbit of G. Assume that G acts freely on $\mu^{-1}(\mathcal{O})$, and that the joint level surfaces of H_i are transversal to G-orbits along $\mu^{-1}(\mathcal{O})$. In this case, the symplectic manifold $R(M, G, \mathcal{O})$ carries a collection of functions $H_1, ..., H_n$ such that $\{H_i, H_j\} = 0$ and dH_i are linearly independent everywhere. It sometimes happens that while $n < \frac{1}{2} \dim M$ (so $H_1, ..., H_n$ is **not** an integrable system on M), one has $n = \frac{1}{2} \dim R(M, G, \mathcal{O})$, so that $H_1, ..., H_n$ is an integrable system on $R(M, G, \mathcal{O})$. In this case the Hamiltonian flow defined by any Hamiltonian of the form $F(H_1, ..., H_n)$ on $R(M, G, \mathcal{O})$ can be solved in quadratures.

Remark. For this construction to work, it suffices to require that dH_i are linearly independent on a dense open set of $\mu^{-1}(\mathcal{O})$.

2.6. **The Calogero-Moser system.** A vivid example of constructing an integrable system by Hamiltonian reduction is the Kazhdan-Kostant-Sternberg construction of the Calogero-Moser system. In this case $M = T^*\mathrm{Mat}_n(\mathbb{C})$ (regarded as the set of pairs of matrices (X, Y) as in Subsection 1.5), with the usual symplectic form $\omega = \mathrm{Tr}(dY \wedge dX)$. Let $H_i = \mathrm{Tr}(Y^i)$, $i = 1, ...n$. These functions are obviously in involution, but they don't form an integrable system, because there are too few of them. However, let $G = PGL_n(\mathbb{C})$ act on M by conjugation, and let $\mathcal{O}$ be the coadjoint orbit of G considered in Subsection 1.5 (consisting of traceless matrices T such that $T + 1$ has rank 1). Then the system $H_1, ..., H_n$ descends to a system of functions in involution on $R(M, G, \mathcal{O})$, which is the Calogero-Moser space $\mathcal{C}_n$. Since this space is $2n$-dimensional, $H_1, ..., H_n$ is an integrable system on $\mathcal{C}_n$. It is called the (rational) Calogero-Moser system.

The Calogero-Moser flow is, by definition, the Hamiltonian flow on $\mathcal{C}_n$ defined by the Hamiltonian $H = H_2 = \mathrm{Tr}(Y^2)$. Thus this flow is integrable, in the sense that it can be included in an integrable system. In particular, its solutions can be found in quadratures using the inductive procedure of reduction of order. However (as often happens with systems obtained by reduction), solutions can also be found by a much simpler procedure, since they can be found already on the "non-reduced" space M: indeed, on M the Calogero-Moser flow is just the motion of a free particle in the space of matrices, so it has the form $g_t(X,Y) = (X + 2Yt, Y)$. The same formula is valid on $\mathcal{C}_n$. In fact, we can use the same method to compute the flows corresponding to all the Hamiltonians $H_i = \mathrm{Tr}(Y^i)$, $i \in \mathbb{N}$: these flows are given by the formulas

$$g_t^{(i)}(X,Y) = (X + iY^{i-1}t, Y).$$

2.7. **Coordinates on $\mathcal{C}_n$ and the explicit form of the Calogero-Moser system.** It seems that the result of our consideratons is trivial and we've gained nothing. To see that this is, in fact, not so, let us write the Calogero-Moser system explicitly in coordinates. To do so, we first need to introduce local coordinates on the Calogero-Moser space $\mathcal{C}_n$.

To this end, let us restrict our attention to the open set $U_n \subset \mathcal{C}_n$ which consists of conjugacy classes of those pairs (X,Y) for which the matrix X is diagonalizable, with distinct eigenvalues; by Wilson's theorem, this open set is dense in $\mathcal{C}_n$.

A point $P \in U_n$ may be represented by a pair (X,Y) such that $X = \mathrm{diag}(x_1, ..., x_n)$, $x_i \neq x_j$. In this case, the entries of $T := XY - YX$ are $(x_i - x_j)y_{ij}$. In particular, the diagonal entries are zero. Since the matrix $T + 1$ has rank 1, its entries κ_{ij} have the form $a_i b_j$ for some numbers a_i, b_j. On the other hand, $\kappa_{ii} = 1$, so $b_j = a_j^{-1}$ and hence $\kappa_{ij} = a_i a_j^{-1}$. By conjugating (X,Y) by the matrix $\mathrm{diag}(a_1, ..., a_n)$, we can reduce to the situation when $a_i = 1$, so $\kappa_{ij} = 1$. Hence the matrix T has entries $1 - \delta_{ij}$ (zeros on the diagonal, ones off the diagonal). Moreover, the representative of P with diagonal X and T as above is unique up to the action of the symmetric group S_n. Finally, we have $(x_i - x_j)y_{ij} = 1$ for $i \neq j$, so the entries of the matrix Y are $y_{ij} = \frac{1}{x_i - x_j}$ if $i \neq j$. On the other hand, the diagonal entries y_{ii} of Y are unconstrained. Thus we have obtained the following result.

Proposition 2.6. *Let $\mathbb{C}^n_{\mathrm{reg}}$ be the open set of $(x_1, ..., x_n) \in \mathbb{C}^n$ such that $x_i \neq x_j$ for $i \neq j$. Then there exists an isomorphism of algebraic varieties $\xi : T^*(\mathbb{C}^n_{\mathrm{reg}}/S_n) \to U_n$ given by the formula $(x_1, ...x_n, p_1, ..., p_n) \to (X,Y)$, where $X = \mathrm{diag}(x_1, ..., x_n)$, and $Y = Y(x,p) := (y_{ij})$,*

$$y_{ij} = \frac{1}{x_i - x_j}, i \neq j, \;\; y_{ii} = p_i.$$

In fact, we have a stronger result:

Proposition 2.7. *ξ is an isomorphism of symplectic varieties (where the cotangent bundle is equipped with the usual symplectic structure).*

Proof. Let $a_k = \mathrm{Tr}(X^k)$, $b_k = \mathrm{Tr}(X^k Y)$. It is easy to check that on M we have

$$\{b_m, a_k\} = m a_{m+k-1}.$$

On the other hand, $\xi^* a_k = \sum x_i^k$, $\xi^* b_k = \sum x_i^k p_i$. Thus we see that

$$\{b_m, a_k\} = \{\xi^* b_m, \xi^* a_k\}.$$

On the other hand, the functions a_k, b_k, $k = 0, ..., n-1$, form a local coordinate system near a generic point of U_n, so we are done. □

Now let us write the Hamiltonian of the Calogero-Moser system in coordinates. It has the form

$$H = \mathrm{Tr}(Y(x,p)^2) = \sum_i p_i^2 - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}. \tag{1}$$

Thus the Calogero-Moser Hamiltonian describes the motion of a system of n particles on the line with interaction potential $-1/x^2$. This is the form of the Calogero-Moser Hamiltonian in which it originally occured in the work of F. Calogero.

Now we finally see the usefulness of the Hamiltonian reduction procedure. The point is that it is not clear at all from formula (1) why the Calogero-Moser Hamiltonian should be completely integrable. However, our reduction procedure implies the complete integrability of H, and gives an explicit formula for the first integrals:

$$H_i = \mathrm{Tr}(Y(x,p)^i).$$

Moreover, this procedure immediately gives us an explicit solution of the system. Namely, assume that $x(t), p(t)$ is the solution with initial condition $x(0), p(0)$. Let $(X_0, Y_0) = \xi(x(0), p(0))$. Then $x_i(t)$ are the eigenvalues of the matrix $X_t := X_0 + 2tY_0$, and $p_i(t) = x_i'(t)$.

Remark. In fact, the reduction procedure not only allows us to solve the Calogero-Moser system, but also provides a "partial compactification" of its phase space $T^*\mathbb{C}^n_{\mathrm{reg}}$, namely the Calogero-Moser space $\mathcal{C}_n$, to which the Calogero-Moser flow smoothly extends. I think it is fair to say that $\mathcal{C}_n$ is "the right" phase space for the Calogero-Moser flow.

Exercise 2.8. Compute explicitly the integral H_3.

2.8. **The trigonometric Calogero-Moser system.** Another integrable system which can be obtained by a similar reduction procedure is the trigonometric Calogero-Moser system. To obtain it, take the same $M, G, \mathcal{O}$ as in the case of the rational Calogero-Moser system, but define $H_i^* := \mathrm{Tr}(Y_*^i)$, where $Y_* = XY$.

Proposition 2.9. *The functions H_i^* are in involution.*

Proposition 2.9 is a direct consequence of the following general and important but easy theorem.

Theorem 2.10. *(The necklace bracket formula) Let $a_1, ..., a_r$ and $b_1, ..., b_s$ be either X or Y. Then on M we have*

$$\{\,Tr(a_1...a_r),\, Tr(b_1...b_s)\} = \tag{2}$$
$$\sum_{(i,j):a_i=Y,b_j=X} Tr(a_{i+1}...a_r a_1...a_{i-1} b_{j+1}...b_s b_1...b_{j-1}) -$$
$$\sum_{(i,j):a_i=X,b_j=Y} Tr(a_{i+1}...a_r a_1...a_{i-1} b_{j+1}...b_s b_1...b_{j-1}).$$

Exercise 2.11. Prove Theorem 2.10 and deduce Proposition 2.9.

Thus the functions $H_1^*, ..., H_n^*$ define an integrable system on the Calogero-Moser space $\mathcal{C}_n$.

The trigonometric Calogero-Moser system is defined by the Hamiltonian $H = H_2^*$. Let us compute it more explicitly on the open set U_n using the coordinates x_i, p_i. We get $Y_* = (y_{*ij})$, where

$$y_{*ij} = \frac{x_i}{x_i - x_j}, i \neq j, \quad y_{*ii} = x_i p_i.$$

Thus we have

$$H = \sum_i (x_i p_i)^2 - \sum_{i\neq j} \frac{x_i x_j}{(x_i - x_j)^2}.$$

Let us introduce "additive" coordinates $x_{i*} = \log x_i$, $p_{i*} = x_i p_i$. It is easy to check that these coordinates are canonical. In them, the trigonometric Calogero-Moser Hamiltonian looks like

$$H = \sum_i p_{i*}^2 - \sum_{i\neq j} \frac{4}{\sinh^2((x_{i*} - x_{j*})/2)},$$

This Hamiltonian describes the motion of a system of n particles on the line with interaction potential $-4/\sinh^2(x/2)$.

Remark. By replacing x_{j*} with ix_{j*}, we can also integrate the system with Hamiltonian

$$H = \sum_i p_{i*}^2 + \sum_{i\neq j} \frac{4}{\sin^2((x_{i*} - x_{j*})/2)},$$

which corresponds to a system of n particles on the circle of length 2π with interaction potential $+4/\sin^2(x/2)$.

2.9. **Notes.** 1. For generalities on classical mechanics, symmetries of a mechanical system, reduction of order using symmetries, integrable systems, action-angle variables we refer the reader to [Ar]. Classical Calogero-Moser systems go back to the papers [Ca], [Mo]; their construction using reduction along orbit is due to Kazhdan, Kostant, and Sternberg, [KKS].

2. The necklace bracket formula is a starting point of noncommutative symplectic geometry; it appears in [Ko3]. This formula was generalized to the case of quivers in [BLB],[Gi].

3. Deformation theory

Before developing the quantum analogs of the notions and results of lectures 1 and 2, we need to discuss the general theory of quantization of Poisson manifolds. We start with an even more general discussion – the deformation theory of associative algebras.

3.1. **Formal deformations of associative algebras.** Let k be a field, and $K = k[[\hbar_1, ..., \hbar_n]]$. Let $\mathfrak{m} = (\hbar_1, ..., \hbar_n)$ be the maximal ideal in K.

Definition 3.1. A topologically free K-module is a K-module isomorphic to $V[[\hbar_1, ..., \hbar_n]]$ for some vector space V over k.

Let A_0 be an algebra[3] over k.

Definition 3.2. A (flat) formal n-parameter deformation of A_0 is an algebra A over K which is topologically free as a K-module, together with an algebra isomorphism $\eta_0 : A/\mathfrak{m}A \to A_0$.

When no confusion is possible, we will call A a deformation of A_0 (omitting "formal").

Let us restrict ourselves to one-parameter deformations with parameter $\hbar$. Let us choose an identification $\eta : A \to A_0[[\hbar]]$ as K-modules, such that $\eta = \eta_0$ modulo $\hbar$. Then the product in A is completely determined by the product of elements of A_0, which has the form of a "star-product"

$$\mu(a, b) = a * b = \mu_0(a, b) + \hbar\mu_1(a, b) + \hbar^2\mu_2(a, b) + ...,$$

where $\mu_i : A_0 \otimes A_0 \to A_0$ are linear maps, and $\mu_0(a, b) = ab$.

[3]In these lectures, by an algebra we always mean an associative algebra with unit

3.2. **Hochschild cohomology.** The main tool in deformation theory of associative algebras is Hochschild cohomology. Let us recall its definition.

Let A be an associative algebra. Let M be a bimodule over A. A Hochschild n-cochain of A with coefficients in M is a linear map $A^{\otimes n} \to M$. The space of such cochains is denoted by $C^n(A, M)$. The differential $d : C^n(A, M) \to C^{n+1}(A, M)$ is defined by the formula

$$df(a_1, ..., a_{n+1}) = f(a_1, ..., a_n)a_{n+1} - f(a_1, ..., a_n a_{n+1})$$

$$+f(a_1, ..., a_{n-1}a_n, a_{n+1}) - ... + (-1)^n f(a_1 a_2, ..., a_{n+1})+$$

$$(-1)^{n+1} a_1 f(a_2, ..., a_{n+1}).$$

It is easy to show that $d^2 = 0$.

Definition 3.3. The Hochschild cohomology $H^\bullet(A, M)$ to be the cohomology of the complex $(C^\bullet(A, M), d)$.

Proposition 3.4. *One has a natural isomorphism*

$$H^i(A, M) \to \mathrm{Ext}^i_{A-\mathrm{bimod}}(A, M),$$

where $A - \mathrm{bimod}$ *denotes the category of* A*-bimodules.*

Proof. The proof is obtained immediately by considering the bar resolution of the bimodule A:

$$... \to A \otimes A \otimes A \to A \otimes A \to A,$$

where the the bimodule structure on $A^{\otimes n}$ is given by

$$b(a_1 \otimes a_2 \otimes ... \otimes a_n)c = ba_1 \otimes a_2 \otimes ... \otimes a_n c,$$

and the map $\partial_n : A^{\otimes n} \to A^{\otimes n-1}$ is given by the formula

$$\partial_n(a_1 \otimes a_2 \otimes ... \otimes a_n) = a_1 a_2 \otimes ... \otimes a_n - ... + (-1)^{n-1} a_1 \otimes ... \otimes a_{n-1} a_n.$$

□

Note that we have the associative Yoneda product

$$H^i(A, M) \otimes H^j(A, N) \to H^{i+j}(A, M \otimes_A N),$$

induced by tensoring of cochains.

If $M = A$, the algebra itself, then we will denote $H^\bullet(A, M)$ by $H^\bullet(A)$. For example, $H^0(A)$ is the center of A, and $H^1(A)$ is the quotient of the Lie algebra of derivations of A by inner derivations. The Yoneda product induces a graded algebra structure on $H^\bullet(A)$; it can be shown that this algebra is supercommutative.

3.3. **Hochschild cohomology and deformations.** Let A_0 be an algebra, and let us look for 1-parameter deformations $A = A_0[[\hbar]]$ of A_0. Thus, we look for such series μ which satisfy the associativity equation, modulo the automorphisms of the $k[[\hbar]]$-module $A_0[[\hbar]]$ which are the identity modulo $\hbar$. [4]

The associativity equation $\mu \circ (\mu \otimes Id) = \mu \circ (Id \otimes \mu)$ reduces to a hierarchy of linear equations:

$$\sum_{s=0}^{N} \mu_s(\mu_{N-s}(a,b),c) = \sum_{s=0}^{N} \mu_s(a, \mu_{N-s}(b,c)). \tag{3}$$

(These equations are linear in μ_N if μ_i, $i < N$, are known).

To study these equations, one can use Hochschild cohomology. Namely, we have the following are standard facts (due to Gerstenhaber, [Ge2]), which can be checked directly.

1. The linear equation for μ_1 says that μ_1 is a Hochschild 2-cocycle. Thus algebra structures on $A_0[\hbar]/\hbar^2$ deforming μ_0 are parametrized by the space $Z^2(A_0)$ of Hochschild 2-cocycles of A_0 with values in $M = A_0$.

2. If μ_1, μ_1' are two 2-cocycles such that $\mu_1 - \mu_1'$ is a coboundary, then the algebra structures on $A_0[\hbar]/\hbar^2$ corresponding to μ_1 and μ_1' are equivalent by a transformation of $A_0[\hbar]/\hbar^2$ that equals the identity modulo $\hbar$, and vice versa. Thus equivalence classes of multiplications on $A_0[\hbar]/\hbar^2$ deforming μ_0 are parametrized by the cohomology $H^2(A_0)$.

3. The linear equation for μ_N says that $d\mu_N$ is a certain quadratic expression b_N in $\mu_0, \mu_1, ..., \mu_{N-1}$. This expression is always a Hochschild 3-cocycle, and the equation is solvable iff it is a coboundary. Thus the cohomology class of b_N in $H^3(A_0)$ is the only obstruction to solving this equation.

3.4. **Universal deformation.** In particular, if $H^3(A_0) = 0$ then the equation for μ_n can be solved for all n, and for each n the freedom in choosing the solution, modulo equivalences, is the space $H := H^2(A_0)$. Thus there exists an algebra structure over $k[[H]]$ on the space $A_u := A_0[[H]]$ of formal functions from H to A_0, $a, b \mapsto \mu_u(a,b) \in A_0[[H]]$, $(a, b \in A_0)$, such that $\mu_u(a,b)(0) = ab \in A_0$, and every 1-parameter flat formal deformation A of A_0 is given by the formula $\mu(a,b)(\hbar) = \mu_u(a,b)(\gamma(\hbar))$ for a unique formal series $\gamma \in \hbar H[[\hbar]]$, with the property that $\gamma'(0)$ is the cohomology class of the cocycle μ_1.

[4]Note that we don't have to worry about the existence of a unit in A since a formal deformation of an algebra with unit always has a unit.

Such an algebra A_u is called a **universal deformation** of A_0. It is unique up to an isomorphism (which may involve an automorphism of $k[[\hbar]]$).

Thus in the case $H^3(A_0) = 0$, deformation theory allows us to completely classify 1-parameter flat formal deformations of A_0. In particular, we see that the "moduli space" parametrizing formal deformations of A_0 is a smooth space – it is the formal neighborhood of zero in H.

If $H^3(A_0)$ is nonzero then in general the universal deformation parametrized by H does not exist, as there are obstructions to deformations. In this case, the moduli space of deformations will be a closed subscheme of H, which is often singular. On the other hand, even when $H^3(A_0) \neq 0$, the universal deformation parametrized by H may exist (although it may be more difficult to prove than in the vanishing case). In this case one says that the deformations of A_0 are **unobstructed** (since all obstructions vanish even though the space of obstructions doesn't).

3.5. **Quantization of Poisson algebras and manifolds.** An example when there are obstructions to deformations is the theory of quantization of Poisson algebras and manifolds.

Let M be a smooth C^∞-manifold or a smooth affine algebraic variety over $\mathbb{C}$, and $A_0 = \mathcal{O}(M)$ the structure algebra of M.

Remark. In the C^∞-case, we will consider only local maps $A_0^{\otimes n} \to A_0$, i.e. those given by polydifferential operators, and all deformations and the Hochschild cohomology is defined using local, rather than general, cochains.

Theorem 3.5. *(Hochschild-Kostant-Rosenberg)* [HKR] *One has* $H^i(A_0) = \Gamma(M, \wedge^i TM)$ *as a module over* $A_0 = H^0(A_0)$.

In particular, H^2 is the space of bivector fields, and H^3 the space of trivector fields. So the cohomology class of μ_1 is a bivector field; in fact, it is $\frac{1}{2}\pi$, where $\pi(a,b) := \mu_1(a,b) - \mu_1(b,a)$, since any 2-coboundary in this case is symmetric. The equation for μ_2 says that the $d\mu_2$ is a certain expression that depends quadratically on μ_1. It is easy to show that the cohomology class of this expression is the Schouten bracket $\frac{1}{4}[\pi,\pi]$. Thus, for the existence of μ_2 it is necessary that $[\pi,\pi] = 0$, i.e. that π be a **Poisson bracket**. In other words, the trivector field $[\pi,\pi] \in H^3(A_0)$ is an obstruction to extending the first order deformation $a * b = ab + \hbar\mu_1(a,b)$ to higher orders.

More generally, let A_0 be any commutative algebra, and $A = A_0[[\hbar]]$ be a not necessarily commutative (but associative) deformation of A_0. In this case, A_0 has a natural Poisson structure, given by the formula $\{a,b\} = [a',b']/\hbar \bmod \hbar$, where a', b' are any lifts of a, b to A. It is easy to check that this expression is independent on the choice of the lifts. In terms of the star-product, this bracket is given by the formula $\{a, b\} = \mu_1(a, b) - \mu_1(b, a)$.

Definition 3.6. In this situation, (A, μ) is said to be **a quantization** of $(A_0, \{,\})$, and $(A_0, \{,\})$ is said to be the **quasiclassical limit** of (A, μ).

Remark. If $A_0 = \mathcal{O}(M)$, one says that A is a quantization of M, and M the quasiclassical limit of A.

This raises the following important question. Suppose $A_0 = \mathcal{O}(M)$. Given a Poisson bracket π on M, is it always possible to construct its quantization?

By the above arguments, μ_2 exists (and a choice of μ_2 is unique up to adding an arbitrary bivector). So there arises the question of existence of μ_3 etc., i.e. the question whether there are other obstructions.

The answer to this question is yes and no. Namely, if you don't pick μ_2 carefully, you may be unable to find μ_3, but you can always pick μ_2 so that μ_3 exists, and there is a similar situation in higher orders. This subtle fact is a consequence of the following deep theorem of Kontsevich:

Theorem 3.7. [Ko1, Ko2] *Any Poisson structure π on A_0 can be quantized. Moreover, there is a natural bijection between products μ up to an isomorphism and Poisson brackets $\pi_0 + \hbar\pi_1 + \hbar^2\pi_2 + ...$, such that the quasiclassical limit of μ is π_0.*

Remarks. 1. The Kontsevich deformation quantization has an additional property called locality: the maps $\mu_i(f, g)$ are differential operators with respect to both f and g.

2. Note that, as was shown by O. Mathieu, a Poisson bracket on a general commutative $\mathbb{C}$-algebra may fail to admit a quantization.

Let us consider the special case of symplectic manifolds, i.e. the case when π is a nondegenerate bivector. In this case we can consider $\pi^{-1} = \omega$, which is a closed, nondegenerate 2-form (=symplectic structure) on M. In this case, Kontsevich's theorem is easier, and was proved by De Wilde - Lecomte, and later Deligne and Fedosov. Moreover, in this case there is the following additional result, also due to Kontsevich, [Ko1, Ko2].

Theorem 3.8. *If M is symplectic and A is a quantization of M, then the Hochschild cohomology $H^i(A[\hbar^{-1}])$ is isomorphic to $H^i(M, \mathbb{C}((\hbar)))$.*

Remark. Here the algebra $A[\hbar^{-1}]$ is regarded as a (topological) algebra over the field of Laurent series $\mathbb{C}((\hbar))$, so Hochschild cochains are, by definition, linear maps $A_0^{\otimes n} \to A_0((\hbar))$.

Example 3.9. The algebra $B = A[\hbar^{-1}]$ provides an example of an algebra with possibly nontrivial $H^3(B)$, for which the universal deformation parametrized by $H = H^2(B)$ exists. Namely, this deformation is attached through the correspondence of Theorem 3.7 (and inversion of $\hbar$) to the Poisson bracket $\pi = (\omega + t_1\omega_1 + ... + t_r\omega_r)^{-1}$, where $\omega_1, ..., \omega_r$ are closed 2-forms on M which represent a basis of $H^2(M, \mathbb{C})$, and $t_1, ..., t_r$ are the coordinates on H corresponding to this basis.

3.6. **Algebraic deformations.** Formal deformations of algebras often arise from algebraic deformations. The most naive definition of an algebraic deformation is as follows.

Let Σ be a smooth affine algebraic curve over k (often $\Sigma = k$ or $\Sigma = k^*$), and let $0 \in \Sigma(k)$. Let I_0 be the maximal ideal corresponding to 0.

Definition 3.10. An algebraic deformation over $B := k[\Sigma]$ of an algebra A_0 is a B-algebra A which is a free B-module, together with the identification $\eta_0 : A/I_0A \to A_0$ of the zero-fiber of A with A_0 as algebras.

Any algebraic deformation defines a formal deformation. Indeed, let $\hbar$ be a formal parameter of Σ near 0, and let $\widehat{A}$ be the completion of A with respect to I_0 (i.e., $\widehat{A} = \mathrm{limproj}_{n\to\infty} A/I_0^n A$). Then $\widehat{A}$ is a topologically free $k[[\hbar]]$-module which is a deformation of A_0.

Definition 3.11. An algebraic quantization of a Poisson algebra A_0 is an algebraic deformation A of A_0 such that the completion $\widehat{A}$ is a deformation quantization of A_0.

Example 3.12. 1. (Weyl algebra) $A_0 = k[x, p]$ with the usual Poisson structure, $\Sigma = k$, $A = k[\hbar, x, \hbar\partial_x]$.

2. (Generalization of 1) Let $\overline{A}$ be a filtered algebra: $k = F^0\overline{A} \subset F^1\overline{A} \subset ...$, $\cup_i F^i\overline{A} = \overline{A}$. Assume that $\mathrm{gr}\overline{A} = A_0$. Assume that A_0 is commutative. Then A_0 has a natural Poisson structure (why?). In this case, an algebraic quantization of A_0 is given by the so called *Rees algebra* A of $\overline{A}$. Namely, the algebra $A = \mathrm{Rees}(\overline{A})$ is defined by the formula $A = \oplus_{n=0}^{\infty} F^n\overline{A}$. This is an algebra over $k[\hbar]$, where $\hbar$ is the element 1 of the summand $F^1\overline{A}$. It is easy to see that A is an algebraic deformation of A_0 (with $\Sigma = k$).

An important example of this: X is a manifold, $\overline{A} = D(X)$, the algebra of differential operators on X, $F^\bullet$ is the filtration by order. In this case $A_0 = C^\infty_{pol}(T^*X)$, the space of smooth functions on T^*X which are polynomial along fiber (of uniformly bounded degree), with the usual Poisson structure.

Another important example: $\mathfrak{g}$ is a Lie algebra, $\overline{A} = U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, $F^\bullet$ is the natural filtration on $U(\mathfrak{g})$ which is defined by the condition that $\deg(x) = 1$ for $x \in \mathfrak{g}$. In this case A_0 is the Poisson algebra $S\mathfrak{g}$ with the Lie Poisson structure, and $A = U(\mathfrak{g}_\hbar)$, where $\mathfrak{g}_\hbar$ is the Lie algebra over $\mathbb{C}[\hbar]$ which is equal to $\mathfrak{g}[\hbar]$ as a vector space, with bracket $[a, b]_\hbar := \hbar[a, b]$.

3. (Quantum torus) $A_0 = k[x^{\pm 1}, y^{\pm 1}]$, with Poisson bracket $\{x, y\} = xy$, $A = k[q, q^{-1}] < x^{\pm 1}, y^{\pm 1} > /(xy = qyx)$, $\Sigma = k^*$.

3.7. **Notes.** 1. For generalities on Hochschild cohomology, see the book [Lo]. The basics of deformation theory of algebras are due to Gerstenhaber [Ge2].

2. The notion of deformation quantization was proposed in the classical paper [BFFLS]; in this paper the authors ask the question whether every Poisson manifold admits a quantization, which was solved positively by Kontsevich.

4. Quantum moment maps, quantum Hamiltonian reduction, and the Levasseur-Stafford theorem

4.1. **Quantum moment maps and quantum Hamiltonian reduction.** Now we would like to quantize the notion of a moment map. For simplicity let us work over $\mathbb{C}$. Recall that a classical moment map is defined for a Poisson manifold M with a Poisson action of a Lie group G, as a map $M \to \mathfrak{g}^*$ whose components are Hamiltonians defining the action of $\mathfrak{g} = Lie(G)$ on M. In algebraic terms, a Poisson manifold M with a G-action defines a Poisson algebra A_0 (namely, $C^\infty(M)$) together with a Lie algebra map $\phi_0 : \mathfrak{g} \to \mathrm{Der}_\Pi(A_0)$ from $\mathfrak{g}$ to the Lie algebra of derivations of A_0 preserving its Poisson bracket. A classical moment map is then a homomorphism of Poisson algebras $\mu_0 : S\mathfrak{g} \to A_0$ such that for any $a \in S\mathfrak{g}$, $b \in A_0$ one has $\{\mu_0(a), b\} = \phi_0(a)b$.

This algebraic reformulation makes it perfectly clear how one should define a quantum moment map.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra, and A be an associative algebra equipped with a $\mathfrak{g}$-action, i.e. a Lie algebra map $\phi : \mathfrak{g} \to \mathrm{Der}A$.

(i) A quantum moment map for (A, ϕ) is an associative algebra homomorphism $\mu : U(\mathfrak{g}) \to A$ such that for any $a \in \mathfrak{g}$, $b \in A$ one has $[\mu(a), b] = \phi(a)b$.

(ii) Suppose that A is a filtered associative algebra, such that $\mathrm{gr}A$ is a Poisson algebra A_0, equipped with a $\mathfrak{g}$-action ϕ_0 and a classical moment map μ_0. Suppose that $\mathrm{gr}\phi = \phi_0$. A quantization of μ_0 is a quantum moment map $\mu : U(\mathfrak{g}) \to A$ such that $\mathrm{gr}\mu = \mu_0$.

(iv) More generally, suppose that A is a deformation quantization of a Poisson algebra A_0 equipped with a $\mathfrak{g}$-action ϕ_0 and a classical moment map μ_0. Suppose that $\phi = \phi_0 \mathrm{mod} \hbar$. A quantization of μ_0 is a quantum moment map $\mu : U(\mathfrak{g}) \to A[\hbar^{-1}]$ such that for $a \in \mathfrak{g}$ we have $\mu(a) = \hbar^{-1}\mu_0(a) + O(1)$.

Thus, a quantum moment map is essentially a homomorphism of Lie algebras $\mu : \mathfrak{g} \to A$. Note that like in the classical case, the action ϕ is determined by the moment map μ.

Example 4.2. Let X be a manifold with an action of a Lie group G, and $A = D(X)$ be the algebra of differential operators on X. There is a natural homomorphism $\mu : \mathfrak{g} \to \mathrm{Vect}X \subset D(X)$ which is a quantum moment map for the natural action of $\mathfrak{g}$ on A. It is a quantization of the classical moment map for the action of G on T^*X defined in Section 1.

4.2. **Quantum hamiltonian reduction.** The algebraic definition of Hamiltonian reduction given in Section 1 is easy to translate to the quantum situation. Namely, let A be an algebra with a $\mathfrak{g}$-action and a quantum moment map $\mu : U(\mathfrak{g}) \to A$. The space of $\mathfrak{g}$-invariants $A^{\mathfrak{g}}$, i.e. elements $b \in A$ such that $[\mu(a), b] = 0$ for all $a \in \mathfrak{g}$, is a subalgebra of A. Let $J \subset A$ be the left ideal generated by $\mu(a)$, $a \in \mathfrak{g}$. Then J is not a 2-sided ideal, but $J^{\mathfrak{g}} := J \cap A^{\mathfrak{g}}$ is a 2-sided ideal in $A^{\mathfrak{g}}$.

Indeed, let $c \in A^{\mathfrak{g}}$, and $b \in J^{\mathfrak{g}}$, $b = \sum_i b_i\mu(a_i)$, $b_i \in A, a_i \in \mathfrak{g}$. Then $bc = \sum b_i\mu(a_i)c = \sum b_i c\mu(a_i) \in J^{\mathfrak{g}}$.

Thus, the algebra $A//\mathfrak{g} := A^{\mathfrak{g}}/J^{\mathfrak{g}}$ is an associative algebra, which is called the quantum Hamiltonian reduction of A with respect to the quantum moment map μ. [5]

An easy example of quantum Hamiltonian reduction is given by the following exercise.

Exercise 4.3. Let X be a smooth affine algebraic variety with a free action of a connected reductive algebraic group G. Let $A = D(X)$, and $\mu : \mathfrak{g} \to \mathrm{Vect}X \to D(X)$ be the usual action map. Show that $A//\mathfrak{g} = D(X/G)$.

[5] If the Lie algebra $\mathfrak{g}$ is not reductive, or its action on A is not locally finite, this definition may be too naive to give good results. In this case, instead of taking $\mathfrak{g}$-invariants (which is not an exact functor) one may need to include all its derived functors, i.e. cohomology (Ext^i). Moreover, if $\mathfrak{g}$ is infinite dimensional (Virasoro, affine Kac-Moody), and A is a vertex algebra rather than a usual associative algebra, which is an important case in string theory, then one needs to consider "semi-infinite" cohomology, i.e. $i = \infty/2 + j$, $j \in \mathbb{Z}$.

4.3. **Quantum reduction in the deformational setting.** In the deformation theoretical setting, the construction of the quantum Hamiltonian reduction should be slightly modified. Namely, assume that A is a deformation quantization of an algebra $A_0 = C^\infty(M)$, where M is a Poisson manifold, and $\mu_0 : \mathfrak{g} \to A_0$ is a classical moment map. Suppose that μ is a quantum moment map which is a quantization of μ_0. Let $\mathcal{O} \subset \mathfrak{g}^*$ be an invariant closed set, and $R(M, G, \mathcal{O})$ be the corresponding classical reduction.

Let $\mathfrak{g}_\hbar$ be the Lie algebra over $\mathbb{C}[[\hbar]]$, which is $\mathfrak{g}[[\hbar]]$ as a vector space, with Lie bracket $[a, b]_\hbar = \hbar[a, b]$ $(a, b \in \mathfrak{g})$. Let $U(\mathfrak{g}_\hbar)$ be the enveloping algebra of $\mathfrak{g}_\hbar$; it is a deformation quantization of $S\mathfrak{g}$ (which is a completion of the Rees algebra of $U(\mathfrak{g})$). We have a modified quantum moment map $\mu_\hbar : U(\mathfrak{g}_\hbar) \to A$ given by $\mu_\hbar(a) = \hbar\mu(a)$ for $a \in \mathfrak{g}$.

Let $I \subset U(\mathfrak{g}_\hbar)$ be an ideal deforming the ideal $I_0 \subset S\mathfrak{g}$ of functions vanishing on $\mathcal{O}$. Note that I does not necessarily exist, but it does exist in many important cases, e.g. when $\mathcal{O}$ is a semisimple orbit of a reductive Lie algebra. Then we define the quantum reduction $R(A, \mathfrak{g}, I) := A^\mathfrak{g}/(A\mu_\hbar(I))^\mathfrak{g}$ (quotient by an $\hbar$-adically closed ideal).

It is clear that the algebra $R(A, \mathfrak{g}, I)$ is a deformation of the function algebra on $R(M, G, \mathcal{O})$, but this deformation may not be flat. If it is flat (which happens in nice cases), one says that "quantization commutes with reduction".

4.4. **The Levasseur-Stafford theorem.** In general, similarly to the classical case, it is rather difficult to compute the quantum reduction $A//\mathfrak{g}$. For example, in this subsection we will describe $A//\mathfrak{g}$ in the case when $A = D(\mathfrak{g})$ is the algebra of differential operators on a reductive Lie algebra $\mathfrak{g}$, and $\mathfrak{g}$ acts on A through the adjoint action on itself. This description is a very nontrivial result of Levasseur and Stafford.

To describe $D(\mathfrak{g})//\mathfrak{g}$, we will construct a homomorphism $HC : D(\mathfrak{g})^\mathfrak{g} \to D(\mathfrak{h})^W$, called the Harish-Chandra homomorpism (as it was first constructed by Harish-Chandra). Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Recall that we have the classical Harish-Chandra isomorphism $\zeta : \mathbb{C}[\mathfrak{g}]^\mathfrak{g} \to \mathbb{C}[\mathfrak{h}]^W$, defined simply by restricting $\mathfrak{g}$-invariant functions on $\mathfrak{g}$ to the Cartan subalgebra $\mathfrak{h}$. Using this isomorphism, we can define an action of $D(\mathfrak{g})^\mathfrak{g}$ on $\mathbb{C}[\mathfrak{h}]^W$, which is clearly given by W-invariant differential operators. However, these operators will, in general, have poles on the reflection hyperplanes. Thus we get a homomorphism $HC' : D(\mathfrak{g})^\mathfrak{g} \to D(\mathfrak{h}_{\text{reg}})^W$, where $\mathfrak{h}_{\text{reg}}$ is the complement of the reflection hyperplanes in $\mathfrak{h}$.

The homomorphism HC' is called the radial part homomorphism, as for example for $\mathfrak{g} = \mathfrak{su}(2)$ it computes the radial parts of rotationally invariant differential operators on $\mathbb{R}^3$ in spherical coordinates. This homomorphism is not yet what we want, since it does not actually land in $D(\mathfrak{h})^W$ (the radial parts have poles).

Thus we define the Harish-Chandra homomorphism by twisting HC' by the discriminant $\delta(x) = \prod_{\alpha>0}(\alpha, x)$ ($x \in \mathfrak{h}$, and α runs over positive roots of $\mathfrak{g}$):

$$HC(D) := \delta \circ HC'(D) \circ \delta^{-1} \in D(\mathfrak{h}_{\mathrm{reg}})^W.$$

Let $\mathfrak{h}_{reg}$ denote the set of regular points in $\mathfrak{h}$, i.e. the complement of the reflection hyperhplanes.

Theorem 4.4. *(i) (Harish-Chandra,* [HC]*) For any reductive $\mathfrak{g}$, HC lands in $D(\mathfrak{h})^W \subset D(\mathfrak{h}_{reg})^W$.*

(ii) (Levasseur-Stafford [LS]*) The homomorphism HC defines an isomorphism $D(\mathfrak{g})//\mathfrak{g} = D(\mathfrak{h})^W$.*

Remarks. 1. Part (i) of the theorem says that the poles magically disappear after conjugation by δ.

2. Both parts of this theorem are quite nontrivial. The first part was proved by Harish-Chandra using analytic methods, and the second part by Levasseur and Stafford using the theory of D-modules.

In the case $\mathfrak{g} = \mathfrak{gl}_n$, Theorem 4.4 is a quantum analog of Theorem 1.17. The remaining part of this subsection is devoted to the proof of Theorem 4.4 in this special case, using Theorem 1.17.

We start the proof with the following proposition, valid for any reductive Lie algebra.

Proposition 4.5. *If $D \in (S\mathfrak{g})^{\mathfrak{g}}$ is a differential operator with constant coefficients, then $HC(D)$ is the W-invariant differential operator with constant coefficients on $\mathfrak{h}$, obtained from D via the classical Harish-Chandra isomorphism $\eta : (S\mathfrak{g})^{\mathfrak{g}} \to (S\mathfrak{h})^W$.*

Proof. Without loss of generality, we may assume that $\mathfrak{g}$ is simple.

Lemma 4.6. *Let D be the Laplacian $\Delta_{\mathfrak{g}}$ of $\mathfrak{g}$, corresponding to an invariant form. Then $HC(D)$ is the Laplacian $\Delta_{\mathfrak{h}}$.*

Proof. let us calculate $HC'(D)$. We have

$$D = \sum_{i=1}^{r} \partial_{x_i}^2 + 2 \sum_{\alpha>0} \partial_{f_\alpha} \partial_{e_\alpha},$$

where x_i is an orthonormal basis of $\mathfrak{h}$, and e_α, f_α are root elements such that $(e_\alpha, f_\alpha) = 1$. Thus if $F(x)$ is a $\mathfrak{g}$-invariant function on $\mathfrak{g}$, then we get

$$(DF)|_{\mathfrak{h}} = \sum_{i=1}^{r} \partial_{x_i}^2 (F|_{\mathfrak{h}}) + 2\sum_{\alpha>0} (\partial_{f_\alpha}\partial_{e_\alpha} F)|_{\mathfrak{h}}.$$

Now let $x \in \mathfrak{h}$, and consider $(\partial_{f_\alpha}\partial_{e_\alpha}F)(x)$. We have

$$(\partial_{f_\alpha}\partial_{e_\alpha}F)(x) = \partial_s\partial_t|_{s=t=0}F(x + tf_\alpha + se_\alpha).$$

On the other hand, we have

$$\mathrm{Ad}(e^{s\alpha(x)^{-1}e_\alpha})(x + tf_\alpha + se_\alpha) = x + tf_\alpha + ts\alpha(x)^{-1}h_\alpha + ...,$$

where $h_\alpha = [e_\alpha, f_\alpha]$. Hence,

$$\partial_s\partial_t|_{s=t=0}F(x + tf_\alpha + se_\alpha) = \alpha(x)^{-1}(\partial_{h_\alpha}F)(x).$$

This implies that

$$HC'(D)F(x) = \Delta_{\mathfrak{h}}F(x) + 2\sum_{\alpha>0} \alpha(x)^{-1}\partial_{h_\alpha}F(x).$$

Now the statement of the Lemma reduces to the identity

$$\delta^{-1} \circ \Delta_{\mathfrak{h}} \circ \delta = \Delta_{\mathfrak{h}} + 2\sum_{\alpha>0} \alpha(x)^{-1}\partial_{h_\alpha}.$$

This identity follows immediately from the identity

$$\Delta_{\mathfrak{h}}\delta = 0.$$

To prove the latter, it suffices to note that δ is the lowest degree nonzero polynomial on $\mathfrak{h}$, which is antisymmetric under the action of W. The lemma is proved. $\square$

Now let D be any element of $(S\mathfrak{g})^{\mathfrak{g}} \subset D(\mathfrak{g})^{\mathfrak{g}}$ of degree d (operator with constant coefficients). It is obvious that the leading order part of the operator $HC(D)$ is the operator $\eta(D)$ with constant coefficients, whose symbol is just the restriction of the symbol of D from $\mathfrak{g}^*$ to $\mathfrak{h}^*$. Our job is to show that in fact $HC(D) = \eta(D)$. To do so, denote by Y the difference $HC(D) - \eta(D)$. Assume $Y \neq 0$. By Lemma 4.6, the operator $HC(D)$ commutes with $\Delta_{\mathfrak{h}}$. Therefore, so does Y. Also Y has homogeneity degree d but order $m \leq d - 1$. Let $S(x, p)$ be the symbol of Y ($x \in \mathfrak{h}, p \in \mathfrak{h}^*$). Then S is a homogeneous function of homogeneity degree d under the transformations $x \to t^{-1}x$, $p \to tp$, polynomial in p of degree m. From these properties of S it is clear that S is not a polynomial (its degree in x is $m - d < 0$). On the other hand, since Y commutes with $\Delta_{\mathfrak{h}}$, the Poisson bracket of S with p^2 is zero. Thus Proposition 4.5 follows from the following lemma.

Lemma 4.7. *Let* $S : (x,p) \mapsto S(x,p)$ *be a rational function on* $\mathfrak{h} \oplus \mathfrak{h}^*$*, which is polynomial in* $p \in \mathfrak{h}^*$*. Suppose that the Poisson bracket* $\{p^2, S\}$ *equals to zero. Then* S *is a polynomial:* $S \in \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$.

Proof. We may assume without loss of generality that the function S is homogeneous in the p-variable, say of degree N. By shifting x, we may assume further that S is regular at $x = 0$. Let $S = \sum_{k=0}^{\infty} S_k$ be the Taylor expansion of S at the point $(0,0) \in \mathfrak{h} \oplus \mathfrak{h}^*$. Thus, S_k is a homogeneous polynomial on $\mathfrak{h} \oplus \mathfrak{h}^*$ of total degree k. Clearly, S_k has degree N in the p-variable. Further, separating homogeneous components in the equation $\{p^2, S\} = 0$ yields: $\{p^2, S_k\} = 0$, for any $k \geq 0$.

The quadratic functions: $E(x,p) = p^2$, $H(x,p) = \langle x, p\rangle$, and $F(x,p) = x^2$, $(x,p) \in \mathfrak{h} \times \mathfrak{h}^*$, are well-known to form an sl_2-triple: $\{E, H, F\}$, with respect to the Poisson bracket on $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$. Hence, taking the brackets with these functions gives, for each $m \geq 0$, an sl_2-action on $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{(m)}$, the space of degree m homogeneous polynomials. By representation theory of sl_2, any vector annihilated by E (i.e. a highest weight vector) has to have a nonnegative weight. This means that the p-degree of such a polynomial is greater than or equal to its x-degree. In particular, the x-degree of S_k is $\leq N$, which means that its total degree is $\leq 2N$. Therefore, for $k > 2N$, we have $S_k = 0$. This implies the lemma. ∎

Thus Proposition 4.5 is proved. ∎

Now we continue the proof of Theorem 4.4. Consider the filtration on $D(\mathfrak{g})$ in which $\deg(\mathfrak{g}) = \deg(\mathfrak{g}^*) = 1$ (the Bernstein filtration), and the associated graded map $\mathrm{gr}HC : \mathbb{C}[\mathfrak{g} \times \mathfrak{g}^*]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*]^W$, which attaches to every differential operator the symbol of its radial part. It is easy to see that this map is just the restriction map to $\mathfrak{h} \oplus \mathfrak{h}^* \subset \mathfrak{g} \oplus \mathfrak{g}^*$, so it actually lands in $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$.

Moreover, $\mathrm{gr}HC$ is a map **onto** $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. Indeed, $\mathrm{gr}HC$ is a Poisson map, so the surjectivity follows from the following Lemma.

Lemma 4.8. $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ *is generated as a Poisson algebra by* $\mathbb{C}[\mathfrak{h}]^W$ *and* $\mathbb{C}[\mathfrak{h}^*]^W$*, i.e. by functions* $f_m = \sum x_i^m$ *and* $f_m^* = \sum p_i^m$*,* $m \geq 1$.

Proof. For the proof we need the following theorem due to H. Weyl (from his book "Classical groups").

Theorem 4.9. *Let* B *be an algebra over* $\mathbb{C}$*. Then the algebra* $S^n B$ *is generated by elements of the form*

$$b \otimes 1 \otimes ... \otimes 1 + 1 \otimes b \otimes ... \otimes 1 + ... + 1 \otimes 1 \otimes ... \otimes b.$$

Proof. Since S^nB is linearly spanned by elements of the form $x\otimes ...\otimes x$, $x \in B$, it suffices to prove the theorem in the special case $B = \mathbb{C}[x]$. In this case, the result is simply the fact that the ring of symmetric functions is generated by power sums, which is well known. □

Now, we have $\{f_m^*, f_r\} = mr\sum x_i^{r-1}p_i^{m-1}$, and by Weyl's theorem (applied to $B = \mathbb{C}[x, y]$, such functions generate $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ as an algebra). The lemma is proved. □

Let K_0 be the kernel of grHC. Then by Theorem 1.17, K_0 is the ideal of the commuting scheme Comm$(\mathfrak{g})/G$.

Now consider the kernel K of the homomorphism HC. It is easy to see that $K \supset J^{\mathfrak{g}}$, so $\mathrm{gr}(K) \supset \mathrm{gr}(J)^{\mathfrak{g}}$. On the other hand, since K_0 is the ideal of the commuting scheme, we clearly have $\mathrm{gr}(J)^{\mathfrak{g}} \supset K_0$, and $K_0 \supset \mathrm{gr}K$. This implies that $K_0 = \mathrm{gr}K = \mathrm{gr}(J)^{\mathfrak{g}}$, and $K = J^{\mathfrak{g}}$.

It remains to show that $\mathrm{Im}HC = D(\mathfrak{h})^W$. Since $\mathrm{gr}K = K_0$, we have $\mathrm{grIm}HC = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. Therefore, to finish the proof of the Harish-Chandra and Levasseur-Stafford theorems, it suffices to prove the following proposition.

Proposition 4.10. $\mathrm{Im}HC \supset D(\mathfrak{h})^W$.

Proof. We will use the following Lemma.

Lemma 4.11. *(N. Wallach,* [Wa]*) $D(\mathfrak{h})^W$ is generated as an algebra by W-invariant functions and W-invariant differential operators with constant coefficients.*

Proof. The lemma follows by taking associated graded algebras from Lemma 4.8. □

Remark 4.12. Levasseur and Stafford showed [LS] that this lemma is valid for any finite group W acting on a finite dimensional vector space $\mathfrak{h}$. However, the above proof does not apply, since, as explained in [Wa], Lemma 4.8 fails for many groups W, including Weyl groups of exceptional Lie algebras E_6, E_7, E_8 (in fact it even fails for the cyclic group of order > 2 acting on a 1-dimensional space!). The general proof is more complicated and uses some results in noncommutative algebra.

Lemma 4.11 and Proposition 4.5 imply Proposition 4.10. □

Thus, Theorem 4.4 is proved.

Exercise 4.13. Let $\mathfrak{g}_{\mathbb{R}}$ be the compact form of $\mathfrak{g}$, and $\mathcal{O}$ a regular coadjoint orbit in $\mathfrak{g}_{\mathbb{R}}^*$. Consider the function

$$\psi_{\mathcal{O}}(x) = \int_{\mathcal{O}} e^{i(b,x)}db,\ x \in \mathfrak{h},$$

where db is the measure on the orbit coming from the Kirillov-Kostant symplectic structure. Prove the Harish-Chandra formula

$$\psi_{\mathcal{O}}(x) = \delta^{-1}(x) \sum_{w \in W} (-1)^{l(w)} e^{i(w\lambda, x)},$$

where λ is the intersection of $\mathcal{O}$ with the dominant chamber in the dual Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}^* \subset \mathfrak{g}_{\mathbb{R}}^*$, and $l(w)$ is the length of an element $w \in W$.

Deduce from this the Kirillov character formula for finite dimensional representations [Ki]: If λ is a dominant integral weight, and L_λ is the corresponding representation of G, then

$$\mathrm{Tr}_{L_\lambda}(e^x) = \frac{\delta(x)}{\delta_{tr}(x)} \int_{\mathcal{O}_{\lambda+\rho}} e^{i(b,x)} db,$$

where $\delta_{tr}(x)$ is the trigonometric version of $\delta(x)$, i.e. the Weyl denominator $\prod_{\alpha>0}(e^{\alpha(x)/2} - e^{-\alpha(x)/2})$, ρ is the half-sum of positive roots, and $\mathcal{O}_\mu$ denotes the coadjoint orbit passing through μ.

Hint. Use Proposition 4.5 and the fact that $\psi_{\mathcal{O}}$ is an eigenfunction of $D \in (S\mathfrak{g})^{\mathfrak{g}}$ with eigenvalue $\chi_{\mathcal{O}}(D)$, where $\chi_{\mathcal{O}}(D)$ is the value of the invariant polynomial D at the orbit $\mathcal{O}$.

4.5. **Hamiltonian reduction with respect to an ideal in $U(\mathfrak{g})$.** In Section 1, we defined Hamiltonian reduction along an orbit or, more generally, closed G-invariant subset of $\mathfrak{g}^*$. Algebraically a closed G-invariant subset of $\mathfrak{g}^*$ corresponds to a Poisson ideal in $S\mathfrak{g}$. Thus the quantum analog of this construction should be quantum Hamiltonian reduction with respect to a two-sided ideal $I \subset U(\mathfrak{g})$.

The Hamiltonian reduction with respect to $I \subset U(\mathfrak{g})$ is defined as follows. Let $\mu : U(\mathfrak{g}) \to A$ be a quantum moment map, and $J(I) = A\mu(I) \subset A$. Then $J(I)^{\mathfrak{g}}$ is a 2-sided ideal in $A^{\mathfrak{g}}$ (this is shown similarly to the case of the usual quantum hamiltonian reduction), and we set $R(A, \mathfrak{g}, I) := A^{\mathfrak{g}}/J(I)^{\mathfrak{g}}$. The usual quantum reduction described above is the special case of this, when I is the augmentation ideal.

The following example is a quantization of the Kazhdan-Kostant-Sternberg construction of the Calogero-Moser space given in Section 1.

Example 4.14. Let $\mathfrak{g} = \mathfrak{gl}_n$, $A = D(\mathfrak{g})$ as above. Let k be a complex number, and W_k be the representation of $\mathfrak{sl}_n$ on the space of functions of the form $(x_1 ... x_n)^k f(x_1, ..., x_n)$, where f is a Laurent polynomial of degree 0. We regard W_k as a $\mathfrak{g}$-module by pulling it back to $\mathfrak{g}$ under the natural projection $\mathfrak{g} \to \mathfrak{sl}_n$. Let I_k be the annihilator of W_k in $U(\mathfrak{g})$.

Let $B_k = R(A, \mathfrak{g}, I_k)$. Then B_k has a filtration induced from the Bernstein filtration of $D(\mathfrak{g})^{\mathfrak{g}}$. Let $HC_k : D(\mathfrak{g})^{\mathfrak{g}} \to B_k$ be the natural homomorphism, and $K(k)$ be the kernel of HC_k.

Theorem 4.15. *(Etingof-Ginzburg,* [EG]*) (i)* $K(0) = K$, $B_0 = D(\mathfrak{h})^W$, $HC_0 = HC$.

(ii) $\mathrm{gr} K(k) = \mathrm{Ker}\,\mathrm{gr} HC_k = K_0$ *for all complex* k*. Thus,* HC_k *is a flat family of homomorphisms.*

(iii) The algebra $\mathrm{gr} B_k$ *is commutative and isomorphic to* $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ *as a Poisson algebra.*

Definition 4.16. The algebra B_k is called the spherical rational Cherednik algebra. The homomorphism HC_k is called the deformed Harish-Chandra homomorphism.

The algebra B_k has an important representation on the space $\mathbb{C}[\mathfrak{h}_{reg}]$ of regular functions on the set $\mathfrak{h}_{reg}$ of diagonal matrices with different eigenvalues. To construct it, note that the algebra B_k acts naturally on the space E_k of $\mathfrak{g}$-equivariant functions with values in W_k on the formal neighborhood of $\mathfrak{h}_{reg}$ in $\mathfrak{g}_{reg}$ (where $\mathfrak{g}_{reg}$ is the set of all elements in $\mathfrak{g}$ conjugate to an element of $\mathfrak{h}_{reg}$, i.e. the set of matrices with different eigenvalues). Namely, the algebra $D(\mathfrak{g})^{\mathfrak{g}}$ acts on E_k, and the ideal I_k is clearly annihilated under this action.

Now, an equivaraint function on the formal neighborhood of $\mathfrak{h}_{reg}$ with values in W_k is completely determined by its values on $\mathfrak{h}_{reg}$ itself, and the only restriction for these values is that they are in $W_k[0]$, the zero-weight subspace of W_k. But the space $W_k[0]$ is 1-dimensional, spanned by the function $(x_1, ..., x_n)^k$. Thus the space E_k is naturally isomorphic to $\mathbb{C}[\mathfrak{h}_{reg}]$. Thus we have defined an action of B_k on $\mathbb{C}[\mathfrak{h}_{reg}]$.

The algebra B_k is one of the main objects of this course. It is, in an appropriate sense, a quantization of the Calogero-Moser space. We will discuss the precise formulation of this statement later.

4.6. **Notes.** 1. Quantum moment maps and quantum reduction have been considered for more than 20 years by many authors, in particular, in physics literature; this notion arises naturally when one considers BRST quantization of gauge theories. A convenient reference for us is the paper [Lu], which considers a more general setting of quantum group actions.

2. The "quantization commutes with reduction" conjecture was formulated by Guillemin and Sternberg in [GS]. In the case of a compact Lie group action and deformation quantization, it was proved by Fedosov [Fe].

5. Quantum mechanics, quantum integrable systems, and quantization of the Calogero-Moser system

5.1. **Quantum mechanics.** Before explaining the basic setting of quantum mechanics, let us present the basic setting of classical mechanics in a form convenient for quantization. In classical mechanics, we have the algebra of observables $C^\infty(M)$, where M is the phase space (a Poisson, usually symplectic, manifold) and the motion of a point $x = x(t) \in M$ is described by Hamilton's equation for observables:

$$\dot{f} = \{H, f\},$$

where $f = f(x(t))$ is an observable $f \in C^\infty(M)$, evaluated at $x(t)$.

Similarly, the basic setting of Hamiltonian quantum mechanics is as follows. We have a (noncommutative) algebra A of quantum observables, which acts (faithfully) in a space of states $\mathcal{H}$ (a complex Hilbert space). The Hamiltonian is an element H of A (i.e. an operator on $\mathcal{H}$, self-adjoint, and usually unbounded). The dynamics of the system $\psi = \psi(t)$, $\psi \in H$, is governed by the Schrödinger equation

$$\dot{\psi} = -\frac{iH\psi}{h},$$

where $h > 0$ is the Planck constant. Solutions of the Schrödinger equation have the form

$$\psi(t) = U(t)\psi(0),$$

where $U(t)$ is the evolution operator $e^{-iHt/h}$. This means that if $F \in A$ is any observable then its observed value at the state $\psi(t)$ is given by the formula

$$< \psi(t)|F|\psi(t) >=< \psi(0)|F(t)|\psi(0) >,$$

where $F(t) := e^{iHt/h} F e^{-iHt/h}$. The operator valued function $F(t)$ satisfies the Heisenberg equation

$$\dot{F}(t) = i[H, F(t)]/h.$$

Quantum systems considered in quantum mechanics are usually deformations of certain classical mechanical systems, which are recovered when the Planck constant goes to zero. To study this limit, it is convenient to introduce [6]$\hbar = -ih$, and assume that $\hbar$ is a formal parameter rather than a numerical constant. (In physics such approach is called "perturbation theory"). More specifically, we assume that the algebra

[6]This notation is sacrilegious, as in all textbooks on quantum mechanics h and $\hbar$ differ by 2π

A of quantum observables is a formal deformation of the algebra of classical observables $A_0 = C^\infty(M)$ (i.e., A is a deformation quantization of the phase space M), and the Hamiltonian $H \in A$ is a deformation of a classical Hamiltonian $H_0 \in A_0$. In this case, we see that the Heisenberg equation is nothing but a deformation of the Hamilton's equation for observables. Thus this framework can indeed be used to regard quantum mechanics as a deformation of classical mechanics.

In many situations, $M = T^*X$, where X is a Riemannian manifold, $H_0 = \frac{p^2}{2} + U(x)$, where $U(x)$ is a potential. In this case, in the classical setting we can restrict ourselves to fiberwise polynomial functions: $A_0 = C^\infty_{pol}(X)$. Then in the quantum setting we can work over polynomials $\mathbb{C}[\hbar]$ rather than formal series $\mathbb{C}[[\hbar]]$ (which is good since we can then specialize $\hbar$ to a numerical value), and we have: $A = \mathrm{Rees}(D(X))$, and $H = -\frac{h^2}{2}\Delta + U(x)$, where Δ is the Laplacian. Then A is a quantization of $A_0 = C^\infty_{pol}(T^*X)$, and H is a quantization of H_0 (the minus sign comes from the fact that i has been absorbed into $\hbar$). Finally, the space $\mathcal{H}$ is $L^2(X)$; the algebra $A/(\hbar = -ih_0) = D(X)$, $h_0 \in \mathbb{R}_+$ acts in this space and is the algebra of quantum observables.

With these conventions, the Schrödinger equation takes the form

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2}\Delta\psi + U(x)\psi,$$

which is the classical form of the Schrödinger equation (for a particle of unit mass).

5.2. **Quantum integrable systems.** The above interpretation of quantum mechanics as a deformation of classical mechanics motivates the following definition of a quantum integrable system.

Let $A_0 = C^\infty(M)$, where M is a symplectic manifold (or $A_0 = \mathbb{C}[M]$, where M is a symplectic affine algebraic variety [7]). Assume that M has dimension $2n$. Let A be a quantization of A_0 (formal or algebraic).

Definition 5.1. A quantum integrable system in A is a pairwise commuting system of elements $H_1, ..., H_n$ such that their reductions $H_{1,0}, ..., H_{n,0}$ to A_0 form a classical integrable system on M.

In this situation, we say that the quantum integrable $H_1, ..., H_n$ is a quantization of the classical integrable system $H_{1,0}, ..., H_{n,0}$, and conversely, the classical system is the quasiclassical limit of the quantum system. Also, if we have a quantum mechanical system defined by a Hamiltonian H which is included in a quantum integrable system

[7]If $M = T^*X$, we will often take the subalgebra $A_0 = C^\infty_{pol}(T^*X)$ instead of the full algebra $C^\infty(X)$.

$H = H_1, ..., H_n$ then we say that $H_1, ..., H_n$ are quantum integrals of H.

Remark. It is obvious that if $H_1, ..., H_n$ is an integrable system, then they are algebraically independent.

Example 5.2. $M = T^*X$, $A_0 = C^\infty_{pol}(X)$, $A = \mathrm{Rees}(D(X)) \subset D(X)[\hbar]$. Thus a quantum integrable system in A, upon evaluation $\hbar \to 1$, is just a collection of commuting differential operators $H_1, ..., H_n$ on X.

The motivation for this definition is the same as in the classical case. Namely, if $H = H_1, H_2, ..., H_n$ is an integrable system, then the Schrödinger equation $\dot{\psi} = -H\psi/\hbar$ can usually be solved "explicitly".

More specifically, recall that to solve the Schrödinger equation, it is sufficient to find an eigenbasis ψ_m for H. Indeed, if λ_m are the eigenvalues of H at ψ_m, and $\psi(0) = \sum_m a_m\psi_m$, $a_m \in \mathbb{C}$, then $\psi(t) = \sum_m a_m e^{-i\lambda_m t/h}\psi_m$ (the index m may be continuous, and then instead of the sum we will have to use integral).

But in the problem of finding an eigenbasis, it is very useful to have operators commuting with H: then we can replace the eigenvalue problem $H_1\psi = \lambda_1\psi$ with the joint eigenvalue problem

$$H_i\psi = \lambda_i\psi, i = 1, ..., n, \tag{4}$$

whose solutions are easier to find.

To illustrate this, look at the situation of Example 5.2. An integrable system in this example is an algebraically independent collection of differential operators $H_1, ..., H_n$ on X. Let $S_i(x) := S_x(H_i) \in \mathbb{C}[T^*_xX] = S(T_xX)$ be the symbols of the operators H_i (homogeneous polynomials), and assume that the algebra $\mathbb{C}[T^*_xX]$ is finitely generated as a module over $\mathbb{C}[S_1(x), ..., S_n(x)]$ (this is satisfied in interesting cases). In this case by a standard theorem in commutative algebra (see Section 10.5), $\mathbb{C}[T^*_xX]$ is a free module over $\mathbb{C}[S_1(x), ..., S_n(x)]$ of some rank r.

Proposition 5.3. *The system (4) has an r-dimensional space of solutions near each point of X.*

Proof. Let $P_1, ..., P_r$ be the free homogeneous generators of $\mathbb{C}[T^*_xX]$ over $\mathbb{C}[S_1(x_0), ..., S_n(x_0)]$, and $D_1, ..., D_r \in D(X)$ be liftings of $P_1, ..., P_r$. Then ψ is a solution of (4) iff the functions $D_1\psi$,...,$D_r\psi$ satisfy a first order linear holonomic system of differential equations near x_0 (i.e., represent a horizontal section of a flat connection):

$$d(D_i\psi) = \sum_j \omega_{ij} D_j\psi,$$

where $\omega = (\omega_{ij})$ is a matrix of 1-forms on X satisfying the Maurer-Cartan equation. Therefore, the space of solutions is r-dimensional (as solution is uniquely determined by the values of $D_i\psi(x_0)$). $\square$

Now we see the main difference between integrable and nonintegrable Hamiltonians H. Namely, we see from the proof of Proposition 5.3 that solutions of the eigenvalue problem (4) can be found by solving **ordinary** differential equations (computing holonomy of a flat connection), while in the nonitegrable situation $H\psi = \lambda\psi$ is a **partial** differential equation, which in general does not reduce to ODE. In the theory of PDE, we always regard reduction to ODE as an explicit solution. This justifies the above statement that the qunatum integrable systems, like the classical ones, can be solved "explicitly".

Remark. It can be argued that finding solutions of the eigenvalue problem (4) is the quantum analog of finding the action-angle variables of the quantum integrable system.

5.3. **Constructing quantum integrable systems by quantum Hamiltonian reduction.** As in the classical case, an effective way of constructing quantum integrable systems is quantum Hamiltonian reduction. We will describe this procedure in the setting of formal deformations; the case of algebraic deformations and deformations coming from filtrations is similar.

Namely, suppose we are in the setting of Subsection 2.5. That is, let A_0 be the function algebra on a symplectic (C^∞ or algebraic) manifold M, and suppose that $\mu_0 : \mathfrak{g} \to A_0$ is a classical moment map. Let $H_{1,0}$,..., $H_{n,0}$ be a Poisson commuting family of $\mathfrak{g}$-invariant elements of A_0, which reduces to an integrable system on $R(M, G, \mathcal{O})$. Suppose that A is a deformation quantization of A_0, and $\mu : \mathfrak{g} \to h^{-1}A$ is a quantum moment map quantizing μ_0 (so $\mu = \hbar^{-1}\mu_0 + O(1)$ on $\mathfrak{g}$).

Let I_0 be the ideal in A_0 of functions vanishing on $\mathcal{O}$, and I a deformation of I_0 to an ideal in $U(\mathfrak{g}_\hbar)$. Assume that $H_1, ..., H_n$ is a commuting system of $\mathfrak{g}$-invariant elements of A quantizing $H_{1,0}, ..., H_{n,0}$. Suppose that quantization commutes with reduction, i.e. that the algebra $R(A, \mathfrak{g}, I)$ is a quantization of the symplectic manifold $R(M, G, \mathcal{O})$. In this case, $H_1, ..., H_n$ descend to commuting elements in $R(A, \mathfrak{g}, I)$, which quantize $H_{1,0}, ..., H_{n,0}$, so they are a quantum integrable system. This system is called the quantum reduction of the original system $H_1, ..., H_n$ in A.

5.4. **The quantum Calogero-Moser system.** We now turn to the situation of Example 4.14. Thus $\mathfrak{g} = \mathfrak{gl}_n$ and $M = T^*\mathfrak{g} = \{(X, P)|X, P \in \mathfrak{g}\}$ (we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the trace form). Set H_i, $i = 1, ..., n$, to be the homogeneous differential operators with constant coefficients on $\mathfrak{g}$ with symbols $\mathrm{Tr}(P^i)$. According to the above, they descend to a quantum integrable system in the Cherednik algebra B_k.

Definition 5.4. This system (with $H = H_2$ being the Hamiltonian) is called the quantized Calogero-Moser system.

To make the quantized Calogero-Moser system more explicit, let us recall that the algebra B_k naturally acts on the space $\mathbb{C}[\mathfrak{h}_{reg}]$. Thus $H_1, ..., H_n$ define commuting differential operators on $\mathfrak{h}$ with poles along the reflection hyperplanes (it is easy to see that these operators are W-invariant). Denote these operators by $L'_1, ..., L'_n$. By analogy with the definition of the Harish-Chandra homomorphism, define $L_i = \delta \circ L'_i \circ \delta^{-1}$. It follows from the above that for $1 \le m \le n$, one has

$$L_m = \sum_{j=1}^{n} \partial_j^m + \text{lower order terms},$$

where the lower order terms are polynomial in k, and vanish as $k = 0$ (as for $k = 0$ one has $L_m = HC(H_m)$).

It is easy to see that $L_1 = \sum_{j=1}^n \partial_j$. Now let us calculate more explicitly the Hamiltonian $H = L_2$.

Theorem 5.5.

$$L_2 = \sum_{j=1}^{n} \partial_j^2 - \sum_{i \neq j} \frac{k(k+1)}{(x_i - x_j)^2}.$$

Proof. The proof is analogous to the proof of Lemma 4.6. Namely, let us first calculate L'_2.

Recall that the Casimir D of $\mathfrak{g}$ is given by the formula

$$D = \sum_{i=1}^{n} \partial_{x_i}^2 + 2 \sum_{\alpha > 0} \partial_{f_\alpha} \partial_{e_\alpha}.$$

Thus if $F(x)$ is a $\mathfrak{g}$-equivariant function on the formal neighborhood of $\mathfrak{h}_{reg}$ in $\mathfrak{g}_{reg}$ with values in W_k then we get

$$(DF)|_{\mathfrak{h}} = \sum_{i=1}^{n} \partial_{x_i}^2 (F|_{\mathfrak{h}}) + 2 \sum_{\alpha > 0} (\partial_{f_\alpha} \partial_{e_\alpha} F)|_{\mathfrak{h}}.$$

Now let $x \in \mathfrak{h}$, and consider $(\partial_{f_\alpha} \partial_{e_\alpha} F)(x)$. We have

$$(\partial_{f_\alpha} \partial_{e_\alpha} F)(x) = \partial_s \partial_t|_{s=t=0} F(x + t f_\alpha + s e_\alpha).$$

On the other hand, we have

$$\mathrm{Ad}(e^{s\alpha(x)^{-1} e_\alpha})(x + t f_\alpha + s e_\alpha) = x + t f_\alpha + t s \alpha(x)^{-1} h_\alpha + ...,$$

where $h_\alpha = [e_\alpha, f_\alpha]$. Hence,

$$F(x + tf_\alpha + se_\alpha) = e^{-s\alpha(x)^{-1}e_\alpha} F(x + tf_\alpha + ts\alpha(x)^{-1}h_\alpha + ...) =$$

$$e^{-s\alpha(x)^{-1}e_\alpha} e^{t\alpha(x)^{-1}f_\alpha} F(x + ts\alpha(x)^{-1}h_\alpha + ...).$$

Thus

$$\partial_s \partial_t|_{s=t=0} F(x + tf_\alpha + se_\alpha) = \alpha(x)^{-1}(\partial_{h_\alpha} F)(x) - \alpha(x)^{-2} e_\alpha f_\alpha F(x).$$

But $e_\alpha f_\alpha|_{W_k[0]} = k(k+1)$ (this is obtained by a direct computation using that $e_\alpha = x_i\partial_j, f_\alpha = x_j\partial_i$). This implies that

$$L_2' F(x) = \Delta_{\mathfrak{h}} F(x) + 2\sum_{\alpha>0} (\alpha(x)^{-1}\partial_{h_\alpha} F(x) - k(k+1)\alpha(x)^{-2} F(x)).$$

The rest of the proof is the same as in Lemma 4.6. □

Thus we see that the quantum Calogero-Moser system indeed describes a system of n quantum particles on the line with interaction potential $k(k+1)/x^2$.

Remark. Note that unlike the classical case, the coefficient in front of the potential is an essential parameter, and cannot be removed by rescaling.

5.5. **Notes.** 1. Quantum integrable systems have been studied for more than twenty years; let us mention, for instance, the paper [OP] which is relevant to the subject of these lectures. The construction of such systems by quantum reduction has also been known for a long time: a famous instance of such a construction is the Harish-Chandra-Helgason theory of radial parts of Laplace operators on symmetric spaces, see e.g. [He]. The quantum Calogero-Moser system (more precisely, its trigonometric deformation), appeared in [Su].

2. The construction of the quantum Calogero-Moser system by quantum reduction, which is parallel to the Kazhdan-Kostant-Strenberg construction, is discussed, for instance, in [EG]. The trigonometric version of the Calogero-Moser system (the Sutherland system) with Hamiltonian

$$L_2 = \sum_{j=1}^{n} \partial_j^2 - \sum_{i\neq j} \frac{k(k+1)}{\sin^2(x_i - x_j)}$$

is also integrable, and may be obtained by performing reduction from T^*G (rather than $T^*\mathfrak{g}$), both classically and quantum mechanically, mimicking the Kazhdan-Kostant-Sternberg construction. Another class of quantum integrable systems which can be obtained by reduction is the Toda systems; this is done in the paper by Kostant [Kos].

6. Calogero-Moser systems associated to finite Coxeter groups

It turns out that both the classical and the quantum Calogero-Moser system can be generalized to the case of any finite Coxeter group W, so that the case we have considered corresponds to the symmetric group S_n. This is done using Dunkl operators.

6.1. Dunkl operators.

Let W be a finite Coxeter group, and $\mathfrak{h}$ be its reflection representation, $\dim \mathfrak{h} = r$. Let S be the set of reflections in W. For any reflection $s \in S$, let $\alpha_s \in \mathfrak{h}^*$ be an eigenvector of s with eigenvalue -1. Then the reflection hyperplane of s is given by the equation $\alpha_s = 0$. Let $\alpha_s^\vee \in \mathfrak{h}$ be the -1-eigenvector of s such that $(\alpha_s, \alpha_s^\vee) = 2$.

Let $c : S \to \mathbb{C}$ be a function invariant with respect to conjugation. Let $a \in \mathfrak{h}$.

Definition 6.1. The Dunkl operator $D_a = D_a(c)$ on $\mathbb{C}(\mathfrak{h})$ is defined by the formula

$$D_a = D_a(c) := \partial_a - \sum_{s \in S} \frac{c_s \alpha_s(a)}{\alpha_s}(1 - s)$$

Clearly, $D_a \in \mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$.

Example 6.2. Let $W = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$. Then there is only one Dunkl operator up to scaling, and it equals to

$$D = \partial_x - \frac{c}{x}(1 - s),$$

where the operator s is given by the formula $(sf)(x) = f(-x)$.

Exercise 6.3. Show that D_a maps the space of polynomials $\mathbb{C}[\mathfrak{h}]$ to itself.

Proposition 6.4. *(i) For any $x \in \mathfrak{h}^*$, one has*

$$[D_a, x] = (a, x) - \sum_{s \in S} c_s (a, \alpha_s)(x, \alpha_s^\vee) s.$$

(ii) If $g \in W$ then $g D_a g^{-1} = D_{ga}$.

Proof. (i) The proof follows immediately from the identity $x - sx = (x, \alpha_s^\vee)\alpha_s$.

(ii) The identity is obvious from the invariance of the function c. □

The main result about Dunkl operators, on which all their applications are based, is the following theorem.

Theorem 6.5. *(C. Dunkl,* [Du]*) The Dunkl operators commute:* $[D_a, D_b] = 0$ *for any* $a, b \in \mathfrak{h}$*.*

Proof. Let $x \in \mathfrak{h}^*$. We have

$$[[D_a, D_b], x] = [[D_a, x], D_b] - [[D_b, x], D_a].$$

Now, using Proposition 6.4, we obtain:

$$[[D_a, x], D_b] = -[\sum_s c_s(a, \alpha_s)(x, \alpha_s^\vee)s, D_b] =$$

$$-\sum_s c_s(a, \alpha_s)(x, \alpha_s^\vee)(b, \alpha_s)sD_{\alpha_s^\vee}.$$

Since a and b occur symmetrically, we obtain that $[[D_a, D_b], x] = 0$. This means that for any $f \in \mathbb{C}[\mathfrak{h}]$, $[D_a, D_b]f = f[D_a, D_b]1 = 0$. □

6.2. **Olshanetsky-Perelomov operators.** Assume for simplicity that $\mathfrak{h}$ is an irreducible W-module. Fix an invariant inner product $(,)$ on $\mathfrak{h}$.

Definition 6.6. [OP] The Olshanetsky-Perelomov operator corresponding to a W-invariant function $c : S \to \mathbb{C}$ is the second order differential operator

$$L := \Delta_{\mathfrak{h}} - \sum_{s \in S} \frac{c_s(c_s + 1)(\alpha_s, \alpha_s)}{\alpha_s^2}.$$

It is obvious that L is a W-invariant operator.

Example 6.7. Suppose that $W = S_n$. Then there is only one conjugacy class of reflections, so the function c takes only one value. So L is the quantum Calogero-Moser Hamiltonian, with $c = k$.

Exercise 6.8. Write L explicitly for W of type B_n.

It turns out that the operator L defines a quantum integrable system. This fact was discovered by Olshanetsky and Perelomov (in the case when W is a Weyl group). We are going to give a simple proof of this fact, due to Heckman, based on Dunkl operators.

To do so, note that any element $B \in (\mathbb{C}W \ltimes D(\mathfrak{h}_{reg}))^W$ defines a linear operator on $\mathbb{C}(\mathfrak{h})$. Let $m(B)$ be the restriction of the operator B to the subspace of W-invariant functions, $\mathbb{C}(\mathfrak{h})^W$. It is clear that for any B, $m(B)$ is a differential operator. Therefore the assignment $B \mapsto m(B)$ defines a homomorphism $m : (\mathbb{C}W \ltimes D(\mathfrak{h}_{reg}))^W \to D(\mathfrak{h}_{reg})^W$.

Define the operator

$$\bar{L} := \Delta_{\mathfrak{h}} - \sum_{s \in S} \frac{c_s(\alpha_s, \alpha_s)}{\alpha_s}\partial_{\alpha_s^\vee}$$

Proposition 6.9. *(Heckman,* [Hec]*) Let $\{y_1, \dots y_r\}$ be an orthonormal basis of $\mathfrak{h}$. Then we have*

$$m(\sum_{i=1}^{r} D_{y_i}^2) = \bar{L}.$$

Proof. Let us extend the map m to $\mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$ by defining $m(B)$, $B \in \mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$, to be the (differential) operator $\mathbb{C}(\mathfrak{h})^W \to \mathbb{C}(\mathfrak{h})$ corresponding to B. Then we have $m(D_y^2) = m(D_y \partial_y)$. A simple computation shows that

$$D_y \partial_y = \partial_y^2 - \sum_{s \in S} \frac{c_s \alpha_s(y)}{\alpha_s}(1-s)\partial_y =$$

$$\partial_y^2 - \sum_{s \in S} \frac{c_s \alpha_s(y)}{\alpha_s}(\partial_y(1-s) + \alpha_s(y)\partial_{\alpha_s^\vee} s).$$

This means that

$$m(D_y^2) = \partial_y^2 - \sum_{s \in S} c_s \frac{\alpha_s(y)^2}{\alpha_s} \partial_{\alpha_s^\vee}.$$

So we get

$$m(\sum_{i=1}^{r} D_{y_i}^2) = \sum_i \partial_{y_i}^2 - \sum_{s \in S} c_s \frac{\sum_{i=1}^{r} \alpha_s(y_i)^2}{\alpha_s} \partial_{\alpha_s^\vee} = \bar{L}$$

since $\sum_{i=1}^{r} \alpha_s(y_i)^2 = (\alpha_s, \alpha_s)$. □

Recall that by Chevalley's theorem, the algebra $(S\mathfrak{h})^W$ is free. Let $P_1 = x^2, P_2, ..., P_r$ be homogeneous generators of $(S\mathfrak{h})^W$.

Corollary 6.10. *The operator $\bar{L}$ defines a quantum integrable system. Namely, the operators $\bar{L}_i := m(P_i(D_{y_1}, ..., D_{y_r}))$ are commuting quantum, integrals of $\bar{L} = \bar{L}_1$.*

Proof. The statement follows from Theorem 6.5 and the fact that $m(b_1 b_2) = m(b_1)m(b_2)$. □

To derive from this the quantum integrability of the operator L, we will prove the following proposition.

Proposition 6.11. *Let $\delta_s(x) := \prod_{s \in S} \alpha_s(x)^{c_s}$. Then we have*

$$\delta_c^{-1} \circ \bar{L} \circ \delta_c = L.$$

Proof. We have

$$\sum_i \partial_{y_i}(\log\delta_c)\partial_{y_i} = \sum_{s\in S}\frac{c_s(\alpha_s,\alpha_s)}{2\alpha_s}\partial_{\alpha_s^\vee}.$$

Therefore, we have

$$\delta_c\circ L\circ\delta_c^{-1} = \Delta_{\mathfrak{h}} - \sum_{s\in S}\frac{c_s(\alpha_s,\alpha_s)}{\alpha_s}\partial_{\alpha_s^\vee} + U,$$

where

$$U = \delta_c(\Delta_{\mathfrak{h}}\delta_c^{-1}) - \sum_{s\in S}\frac{c_s(c_s+1)(\alpha_s,\alpha_s)}{\alpha_s^2}.$$

Let us compute U. We have

$$\delta_c(\Delta_{\mathfrak{h}}\delta_c^{-1}) = \sum_{s\in S}\frac{c_s(c_s+1)(\alpha_s,\alpha_s)}{\alpha_s^2} + \sum_{s\neq u\in S}\frac{c_sc_u(\alpha_s,\alpha_u)}{\alpha_s\alpha_u}.$$

We claim that the last sum Σ is actually zero. Indeed, this sum is invariant under the Weyl group, so $\prod_{s\in S}\alpha_s\cdot\Sigma$ is a regular anti-invariant of degree $|S|-2$. But the smallest degree of a nonzero anti-invariant is $|S|$, so $\Sigma = 0$, and we are done. □

Corollary 6.12. *(Heckman* [Hec]*) The Olshanetsky-Perelomov operator L defines a quantum integrable system, namely $\{L_i, i = 1, ..., r\}$, where $L_i = \delta_c^{-1}\circ\bar{L}_i\circ\delta_c$.*

6.3. **Classical Dunkl operators and Olshanetsky-Perelomov hamiltonians.** Let us define the classical analog of Dunkl and Olshanetsky-Perelomov operators. For this purpose we need to introduce the Planck constant $\hbar$. Namely, let us define renormalized Dunkl operators

$$D_a(c,\hbar) := \hbar D_a(c/\hbar).$$

These operators can be regarded as elements of the Rees algebra $A = \mathrm{Rees}(\mathbb{C}W\ltimes D(\mathfrak{h}_{reg}))$, where the filtration is by order of differential operators (and W sits in degree 0). Reducing these operators modulo $\hbar$, we get classical Dunkl operators $D_a^0(c)\in A_0 := A/\hbar A = \mathbb{C}W\ltimes\mathcal{O}(T^*\mathfrak{h}_{reg})$. They are given by the formula

$$D_a^0(c) = p_a - \sum_{s\in S}\frac{c_s\alpha_s(a)}{\alpha_s}(1-s),$$

where p_a is the classical momentum (the linear function on $\mathfrak{h}^*$ corresponding to $a\in\mathfrak{h}$).

It follows from the commutativity of the quantum Dunkl operators D_a that the classical Dunkl operators D_a^0 also commute:

$$[D_a^0, D_b^0] = 0.$$

We also have the following analog of Proposition 6.4:

Proposition 6.13. *(i) For any $x \in \mathfrak{h}^*$, one has*

$$[D_a^0, x] = -\sum_{s \in S} c_s(a, \alpha_s)(x, \alpha_s^\vee)s.$$

(ii) If $g \in W$ then $gD_a^0 g^{-1} = D_{ga}^0$.

Now let us construct the classical Olshanetsky-Perelomov hamiltonians. As in the quantum case, we have a homomorphism $m : (\mathbb{C}W \ltimes \mathcal{O}(T^*\mathfrak{h}_{reg}))^W \to \mathcal{O}(T^*\mathfrak{h}_{reg})^W$, which is given by the formula $\sum f_g \cdot g \to \sum f_g$, $g \in W$, $f \in \mathcal{O}(T^*\mathfrak{h}_{reg})$. We define the hamiltonian

$$\bar{L}^0 := m(\sum_{i=1}^{r} (D_{y_i}^0)^2).$$

It is easy to see by taking the limit from the quantum situation that

$$\bar{L}^0 = p^2 - \sum_{s \in S} \frac{c_s(\alpha_s, \alpha_s)}{\alpha_s} p_{\alpha_s^\vee}.$$

Consider the (outer) automorphism θ_c of the algebra $\mathbb{C}W \ltimes \mathcal{O}(T^*\mathfrak{h}_{reg})$ defined by the formulas

$$\theta_c(x) = x,\ \theta_c(s) = s,\ \theta_c(p_a) = p_a + \partial_a \log \delta_c,$$

for $x \in \mathfrak{h}^*$, $a \in \mathfrak{h}$, $s \in W$. It is easy to see that if $b_0 \in A_0$ and $b \in A$ is a deformation of b_0 then $\theta_c(b_0) = \lim_{\hbar \to 0} \delta_{c/\hbar}^{-1} b \delta_{c/\hbar}$. Therefore, defining

$$L^0 := \theta_c(\bar{L}^0),$$

we find:

$$L^0 = p^2 - \sum_{s \in S} \frac{c_s^2(\alpha_s, \alpha_s)}{\alpha_s^2}.$$

This function is called the classical Olshanetsky-Perelomov hamiltonian for W.

Thus, we obtain the following result.

Theorem 6.14. *The Olshanetsky-Perelomov hamiltonian L^0 defines an integrable system with integrals*

$$L_i^0 := m(\theta_c(P_i(D_{y_1}^0, ..., D_{y_r}^0)))$$

(so that $L_1^0 = L^0$).

6.4. **Notes.** 1. Dunkl operators can be generalized to complex reflection groups, see [DO]. Using them in the same manner as above, one can construct quantum integrable systems whose Hamiltonians have order higher than 2.

2. If W is a Weyl group, then Dunkl operators for W can be extended to the trigonometric setting, see [Op]. The trigonometric Dunkl operators can be used to construct the first integrals of the trigonometric quantum Calogero-Moser system, with Hamiltonian

$$L_{trig} := \Delta_{\mathfrak{h}} - \sum_{s \in S} \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\sin^2(\alpha_s)}.$$

Moreover, these statements generalize to the case of the quantum elliptic Calogero-Moser system, with Hamiltonian

$$L_{ell} := \Delta_{\mathfrak{h}} - \sum_{s \in S} c_s(c_s+1)(\alpha_s, \alpha_s)\wp(\alpha_s, \tau),$$

where $\wp$ is the Weierstrass elliptic function. This is done in the paper [Ch1].

7. The rational Cherednik algebra

7.1. **Definition of the rational Cherednik algebra and the Poincaré-Birkhoff-Witt theorem.** In the previous section we made an essential use of the commutation relations between operators $x \in \mathfrak{h}^*$, $g \in W$, and D_a, $a \in \mathfrak{h}$. This makes it natural to consider the algebra generated by these operators.

Definition 7.1. The rational Cherednik algebra $H_c = H_c(W, \mathfrak{h})$ associated to $(W, \mathfrak{h})$ is the algebra generated inside $A = \mathrm{Rees}(\mathbb{C}W \ltimes D(\mathfrak{h}_{reg}))$ by the elements $x \in \mathfrak{h}^*$, $g \in W$, and $D_a(c, \hbar)$, $a \in \mathfrak{h}$. If $t \in \mathbb{C}$, then the algebra $H_{t,c}$ is the specialization of H_c at $\hbar = t$.

Proposition 7.2. *The algebra H_c is the quotient of the algebra $\mathbb{C}W \ltimes \mathbf{T}(\mathfrak{h} \oplus \mathfrak{h}^*)[\hbar]$ (where $\mathbf{T}$ denotes the tensor algebra) by the ideal generated by the relations*

$$[x, x'] = 0, \;\; [y, y'] = 0, \;\; [y, x] = \hbar(y, x) - \sum_{s \in S} c_s(y, \alpha_s)(x, \alpha_s^\vee)s.$$

Proof. Let us denote the algebra defined in the proposition by H_c'. Then according to the results of the previous section, we have a surjective homomorphism $\phi : H_c' \to H_c$ defined by the formula $\phi(x) = x$, $\phi(g) = g$, $\phi(y) = D_y(c, \hbar)$.

Let us show that this homomorphism is injective. For this purpose assume that y_i is a basis of $\mathfrak{h}$, and x_i is the dual basis of $\mathfrak{h}^*$. Then it is clear from the relations of H'_c that H'_c is spanned over $\mathbb{C}[\hbar]$ by the elements

$$g\prod_{i=1}^{r} y_i^{m_i} \prod_{i=1}^{r} x_i^{n_i}. \tag{5}$$

Thus it remains to show that the images of the elements (5) under the map ϕ, i.e. the elements

$$g\prod_{i=1}^{r} D_{y_i}(c,\hbar)^{m_i} \prod_{i=1}^{r} x_i^{n_i}.$$

are linearly independent. But this follows from the obvious fact that the symbols of these elements in $\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{reg}][\hbar]$ are linearly independent. The proposition is proved. $\square$

It is more convenient to work with algebras defined by generators and relations than with subalgebras of a given algebra generated by a given set of elements. Therefore, from now on we will use the statement of Proposition 7.2 as a definition of the rational Cherednik algebra H_c. According to Proposition 7.2, this algebra comes with a natural embedding $\Theta_c : H_c \to \mathrm{Rees}(\mathbb{C}W \ltimes D(\mathfrak{h}_{reg}))$, defined by the formula $x \to x$, $g \to g$, $y \to D_y(c,\hbar)$. This embedding is called the Dunkl operator embedding.

Let us put a filtration on H_c by declaring $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$ to have degree 1, and $g \in G$ to have degree 0. Let $\mathrm{gr}(H_c)$ denote the associated graded algebra of H_c under this filtration. We have a natural surjective homomorphism $\xi : \mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*][\hbar] \to \mathrm{gr}(H_c)$.

Proposition 7.3. *(the PBW theorem for rational Cherednik algebras) The map ξ is an isomorphism.*

Proof. The statement is equivalent to the claim that the elements (5) are a basis of H_c, which follows from the proof of Proposition 7.2. $\square$

Remark 7.4. 1. It follows from Proposition 7.3 that the algebra H_c is a free module over $\mathbb{C}[\hbar]$, whose basis is the collection of elements (7.2) (In particular, H_c is an algebraic deformation of $H_{0,c}$). This basis is called a PBW basis of H_c (and of $H_{t,c}$).

2. It follows from the definition of $H_{t,c}$ that $H_{t,c}$ is a quotient of $\mathbb{C}W \ltimes \mathbf{T}(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations of Proposition 7.2, with $\hbar$ replaced by t. In particular, for any $\lambda \in \mathbb{C}^*$, the algebra $H_{t,c}$ is naturally isomorphic to $H_{\lambda t, \lambda c}$.

3. The Dunkl operator embedding Θ_c specializes to embeddings $\Theta_{1,c} : H_{1,c} \to \mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$ given by $x \to x$, $g \to g$, $y \to D_y$, and $\Theta_{0,c} : H_{0,c} \to \mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{reg}]$, given by $x \to x$, $g \to g$, $y \to D_y^0$.

4. Since Dunkl operators map polynomials to polynomials, the map $\Theta_{1,c}$ defines a representation of $H_{1,c}$ on $\mathbb{C}[\mathfrak{h}]$. This representation is called the polynomial representation of $H_{1,c}$.

Example 7.5. 1. Let $W = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$. In this case c reduces to one parameter k, and the algebra $H_{t,k}$ is generated by elements x, y, s with defining relations

$$s^2 = 1,\ sx = -xs,\ sy = -ys,\ [y, x] = t - 2ks.$$

2. Let $W = S_n$, $\mathfrak{h} = \mathbb{C}^n$. In this case there is also only one complex parameter $c = k$, and the algebra $H_{t,k}$ is the quotient of $S_n \ltimes \mathbb{C} < x_1, ..., x_n, y_1, ..., y_n >$ by the relations

$$[x_i, x_j] = [y_i, y_j] = 0,\ [y_i.x_j] = s_{ij},\ [y_i, x_i] = t - k \sum_{j \neq i} s_{ij}. \tag{6}$$

Here $\mathbb{C}\langle E \rangle$ denotes the free algebra on a set E, and s_{ij} is the transposition of i and j.

7.2. **The spherical subalgebra.** Let $\mathbf{e} \in \mathbb{C}W$ be the symnmetrizer, $\mathbf{e} = |W|^{-1} \sum_{g \in W} g$. We have $\mathbf{e}^2 = \mathbf{e}$.

Definition 7.6. The spherical subalgebra of H_c is the subalgebra $B_c := \mathbf{e}H_c\mathbf{e}$. The spherical subalgebra of $H_{t,c}$ is $B_{t,c} := B_c/(\hbar = t) = \mathbf{e}H_{t,c}\mathbf{e}$.

Note that $\mathbf{e}\mathbb{C}W \ltimes D(\mathfrak{h}_{reg})\mathbf{e} = D(\mathfrak{h}_{reg})^W$. Therefore, the map $\Theta_{t,c}$ restricts to an embedding $B_{t,c} \to D(\mathfrak{h}_{reg})^W$ for $t \neq 0$, and $B_{0,c} \to \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}_{reg}]^W$ for $t = 0$.

Proposition 7.7. *(i) The spherical subalgebra $B_{0,c}$ is commutative and does not have zero divisors.*

(ii) B_c is an algebraic deformation of $B_{0,c}$.

Proof. (i) Follows immediately from the fact that $B_{0,c} \subset \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}_{reg}]^W$.

(ii) Follows since H_c is an algebraic deformation of $H_{0,c}$. □

Proposition 7.7 implies that the spectrum M_c of $B_{0,c}$ is an irreducible affine algebraic variety. Moreover, M_c has a natural Poisson structure, obtained from the deformation B_c of $B_{0,c}$. In fact, it is clear that this Poisson structure is simply the restriction of the Poisson structure of $\mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{reg}]^W$ to the subalgebra $B_{0,c}$.

Definition 7.8. The Poisson variety M_c is called the Calogero-Moser space of $W, \mathfrak{h}$.

Remark. We will later justify this terminology by showing that in the case $W = S_n$, $\mathfrak{h} = \mathbb{C}^n$ the variety M_c is isomorphic, as a Poisson variety, to the Calogero-Moser space of Kazhdan, Kostant, and Sternberg.

Thus, we may say that the algebra B_c is an algebraic quantization of the Calogero-Moser space M_c.

7.3. **The localization lemma and the basic properties of M_c.** Let $H_{t,c}^{\text{loc}} = H_{t,c}[\delta^{-1}]$ be the localization of $H_{t,c}$ as a module over $\mathbb{C}[\mathfrak{h}]$ with respect to the discriminant δ. Define also $B_{t,c}^{\text{loc}} = \mathbf{e}B_{t,c}\mathbf{e}$.

Proposition 7.9. *(i) For $t \neq 0$ the map $\Theta_{t,c}$ induces an isomorphism of algebras from $H_{t,c}^{\text{loc}} \to \mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$, which restricts to an isomorphism $B_{t,c}^{loc} \to D(\mathfrak{h}_{reg})^W$.*

(ii) The map $\Theta_{0,c}$ induces an isomorphism of algebras from $H_{0,c}^{\text{loc}} \to \mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{reg}]$, which restricts to an isomorphism $B_{0,c}^{loc} \to \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{reg}]^W$.

Proof. This follows immediately from the fact that the Dunkl operators have poles only on the reflection hyperplanes. □

Thus we see that the dependence on c disappears upon localization.

Since $\text{gr}(B_{0,c}) = B_{0,0} = \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}]^W$, we get the following geometric corollary.

Corollary 7.10. *(i) The family of Poisson varieties M_c is a flat deformation of the Poisson variety $M_0 := (\mathfrak{h}^* \oplus \mathfrak{h})/W$. In particular, M_c is smooth outside of a subset of codimension 2.*

(ii) We have a natural map $\beta_c : M_c \to \mathfrak{h}/W$, such that $\beta_c^{-1}(\mathfrak{h}_{reg}/W)$ is isomorphic to $(\mathfrak{h}^* \times \mathfrak{h}_{reg})/W$.

Exercise 7.11. Let $W = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$. Show that M_c is isomorphic to the quadric $pq - r^2 = c^2$ in the 3-dimensional space with coordinates p, q, r. In particular, M_c is smooth for $c \neq 0$.

Exercise 7.12. Show that if $t \neq 0$ then the center of $H_{t,c}$ is trivial.

Hint. Use Proposition 7.9.

7.4. **The SL_2-action on $H_{t,c}$.** It is clear from the definition of $H_{t,c}$ that $\mathfrak{h}$ and $\mathfrak{h}^*$ play completely symmetric roles, i.e. there is a symmetry that exchanges them. In fact, a stronger statement holds.

Proposition 7.13. *The group $SL_2(\mathbb{C})$ acts by automorphisms of $H_{t,c}$, via the formulas $g \to g$, $x_i \to px_i + qy_i$, $y_i \to rx_i + sy_i$, $g \to g$ $(g \in W)$, where $ps - qr = 1$, x_i is an orthonormal basis of $\mathfrak{h}^*$, and y_i the dual basis of $\mathfrak{h}$.*

Proof. The invariant bilinear form on $\mathfrak{h}$ defines an identification $\mathfrak{h} \to \mathfrak{h}^*$, under which $x_i \to y_i$, and α goes to a multiple of $\alpha^\vee$. This implies that

$(\alpha^\vee, x_i)(\alpha, y_j)$ is symmetric in i, j. Thus $[y_i, x_j] = [y_j, x_i]$, and hence for any $p, q \in \mathbb{C}$, the elements $px_i + qy_i$ commute with each other. Also,

$$[rx_i + sy_i, px_j + qy_j] = (ps - qr)[y_i, x_j] = [y_i, x_j].$$

This implies the statement. □

7.5. **Notes.** Rational Cherednik algebras are degenerations of double affine Hecke algebras introduced by Cherednik in early nineties. For introduction to double affine Hecke algebras, see the book [Ch]. The first systematic study of rational Cherednik algebras was undertaken in [EG]. In particular, the results of this lecture can be found in [EG].

8. Symplectic reflection algebras

8.1. **The definition of symplectic reflection algebras.** Rational Cherednik algebras for finite Coxeter groups are a special case of a wider class of algebras called symplectic reflection algebras. To define them, let V be a finite dimensional symplectic vector space over $\mathbb{C}$ with a symplectic form ω, and G be a finite group acting symplectically (linearly) on V. For simplicity let us assume that $(\wedge^2 V)^G = \mathbb{C}\omega$ and that G acts faithfully on V (these assumptions are not essential).

Definition 8.1. A symplectic reflection in G is an element g such that the rank of the operator $g - 1$ on V is 2.

If s is a symplectic reflection, then let $\omega_s(x, y)$ be the form ω applied to the projections of x, y to the image of $1 - s$ along the kernel of $1 - s$; thus ω_s is a skewsymmetric form of rank 2 on V.

Let $S \subset G$ be the set of symplectic reflections, and $c : S \to \mathbb{C}$ be a function which is invariant under the action of G. Let $t \in \mathbb{C}$.

Definition 8.2. The symplectic reflection algebra $H_{t,c} = H_{t,c}[V, G]$ is the quotient of the algebra $\mathbb{C}[G] \ltimes \mathbf{T}(V)$ by the ideal generated by the relation

$$[x, y] = t\omega(x, y) - 2\sum_{s \in S} c_s \omega_s(x, y). \tag{7}$$

Note that if W is a finite Coxeter group with reflection representation $\mathfrak{h}$, then we can set $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\omega(x, x') = \omega(y, y') = 0$, $\omega(y, x) = (y, x)$, for $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$. In this case

1) symplectic reflections are the usual reflections in W;

2) $\omega_s(x, x') = \omega_s(y, y') = 0$, $\omega_s(y, x) = \frac{1}{2}(y, \alpha_s)(\alpha_s^\vee, x)$.

Thus, $H_{t,c}$ is the rational Cherednik algebra defined in the previous subsection.

Note also that for any V, G, $H_{0,0}[V,G] = G \ltimes SV$, and $H_{1,0}[V,G] = G \ltimes \mathrm{Weyl}(V)$, where $\mathrm{Weyl}(V)$ is the Weyl algebra of V, i.e. the quotient of the tensor algebra $\mathbb{T}(V)$ by the relation $xy - yx = \omega(x,y)$, $x, y \in V$.

8.2. **The PBW theorem for symplectic reflection algebras.** To ensure that the symplectic reflection algebras $H_{t,c}$ have good properties, we need to prove a PBW theorem for them, which is an analog of Proposition 7.3. This is done in the following theorem, which also explains the special role played by symplectic reflections.

Theorem 8.3. *Let $\kappa : \wedge^2 V \to \mathbb{C}[G]$ be a G-invariant function. Define the algebra H_κ to be the quotient of the algebra $\mathbb{C}[G] \ltimes \mathbf{T}(V)$ by the relation $[x,y] = \kappa(x,y)$, $x, y \in V$. Put an increasing filtration on H_κ by setting $\deg(V) = 1$, $\deg(G) = 0$, and define $\xi : \mathbb{C}G \ltimes SV \to \mathrm{gr} H_\kappa$ to be the natural surjective homomorphism. Then ξ is an isomorphism if and only if κ has the form*

$$\kappa(x,y) = t\omega(x,y) - 2\sum_{s \in S} c_s \omega_s(x,y),$$

for some $t \in \mathbb{C}$ and G-invariant function $c : S \to \mathbb{C}$.

Unfortunately, for a general symplectic reflection algebra we don't have a Dunkl operator representation, so the proof of the more difficult "if" part of this Theorem is not as easy as the proof of Proposition 7.3. Instead of explicit computations with Dunkl operators, it makes use of the deformation theory of Koszul algebras, which we will now discuss.

8.3. **Koszul algebras.** Let R be a finite dimensional semisimple algebra (over $\mathbb{C}$). Let A be a $\mathbb{Z}_+$-graded algebra, such that $A[0] = R$.

Definition 8.4. (i) The algebra A is said to be quadratic if it is generated over R by $A[1]$, and has defining relations in degree 2.

(ii) A is Koszul if all elements of $Ext^i(R,R)$ (where R is the augmentation module over A) have grade degree precisely i.

Remarks. 1. Thus, in a quadratic algebra, $A[2] = A[1] \otimes_R A[1]/E$, where E is the subspace (R-subbimodule) of relations.

2. It is easy to show that a Koszul algebra is quadratic, since the condition to be quadratic is just the Koszulity condition for $i = 1, 2$.

Now let A_0 be a quadratic algebra, $A_0[0] = R$. Let E_0 be the space of relations for A_0. Let $E \subset A_0[1] \otimes_R A_0[1][[\hbar]]$ be a topologically free (over $\mathbb{C}[[\hbar]]$) R-subbimodule which reduces to E_0 modulo $\hbar$ ("deformation of the relations"). Let A be the ($\hbar$-adically complete) algebra generated over $R[[\hbar]]$ by $A[1] = A_0[1][[\hbar]]$ with the space of defining relations E. Thus A is a $\mathbb{Z}_+$-graded algebra.

The following very important theorem is due to Beilinson, Ginzburg, and Soergel, [BGS] (less general versions appeared earlier in the works of Drinfeld [Dr1], Polishchuk-Positselski [PP], Braverman-Gaitsgory [BG]). We will not give its proof.

Theorem 8.5. *(Koszul deformation principle) If A_0 is Koszul then A is a topologically free $\mathbb{C}[[\hbar]]$ module if and only if so is $A[3]$.*

Remark. Note that $A[i]$ for $i < 3$ are obviously topologically free.

We will now apply this theorem to the proof of Theorem 8.3.

8.4. **Proof of Theorem 8.3.** Let $\kappa : \wedge^2 V \to \mathbb{C}[G]$ be an equivariant map. We write $\kappa(x, y) = \sum_{g\in G} \kappa_g(x, y)g$, where $\kappa_g(x, y) \in \wedge^2 V^*$. To apply Theorem 8.5, let us homogenize our algebras. Namely, let $A_0 = (\mathbb{C}G \ltimes SV) \otimes \mathbb{C}[u]$. Also let $\hbar$ be a formal parameter, and consider the deformation $A = H_{\hbar u^2 \kappa}$ of A_0. That is, A is the quotient of $\mathbb{G} \ltimes \mathbf{T}(V)[u][[\hbar]]$ by the relations $[x, y] = \hbar u^2 \kappa(x, y)$. This is a deformation of the type considered in Theorem 8.5, and it is easy to see that its flatness in $\hbar$ is equivalent to Theorem 8.3. Also, the algebra A_0 is Koszul, because the polynomial algebra SV is a Koszul algebra. Thus by Theorem 8.5, it suffices to show that A is flat in degree 3.

The flatness condition in degree 3 is "the Jacobi identity"

$$[\kappa(x, y), z] + [\kappa(y, z), x] + [\kappa(z, x), y] = 0,$$

which must be satisfied in $\mathbb{C}G \ltimes V$. In components, this equation transforms into the system of equations

$$\kappa_g(x, y)(z - z^g) + \kappa_g(y, z)(x - x^g) + \kappa_g(z, x)(y - y^g) = 0$$

for every $g \in G$ (here z^g denotes the result of the action of g on z).

This equation, in particular, implies that if x, y, g are such that $\kappa_g(x, y) \neq 0$ then for any $z \in V$ $z - z^g$ is a linear combination of $x - x^g$ and $y - y^g$. Thus $\kappa_g(x, y)$ is identically zero unless the image of $(1 - g)|_V$ is at most 2, i.e. $g = 1$ or g is a symplectic reflection.

If $g = 1$ then $\kappa_g(x, y)$ has to be G-invariant, so it must be of the form $t\omega(x, y)$, where $t \in \mathbb{C}$.

If g is a symplectic reflection, then $\kappa_g(x, y)$ must be zero for any x such that $x - x^g = 0$. Indeed, if for such an x there had existed y with $\kappa_g(x, y) \neq 0$ then $z - z^g$ for any z would be a multiple of $y - y^g$, which is impossible since $Im(1 - g)|_V$ is 2-dimensional. This implies that $\kappa_g(x, y) = 2c_g\omega_g(x, y)$, and c_g must be invariant.

Thus we have shown that if A is flat (in degree 3) then κ must have the form given in Theorem 8.3. Conversely, it is easy to see that if κ does have such form, then the Jacobi identity holds. So Theorem 8.3 is proved.

8.5. **The spherical subalgebra of the symplectic reflection algebra.** The properties of symplectic reflection algebras are similar to properties of rational Cherednik algebras we have studied before. The main difference is that we no longer have the Dunkl representation and localization results, so some proofs are based on different ideas and are more complicated.

The spherical subalgebra of the symplectic reflection algebra is defined in the same way as in the Coxeter group case. Namely, let $\mathbf{e} = \frac{1}{|G|}\sum_{g\in G} g$, and $B_{t,c} = \mathbf{e}H_{t,c}\mathbf{e}$.

Proposition 8.6. *$B_{t,c}$ is commutative if and only if $t = 0$.*

Proof. Let A be a $\mathbb{Z}_+$-filtered algebra. If A is not commutative, then we can define a Poisson bracket on $\mathrm{gr}(A)$ in the following way. Let m be the minimum of $\deg(a) + \deg(b) - \deg([a, b])$ (over $a, b \in A$ such that $[a, b] \neq 0$). Then for homogeneous elements $a_0, b_0 \in A_0$ of degrees p, q, we can define $\{a_0, b_0\}$ to be the image in $A_0[p + q - m]$ of $[a, b]$, where a, b are any lifts of a_0, b_0 to A. It is easy to check that $\{,\}$ is a Poisson bracket on A_0 of degree $-m$.

Let us now apply this construction to the filtered algebra $A = B_{t,c}$. We have $\mathrm{gr}(A) = A_0 = (SV)^G$.

Lemma 8.7. *A_0 has a unique, up to scaling, Poisson bracket of degree -2, and no nonzero Poisson brackets of degrees < -2.*

Proof. A Poisson bracket on $(SV)^G$ is the same thing as a Poisson bracket on the variety V/G. On the smooth part $(V/G)_s$ of V/G, it is simply a bivector field, and we can lift it to a bivector field on the preimage V_s of $(V/G)_s$ in V, which is the set of points in V with trivial stabilizers. But the codimension on $V \setminus V_s$ in V is 2 (as $V \setminus V_s$ is a union of symplectic subspaces), so the bivector on V_s extends to a regular bivector on V. So if this bivector is homogeneous, it must have degree ≥ -2, and if it has degree -2 then it must be with constant coefficients, so being G-invariant, it is a multiple of ω^{-1}. The lemma is proved. $\square$

Now, for each t, c we have a natural Poisson bracket on A_0 of degree -2, which depends linearly on t, c. So by the lemma, this bracket has to be of the form $f(t, c)\Pi$, where Π is the unique up to scaling Poisson bracket of degree -2, and f a homogeneous linear function. Thus the algebra $A = B_{t,c}$ is not commutative unless $f(t, c) = 0$. On the other hand, if $f(t, c) = 0$, and $B_{t,c}$ is not commutative, then, as we've shown, A_0 has a nonzero Poisson bracket of degree < -2. But By Lemma 8.7, there is no such brackets. So $B_{t,c}$ must be commutative if $f(t, c) = 0$.

It remains to show that $f(t,c)$ is in fact a nonzero multiple of t. First note that $f(1,0)\neq 0$, since $B_{1,0}$ is definitely noncommutative. Next, let us take a point (t,c) such that $B_{t,c}$ is commutative. Look at the $H_{t,c}$-module $H_{t,c}\mathbf{e}$, which has a commuting action of $B_{t,c}$ from the right. Its associated graded is SV as an $(\mathbb{C}\ltimes SV,(SV)^G)$-bimodule, which implies that the generic fiber of $H_{t,c}\mathbf{e}$ as a $B_{t,c}$-module is the regular representation of G. So we have a family of finite dimensional representations of $H_{t,c}$ on the fibers of $H_{t,c}\mathbf{e}$, all realized in the regular representation of G. Computing the trace of the main commutation relation (7) of $H_{t,c}$ in this representation, we obtain that $t=0$ (since $Tr(s)=0$ for any reflection s). The Proposition is proved. □

Note that $B_{0,c}$ has no zero divisors, since its associated graded algebra $(SV)^G$ does not. Thus, like in the Cherednik algebra case, we can define a Poisson variety M_c, the spectrum of $B_{0,c}$, called the Calogero-Moser space of G,V. Moreover, the algebra $B_c:=B_{\hbar,c}$ over $\mathbb{C}[\hbar]$ is an algebraic quantization of M_c.

8.6. **Notes.** 1. Symplectic reflection algebras are a special case of generalized Hecke algebras defined by Drinfeld in Section 4 of [Dr2]. They were systematically studied in [EG], which is a basic reference for the results of this lecture. A generalization of symplectic reflection algebras, in which the action of G on V may be non-faithful and projective, is considered by Chmutova [Chm].

2. The notion of a Koszul algebra is due to Priddy [Pr]. A good reference for the theory of quadratic and Koszul algebras is the book [PP].

9. Deformation-theoretic interpretation of symplectic reflection algebras

9.1. **Hochschild cohomology of semidirect products.** In the previous lectures we saw that the symplectic reflection algebra $H_{1,c}[V,G]$, for formal c, is a flat formal deformation of $G\ltimes \mathrm{Weyl}(V)$. It turns out that this is in fact a universal deformation.

Namely, let $A_0=G\ltimes \mathrm{Weyl}(V)$, where V is a symplectic vector space (over $\mathbb{C}$), and G a finite group acting symplectically on V. For this we need to calculate the Hochschild cohomology of this algebra.

Theorem 9.1. *(Alev, Farinati, Lambre, Solotar,* [AFLS]*) The cohomology space $H^i(G\ltimes \mathrm{Weyl}(V))$ is naturally isomorphic to the space of conjugation invariant functions on the set S_i of elements $g\in G$ such that* $\mathrm{rank}(1-g)|_V=i$.

Since $\mathrm{Im}(1-g)$ is a symplectic vector space, we get the following corollary.

Corollary 9.2. *Odd cohomology of $G \ltimes \mathrm{Weyl}(V)$ vanishes, and $H^2(G \ltimes \mathrm{Weyl}(V))$ is the space $\mathbb{C}[S]^G$ of conjugation invariant functions on the set of symplectic reflections. In particular, there exists a universal deformation A of $A_0 = G \ltimes \mathrm{Weyl}(V)$ parametrized by $\mathbb{C}[S]^G$.*

Proof. (of Theorem 9.1)

Lemma 9.3. *Let B be a $\mathbb{C}$-algebra together with an action of a finite group G. Then*

$$H^*(G \ltimes B, G \ltimes B) = (\oplus_{g\in G} H^*(B, Bg))^G,$$

where Bg is the bimodule isomorphic to B as a space where the left action of B is the usual one and the right action is the usual action twisted by g.

Proof. The algebra $G \ltimes B$ is a projective B-module. Therefore, using the Shapiro lemma, we get

$$H^*(G \ltimes B, G \ltimes B) = \mathrm{Ext}^*_{(G\times G)\ltimes(B\otimes B^{op})}(G \ltimes B, G \ltimes B) =$$

$$\mathrm{Ext}^*_{G_{\mathrm{diagonal}}\ltimes(B\otimes B^{op})}(B, G \ltimes B) = \mathrm{Ext}^*_{B\otimes B^{op}}(B, G \ltimes B)^G$$

$$(\oplus_{g\in G}\mathrm{Ext}^*_{B\otimes B^{op}}(B, Bg))^G = (\oplus_{g\in G} H^*(B, Bg))^G,$$

as desired. □

Now apply the lemma to $B = \mathrm{Weyl}(V)$. For this we need to calculate $H^*(B, Bg)$, where g is any element of G. We may assume that g is diagonal in some symplectic basis: $g = \mathrm{diag}(\lambda_1, \lambda_1^{-1}, ...\lambda_n, \lambda_n^{-1})$. Then by the Künneth formula we find that $H^*(B, Bg) = \otimes_{i=1}^n H^*(\mathbb{A}_1, \mathbb{A}_1 g_i)$, where $\mathbb{A}_1$ is the Weyl algebra of the 2-dimensional space, (generated by x, y with defining relation $xy - yx = 1$), and $g_i = \mathrm{diag}(\lambda_i, \lambda_i^{-1})$.

Thus we need to calculate $H^*(B, Bg)$, $B = \mathbb{A}_1$, $g = \mathrm{diag}(\lambda, \lambda^{-1})$.

Proposition 9.4. *$H^*(B, Bg)$ is 1-dimensional, concentrated in degree 0 if $\lambda = 1$ and in degree 2 otherwise.*

Proof. If $B = \mathbb{A}_1$ then B has the following Koszul resolution as a B-bimodule:

$$B \otimes B \to B \otimes \mathbb{C}^2 \otimes B \to B \otimes B \to B.$$

Here the first map is given by the formula

$$b_1 \otimes b_2 \to b_1 \otimes x \otimes yb_2 - b_1 \otimes y \otimes xb_2$$

$$-b_1 y \otimes x \otimes b_2 + b_1 x \otimes y \otimes b_2,$$

the second map is given by

$$b_1 \otimes x \otimes b_2 \to b_1 x \otimes b_2 - b_1 \otimes x b_2, \ b_1 \otimes y \otimes b_2 \to b_1 y \otimes b_2 - b_1 \otimes y b_2,$$

and the third map is the multiplication.

Thus the cohomology of B with coefficients in Bg can be computed by mapping this resolution into Bg and taking the cohomology. This yields the following complex $C^\bullet$:

$$0 \to Bg \to Bg \oplus Bg \to Bg \to 0 \tag{8}$$

where the first nontrivial map is given by $bg \to [bg, y] \otimes x - [bg, x] \otimes y$, and the second nontrivial map is given by $bg \otimes x \to [x, bg]$, $bg \otimes y \to [y, bg]$.

Consider first the case $g = 1$. Equip the complex $C^\bullet$ with the Bernstein filtration ($\deg(x) = \deg(y) = 1$), starting with $0, 1, 2$, for C^0, C^1, C^2, respectively (this makes the differential perserve the filtartion). Consider the associated graded complex $C^\bullet_{gr}$. In this complex, brackets are replaced with Poisson brackets, and thus it is easy to see that $C^\bullet_{gr}$ is the De Rham complex for the affine plane, so its cohomology is $\mathbb{C}$ in degree 0 and 0 in other degrees. Therefore, the cohomology of $C^\bullet$ is the same.

Now consider $g \neq 1$. In this case, declare that C^0, C^1, C^2 start in degrees 2,1,0 respectively (which makes the differential preserve the filtration), and again consider the graded complex $C^\bullet_{gr}$.

The graded Euler characteristic of this complex is $(t^2 - 2t + 1)(1 - t)^{-2} = 1$.

The cohomology in the C^0_{gr} term is the set of $b \in \mathbb{C}[x, y]$ such that $ab = ba^g$ for all a. This means that $H^0 = 0$.

The cohomology of the C^2_{gr} term is the quotient of $\mathbb{C}[x, y]$ by the ideal generated by $a - a^g$, $a \in \mathbb{C}[x, y]$. Thus the cohomology H^2 of the rightmost term is 1-dimensional, in degree 0. By the Euler characteristic argument, this implies that $H^1 = 0$. The cohomology of the filtered complex $C^\bullet$ is therefore the same, and we are done. □

The proposition implies that in the n-dimensional case $H^*(B, Bg)$ is 1-dimensional, concentrated in degree $\mathrm{rank}(1 - g)$. It is not hard to check that the group G acts on the sum of these 1-dimensional spaces by simply premuting the basis vectors. Thus the theorem is proved. □

9.2. The universal deformation of $G \ltimes \mathrm{Weyl}(V)$.

Theorem 9.5. *The algebra $H_{1,c}[G, V]$, with formal c, is the universal deformation of $H_{1,0}[G, V] = G \ltimes \mathrm{Weyl}(V)$. That is, the map $f : \mathbb{C}[S]^G \to H^2(G \ltimes \mathrm{Weyl}(V))$ induced by this deformation is an isomorphism.*

Proof. The map f is a map between spaces of the same dimension, so it suffices to show that f is injective. For this purpose it suffices to show that that for any $\gamma \in \mathbb{C}[S]^G$, the algebra $H_{1,\hbar\gamma}$ over $\mathbb{C}[\hbar]/\hbar^2$ is a nontrivial deformation of $H_{1,0}$. For this, in turn, it suffices to show that $H_{1,\hbar\gamma}$ does not have a subalgebra isomorphic to $\mathrm{Weyl}(V)[\hbar]/\hbar^2$. This can be checked directly, as explained in [EG], Section 2. □

9.3. **Notes.** The results of this lecture can be found in [EG]. For their generalizations, see [E].

10. The center of the symplectic reflection algebra

10.1. **The module $H_{t,c}\mathbf{e}$.** Let $H_{t,c}$ be a symplectic reflection algebra. Consider the bimodule $H_{t,c}\mathbf{e}$, which has a left action of $H_{t,c}$ and a right commuting action of $\mathbf{e}H_{t,c}\mathbf{e}$. It is obvious that $\mathrm{End}_{H_{t,c}} H_{t,c}\mathbf{e} = \mathbf{e}H_{t,c}\mathbf{e}$ (with opposite product). The following theorem shows that the bimodule $H_{t,c}\mathbf{e}$ has the double centralizer property.

Note that we have a natural map $\xi_{t,c} : H_{t,c} \to \mathrm{End}_{\mathbf{e}H_{t,c}\mathbf{e}} H_{t,c}\mathbf{e}$.

Theorem 10.1. *$\xi_{t,c}$ is an isomorphism for any t, c.*

Proof. The complete proof is given [EG]. We will give the main ideas of the proof skipping straightforward technical details. The first step is to show that the result is true in the graded case, $(t, c) = (0, 0)$. To do so, note the following general fact:

Exercise 10.2. If X is an affine complex algebraic variety with algebra of functions $\mathcal{O}_X$ and G a finite group acting freely on X then the natural map $\xi_X : G \ltimes \mathcal{O}_X \to \mathrm{End}_{\mathcal{O}_X^G} \mathcal{O}_X$ is an isomorphism.

Therefore, the map $\xi_{0,0} : G \ltimes SV \to \mathrm{End}_{(SV)^G}(SV)$ is injective, and moreover becomes an isomorphism after localization to the field of quotients $\mathbb{C}(V)^G$. To show it's surjective, take $a \in \mathrm{End}_{(SV)^G}(SV)$. There exists $a' \in G \ltimes \mathbb{C}(V)$ which maps to a. Moreover, by Exercise 10.2, a' can have poles only at fixed points of G on V. But these fixed points form a subset of codimension ≥ 2, so there can be no poles and we are done in the case $(t, c) = (0, 0)$.

Now note that the algebra $\mathrm{End}_{\mathbf{e}H_{t,c}\mathbf{e}} H_{t,c}\mathbf{e}$ has an increasing integer filtration (bounded below) induced by the filtration on $H_{t,c}$. This is due to the fact that $H_{t,c}\mathbf{e}$ is a finitely generated $\mathbf{e}H_{t,c}\mathbf{e}$-module (since it is true in the associated graded situation, by Hilbert's theorem about invariants). Also, the natural map $\mathrm{gr}\mathrm{End}_{\mathbf{e}H_{t,c}\mathbf{e}} H_{t,c}\mathbf{e} \to \mathrm{End}_{\mathrm{gr}\mathbf{e}H_{t,c}\mathbf{e}} \mathrm{gr} H_{t,c}\mathbf{e}$ is clearly injective. Therefore, our result in the case $(t, c) = (0, 0)$ implies that this map is actually an isomorphism (as so is its composition with the associated garded of $\xi_{t,c}$). Identifying the two algebras by this isomorphism, we find that $\mathrm{gr}(\xi_{t,c}) = \xi_{0,0}$. Since $\xi_{0,0}$ is an isomorphism, $\xi_{t,c}$ is an isomorphism for all t, c, as desired. [8] $\square$

10.2. **The center of $H_{0,c}$.** It turns out that if $t \neq 0$, then the center of $H_{t,c}$ is trivial (it was proved by Brown and Gordon). However, if $t = 0$, $H_{t,c}$ has a nontrivial center, which we will denote Z_c. Also, as before, we denote the spherical subalgebra $\mathbf{e}H_{0,c}\mathbf{e}$ by $B_{0,c}$.

We have a natural homomorphism $\zeta_c : Z_c \to B_{0,c}$, defined by the formula $\zeta(z) = z\mathbf{e}$. We also have a natural injection $\tau_c : \mathrm{gr}(Z_c) \to Z_0 = SV$. These maps are clearly isomorphisms for $c = 0$.

Proposition 10.3. *The maps ζ_c, τ_c are isomorphisms for any c.*

Proof. It is clear that the morphism τ_c is injective. Identifying $\mathrm{gr}B_{0,c}$ with $B_{0,0} = SV$, we get $\mathrm{gr}\zeta_c = \tau_c$. This implies that $\mathrm{gr}(\zeta_c)$ is injective.

Now we show that ζ_c is an isomorphism. To do so, we will construct the inverse homomorphism ζ_c^{-1}. Namely, take an element $b \in B_{0,c}$. Since the algebra $B_{0,c}$ is commutative, it defines an element in $\mathrm{End}_{B_{0,c}}(H_{0,c}\mathbf{e})$. Thus using Theorem 10.1, we can define $a := \xi_{0,c}^{-1}(b) \in H_{0,c}$. Note that a commutes with $H_{0,c}$ as an operator on $H_{0,c}\mathbf{e}$ since it is defined by right multiplication by b. Thus a is central. It is easy to see that $b = z\mathbf{e}$.

Thus ζ_c is an filtration preserving isomorphism whose associated graded is injective. This implies that $\mathrm{gr}\zeta_c = \tau_c$ is an isomorphism. The theorem is proved. $\square$

Thus the spectrum of Z_c is equal to $M_c = \mathrm{Spec}B_{0,c}$.

10.3. **Finite dimensional representations of $H_{0,c}$.** Let $\chi \in M_c$ be a central character: $\chi : Z_c \to \mathbb{C}$. Let (χ) be the ideal in $H_{0,c}$ generated by the kernel of χ.

Proposition 10.4. *If χ is generic then $H_{0,c}/(\chi)$ is the matrix algebra of size $|G|$. In particular, $H_{0,c}$ has a unique irreducible representation V_χ with central character χ. This representation is isomorphic to $\mathbb{C}G$ as a G-module.*

Proof. It is shown by a standard argument (which we will skip) that it is sufficient to check the statement in the associated graded case $c = 0$. In this case, for generic χ $G \ltimes SV/(\chi) = G \ltimes \mathrm{Fun}(\mathcal{O}_\chi)$, where $\mathcal{O}_\chi$ is the

[8] Here we use the fact that the filtration is bounded from below. In the case of an unbounded filtration, it is possible for a map not to be an isomorphism if its associated graded is an isomorphism. An example of this is the operator of multiplication by $1 + t^{-1}$ in the space of Laurent polynomials in t, filtered by degree.

(free) orbit of G consisting of the points of V that map to $\chi \in V/G$, and $\mathrm{Fun}(\mathcal{O}_\chi)$ is the algebra of functions on $\mathcal{O}_\chi$. It is easy to see that this algebra is isomorphic to a matrix algebra, and has a unique irreducible representation, $\mathrm{Fun}(\mathcal{O}_\chi)$, which is a regular representation of G. □

Corollary 10.5. *Any irreducible representation of $H_{0,c}$ has dimension $\leq |G|$.*

Proof. Since for generic χ the algebra $H_{0,c}/(\chi)$ is a matrix algebra, the algebra $H_{0,c}$ satisfies the standard polynomial identity (the Amitsur-Levitzki identity) for matrices $N \times N$ ($N = |G|$):

$$\sum_{\sigma \in S_{2N}} (-1)^\sigma X_{\sigma(1)} ... X_{\sigma(2N)} = 0.$$

Next, note that since $H_{0,c}$ is a finitely generated Z_c-module (by passing to the associated graded and using Hilbert's theorem), every irreducible representation of $H_{0,c}$ is finite dimensional. If $H_{0,c}$ had an irreducible representation E of dimension $m > |G|$, then the by the density theorem the matrix algebra Mat_m would be a quotient of $H_{0,c}$. But the Amitsur-Levitski identity of degree $|G|$ is not satisfied for matrices of bigger size than $|G|$. Contradiction. Thus, $\dim E \leq |G|$, as desired. □

In general, for special central characters there are representations of $H_{0,c}$ of dimension $< |G|$. However, in some cases one can show that all irreducible representations have dimension exactly $|G|$. For example, we have the following result.

Theorem 10.6. *Let $G = S_n$, $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\mathfrak{h} = \mathbb{C}^n$ (the rational Cherednik algebra). Then for $c \neq 0$, every irreducible representation of $H_{0,c}$ has dimension $n!$ and is isomorphic to the regular representation of S_n.*

Proof. Let E be an irreducible representation of $H_{0,c}$. Let us calculate the trace in E of any permutation $\sigma \neq 1$. Let j be an index such that $\sigma(j) = i \neq j$. Then $s_{ij}\sigma(j) = j$. Hence in $H_{0,c}$ we have

$$[y_j, x_i s_{ij} \sigma] = [y_j, x_i] s_{ij} \sigma = c s_{ij}^2 \sigma = c\sigma.$$

Hence $\mathrm{Tr}_E(\sigma) = 0$, and thus E is a multiple of the regular representation of S_n. But by Theorem 10.5, $\dim E \leq n!$, we get that E is the regular representation, as desired. □

10.4. **Azumaya algebras.** Let Z be a finitely generated commutative algebra over $\mathbb{C}$ (for simplicity without nilpotents), $M = \mathrm{SpecZ}$ is the corresponding algebraic variety, and A a finitely generated Z-algebra.

Definition 10.7. A is said to be an Azumaya algebra of degree N if the completion $\hat{A}_\chi$ of A at every maximal ideal χ in Z is the matrix algebra of size N over the completion $\hat{Z}_\chi$ of Z.

For example, if E is an algebraic vector bundle on M then $\mathrm{End}(E)$ is an Azumaya algebra. However, not all Azumaya algebras are of this form.

Exercise 10.8. Let q be a root of unity of order N. Show that the algebra of functions on the quantum torus, generated by $X^{\pm 1}, Y^{\pm 1}$ with defining relation $XY = qYX$ is an Azumaya algebra of degree N. Is this the endomorphism algebra of a vector bundle?

It is clear that if A is an Azumaya algebra then for every central character χ of A, $A/(\chi)$ is the algebra $Mat_N(\mathbb{C})$ of complex N by N matrices, and every irreducible representation of A has dimension N.

The following important result is due to M. Artin.

Theorem 10.9. *Let A be a finitely generated (over $\mathbb{C}$) polynomial identity (PI) algebra of degree N (i.e. all the polynomial relations of the matrix algebra of size N are satisfied in A). Then A is an Azumaya algebra if every irreducible representation of A has dimension exactly N.*

Thus, by Theorem 10.6, for $G = S_n$, $H_{0,c}$ for $c \neq 0$ is an Azumaya algebra of degree $n!$. Indeed, this algebra is PI of degree $n!$ because the classical Dunkl representation embeds it into matrices of size $n!$ over $\mathbb{C}(x_1, ..., x_n, p_1, ..., p_n)^{S_n}$.

Let us say that $\chi \in M$ is an Azumaya point if for some affine neighborhood U of χ the localization of A to U is an Azumaya algebra. Obviously, the set $Az(M)$ of Azumaya points of M is open.

Now we come back to the study the space M_c corresponding to a symplectic reflection algebra $H_{0,c}$.

Theorem 10.10. *The set $Az(M_c)$ coincides with the set of smooth points of M_c.*

The proof of this theorem is given in the following two subsections.

Corollary 10.11. *If $G = S_n$ and $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\mathfrak{h} = \mathbb{C}^n$ (the rational Cherednik algebra case) then M_c is a smooth algebraic variety for $c \neq 0$.*

10.5. **Cohen-Macaulay property and homological dimension.** To prove Theorem 10.10, we will need some commutative algebra tools. Let Z be a finitely generated commutative algebra over $\mathbb{C}$ without zero divisors. By Noether's normalization lemma, there exist elements

$z_1, ..., z_n \in Z$ which are algebraically independent, such that Z is a finitely generated module over $\mathbb{C}[z_1, ..., z_n]$.

Definition 10.12. The algebra Z is said to be Cohen-Macaulay if Z is a locally free (=projective) module over $\mathbb{C}[z_1, ..., z_n]$. [9]

Remark 10.13. It was shown by Serre that if Z is locally free over $\mathbb{C}[z_1, ..., z_n]$ for **some** choice of $z_1, ..., z_n$, then it happens for **any** choice of them (such that Z is finitely generated as a nodule over $\mathbb{C}[z_1, ..., z_n]$).

Remark 10.14. Another definition of the Cohen-Macaulay property is that the dualizing complex $\omega_Z^\bullet$ of Z is concentrated in degree zero. We will not discuss this definition here.

It can be shown that the Cohen-Macaulay property is stable under localization. Therefore, it makes sense to make the following definition.

Definition 10.15. An algebraic variety X is Cohen-Macaulay if the algebra of functions on every affine open set in X is Cohen-Macaulay.

Let Z be a finitely generated commutative algebra over $\mathbb{C}$ without zero divisors, and let M be a finitely generated module over Z.

Definition 10.16. M is said to be Cohen-Macaulay if for some algebraically independent $z_1, ..., z_n \in Z$ such that Z is finitely generated over $\mathbb{C}[z_1, ..., z_n]$, M is locally free over $\mathbb{C}[z_1, ..., z_n]$.

Again, if this happens for some $z_1, ..., z_n$, then it happens for any of them. We also note that M can be Cohen-Macaulay without Z being Cohen-Macaulay, and that Z is a Cohen-Macaulay algebra iff it is a Cohen-Macaulay module over itself.

We will need the following standard properties of Cohen-Macaulay algebras and modules.

Theorem 10.17. *(i) Let $Z_1 \subset Z_2$ be a finite extension of finitely generated commutative $\mathbb{C}$-algebras, without zero divisors, and M be a finitely generated module over Z_2. Then M is Cohen-Macaulay over Z_2 iff it is Cohen-Macaulay over Z_1.*

(ii) Suppose that Z is the algebra of functions on a smooth affine variety. Then a Z-module M is Cohen-Macaulay if and only if it is projective.

In particular, this shows that the algebra of functions on a smooth affine variety is Cohen-Macaulay. Algebras of functions on many singular varieties are also Cohen-Macaulay.

[9]It was proved by Quillen that a locally free module over a polynomial algebra is free; this is a difficult theorem, which will not be needed here.

Exercise 10.18. Show that the algebra of functions on the cone $xy = z^2$ is Cohen-Macaulay.

Another tool we will need is **homological dimension.** We will say that an algebra A has homological dimension $\leq d$ if any (left) A-module M has a projective resolution of length $\leq d$. The homological dimension of A is the smallest integer having this property. If such an integer does not exist, A is said to have infinite homological dimension.

It is easy to show that the homological dimension of A is $\leq d$ if and only if for any A-modules M, N one has $\mathrm{Ext}^i(M, N) = 0$ for $i > d$. Also, the homological dimension clearly does not increase under taking associated graded of the algebra under a positive filtration (this is clear from considering the spectral sequence attached to the filtration).

It follows immediately from this definition that homological dimension is Morita invariant. Namely, recall that a Morita equivalence between algebras A and B is an equivalence of categories $A - \mathrm{mod} \to B - \mathrm{mod}$. Such an equivalence maps projective modules to projective ones, since projectivity is a categorical property (P is projective iff $\mathrm{Hom}(P, ?)$ is exact). This implies that if A and B are Morita equivalent then their homological dimensions are the same.

Then we have the following important theorem.

Theorem 10.19. *The homological dimension of a commutative finitely generated $\mathbb{C}$-algebra Z is finite if and only if Z is regular, i.e. is the algebra of functions on a smooth affine variety.*

10.6. **Proof of Theorem 10.10.** First let us show that any smooth point χ of M_c is an Azumaya point. Since $H_{0,c} = \mathrm{End}_{B_{0,c}} H_{0,c}\mathbf{e} = \mathrm{End}_{Z_c}(H_{0,c}\mathbf{e})$, it is sufficient to show that the coherent sheaf on M_c corresponding to the module $H_{0,c}\mathbf{e}$ is a vector bundle near χ. By Theorem 10.17 (ii), for this it suffices to show that $H_{0,c}\mathbf{e}$ is a Cohen-Macaulay Z_c-module.

To do so, first note that the statement is true for $c = 0$. Indeed, in this case the claim is that SV is a Cohen-Macaulay module over $(SV)^G$. but SV is a polynomial algebra, which is Cohen-Macaulay, so the result follows from Theorem 10.17, (i).

Now, we claim that if Z, M are positively filtered and $\mathrm{gr}(M)$ is a Cohen-Macaulay $\mathrm{gr}Z$-module then M is a Cohen-Macaulay Z-module. Indeed, let $z_1, ..., z_n$ be homogeneous algebraically independent elements of $\mathrm{gr}Z$ such that $\mathrm{gr}Z$ is a finite module over the subalgebra generated by them. Let $z_1', ..., z_n'$ be their liftings to Z. Then $z_1', ..., z_n'$ are algebraically independent, and the module M over $\mathbb{C}[z_1', ..., z_n']$ is finitely generated and (locally) free since so is the module $\mathrm{gr}M$ over $\mathbb{C}[z_1, ..., z_n]$.

Recall now that $\mathrm{gr} H_{0,c}\mathbf{e} = SV$, $\mathrm{gr} Z_c = (SV)^G$. Thus the $c = 0$ case implies the general case, and we are done.

Now let us show that any Azumaya point of M_c is smooth. Let U be an affine open set in M_c consisting of Azumaya points. Then the localization $H_{0,c}(U) := H_{0,c} \otimes_{Z_c} \mathcal{O}_U$ is an Azumaya algebra. Moreover, for any $\chi \in U$, the unique irreducible representation of $H_{0,c}(U)$ with central character χ is the regular representation of G (since this holds for generic χ by Proposition 10.4). This means that $\mathbf{e}$ is a rank 1 idempotent in $H_{0,c}(U)/(\chi)$ for all χ. In particular, $H_{0,c}(U)\mathbf{e}$ is a vector bundle on U. Thus the functor $F : \mathcal{O}_U - \mathrm{mod} \to H_{0,c}(U) - \mathrm{mod}$ given by the formula $F(Y) = H_{0,c}(U)\mathbf{e} \otimes_{\mathcal{O}_U} Y$ is an equivalence of categories (the quasi-inverse functor is given by the formula $F^{-1}(N) = \mathbf{e}N$). Thus $H_{0,c}(U)$ is Morita equivalent to $\mathcal{O}_U$, and therefore their homological dimensions are the same.

On the other hand, the homological dimension of $H_{0,c}$ is finite (in fact, it equals to $\dim V$). To show this, note that by the Hilbert syzygies theorem, the homological dimension of SV is $\dim V$. Hence, so is the homological dimension of $G \ltimes SV$ (as $\mathrm{Ext}^*{}_{G\ltimes SV}(M, N) = \mathrm{Ext}^*{}_{SV}(M, N)^G$). Thus, since $\mathrm{gr} H_{0,c} = G \ltimes SV$, we get that $H_{0,c}$ has homological dimension $\leq \dim V$. Hence, the homological dimension of $H_{0,c}(U)$ is also $\leq \dim V$ (as the homological dimension clearly does not increase under the localization). But $H_{0,c}(U)$ is Morita equivalent to $\mathcal{O}_U$, so $\mathcal{O}_U$ has a finite homological dimension. By Theorem 10.19, this implies that U consists of smooth points.

Corollary 10.20. *$Az(M_c)$ is also the set of points at which the Poisson structure of M_c is symplectic.*

Proof. The variety M_c is generically symplectic, because so is M_0. Thus the set S of smooth points of M_c where the top exterior power of the Poisson bivector vanishes is of codimension ≥ 2. So By Hartogs' theorem S is empty. Thus, every smooth point is symplectic, and the corollary follows from the theorem. ☐

10.7. **The space M_c for $G = S_n$.** Let us consider the space M_c for $G = S_n$, $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} = \mathbb{C}^n$. In this case we have only one parameter c corresponding to the conjugacy class of reflections, and there are only two essentially different cases: $c = 0$ and $c \neq 0$.

Theorem 10.21. *For $c \neq 0$ the space M_c is isomorphic to the Calogero-Moser space $\mathcal{C}_n$ as a symplectic manifold.*

Proof. To prove the theorem, we will first construct a map $f : M_c \to \mathcal{C}_n$, and then prove that f is an isomorphism.

Without loss of generality, we may assume that $c = 1$. As we have shown before, the algebra $H_{0,c}$ is an Azumaya algebra. Therefore, M_c can be regarded as the moduli space of irreducible representations of $H_{0,c}$.

Let $E \in M_c$ be an irreducible representation of $H_{0,c}$. We have seen before that E has dimension $n!$ and is isomorphic to the regular representation as a representation of S_n. Let $S_{n-1} \subset S_n$ be the subgroup which preserves the element 1. Then the space of invariants $E^{S_{n-1}}$ has dimension n. On this space we have operators $X, Y : E^{S_{n-1}} \to E^{S_{n-1}}$ obtained by restriction of the operators x_1, y_1 on E to the subspace of invariants. We have

$$[X, Y] = T := \sum_{i=2}^{n} s_{1i}.$$

Let us now calculate the right hand side of this equation explicitly. Let $\mathbf{p}$ be the symmetrizer of S_{n-1}. Let us realize the regular representation E of S_n as $\mathbb{C}[S_n]$ with action of S_n by left multiplication. Then $v_1 = \mathbf{p}, v_2 = \mathbf{p}s_{12}, ..., v_n = \mathbf{p}s_{1n}$ is a basis of $E^{S_{n-1}}$. The element T commutes with $\mathbf{p}$, so we have

$$Tv_i = \sum_{j \neq i} v_j$$

This means that $T + 1$ has rank 1, and hence the pair (X, Y) defines a point on the Calogero-Moser space $\mathcal{C}_n$. [10]

We now set $(X, Y) = f(E)$. It is clear that $f : M_c \to \mathcal{C}_n$ is a regular map. So it remains to show that f is an isomorphism. This is equivalent to showing that the corresponding map of function algebras $f^* : \mathcal{O}(\mathcal{C}_n) \to Z_c$ is an isomorphism.

Let us calculate f and f^* more explicitly. To do so, consider the open set U in M_c consisting of representations in which $x_i - x_j$ act invertibly. These are exactly the representations that are obtained by restricting representations of $S_n \ltimes \mathbb{C}[x_1, ..., x_n, p_1, ..., p_n, \delta(x)^{-1}]$ using the classical Dunkl embedding. Thus representations $E \subset U$ are of the form $E = E_{\lambda,\mu}$ ($\lambda, \mu \in \mathbb{C}^n$, and λ has distinct coordinates), where $E_{\lambda,\mu}$ is the space of complex valued functions on the orbit $O_{\lambda,\mu} \subset \mathbb{C}^{2n}$, with the following action of $H_{0,c}$:

$$(x_i F)(a, b) = a_i F(a, b), (y_i F)(a, b) = b_i F(a, b) + \sum_{j \neq i} \frac{(s_{ij} F)(a, b)}{a_i - a_j}.$$

(the group S_n acts by permutations).

[10]Note that the pair (X, Y) is well defined only up to conjugation, because the representation E is well defined only up to an isomorphism.

Now let us consider the space $E_{\lambda,\mu}^{S_{n-1}}$. A basis of this space is formed by characteristic functions of S_{n-1}-orbits on $O_{\lambda,\mu}$. Using the above presentation, it is straightforward to calculate the matrices of the operators X and Y in this basis:

$$X = \mathrm{diag}(\lambda_1, ..., \lambda_n),$$

and

$$Y_{ij} = \mu_i \text{ if } j = i,\ Y_{ij} = \frac{1}{\lambda_i - \lambda_j} \text{ if } j \neq i.$$

This shows that f induces an isomorphism $f|_U : U \to U_n$, where U_n is the subset of $\mathcal{C}_n$ consisting of pairs (X, Y) for which X has distinct eigenvalues.

From this presentation, it is straigtforward that $f^*(\mathrm{Tr}(X^p)) = x_1^p + ... + x_n^p$ for every positive integer p. Also, f commutes with the natural $SL_2(\mathbb{C})$ action on M_c and $\mathcal{C}_n$ (by $(X, Y) \to (aX + bY, cX + dY)$), so we also get $f^*(\mathrm{Tr}(Y^p)) = y_1^p + ... + y_n^p$, and

$$f^*(Tr(X^pY)) = \frac{1}{p+1} \sum_{m=0}^{p} \sum_i x_i^m y_i x_i^{p-m}.$$

Now, using the necklace bracket formula on $\mathcal{C}_n$ and the commutation relatrions of $H_{0,c}$, we find, by a direct computation, that f^* preserves Poisson bracket on the elements $Tr(X^p)$, $Tr(X^qY)$. But these elements are a local coordinate system near a generic point, so it follows that f is a Poisson map. Since the algebra Z_c is Poisson generated by $\sum x_i^p$ and $\sum y_i^p$ for all p, we get that f^* is a surjective map.

Also, f^* is injective. Indeed, by Wilson's theorem the Calogero-Moser space is connected, and hence the algebra $\mathcal{O}(\mathcal{C}_n)$ has no zero divisors, while and $\mathcal{C}_n$ has the same dimension as M_c. This proves that f^* is an isomorphism, so f is an isomorphism. □

10.8. **Generalizations.** Now let L be a 2-dimensional symplectic vector space, $\Gamma \subset Sp(L)$ is a finite subgroup, and $G = S_n \ltimes \Gamma^n$. Let $V = L \oplus ... \oplus L$ (n times). In this situation c consists of a number k corresponding to the conjugacy class of s_{ij} (for $n > 1$), and numbers c_s corresponding to conjugacy classes of nontrivial elements in Γ. In this case, the above results may be generalized. More specifically, we have the following result (which we will state without proof).

Theorem 10.22. *(i) M_c is smooth for generic c. (It has a matrix realization in terms of McKay's correspondence).*

(ii) M_c is a deformation of the Hilbert scheme of n-tuples of points on the resolution of the Kleinian singularity $\mathbb{C}^2/\Gamma$. In particular, if $n = 1$ then M_c is the versal deformation of the Kleinian singularity.

Part (i) of this theorem is proved in [EG]. The proof of part (ii) of this theorem for $\Gamma = 1$ can be found in Nakajima's book [Na]. The general case is similar.

We also have the following result about the cohomology of M_c. Introduce a filtration on $\mathbb{C}[G]$ by setting the degree of g to be the rank of $g - 1$ on V.

Theorem 10.23. *The cohomology ring of M_c is $H^*(M_c, \mathbb{C}) = \mathrm{gr}(\mathrm{Center}(\mathbb{C}[G]))$.*

The proof is based on the following argument. We know that the algebra $B_{t,c}$ is a quantization of M_c. Therefore, by Kontsevich's "compatibility with cup products" theorem, the cohomology algebra of M_c is the cohomology of $B_{t,c}$ (for generic t). But $B_{t,c}$ is Morita equivalent to $H_{t,c}$, so this cohomology is the same as the Hochschild cohomology of $H_{t,c}$. However, the latter can be computed by using that $H_{t,c}$ is given by generators and relations (by producing explicit representatives of cohomology classes and computing their product).

10.9. **Notes.** The results of this lecture are contained in [EG]. For the commutative algebraic tools (Cohen-Macaulayness, homological dimension) we refer the reader to the textbook [Ei]. The Amitsur-Levitzky identity and Artin's theorem on Azumaya algebras can be found in [Ja]. The center of the double affine Hecke algebra of type A, which is a deformation of the rational Cherednik algebra, was computed by Oblomkov [Ob].

11. Representation theory of rational Cherednik algebras

11.1. **Rational Cherednik algebras for any finite group of linear transformations.** Above we defined rational Cherednik algebras for reflection groups, as a tool for understanding Olshanetsky-Perelomov operators. Actually, it turns out that such algebras can be defined for any finite group of linear transformations.

Namely, let $\mathfrak{h}$ be an ℓ-dimensional complex vector space, and W be a finite group of linear transformations of $\mathfrak{h}$. An element s of W is said to be a *complex reflection* if all eigenvalues of s in $\mathfrak{h}$ are equal to 1 except one eigenvalue $\lambda \neq 1$ (which will of course be a root of unity). For each $s \in S$, let α_s be an eigenvector of s in $\mathfrak{h}^*$ with nontrivial eigenvalue, and $\alpha_s^\vee$ the eigenvector of s in $\mathfrak{h}$ such that $(\alpha_s, \alpha_s^\vee) = 2$. Let S be the set of conjugacy classes of complex reflections in W, and $c : S \to \mathbb{C}$ be a function invariant under conjugation.

Definition 11.1. The rational Cherednik algebra $H_{1,c} = H_{1,c}(\mathfrak{h}, W)$ attached to $\mathfrak{h}, W, c$ is the quotient of the algebra

$\mathbb{C}W \ltimes \mathbf{T}(\mathfrak{h} \oplus \mathfrak{h}^*)$ (where $\mathbf{T}$ denotes the tensor algebra) by the ideal generated by the relations

$$[x, x'] = 0,\ [y, y'] = 0,\ [y, x] = (y, x) - \sum_{s \in S} c_s(y, \alpha_s)(x, \alpha_s^\vee)s.$$

Obviously, this is a generalization of the definition of the rational Cherednik algebra for a reflection group, and a special case of symplectic reflection algebras. In particular, the PBW theorem holds for $H_{1,c}$, and thus we have a tensor product decomposition $H_{1,c} = \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]$ (as a vector space).

11.2. **Verma and irreducible lowest weight modules over $H_{1,c}$.** We see that the structure of $H_{1,c}$ is similar to the structure of the universal enveloping algebra of a simple Lie algebra: $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$. Namely, the subalgebra $\mathbb{C}W$ plays the role of the Cartan subalgebra, and the subalgebras $\mathbb{C}[\mathfrak{h}^*]$ and $\mathbb{C}[\mathfrak{h}]$ play the role of the positive and negative nilpotent subalgebras. This similarity allows one to define and study the category $\mathcal{O}$ analogous to the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for simple Lie algebras, in which lowest weights will be representations of W.

Let us first define the simplest objects of this category – the standard, or Verma modules.

For any irreducible representation τ of W, let $M_c(\tau)$ be the standard representation of $H_{1,c}$ with lowest weight τ, i.e. $M_c(\tau) = H_{1,c} \otimes_{\mathbb{C}[W] \ltimes \mathbb{C}[\mathfrak{h}^*]} \tau$, where τ is the representation of $\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}^*]$, in which $y \in \mathfrak{h}$ act by 0. Thus, as a vector space, $M_c(\tau)$ is naturally identified with $\mathbb{C}[\mathfrak{h}] \otimes \tau$.

An important special case of $M_c(\tau)$ is $M_c = M_c(\mathbb{C})$, the polynomial representation, corresponding to the case when $\tau = \mathbb{C}$ is trivial. The polynomial representation can thus be naturally identified with $\mathbb{C}[\mathfrak{h}]$. Elements of W and $\mathfrak{h}^*$ act in this space in the obvious way, while elements $y \in \mathfrak{h}$ act by Dunkl operators

$$\partial_y - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} \frac{(\alpha_s, y)}{\alpha_s}(1 - s),$$

where λ_s is the nontrivial eigenvalue of s in $\mathfrak{h}^*$.

Introduce an important element $\mathbf{h} \in H_{1,c}$:

$$\mathbf{h} = \sum_i x_i y_i + \frac{\ell}{2} - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} s,$$

where y_i is a basis of $\mathfrak{h}$ and x_i the dual basis of $\mathfrak{h}^*$.

Exercise 11.2. Show that if W preserves an inner product in $\mathfrak{h}$ then the element $\mathbf{h}$ can be included in an sl_2 triple $\mathbf{h}, \mathbf{E} = \frac{1}{2}\sum x_i^2, \mathbf{F} = \frac{1}{2}\sum y_i^2$, where x_i, y_i are orthonormal bases of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively.

Lemma 11.3. *The element* $\mathbf{h}$ *is* W*-invariant and satisfies the equations* $[\mathbf{h}, x] = x$ *and* $[\mathbf{h}, y] = -y$.

Exercise 11.4. Prove Lemma 11.3.

It is easy to show that the lowest eigenvalue of $\mathbf{h}$ in $M_c(\tau)$ is $h(\tau) := \ell/2 - \sum_s \frac{2c_s}{1-\lambda_s} s|_\tau$.

Corollary 11.5. *The sum* $J_c(\tau)$ *of all proper submodules of* $M_c(\tau)$ *is a proper submodule of* $M_c(\tau)$*, and the quotient module* $L_c(\tau) := M_c(\tau)/J_c(\tau)$ *is irreducible.*

Proof. Every eigenvalue μ of $\mathbf{h}$ in a proper submodule of $M_c(\tau)$, and hence in $J_c(\tau)$, satisfies the inequality $\mu - h(\tau) > 0$. Thus $J_c(\tau) \neq M_c(\tau)$. □

The module $L_c(\tau)$ can be characterized in terms of the contragredient standard modules. Namely, let $\hat{M}_c(\tau) = \tau^* \otimes_{\mathbb{C}[W]\ltimes\mathbb{C}[\mathfrak{h}]} H_{1,c}(W)$ be a right $H_{1,c}(W)$-module, and $M_c(\tau)^\vee = \hat{M}_c(\tau)^*$ its restricted dual, which may be called the contragredient standard module. Clearly, there is a natural morphism $\phi : M_c(\tau) \to M_c(\tau)^\vee$.

Lemma 11.6. *The module* $L_c(\tau)$ *is the image of* ϕ.

Exercise 11.7. Prove Lemma 11.6.

Note that the map ϕ can be viewed as a bilinear form $B : \hat{M}_c(\tau) \otimes M_c(\tau) \to \mathbb{C}$. This form is analogous to the Shapovalov form in Lie theory.

If W preserves an inner product on $\mathfrak{h}$, we can define an antiinvolution of $H_c(W)$ by $x_i \to y_i$, $y_i \to x_i$, $g \to g^{-1}$ for $g \in W$ (where x_i, y_i are orthonormal bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ dual to each other). Under this antiinvolution, the right module $\hat{M}_c(\tau)$ turns into the left module $M_c(\tau^*)$, so the form B is a (possibly degenerate) pairing $M_c(\tau^*) \otimes M_c(\tau) \to \mathbb{C}$. Moreover, it is clear that if Y, Y' are any quotients of $M_c(\tau), M_c(\tau^*)$ respectively, then B descends to a pairing $Y' \otimes Y \to \mathbb{C}$ (nondegenerate iff Y, Y' are irreducible). This pairing satisfies the contravariance equations $B(a, x_i b) = B(y_i a, b)$, $B(a, y_i b) = B(x_i a, b)$, and $B(ga, gb) = B(a, b)$ for $g \in W$.

11.3. **Category $\mathcal{O}$.**

Definition 11.8. The category $\mathcal{O} = \mathcal{O}_c$ of $H_{1,c}$-modules is the category of $H_{1,c}$-modules V, such that V is the direct sum of finite dimensional generalized eigenspaces of $\mathbf{h}$, and the real part of the spectrum of $\mathbf{h}$ is bounded below.

Obviously, the standard representations $M_c(\tau)$ and their irreducible quotients $L_c(\tau)$ belong to $\mathcal{O}$.

Exercise 11.9. Show that $\mathcal{O}$ is an abelian subcategory of the category of all $H_{1,c}$-modules, which is closed under extensions, and its set of isomorphism classes of simple objects is $\{L_c(\tau)\}$. Moreover, show that every object of $M_c(\tau)$ has finite length.

Definition 11.10. The character of a module $V \in \mathcal{O}$ is $\chi_V(g,t) = \mathrm{Tr}|_V(gt^{\mathbf{h}})$, $g \in W$ (this is a series in t).

For example, the character of $M_c(\tau)$ is

$$\chi_{M_c(\tau)}(g,t) = \frac{\chi_\tau(g)t^{h(\tau)}}{\det|_{\mathfrak{h}^*}(1-gt)}.$$

On the other hand, determining the characters of $L_c(\tau)$ is in general a hard problem. An equivalent problem is determining the multiplicity of $L_c(\sigma)$ in the Jordan-Hölder series of $M_c(\tau)$.

Exercise 11.11. 1. Show that for generic c (outside of countably many hyperplanes), $M_c(\tau) = L_c(\tau)$, so $\mathcal{O}$ is a semisimple category.

2. Determine the characters of $L_c(\tau)$ for all c in the case $\ell = 1$.

11.4. **The Frobenius property.** Let A be a $\mathbb{Z}_+$-graded commutative algebra. The algebra A is called Frobenius if the top degree $A[d]$ of A is 1-dimensional, and the multiplication map $A[m] \times A[d-m] \to A[d]$ is a nondegenerate pairing for any $0 \le m \le d$. In particular, the Hilbert polynomial of a Frobenius algebra A is palindromic.

Now, let us go back to considering modules over the rational Cherednik algebra $H_{1,c}$. Any submodule J of the polynomial representation $M_c(\mathbb{C}) = M_c = \mathbb{C}[\mathfrak{h}]$ is an ideal in $\mathbb{C}[\mathfrak{h}]$, so the quotient $A = M_c/J$ is a $\mathbb{Z}_+$-graded commutative algebra.

Now suppose that W preserves an inner product in $\mathfrak{h}$.

Theorem 11.12. *If $A = M_c/J$ is finite dimensional, then $A = L_c = L_c(\mathbb{C})$ (i.e. A is irreducible) if and only if it is a Frobenius algebra.*

Proof. (i) Suppose A is an irreducible $H_{1,c}$-module, i.e. $A = L_c(\mathbb{C})$. Recall that $H_{1,c}$ has an $\mathfrak{s}l_2$ subalgebra generated by $\mathbf{h}$, $\mathbf{E} = \sum x_i^2$, $\mathbf{F} = \sum y_i^2$. Thus the $\mathfrak{sl}_2$-representation theory shows that the top degree homogeneous component of A is 1-dimensional. Let $\phi \in A^*$

denote a nonzero linear functional on A which factors through the projection to the top component; it is unique up to scaling.

Further, since A is finite dimensional, the action of $\mathfrak{sl}_2$ in A integrates to an action of the group $SL_2(\mathbb{C})$. Also, as we have explained, this module has a canonical contravariant nondegenerate bilinear form $B(,)$. Let F be the 'Fourier transform' endomorphism of A corresponding to the action of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{C})$. Since the Fourier transform interchanges $\mathfrak{h}$ and $\mathfrak{h}^*$, the bilinear form: $E : (v_1, v_2) \longmapsto B(v_1\,,\, Fv_2)$ on A is compatible with the algebra structure, i.e. for any $x \in \mathfrak{h}^*$, we have $E(xv_1, v_2) = E(v_1, xv_2)$. This means that for any polynomials $p, q \in \mathbb{C}[\mathfrak{h}]$, one has $B(p(x), q(x)) = \phi(p(x)q(x))$ (for a suitable normalization of ϕ). Hence, since the form E is nondegenerate, A is a Frobenius algebra.

(ii) Suppose A is Frobenius. Then the highest degree component of A is 1-dimensional, and the pairing $E : A \otimes A \to \mathbb{C}$ given by $E(a, b) := \phi(ab)$ (where, as before, ϕ stands for the highest degree coefficient) is nondegenerate. This pairing obviously satisfies the condition $E(xa, b) = E(a, xb), x \in \mathfrak{h}^*$. Set $\tilde{B}(a, b) := E(a, Fb)$. Then B satisfies the equations $\tilde{B}(a, x_i b) = \tilde{B}(y_i a, b)$. So for any $f_1, f_2 \in \mathbb{C}[\mathfrak{h}]$, one has $\tilde{B}(p(x)v, q(x)v) = \tilde{B}(q(y)p(x)v, v)$, where $v = 1$ is the lowest weight vector of A. This shows that $\tilde{B}$ coincides with the Shapovalov form B on A. Thus A is an irreducible representation of $H_{1,c}$. □

Remark. It easy to see by considering the rank 1 case that for complex reflection groups Theorem 11.12 is, in general, false.

Let us now consider the Frobenius property of L_c in the case of a general group $W \subset GL(\mathfrak{h})$.

Theorem 11.13. *Let $U \subset M_c = \mathbb{C}[\mathfrak{h}]$ be a W-subrepresentation of dimension $\ell = \dim(\mathfrak{h})$ sitting in degree r, consisting of singular vectors (i.e. those killed by $y \in \mathfrak{h}$). Let J be the ideal (=submodule) generated by U. Assume that the quotient representation $A = M_c/J$ is finite dimensional. Then*

(i) The algebra A is Frobenius.

(ii) The representation A admits a BGG type resolution

$$A \leftarrow M_c(\mathbb{C}) \leftarrow M_c(U) \leftarrow M_c(\wedge^2 U) \leftarrow \ldots \leftarrow M_c(\wedge^\ell U) \leftarrow 0.$$

(iii) The character of A is given by the formula

$$\chi_A(g, t) = t^{\frac{\ell}{2} - \sum_s \frac{2c_s}{1-\lambda_s}} \frac{\det|_U(1 - gt^r)}{\det_{\mathfrak{h}^*}(1 - gt)}.$$

In particular, the dimension of A is r^ℓ.

(iv) If W preserves an inner product on $\mathfrak{h}$, then A is irreducible.

Proof. (i) Since Spec(A) is a complete intersection, A is Frobenius (=Gorenstein of dimension 0) (see [Ei], p.541).

(ii) Consider the subring $\mathbb{C}[U]$ in $\mathbb{C}[\mathfrak{h}]$. Then $\mathbb{C}[\mathfrak{h}]$ is a finitely generated $\mathbb{C}[U]$-module. A standard theorem of Serre which we have discussed above (Section 10.5) says that if $B = \mathbb{C}[t_1, ..., t_n]$, $f_1, ..., f_n \in B$ are homogeneous, $A = \mathbb{C}[f_1, ..., f_n] \subset B$, and B is a finitely generated module over A, then B is a free module over A. Applying this in our situation, we see that $\mathbb{C}[\mathfrak{h}]$ is a free $\mathbb{C}[U]$-module. It is easy to see by computing the Hilbert series that the rank of this free module is r^ℓ.

Consider the Koszul complex attached to the module $\mathbb{C}[\mathfrak{h}]$ over $\mathbb{C}[U]$. Since this module is free, the Koszul complex is exact (i.e. it is a resolution of the zero-fiber). At the level of $\mathbb{C}[\mathfrak{h}]$ modules, this resolution looks exactly as we want in (ii). So we need to show that the maps of the resolution are in fact morphisms of $H_{1,c}$-modules and not only $\mathbb{C}[\mathfrak{h}]$-modules. This is easily established by induction (going from left to right), cf. proof of Proposition 2.2 in Yu. Berest, P.Etingof, V. Ginzburg, "Finite dimensional representations of rational Cherednik algebras".

(iii) Follows from (ii) by the Euler-Poincare principle.

(iv) Follows from Theorem 11.12. □

11.5. **Representations of the rational Cherednik algebra of type A.**

11.5.1. *The results.* Let $W = S_n$, and $\mathfrak{h}$ be its reflection representation. In this case the function c reduces to one number k. We will denote the rational Cherednik algebra $H_{1,k}(S_n)$ by $H_k(n)$. The polynomial representation M_k of this algebra is the space of $\mathbb{C}[x_1, ..., x_n]^T$ of polynomials of $x_1, ..., x_n$, which are invariant under simultaneous translation $x_i \mapsto x_i + a$. In other words, it is the space of regular functions on $\mathfrak{h} = \mathbb{C}^n/\Delta$, where Δ is the diagonal.

Proposition 11.14. *(C. Dunkl) Let r be a positive integer not divisible by n, and $k = r/n$. Then M_k contains a copy of the reflection representation $\mathfrak{h}$ of S_n, which consists of singular vectors (i.e. those killed by $y \in \mathfrak{h}$). This copy sits in degree r and is spanned by the functions*

$$f_i(x_1, ..., x_n) = \mathrm{Res}_\infty[(z - x_1)...(z - x_n)]^{\frac{r}{n}} \frac{dz}{z - x_i}.$$

(the symbol Res_∞ denotes the residue at infinity).

Remark. The space spanned by f_i is $n - 1$-dimensional, since $\sum_i f_i = 0$ (this sum is the residue of an exact differential).

Proof. This proposition can be proved by a straightforward computation. The functions f_i are a special case of Jack polynomials. $\square$

Exercise 11.15. Do this computation.

Let I_k be the submodule of M_k generated by f_i. Consider the $H_k(n)$-module $V_k = M_k/I_k$, and regard it as a $\mathbb{C}[\mathfrak{h}]$-module.

Our result is

Theorem 11.16. *Let $d = (r, n)$ denote the greatest common divisor of r and n. Then the (set-theoretical) support of V_k is the union of S_n-translates of the subspaces of $\mathbb{C}^n/\Delta$, defined by the equations*

$$
\begin{aligned}
x_1 = x_2 = \cdots = x_{\frac{n}{d}};\\
x_{\frac{n}{d}+1} = \cdots = x_{2\frac{n}{d}};\\
\cdots\\
x_{(d-1)\frac{n}{d}+1} = \cdots = x_n.
\end{aligned}
$$

In particular, the Gelfand-Kirillov dimension of V_k is $d - 1$.

The theorem allows us to give a simple proof of the following result of Berest, Etingof, Ginzburg (without the use of the KZ functor and Hecke algebras).

Corollary 11.17. *If $d = 1$ then the module $V_k := M_k/I_k$ is finite dimensional, irreducible, admits a BGG type resolution, and its character is*

$$\chi_{V_k}(g, t) = t^{(1-r)(n-1)/2} \frac{\det|_{\mathfrak{h}}(1 - gt^r)}{\det|_{\mathfrak{h}}(1 - gt)}.$$

Proof. For $d = 1$ Theorem 11.16 says that the support of M_k/I_k is $\{0\}$. This implies that M_k/I_k is finite dimensional. The rest follows from Theorem 11.13. $\square$

11.5.2. *Proof of Theorem 11.16.* The support of V_k is the zero-set of I_k, i.e. the common zero set of f_i. Fix $x_1, ..., x_n \in \mathbb{C}$. Then $f_i(x_1, ..., x_n) = 0$ for all i iff $\sum_{i=1}^{n} \lambda_i f_i = 0$ for all λ_i, i.e.

$$\mathrm{Res}_\infty \left(\prod_{j=1}^{n} (z - x_j)^{\frac{r}{n}} \sum_{i=1}^{n} \frac{\lambda_i}{z - x_i} \right) dz = 0.$$

Assume that $x_1, \ldots x_n$ take distinct values $y_1, \ldots, y_p$ with positive multiplicities $m_1, \ldots, m_p$. The previous equation implies that the point

$(x_1, ..., x_n)$ is in the zero set iff

$$\mathrm{Res}_\infty \prod_{j=1}^{p}(z-y_j)^{m_j\frac{r}{n}-1}\left(\sum_{i=1}^{p}\nu_i(z-y_1)\dots\widehat{(z-y_i)}\dots(z-y_p)\right)dz=0 \quad \forall\nu_i.$$

Since ν_i are arbitrary, this is equivalent to the condition

$$\mathrm{Res}_\infty \prod_{j=1}^{p}(z-y_j)^{m_j\frac{r}{n}-1}z^i dz=0, \quad i=0,\dots,p-1.$$

We will now need the following lemma.

Lemma 11.18. *Let* $a(z)=\prod_{j=1}^{p}(z-y_j)^{\mu_j}$*, where* $\mu_j\in\mathbb{C}$*,* $\sum_j\mu_j\in\mathbb{Z}$ *and* $\sum_j\mu_j>-p$*. Suppose*

$$\mathrm{Res}_\infty a(z)z^i dz=0, \quad i=0,1,\dots,p-2.$$

Then $a(z)$ *is polynomial.*

Proof. Let g be a polynomial. Then

$$0=\mathrm{Res}_\infty d(g(z)\cdot a(z))=\mathrm{Res}_\infty(g'(z)a(z)+a'(z)g(z))dz$$

and hence

$$\mathrm{Res}_\infty\left(g'(z)+\sum_i\frac{\mu_j}{z-y_j}g(z)\right)a(z)dz=0.$$

Let $g(z)=z^l\prod_j(z-y_j)$. Then $g'(z)+\sum\frac{\mu_j}{z-y_j}g(z)$ is a polynomial of degree $l+p-1$ with highest coefficient $l+p+\sum\mu_j\neq0$ (as $\sum\mu_j>-p$). This means that for every $l\geq0$, $\mathrm{Res}_\infty z^{l+p-1}a(z)dz$ is linear combination of residues of $z^q a(z)dz$ with $q<l+p-1$. By the assumption of the lemma, this implies by induction in l that all such residues are 0 and hence a is a polynomial. □

In our case $\sum(m_j\frac{r}{n}-1)=r-p$ (since $\sum m_j=n$) and the conditions of the lemma are satisfied. Hence $(x_1, ..., x_n)$ is in the zero set of I_k iff $\prod_{j=1}^{p}(z-y_j)^{m_j\frac{r}{n}-1}$ is a polynomial. This is equivalent to saying that all m_j are divisible by $\frac{n}{d}$.

We have proved that $(x_1,\dots x_n)$ is in the zero set of I_k iff $(z-x_1)\dots(z-x_n)$ is the n/d-th power of a polynomial of degree d. This implies the theorem.

11.6. **Notes.** In this lecture we have followed the paper [CE]. The classification of finite dimensional representations of the rational Cherednik algebra of type A was obtained in [BEG] (the corresponding result for double affine Hecke algebra is due to Cherednik [Ch2]).

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