Corrections to the book "Tensor categories" by Etingof, Gelaki, Nikshych and Ostrik, AMS, 2015

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Here are some corrections to the book "Tensor categories". We thank Johannes Berger, Ryan Kinser, Vanessa Miemietz, Ivan Motorin, and Ulrich Thiel for pointing out many of the corrections below.

Many useful comments and explanations can also be found at `https://ulthiel.com/math/wp-content/uploads/lecture-notes/Comments-on-EGNO.pdf`

If you see other mistakes, please let us know!

1. General comments

1. When working with an additive category $\mathcal{C}$ over a field $\Bbbk$, we often use the fact that for any finite dimensional $\Bbbk$-vector space V we have a natural functor $V\otimes : \mathcal{C} \to \mathcal{C}$. Namely, for $X \in \mathcal{C}$ the object $V \otimes X$ is uniquely defined through the Yoneda lemma by the formula $\mathrm{Hom}(Y, V\otimes X) = V\otimes \mathrm{Hom}(Y, X)$ (and the existence of this object is checked by choosing a basis in V).

2. We often abuse terminology and refer to epimorphisms in abelian categories as surjections and to monomorphisms as injections.

3. We use the terms "natural morphism" and "functorial morphism" interchangeably.

2. Other comments.

2.1. Chapter 1.

Definition 1.8.3, typo: "any simple object" should be "any object".

Definition 1.8.13: $\Bbbk$ should be replaced by another field $\mathbf{k}$, usually of characteristic zero, and not necessarily equal to the field $\Bbbk$ of definition of $\mathcal{C}$.

Subsection 1.10 (Coend), clarification. In (1.9), the direct sum is taken over isomorphism classes of objects. Also the (co)limits in lines 6,7 of the section are taken for the following diagram J: the vertices v_X of J are labeled by representatives X of isomorphism classes of objects, and arrows are $\phi_f : v_X \to v_{X\oplus Y}$ and $\psi_f : v_Y \to v_{X\oplus Y}$ for every morphism $f : X \to Y$. The map $E_F : J \to \mathcal{C}$ is given by the formula $E_F(v_X) = \mathrm{End}(F(X))$, $E_F(\phi_f)(a) = f \circ a$, $E_F(\psi_f)(a) = a \circ f$, where we view $\mathrm{Hom}(X, Y)$ naturally as a subgroup of $\mathrm{End}(X \oplus Y)$. Then $\varinjlim \mathrm{End}(F(X))$ reproduces the usual definition of $\mathrm{End}(F)$: the set of collections $\{a_X \in \mathrm{End}(F(X))\}$ such that $a_Y \circ f = f \circ a_Y$ for all X, Y and $f : X \to Y$.

2.2. Chapter 2.

Proof of Corollary 2.2.5, typo at the beginning: $Y = Z = \mathbf{1}$ should be $Y = X = \mathbf{1}$.

Remark 2.4.2: Section 2.5 should be Section 2.6.

Subsection 2.6, l. 3, typo: $\mathcal{C}_i$ are not the categories of graded vector spaces (they are not linear). l. 5: the word "simple" should be deleted. Also we consider only monoidal functors which act trivially on A.

Remark 2.8.6, clarification. In the last sentence of p.38, "the same category with the trivial associativity morphism" should read "the same category with the trivial associativity morphism and a modified tensor product of morphisms". Namely, let $\mu : \widetilde{G} \times \widetilde{G} \to A$ be a 2-cochain trivializing the cocycle $f^*\omega$. Then the tensor product of morphisms in the strict category isomorphic to $\mathcal{C}^l$ is defined as follows. Given elements $g, g', h, h' \in \widetilde{G}$ with $f(g) = f(g')$, $f(h) = f(h')$ and $a \in \mathrm{Hom}(g, g') = A$, $b \in \mathrm{Hom}(h, h') = A$, we set

$$a \otimes b := ab\mu(g, h)\mu(g', h')^{-1} \in \mathrm{Hom}(gh, g'h') = A.$$

Exercise 2.9.1, typo: the correct answer is the $n-1$-th Catalan number, $\frac{1}{n}\binom{2(n-1)}{n-1}$.

Proposition 2.10.5: "...up to a unique isomorphism preserving the evaluation and coevaluation morphisms".

Remark 2.10.9, typo: in line 3, V^* should be ${}^*V$.

Example 2.10.14, $\omega(g, g^{-1}, g)$ should be $\omega(g, g^{-1}, g)^{-1}$.

Paragraph after Example 2.10.14, typo: 2.10.7(ii) should be 2.10.7(b).

Exercise 2.10.16, typo: "left (respectively, right)" should be replaced by "right (respectively, left)".

2.3. **Chapter 3.** *Proposition 3.1.4* (claiming that $1 = \sum_{i \in I_0} b_i$, i.e., that all coefficients in the decomposition of the unit equal 1) holds not only for based rings but for general $\mathbb{Z}_+$-rings. Proof: If $1 = \sum_{i \in I_0} n_i b_i$ with $n_i > 0$ then for all $j \in I$ $b_j = \sum_{i \in I_0} n_i b_i b_j$, so for every $j \in I$ there exists $i(j) \in I_0$ such that $b_{i(j)} b_j = b_j$ and $b_i b_j = 0$ if $i(j) \neq i \in I_0$. Let I_0' be the image of the map $i : I \to I_0$ and $1' := \sum_{i \in I_0'} b_i$. Then $1' = 1'1 = 1$, so $I_0' = I_0$ and $n_i = 1$ for all i.

Proposition 3.3.6. In the penultimate paragraph of the proof, "left multiplication by the element $\sum_{X \in I} X$" should be replaced with "right multiplication by the element $\sum_{X \in I} X$".

*Section 3.4.*The definition of a *based* $\mathbb{Z}_+$-module over a based ring is accidentally omitted, although it is used further in the text (first time in Proposition 3.6.2).

By definition, a based $\mathbb{Z}_+$-module over a based ring A is one where if $x \in A$ acts by a matrix X then x^* acts by X^T.

There exist non-based finite dimensional $\mathbb{Z}_+$-modules, even if A is finite dimensional, e.g. for the 2-dimensional ring A with basis $1, b$ and $b^2 = 1 + 2b$, $b^* = b$ we may take a 2-dimensional module with basis v, w with $bv = v + w$, $bw = 2v + w$.

Exercise 3.4.3(i). The $\mathbb{Z}_+$-module should assumed be based. Otherwise there is a counterexample: take the $\mathbb{Z}_+$-ring A of representations of $\mathfrak{sl}_2(\mathbb{C})$ with basis $b_j, j \geq 0$ (the irreducible $j+1$-dimensional representation, so $b_j^* = b_j$) and consider the (non-based) 2-dimensional $\mathbb{Z}_+$-module with basis

v, w and $b_{j-1}v = jv, b_{j-1}w = \frac{j^3-j}{6}v + jw$. This module is indecomposable but not irreducible.[1]

Another way to make this exercise correct is to require that the based ring A be finite dimensional. Namely, we have

Lemma 2.1. *Let A be a based ring of finite rank and M an indecomposable $\mathbb{Z}_+$-module over A. Then M is irreducible and has finite rank.*

Proof. We may assume that A is indecomposable. Let $m \in M$ be a basis vector, and let B_m be the subset of the $\mathbb{Z}_+$-basis B_M of M consisting of summands in bm where $b \in B$ runs through the $\mathbb{Z}_+$-basis of A. It is clear that B_m spans a finite rank submodule $M_m \subset M$. Pick an irreducible $\mathbb{Z}_+$-submodule $N \subset M_m$. It suffices to show that $N = M$.

Assume the contrary. Then, since M is indecomposable, we can find a basis element $p \notin B_N$ of M such that $B_p \cap B_N \neq \emptyset$. Let $M_{p,N}$ be the submodule of M spanned by B_N and B_p. On the basis vectors of $M_{p,N}$ we have a preorder in which $m' \leq m''$ if m' is a summand in bm'' for some $b \in B$. The set S of equivalence classes of this preorder is a partially ordered set, and B_N maps to a single element $s_N \in S$. By assumption, there exists $s \in S$ such that $s_N < s$, so let us choose a minimal s with this property. Let $M_* \subset M$ be the span of the basis elements in s and s_N. The matrix of $x \in A$ in M_* looks like

$$\rho(x) := \begin{pmatrix} P(x) & Q(x) \\ 0 & R(x) \end{pmatrix}$$

where $R(x)$ is the matrix by which x acts on N and $P(x)$ the matrix by which it acts on the (irreducible) $\mathbb{Z}_+$-module $(M_* + N)/N$. Note that $Q(b) \neq 0$ for $b \in B$.

Now take $x = \sum_i b_i$. Then all entries of $P(x)$ and $R(x)$ are strictly positive integers. Thus if $\lambda = \mathrm{FPdim}(x)$ then there exists $C > 0$ such that all entries of $P(x)^n$ and $R(x)^n$ are $\geq C\lambda^n$. The upper right corner of $\rho(x)^n$ is $\sum_{k=0}^{n-1} P(x)^k Q(x) R(x)^{n-1-k}$, so every entry of this matrix is $\geq C^2 n \lambda^{n-1}$ (as $Q(x) \neq 0$ and has nonnegative integer entries). On the other hand, the sum of the coefficients of x^n is $O(\lambda^n)$ as $n \to \infty$, hence so is $\rho(x)^n = \rho(x^n)$. Contradiction.

□

2.4. **Chapter 4.** *Proof of Proposition 4.2.8*, line 4: "and using Proposition 4.2.1" should be replaced by "by Definition 4.2.3."

Corollary 4.3.9: "monomorphisms" and "epimorphisms" should be interchanged.

Paragraph before Example 4.5.5: the ring category should be assumed finite.

Remark 4.5.6: "homomorphism of unital $\mathbb{Z}_+$-rings" should be "unital homomorphism of $\mathbb{Z}_+$-rings".

[1] We thank Ivan Motorin for pointing out this issue.

Definition 4.7.11: Tr^L in the first line is not needed, but Tr should be Tr^L in the formula right below.

Proof of Theorem 4.7.15, line 3: Y_i and V_i should be switched (I.e., V_i are objects and Y_i are vector spaces).

Exercise 4.7.16: the end of the last sentence should say "...simple objects **of nonzero dimension**". Without this correction, the statement is false. Indeed, consider the Hopf algebra H over $\mathbb{C}$ with generators g, x and relations $gx = -xg, g^2 = 1$ and coproduct $\Delta(g) = g \otimes g, \Delta(x) = x \otimes 1 + g \otimes x$. The antipode is $S(g) = g^{-1}$, $S(x) = -gx$, so $S^2 = \mathrm{Ad}g$ and g defines a spherical structure on $\mathrm{Rep}\, H$. Note that $x^2 \in H$ is a primitive central element. Let $\mathcal{C}$ be the tensor subcategory of $\mathrm{Rep}\, H$ consisting of representations where x^2 acts semisimply with integer eigenvalues. Then $\mathcal{C}$ is $\mathbb{Z}$-graded by eigenvalues of x^2: $\mathcal{C} = \oplus_{n\in\mathbb{Z}}\mathcal{C}_n$, and for any $n \neq 0$, $\mathcal{C}_n$ is semisimple with one simple object X_n (a 2-dimensional representation) of dimension 0 with respect to g. Thus any $c \in \mathbb{C}^\times$ defines a tensor automorphism of the identity functor of $\mathcal{C}$ acting by c^n on $\mathcal{C}_n$, and gc is a spherical structure for any c.

Exercise 4.9.6 is an example.

2.5. **Chapter 5.** *Exercise 5.3.7(ii)*: "is a commutative" should be "is commutative".

Exercise 5.3.13. The exercise is to prove the given statement.

Remark 5.4.3: Corollary 5.3.15 should be Proposition 5.3.15.

Theorem 5.6.2, clarification: there is a unique Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ *such that...*

Theorem 5.10.2, proof, last two lines, typo: A_0 should be $S\mathfrak{g}$ in three places.

2.6. **Chapter 6.** *Subsection 6.1*, line 3 should start as: "Let $\mathcal{C}$ be a finite multitensor category".

Exercise 6.5.10(i): "show that H is not unimodular" should be replaced with "show that H is unimodular if and only if n is even."

Subsection 6.6. In the displayed formula ev_P should be coev_P, coev_{P^*} should be ev_{P^*}.

2.7. **Chapter 7.** *Definition 7.2.2*: the diagram (7.8) defining morphisms of $\mathcal{C}$-module functors should read

$$\begin{array}{ccc} F(X \otimes M) & \xrightarrow{s_{X,M}} & X \otimes F(M) \\ {\scriptstyle \nu_{X\otimes M}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X \otimes \nu_M} \\ G(X \otimes M) & \xrightarrow{t_{X,M}} & X \otimes G(M). \end{array}$$

Subsection 7.3, p.135, footnote 1: $\mathrm{End}_{\mathcal{C}}$ should be End_l.

Example 7.4.6: $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ should be $\mathrm{End}(\mathcal{M})$. Also Proposition 7.1.3 should be Proposition 7.3.3.

Example 7.8.3(4), typos: H should be L (in several places) to indicate it's a subgroup as in (3) and two instances of k should be $\Bbbk$.

Remark 7.8.6(i), line 3, typo: the target of q should be ${}^*M$.

Definition 7.8.20: one should assume the multitensor category $\mathcal{C}$ to have enough projective objects (to agree with Definition 7.5.1).

Lemma 7.8.24: "let M, N be left A-modules" should be "let M, N be right A-modules".

Exercise 7.8.27, typo: $\otimes_A$ should be $\otimes_B$.

Exercise 7.10.4: "condition (2) above" should be "condition (ii) of Theorem 7.10.1".

Proof of Proposition 7.11.6, clarification: The category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ has finitely many simple objects since any simple object of this category is a quotient of $A_1 \otimes P \otimes A_2$ for some indecomposable projective object $P \in \mathcal{C}$.

Example 7.12.26, last 4 lines: H^* should be $H^{*\mathrm{cop}}$ and vice versa.

Subsection 7.18, p.175, line 5: $\mathbf{1} \boxtimes \mathbf{1}$ is a direct summand of $A \otimes A^*$.

Subsection 7.19, p.178, line 9, typo: "V in the category of..."

Subsection 7.21, p.181, lines 27-28: "positive" instead of "non-negative" twice.

Proposition 7.22.3 has typos, A-invariance should be removed. It should read:

If $\mathcal{C} = A-\mathrm{comod}$ *for a Hopf algebra A, and F is the forgetful functor, then* $C_F^i(\mathcal{C}) = (A^{\otimes i})^*$ *and the differential is the usual Hochschild differential (for trivial coefficients). In particular,* $H_F^i(\mathcal{C})$ *is the usual Hochschild cohomology* $HH^i(A, \mathbf{k})$ *of A with trivial coefficients.*

2.8. **Chapter 8.** *p.201, two lines below Exercise 8.3.20*: formula should read $c_{X,Y} = \tau \circ R^{13} \circ (\rho_X \otimes \rho_Y)$ (i.e., permutation of components τ should be added).

p.201, line above Exercise 8.3.21: "triangular" should read "cotriangular".

Example 8.10.2. The power of q at the end should be $q^{-N(N+2)/2}$.

Lemma 8.10.5 is incorrect. For example, if $X \in \mathrm{Rep}_{\Bbbk}(G)$ is an irreducible representation of a finite group G over a field $\Bbbk$ of characteristic p such that $\dim V$ is divisible by p (e.g. the Steinberg representation of $SL_2(\mathbb{F}_p)$) then the composition in the lemma is, in fact, zero. In the proof, the map $\mathrm{Hom}(\mathbf{1}, X^* \otimes X) \to \mathrm{End}(\mathbf{1})$ is zero, even though $X^* \otimes X \to \mathbf{1}$ is surjective (which is possible since the functor $\mathrm{Hom}(\mathbf{1}, ?)$ is not right exact).

Lemma 8.10.5 is only invoked in the proof of Proposition 8.10.6, but it is not needed. Namely, the second displayed equation in this proof is incorrect, and instead we simply have $v_X \circ u_X = \mathrm{Id}_X$, for any object X. This is easy to prove by a direct computation, and the proofs can be found, say, in [BakK] and [Tu4].

Remark 8.10.16 applies to fusion categories.

Example 8.13.6: the last G on the page should be C.

Proof of Theorem 8.14.7, line 2, typo: K_O should be K_0.

Definition 8.17.1(3), typo: "coming from (2)" (not (Z)).

Subsection 8.18, p. 233 line 7 from bottom: T_s/T_pT_q should be T_pT_q/T_s.

Exercise 8.18.9(vii). This exercise requires an extension of the semisimplification procedure to non-spherical pivotal categories, see P. Etingof and V. Ostrik, Semisimplification of tensor categories, arXiv:1801.04409, Subsection 2.3. Also the answer is incorrect for even n. The correct answer is the subring of $\mathbb{Z}[\mathbb{Z}/2n]\otimes \mathrm{Ver}_{n-2}$ generated by $g\otimes L$, where g is a generator of $\mathbb{Z}/2n$ and L the tautological object of the Verlinde category (corresponding to the 2-dimensional irreducible representation of the quantum $\mathfrak{sl}_2$).

Subsection 8.18, p. 237, line 1, typo: "categories" should be "categorifies".

Proof of Lemma 8.20.8, p. 242, line 13, typo: $d_-(Z)^{-1}$ should be $d_+(Z)^{-1}$.

Proof of Lemma 8.20.9, p. 242, line 22, typo: $\dim(\mathcal{C})$ should be $\dim(\mathcal{D})$.

Proposition 8.20.16, line 5, typo: "the sum is over the simple objects X" (not of X).

p. 251, diagram (8.101), typo: $c_{M,N}\otimes A$ should be $c_{M,N\otimes A}$ on the left side of the diagram.

Subsection 8.25, p. 257, line 15 from bottom, typo: "find a basis x_i of V" (instead of "find a biproduct of V").

2.9. **Chapter 9.** *Remark 9.7.3* applies to the situation when $\omega = 1$. The general case is discussed in detail by S. Natale in arXiv:1608.04435.

Subsection 9.11, p. 303, line 18: $\mathrm{Coend}(F)$ should be $\mathrm{Coend}(F_1)$.

p.306, line 10: "$P \geq P'$" should be replaced with "$P \leq P'$".