

**Corrections to the book “Tensor categories”**  
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Here are some corrections to the book “Tensor categories”. We thank Johannes Berger, Ryan Kinser, Vanessa Miemietz, and Ulrich Thiel for pointing out many of the corrections below.

**If you see other mistakes, please let us know!**

1. GENERAL COMMENTS

1. When working with an additive category  $\mathcal{C}$  over a field  $\mathbb{k}$ , we often use the fact that for any finite dimensional  $\mathbb{k}$ -vector space  $V$  we have a natural functor  $V \otimes : \mathcal{C} \rightarrow \mathcal{C}$ . Namely, for  $X \in \mathcal{C}$  the object  $V \otimes X$  is uniquely defined through the Yoneda lemma by the formula  $\text{Hom}(Y, V \otimes X) = V \otimes \text{Hom}(Y, X)$  (and the existence of this object is checked by choosing a basis in  $V$ ).

2. We often abuse terminology and refer to epimorphisms in abelian categories as surjections and to monomorphisms as injections.

3. We use the terms “natural morphism” and “functorial morphism” interchangeably.

2. OTHER COMMENTS.

2.1. **Chapter 1.** Definition 1.8.3, typo: “any simple object” should be “any object”.

Definition 1.8.13:  $\mathbb{k}$  should be replaced by another field  $\mathbf{k}$ , usually of characteristic zero, and not necessarily equal to the field  $\mathbb{k}$  of definition of  $\mathcal{C}$ .

Subsection 1.10 (Coend), clarification. In (1.9), the direct sum is taken over isomorphism classes of objects. Also the (co)limits in lines 6,7 of the section are taken for the following diagram  $J$ : the vertices  $v_X$  of  $J$  are labeled by representatives  $X$  of isomorphism classes of objects, and arrows are  $\phi_f : v_X \rightarrow v_{X \oplus Y}$  and  $\psi_f : v_Y \rightarrow v_{X \oplus Y}$  for every morphism  $f : X \rightarrow Y$ . The map  $E_F : J \rightarrow \mathcal{C}$  is given by the formula  $E_F(v_X) = \text{End}(F(X))$ ,  $E_F(\phi_f)(a) = f \circ a$ ,  $E_F(\psi_f)(a) = a \circ f$ , where we view  $\text{Hom}(X, Y)$  naturally as a subgroup of  $\text{End}(X \oplus Y)$ . Then  $\varinjlim \text{End}(F(X))$  reproduces the usual definition of  $\text{End}(F)$ : the set of collections  $\{a_X \in \text{End}(F(X))\}$  such that  $a_Y \circ f = f \circ a_X$  for all  $X, Y$  and  $f : X \rightarrow Y$ .

2.2. **Chapter 2.** Proof of Corollary 2.2.5, typo at the beginning:  $Y = Z = \mathbf{1}$  should be  $Y = X = \mathbf{1}$ .

Remark 2.4.2: Section 2.5 should be Section 2.6.

Subsection 2.6, l. 3, typo:  $\mathcal{C}_i$  are not the categories of graded vector spaces (they are not linear). l. 5: the word “simple” should be deleted. Also we consider only monoidal functors which act trivially on  $A$ .

Remark 2.8.6, clarification. In the last sentence of p.38, “the same category with the trivial associativity morphism” should read “the same category with the trivial associativity morphism and a modified tensor product

of morphisms”. Namely, let  $\mu : \tilde{G} \times \tilde{G} \rightarrow A$  be a 2-cochain trivializing the cocycle  $f^*\omega$ . Then the tensor product of morphisms in the strict category isomorphic to  $\mathcal{C}^3$  is defined as follows. Given elements  $g, g', h, h' \in \tilde{G}$  with  $f(g) = f(g')$ ,  $f(h) = f(h')$  and  $a \in \text{Hom}(g, g') = A$ ,  $b \in \text{Hom}(h, h') = A$ , we set

$$a \otimes b := ab\mu(g, h)\mu(g', h')^{-1} \in \text{Hom}(gh, g'h') = A.$$

Exercise 2.9.1, typo: the correct answer is the  $n - 1$ -th Catalan number,  $\frac{1}{n} \binom{2(n-1)}{n-1}$ .

Proposition 2.10.5: “...up to a unique isomorphism preserving the evaluation and coevaluation morphisms”.

Remark 2.10.9, typo: in line 3,  $V^*$  should be  ${}^*V$ .

Example 2.10.14,  $\omega(g, g^{-1}, g)$  should be  $\omega(g, g^{-1}, g)^{-1}$ .

Paragraph after Example 2.10.14, typo: 2.10.7(ii) should be 2.10.7(b).

Exercise 2.10.16, typo: ”left (respectively, right)” should be replaced by ”right (respectively, left)”.

**2.3. Chapter 3.** *Proposition 3.1.4* (claiming that  $1 = \sum_{i \in I_0} b_i$ , i.e., that all coefficients in the decomposition of the unit equal 1) holds not only for based rings but for general  $\mathbb{Z}_+$ -rings. Proof: If  $1 = \sum_{i \in I_0} n_i b_i$  with  $n_i > 0$  then for all  $j \in I$   $b_j = \sum_{i \in I_0} n_i b_i b_j$ , so for every  $j \in I$  there exists  $i(j) \in I_0$  such that  $b_{i(j)} b_j = b_j$  and  $b_i b_j = 0$  if  $i(j) \neq i \in I_0$ . Let  $I'_0$  be the image of the map  $i : I \rightarrow I_0$  and  $1' := \sum_{i \in I'_0} b_i$ . Then  $1' = 1'1 = 1$ , so  $I'_0 = I_0$  and  $n_i = 1$  for all  $i$ .

*Proposition 3.3.6.* In the penultimate paragraph of the proof, “left multiplication by the element  $\sum_{X \in I} X$ ” should be replaced with “right multiplication by the element  $\sum_{X \in I} X$ ”.

*Section 3.4.* The definition of a *based*  $\mathbb{Z}_+$ -module over a based ring is accidentally omitted, although it is used further in the text (first time in Proposition 3.6.2).

By definition, a based  $\mathbb{Z}_+$ -module over a based ring  $A$  is one where if  $x \in A$  acts by a matrix  $X$  then  $x^*$  acts by  $X^T$ .

There exist non-based finite dimensional  $\mathbb{Z}_+$ -modules, even if  $A$  is finite dimensional, e.g. for the 2-dimensional ring  $A$  with basis  $1, b$  and  $b^2 = 1 + 2b$ ,  $b^* = b$  we may take a 2-dimensional module with basis  $v, w$  with  $bv = v + w$ ,  $bw = 2v + w$ .

*Exercise 3.4.3(i).* The  $\mathbb{Z}_+$ -module should assumed be based. Otherwise there is a counterexample: take the  $\mathbb{Z}_+$ -ring  $A$  of representations of  $\mathfrak{sl}_2(\mathbb{C})$  with basis  $b_j, j \geq 0$  (the irreducible  $j + 1$ -dimensional representation, so  $b_j^* = b_j$ ) and consider the (non-based) 2-dimensional  $\mathbb{Z}_+$ -module with basis  $v, w$  and  $b_{j-1}v = jv, b_{j-1}w = \frac{j^3-j}{6}v + jw$ . This module is indecomposable but not irreducible.<sup>1</sup>

Another way to make this exercise correct is to require that the based ring  $A$  be finite dimensional. Namely, we have

<sup>1</sup>We thank Ivan Motorin for pointing out this issue.

**Lemma 2.1.** *Let  $A$  be a based ring of finite rank and  $M$  an indecomposable  $\mathbb{Z}_+$ -module over  $A$ . Then  $M$  is irreducible and has finite rank.*

*Proof.* We may assume that  $A$  is indecomposable. Let  $m \in M$  be a basis vector, and let  $B_m$  be the subset of the  $\mathbb{Z}_+$ -basis  $B_M$  of  $M$  consisting of summands in  $bm$  where  $b \in B$  runs through the  $\mathbb{Z}_+$ -basis of  $A$ . It is clear that  $B_m$  spans a finite rank submodule  $M_m \subset M$ . Pick an irreducible  $\mathbb{Z}_+$ -submodule  $N \subset M_m$ . It suffices to show that  $N = M$ .

Assume the contrary. Then, since  $M$  is indecomposable, we can find a basis element  $p \notin B_N$  of  $M$  such that  $B_p \cap B_N \neq \emptyset$ . Let  $M_{p,N}$  be the submodule of  $M$  spanned by  $B_N$  and  $B_p$ . On the basis vectors of  $M_{p,N}$  we have a preorder in which  $m' \leq m''$  if  $m'$  is a summand in  $bm''$  for some  $b \in B$ . The set  $S$  of equivalence classes of this preorder is a partially ordered set, and  $B_N$  maps to a single element  $s_N \in S$ . By assumption, there exists  $s \in S$  such that  $s_N < s$ , so let us choose a minimal  $s$  with this property. Let  $M_* \subset M$  be the span of the basis elements in  $s$  and  $s_N$ . The matrix of  $x \in A$  in  $M_*$  looks like

$$\rho(x) := \begin{pmatrix} P(x) & Q(x) \\ 0 & R(x) \end{pmatrix}$$

where  $R(x)$  is the matrix by which  $x$  acts on  $N$  and  $P(x)$  the matrix by which it acts on the (irreducible)  $\mathbb{Z}_+$ -module  $(M_* + N)/N$ . Note that  $Q(b) \neq 0$  for  $b \in B$ .

Now take  $x = \sum_i b_i$ . Then all entries of  $P(x)$  and  $R(x)$  are strictly positive integers. Thus if  $\lambda = \text{FPdim}(x)$  then there exists  $C > 0$  such that all entries of  $P(x)^n$  and  $R(x)^n$  are  $\geq C\lambda^n$ . The upper right corner of  $\rho(x)^n$  is  $\sum_{k=0}^{n-1} P(x)^k Q(x) R(x)^{n-1-k}$ , so every entry of this matrix is  $\geq C^2 n \lambda^{n-1}$  (as  $Q(x) \neq 0$  and has nonnegative integer entries). On the other hand, the sum of the coefficients of  $x^n$  is  $O(\lambda^n)$  as  $n \rightarrow \infty$ , hence so is  $\rho(x)^n = \rho(x^n)$ . Contradiction. □

**2.4. Chapter 4.** Proof of Proposition 4.2.8, line 4: “and using Proposition 4.2.1” should be replaced by “by Definition 4.2.3.”

Corollary 4.3.9: “monomorphisms” and “epimorphisms” should be interchanged.

Paragraph before Example 4.5.5: the ring category should be assumed finite.

Remark 4.5.6: “homomorphism of unital  $\mathbb{Z}_+$ -rings” should be “unital homomorphism of  $\mathbb{Z}_+$ -rings”.

Definition 4.7.11:  $\text{Tr}^L$  in the first line is not needed, but  $\text{Tr}$  should be  $\text{Tr}^L$  in the formula right below.

Proof of Theorem 4.7.15, line 3:  $Y_i$  and  $V_i$  should be switched (I.e.,  $V_i$  are objects and  $Y_i$  are vector spaces).

Exercise 4.9.6 is an example.

2.5. **Chapter 5.** Exercise 5.3.7(ii): “is a commutative” should be “is commutative”.

Exercise 5.3.13. The exercise is to prove the given statement.

Remark 5.4.3: Corollary 5.3.15 should be Proposition 5.3.15.

Theorem 5.6.2, clarification: there is a unique Hopf algebra structure on  $U_q(\mathfrak{sl}_2)$  such that...

Theorem 5.10.2, proof, last two lines, typo:  $A_0$  should be  $Sg$  in three places.

2.6. **Chapter 6.** Subsection 6.1, line 3 should start as: “Let  $\mathcal{C}$  be a finite multitensor category”.

2.7. **Chapter 7.** Definition 7.2.2: the diagram (7.8) defining morphisms of  $\mathcal{C}$ -module functors should read

$$\begin{array}{ccc} F(X \otimes M) & \xrightarrow{s_{X,M}} & X \otimes F(M) \\ \nu_{X \otimes M} \downarrow & & \downarrow \text{id}_X \otimes \nu_M \\ G(X \otimes M) & \xrightarrow{t_{X,M}} & X \otimes G(M). \end{array}$$

Subsection 7.3, p.135, footnote 1:  $\text{End}_{\mathcal{C}}$  should be  $\text{End}_l$ .

Example 7.4.6:  $\text{End}_{\mathcal{C}}(\mathcal{M})$  should be  $\text{End}(\mathcal{M})$ . Also Proposition 7.1.3 should be Proposition 7.3.3.

Example 7.8.3(4), typos:  $H$  should be  $L$  (in several places) to indicate it’s a subgroup as in (3) and two instances of  $k$  should be  $\mathbb{k}$ .

Remark 7.8.6(i), line 3, typo: the target of  $q$  should be  $*M$ .

In Definition 7.8.20, one should assume the multitensor category  $\mathcal{C}$  to have enough projective objects (to agree with Definition 7.5.1).

Lemma 7.8.24: “let  $M, N$  be left  $A$ -modules” should be “let  $M, N$  be right  $A$ -modules”.

Exercise 7.8.27, typo:  $\otimes_A$  should be  $\otimes_B$ .

Exercise 7.10.4: “condition (2) above” should be “condition (ii) of Theorem 7.10.1”.

Proof of Proposition 7.11.6, clarification: The category  $\text{Func}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$  has finitely many simple objects since any simple object of this category is a quotient of  $A_1 \otimes P \otimes A_2$  for some indecomposable projective object  $P \in \mathcal{C}$ .

Example 7.12.26, last 4 lines:  $H^*$  should be  $H^{*\text{cop}}$  and vice versa.

Subsection 7.18, p.175, line 5:  $\mathbf{1} \boxtimes \mathbf{1}$  is a direct summand of  $A \otimes A^*$ .

Subsection 7.19, p.178, line 9, typo: “ $V$  in the category of...”

Subsection 7.21, p.181, lines 27-28: “positive” instead of “non-negative” twice.

Proposition 7.22.3 has typos,  $A$ -invariance should be removed. It should read:

*If  $\mathcal{C} = A\text{-comod}$  for a Hopf algebra  $A$ , and  $F$  is the forgetful functor, then  $C_F^i(\mathcal{C}) = (A^{\otimes i})^*$  and the differential is the usual Hochschild differential (for*

trivial coefficients). In particular,  $H_F^i(\mathcal{C})$  is the usual Hochschild cohomology  $HH^i(A, \mathbf{k})$  of  $A$  with trivial coefficients.

2.8. **Chapter 8.** p.201, two lines below Exercise 8.3.20: formula should read  $c_{X,Y} = \tau \circ R^{13} \circ (\rho_X \otimes \rho_Y)$  (i.e., permutation of components  $\tau$  should be added).

p.201, line above Exercise 8.3.21: “triangular” should read “cotriangular”.

Lemma 8.10.5 is incorrect. For example, if  $X \in \text{Rep}_{\mathbf{k}}(G)$  is an irreducible representation of a finite group  $G$  over a field  $\mathbf{k}$  of characteristic  $p$  such that  $\dim V$  is divisible by  $p$  (e.g. the Steinberg representation of  $SL_2(\mathbb{F}_p)$ ) then the composition in the lemma is, in fact, zero. In the proof, the map  $\text{Hom}(\mathbf{1}, X^* \otimes X) \rightarrow \text{End}(\mathbf{1})$  is zero, even though  $X^* \otimes X \rightarrow \mathbf{1}$  is surjective (which is possible since the functor  $\text{Hom}(\mathbf{1}, ?)$  is not right exact).

Lemma 8.10.5 is only invoked in the proof of Proposition 8.10.6, but it is not needed. Namely, the second displayed equation in this proof is incorrect, and instead we simply have  $v_X \circ u_X = \text{Id}_X$ , for any object  $X$ . This is easy to prove by a direct computation, and the proofs can be found, say, in [BakK] and [Tu4].

Proof of Theorem 8.14.7, line 2, typo:  $K_O$  should be  $K_0$ .

Definition 8.17.1(3), typo: “coming from (2)” (not (Z)).

Subsection 8.18, p. 233 line 7 from bottom:  $T_s/T_p T_q$  should be  $T_p T_q/T_s$ .

Exercise 8.18.9(vii). This exercise requires an extension of the semisimplification procedure to non-spherical pivotal categories, see P. Etingof and V. Ostrik, Semisimplification of tensor categories, arXiv:1801.04409, Subsection 2.3. Also the answer is incorrect for even  $n$ . The correct answer is the subring of  $\mathbb{Z}[\mathbb{Z}/2n] \otimes \text{Ver}_{n-2}$  generated by  $g \otimes L$ , where  $g$  is a generator of  $\mathbb{Z}/2n$  and  $L$  the tautological object of the Verlinde category (corresponding to the 2-dimensional irreducible representation of the quantum  $\mathfrak{sl}_2$ ).

Subsection 8.18, p. 237, line 1, typo: “categories” should be “categorifies”.

Proof of Lemma 8.20.8, p. 242, line 13, typo:  $d_-(Z)^{-1}$  should be  $d_+(Z)^{-1}$ .

Proof of Lemma 8.20.9, p. 242, line 22, typo:  $\dim(\mathcal{C})$  should be  $\dim(\mathcal{D})$ .

Proposition 8.20.16, line 5, typo: “the sum is over the simple objects  $X$ ” (not of  $X$ ).

p. 251, diagram (8.101), typo:  $c_{M,N} \otimes A$  should be  $c_{M,N \otimes A}$  on the left side of the diagram.

Subsection 8.25, p. 257, line 15 from bottom, typo: “find a basis  $x_i$  of  $V$ ” (instead of “find a biproduct of  $V$ ”).

2.9. **Chapter 9.** Remark 9.7.3 applies to the situation when  $\omega = 1$ . The general case is discussed in detail by S. Natale in arXiv:1608.04435.

Subsection 9.11, p. 303, line 18:  $\text{Coend}(F)$  should be  $\text{Coend}(F_1)$ .

p.306, line 10: “ $P \geq P'$ ” should be replaced with “ $P \leq P'$ ”.