18.745: LIE GROUPS AND LIE ALGEBRAS, I

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Contents

1. Lecture 1

The purpose of **group theory** is to give a mathematical treatment of **symmetries**. For example, symmetries of a set of n elements form the symmetric group S_n, and symmetries of a regular n-gon – the dihedral group D_n. Likewise, **Lie group theory** serves to give a mathematical treatment of **continuous symmetries**, i.e., families of symmetries continuously depending on several real parameters.

The theory of Lie groups was founded in the second half of the 19th century by the Norwegian mathematician **Sophus Lie**, after whom it is named. It was then developed by many mathematicians over the last 150 years, and has numerous applications in mathematics and science, especially physics.

A prototypical example of a Lie group is the group $SO(3)$ of rotational symmetries of the 2-dimensional sphere; in this case the parameters are the Euler angles ϕ, θ, ψ.

It turns out that unlike ordinary parametrized curves and surfaces, Lie groups are determined by their linear approximation at the identity element. This leads to the notation of the **Lie algebra** of a Lie group. This notion allows one to reformulate the theory of continuous symmetries in purely algebraic terms, which provides an extremely effective way of studying such symmetries. The goal of this course is to give a detailed study of Lie groups and Lie algebras and interactions between them, with numerous examples.

1.1. **Topological groups.** Recall that the mathematical notion responsible for describing continuity is that of a **topological space**. Thus, to describe continuous symmetries, we should put this notion together with the notion of a group. This leads to the concept of a **topological group**.

Recall:

• A **topological space** is a set X certain subsets of which (including $\emptyset$ and X) are declared to be **open**, so that an arbitrary union and finite intersection of open sets is open.

• The collection of open sets in X is called **the topology** of X.

• If X, Y are topological spaces then the Cartesian product $X \times Y$ has a natural **product topology** in which open sets are (possibly infinite) unions of products $U \times V$, where $U \subset X, V \subset Y$ are open.

• Every subset $Z \subset X$ of a topological space X carries a natural **induced topology**, in which open sets are intersections of open sets in X with Z.

• A map $f : X \to Y$ between topological spaces is **continuous** if for every open set $V \subset Y$, the preimage $f^{-1}(V)$ is open in X.

Definition 1.1. A **topological group** is a group G which is also a topological space, so that the multiplication map $m : G \times G \to G$ and the inversion map $\iota : G \to G$ are continuous.

For example, the group $(\mathbb{R}, +)$ of real numbers with the operation of addition and the usual topology of the real line is a topological group, since the functions $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous. Also a subgroup of a topological group is itself a topological group, so another example is rational numbers with addition, $(\mathbb{Q}, +)$. This example is not a very good model for continuity, and shows that general topological groups are not well behaved. Thus, we will focus on a special class of topological groups called **Lie groups**.

Lie groups are distinguished among topological groups by the property that as topological spaces they belong to a very special class called **topological manifolds.** So we need to start with reviewing this notion.

1.2. **Topological manifolds.** Recall:

• A **neighborhood** of a point $x \in X$ in a topological space X is an open set containing x.

• A **base** for a topological space X is a collection $\mathcal{B}$ of open sets in X such that for every neighborhood U of a point $x \in X$ there exists a smaller (or the same) neighborhood $V \subset U$ of x which belongs to $\mathcal{B}$. Equivalently, every open set in X is a union of members of $\mathcal{B}$.

• X is **Hausdorff** if any two distinct points have disjoint neighborhoods.

• In this case we say that a sequence of points $x_n \in X$ **converges** to $x \in X$ ($x_n \to x$) if every neighborhood of x contains almost all terms of this sequence. Then one also says that the **limit** of x_n is x and writes

$$\lim_{n\to\infty} x_n = x,$$

and it is easy to show that the limit is unique when exists.

• A continuous map $f : X \to Y$ is a **homeomorphism** if it is a bijection and $f^{-1} : Y \to X$ is continuous.

Definition 1.2. A Hausdorff topological space X is said to be an **n-dimensional topological manifold** if it has a countable base and is locally homeomorphic to $\mathbb{R}^n$; namely, for every $x \in X$ there is a neighborhood $U \subset X$ of x and a continuous map $\phi : U \to \mathbb{R}^n$ such that $\phi : U \to \phi(U)$ is a homeomorphism and $\phi(U) \subset \mathbb{R}^n$ is open.

It is true (although not obvious) that if a nonempty open set in $\mathbb{R}^n$ is homeomorphic to one in $\mathbb{R}^m$ then $n = m$. Therefore, the number n is uniquely determined by X as long as $X \neq \emptyset$. It is called **the dimension** of X. (By convention, $\emptyset$ is a manifold of any integer dimension).

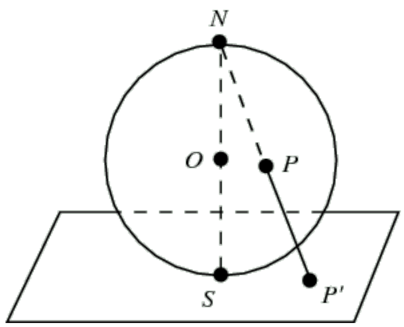
Example 1.3. 1. Obviously $X = \mathbb{R}^n$ is an n-dimensional topological manifold: we can take $U = X$ and $\phi = \mathrm{Id}$.

2. An open subset of a topological manifold is itself a topological manifold of the same dimension.

3. The circle $S^1 \subset \mathbb{R}^2$ defined by the equation $x^2 + y^2 = 1$ is a topological manifold: for example, the point $(1, 0)$ has a neighborhood $U = S^1 \setminus \{(-1, 0)\}$ and a map $\phi : U \to \mathbb{R}$ given by the stereographic projection:

$$\phi(\theta) = \tan(\theta/2), \ -\pi < \theta < \pi.$$

and similarly for every other point. More generally, the sphere $S^n \subset \mathbb{R}^{n+1}$ defined by the equation $x_0^2 + ... + x_n^2 = 1$ is a topological manifold, for the same reason.



4. The curve ∞ is not a manifold, since it is not locally homeomorphic to $\mathbb{R}$ at the self-intersection point (show it!)

A pair (U, ϕ) with the above properties is called a **local chart**. An **atlas** of local charts is a collection of charts $(U_\alpha, \phi_\alpha), \alpha \in A$ such that

$$\cup_{\alpha \in A} U_\alpha = X;$$

i.e., $\{U_\alpha, \alpha \in A\}$ is an open cover of X. Thus any topological manifold X admits an atlas labeled by points of X. There are also much smaller atlases. For instance, an open set in $\mathbb{R}^n$ has an atlas with just one chart, while the sphere S^n has an atlas with two charts. Very often X admits an atlas with finitely many charts. For example, if X is **compact** (i.e., every open cover of X has a finite subcover) then there is a finite atlas, since every atlas has a finite subatlas. Moreover, there is always a countable atlas.

Now let (U, ϕ) and (V, ψ) be two charts such that $V \cap U \neq \emptyset$. Then we have the **transition map**

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),$$

which is a homeomorphism between open subsets in $\mathbb{R}^n$. For example, consider the atlas of two charts for the circle S^1 (Example 1.3(3)), one missing the point $(-1, 0)$ and the other $(1, 0)$. Then $\phi(\theta) = \tan(\theta/2)$ and $\psi(\theta) = \cot(\theta/2)$, $\phi(U \cap V) = \psi(U \cap V) = \mathbb{R} \setminus 0$, and $(\phi \circ \psi^{-1})(x) = \frac{1}{x}$.

1.3. C^k**, real analytic and complex analytic manifolds.** The notion of topological manifold is not convenient for us, since continuous functions on which it is based in general do not admit a linear approximation. To develop the theory of Lie groups, we need more regularity. So we make the following definition.

Definition 1.4. An atlas on X is said to be **of regularity class** C^k, $1 \le k \le \infty$, if all transition maps between its charts are of class C^k (k times continuously differentiable). An atlas of class C^∞ is called **smooth**. Also an atlas is said to be **real analytic** if all transition maps are real analytic. Finally, if $n = 2m$ is even, so that $\mathbb{R}^n = \mathbb{C}^m$, then an atlas is called **complex analytic** if all its transition maps are complex analytic (i.e., holomorphic).

Example 1.5. The two-chart atlas for the circle S^1 defined by stereographic projections (Example 1.3(3)) is real analytic, since the function $f(x) = \frac{1}{x}$ is analytic. The same applies to the sphere S^n for any n. For example, for S^2 it is easy to see that the transition map $\mathbb{R}^2 \setminus 0 \to \mathbb{R}^2 \setminus 0$ is given by the formula

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Using the complex coordinate $z = x + iy$, we get

$$f(z) = z/|z|^2 = 1/\overline{z}.$$

So this atlas is not complex analytic. But it can be easily made complex analytic by replacing one of the stereographic projections (ϕ or ψ) by its complex conjugate. Then we will have $f(z) = \frac{1}{z}$. On the other hand, it is known (although hard to prove) that S^n does not admit a complex analytic atlas for $n \neq 2, 6$. For $n = 6$ this is a famous conjecture.

Definition 1.6. Two C^k, real analytic, or complex analytic atlases U_α, V_β are said to be **compatible** if the transition maps between U_α and V_β are of the same class (C^k, real analytic, or complex analytic).

It is clear that compatibility is an equivalence relation.

Definition 1.7. A C^k, real analytic, resp. complex analytic structure on a topological manifold X is an equivalence class of C^k, real analytic, or complex analytic atlases. If X is equipped with such a structure, it is said to be a C^k**, real analytic, or complex analytic manifold**. Complex analytic manifolds are also called **complex manifolds**, and a C^∞-manifold is also called **smooth**. A **diffeomorphism** (or **isomorphism**) between such manifolds is a homeomorphism which respects the corresponding classes of atlases.

Remark 1.8. This is really a **structure** and not a **property**. For example, consider $X = \mathbb{C}$ and $Y = D \subset \mathbb{C}$ the open unit disk, with the usual complex coordinate z. It is easy to see that X, Y are isomorphic as real analytic manifolds. But they are not isomorphic as complex analytic manifolds: a complex isomorphism would be a holomorphic function $f : \mathbb{C} \to D$, hence bounded, but by Liouville's theorem any bounded holomorphic function is a constant. Thus we have two different complex structures on $\mathbb{R}^2$ (Riemann showed that there are no others). Also, it is true, but much harder to show that there are uncountably many different non-standard smooth structures on $\mathbb{R}^4$, and there are 28 (oriented) smooth structures on S^7.

Note that the Cartesian product $X \times Y$ of manifolds X, Y is naturally a manifold (of the same regularity type) of dimension $\dim X + \dim Y$.

Exercise 1.9. Let $f_1, ..., f_m$ be functions $\mathbb{R}^n \to \mathbb{R}$ or $\mathbb{C}^n \to \mathbb{C}$ which are C^k, real analytic or complex analytic. Let $X \subset \mathbb{R}^n$ be the set of points P such that $f_i(P) = 0$ for all i and $df_i(P)$ are linearly independent. Use the implicit function theorem to show that X is a topological manifold of dimension $n - m$ and equip it with a natural C^k, real analytic or complex analytic structure.

1.4. **Regular functions.** Now let $P \in X$ and (U, ϕ) be a local chart such that $P \in U$ and $\phi(P) = 0$. Such a chart is called a **coordinate chart** around P. In particular, we have **local coordinates** $x_1, ..., x_n : U \to \mathbb{R}$ (or $\mathbb{C}$ for complex manifolds). Note that $x_i(P) = 0$, and $x_i(Q)$ determine Q if $Q \in U$.

Definition 1.10. A **regular function** on an open set $V \subset X$ in a C^k, real analytic, or complex analytic manifold X is a function $f : V \to \mathbb{R}, \mathbb{C}$ such that $f \circ \phi_\alpha^{-1} : \phi_\alpha(V \cap U_\alpha) \to \mathbb{R}, \mathbb{C}$ is of the corresponding regularity class, for some (and then any) atlas (U_α, ϕ_α) defining the corresponding structure on X.

In other words, f is regular if it is expressed as a regular function in local coordinates near every point of V. Clearly, this is independent on the choice of coordinates.

The space (in fact, algebra) of regular functions on V will be denoted by $O(V)$.

Definition 1.11. Let V, U be neighborhoods of $P \in X$. Let us say that $f \in O(V)$, $g \in O(U)$ are **equal near** P if there exists a neighborhood $W \subset U \cap V$ of P such that $f|_W = g|_W$.

It is clear that this is an equivalence relation.

Definition 1.12. A **germ** of a regular function at P is a class of regular functions on neighborhoods of P which are equal near P.

The algebra of germs of regular functions at P is denoted by O_P. Thus we have $O_P = \varinjlim O(U)$, where the direct limit is taken over neighborhoods of P.

1.5. **Tangent spaces.** From now on we will only consider smooth, real analytic and complex analytic manifolds. By a **derivation at** P we will mean a linear map $D : O_P \to \mathbb{R}, \mathbb{C}$ satisfying the Leibniz rule

$$D(fg) = D(f)g(P) + f(P)D(g).$$

Note that for any such D we have $D(1) = 0$.

Let T_PX be the space of all such derivations.

Lemma 1.13. *Let $x_1, ..., x_n$ be local coordinates at P. Then T_PX has basis $D_1, ..., D_n$, where*

$$D_i(f) := \frac{\partial f}{\partial x_i}(0).$$

Proof. We may assume $X = \mathbb{R}^n$ or $\mathbb{C}^n$, $P = 0$. Clearly, $D_1, ..., D_n$ is a linearly independent set in T_PX. Also let $D \in T_PX$, $D(x_i) = a_i$, and consider $D_* := D - \sum_i a_i D_i$. Then $D_*(x_i) = 0$ for all i. Now given a regular function f near 0, for small $x_1, ..., x_n$ by the fundamental theorem of calculus and the chain rule we have:

$$f(x_1, ..., x_n) = f(0)+\int_0^1 \frac{d}{dt} f(tx_1, ..., tx_n)dt = f(0)+\sum_{i=1}^n x_i h_i(x_1, ..., x_n),$$

where

$$h_i(x_1, ..., x_n) := \int_0^1 \partial_i f(tx_1, ..., tx_n)dt$$

are regular near 0. So by the Leibniz rule $D_*(f) = 0$, hence $D_* = 0$. □

Definition 1.14. The space T_PX is called the **tangent space** to X at P. Elements $v \in T_PX$ are called **tangent vectors** to X at P

Observe that every tangent vector $v \in T_PX$ defines a derivation $\partial_v : O(U) \to \mathbb{R}, \mathbb{C}$ for every neighborhood U of P, satisfying the above Leinniz rule. The number $\partial_v f$ is called the **derivative of f in the direction of v**. For usual curves and surfaces in $\mathbb{R}^3$ these coincide with the familiar notions from calculus.

1.6. **Regular maps.**

Definition 1.15. A continuous map $F : X \to Y$ between manifolds is **regular** if for any regular function h on an open set $U \subset Y$ the function $h \circ F$ on $F^{-1}(U)$ is regular. In other words, F is regular if it is expressed by regular functions in local coordinates.

It is easy to see that the composition of regular maps is regular, and that a homeomorphism F such that F, F^{-1} are both regular is the same thing as a diffeomorphism (=isomorphism).

Let $F : X \to Y$ be a regular map and $P \in X$. Then we can define the **differential** of F at P, $d_P F$, which is a linear map $T_P X \to T_{F(P)} Y$. Namely, for $f \in O_{F(P)}$ and $v \in T_P X$, the vector $d_P F \cdot v$ is defined by the formula

$$(d_P F \cdot v)(f) := v(f \circ F).$$

The differential of F is also denoted by F_*; namely, for $v \in T_P X$ one writes $dF_P v = F_* v$.

Moreover, if $G : Y \to Z$ is another regular map, then we have the usual chain rule,

$$d(G \circ F)_P = dG_{F(P)} \circ dF_P.$$

In particular, if $\gamma : (a, b) \to X$ is a regular **parametrized curve** then for $t \in (a, b)$ we can define the **velocity vector**

$$d_t \gamma \cdot 1 = \gamma'(t) \in T_{\gamma(t)} X$$

(where $1 \in \mathbb{R} = T_t(a, b)$).

1.7. **Submersions and immersions, submanifolds.**

Definition 1.16. A regular map of manifolds $F : X \to Y$ is a **submersion** if $dF_P : T_P X \to T_{F(P)} Y$ is surjective for all $P \in X$.

The following proposition is a version of the implicit function theorem for manifolds.

Proposition 1.17. *If F is a submersion then for any $Q \in Y$, $F^{-1}(Q)$ is a manifold of dimension $\dim X - \dim Y$.*

Proof. This is a local question, so it reduces to the case when X, Y are open subsets in Euclidean spaces. In this case it reduces to Exercise 1.9. □

Definition 1.18. A regular map of manifolds $f : X \to Y$ is an **immersion** if $d_P F : T_P X \to T_{F(P)} Y$ is injective for all $P \in X$.

Example 1.19. The inclusion of the sphere S^n into $\mathbb{R}^{n+1}$ is an immersion. The map $F : S^1 \to \mathbb{R}^2$ given by

$$x(t) = \frac{\cos\theta}{1+\sin^2\theta},\ y(t) = \frac{\sin\theta\cos\theta}{1+\sin^2\theta}$$

is also an immersion; its image is the lemniscate (shaped as ∞). This shows that an immersion need not be injective. On the other hand, the map $F : \mathbb{R} \to \mathbb{R}^2$ given by $F(T) = (t^2, t^3)$ parametrizing a semicubic parabola $\prec$ is injective, but not an immersion, since $F'(0) = (0, 0)$.

Definition 1.20. An immersion $f : X \to Y$ is an **embedding** if the map $F : X \to F(X)$ is a homeomorphism (where $F(X)$ is equipped with the induced topology from Y). In this case, $F(X) \subset Y$ is said to be an **(embedded) submanifold.**

Example 1.21. The immersion of S^n into $\mathbb{R}^{n+1}$ and of $(0, 1)$ into $\mathbb{R}$ are embeddings, but the above parametrization of the lemniscate by the circle is not. The parametrization of the curve ρ by $\mathbb{R}$ is also not an embedding; it is injective but the inverse is not a homeomorphism.

Recall that a subset $Z \subset X$ of a topological space X is **closed** if its complement is open. In a Hausdorff space, a closed set is one that is closed under taking limits of sequences.

Definition 1.22. An embedding $F : X \to Y$ of manifolds is **closed** if $F(X) \subset Y$ is a closed subset. In this case we say that $F(X)$ is a **closed (embedded) submanifold** of Y.

Example 1.23. The embedding of S^n into $\mathbb{R}^{n+1}$ is closed but of $(0, 1)$ into $\mathbb{R}$ is not. Also in Proposition 1.17, $f^{-1}(Q)$ is a closed submanifold of X.

2. Lecture 2

2.1. **Lie groups.**

Definition 2.1. A C^k, real or complex analytic **Lie group** is a manifold G of the same class, with a group structure such that the multiplication map $m : G \times G \to G$ is regular.

Thus, in a Lie group G for any $g \in G$ the left and right translation maps $L_g, R_g : G \to G$, $L_g(x) := gx$, $R_g(x) := xg$, are diffeomorphisms.

Proposition 2.2. *In a Lie group G, the inversion map $\iota : G \to G$ is a diffeomorphism, and $d\iota_1 = -\mathrm{Id}$.*

Proof. For the first statement it suffices to show that ι is regular near 1, the rest follows by translation. So let us pick a coordinate chart near $1 \in G$ and write the map m in this chart in local coordinates. Note that in these coordinates, $1 \in G$ corresponds to $0 \in \mathbb{R}^n$. Since $m(x,0) = x$ and $m(0,y) = y$, the linear approximation of $m(x,y)$ at 0 is $x+y$. Thus by the implicit function theorem, the equation $m(x,y) = 0$ is solved near 0 by a regular function $y = \iota(x)$ with $d\iota(0) = -\mathrm{Id}$. This proves the proposition. □

Remark 2.3. A C^0 **Lie group** is a topological group which is a topological manifold. The **Hilbert 5th problem** was to show that any such group is actually a real analytic Lie group (i.e., the regularity class does not matter). This problem is solved by the deep **Gleason-Yamabe theorem**, proved in 1950s. So from now on we will not pay attention to regularity class and consider only real and complex Lie groups.

Note that any complex Lie group of dimension n can be regarded as a real Lie group of dimension $2n$. Also the Cartesian product of real (complex) Lie groups is a real (complex) Lie group.

2.2. **Homomorphisms.**

Definition 2.4. A **homomorphism of Lie groups** $f : G \to H$ is a group homomorphism which is also a regular map. An **isomorphism of Lie groups** is a homomorphism f which is a group isomorphism, such that $f^{-1} : H \to G$ is regular.

We will see later that the last condition is in fact redundant.

2.3. **Examples.**

Example 2.5. 1. $(\mathbb{R}^n, +)$ is a real Lie group and $(\mathbb{C}^n, +)$ is a complex Lie group (both n-dimensional).

2. $(\mathbb{R}^\times, \times)$, $(\mathbb{R}_{>0}, \times)$ are real Lie groups, $(\mathbb{C}^\times, \times)$ is a complex Lie group (all 1-dimensional)

3. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a 1-dimensional real Lie group under multiplication of complex numbers.

Note that $\mathbb{R}^\times \cong \mathbb{R}_{>0} \times \mathbb{Z}/2$, $\mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1$ as real Lie groups (trigonometric form of a complex number) and $(\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \times)$ via $x \mapsto e^x$.

4. The groups of invertible n by n matrices: $GL_n(\mathbb{R})$ is a real Lie group and $GL_n(\mathbb{C})$ is a complex Lie group. These are open sets in the corresponding spaces of all matrices and have dimension n^2.

5. $SU(2)$, the special unitary group of size 2. This is the set of complex 2 by 2 matrices A such that

$$AA^\dagger = \mathbf{1},\ \det A = 1.$$

So writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix},$$

we get

$$a\bar{a} + b\bar{b} = 1,\ a\bar{c} + b\bar{d} = 0,\ c\bar{c} + d\bar{d} = 1.$$

The second equation implies that $(c, d) = \lambda(-\bar{b}, \bar{a})$. Then we have

$$1 = \det A = ad - bc = \lambda(a\bar{a} + b\bar{b}) = \lambda,$$

so $\lambda = 1$. Thus $SU(2)$ is identified with the set of $(a, b) \in \mathbb{C}^2$ such that $a\bar{a} + b\bar{b} = 1$. Writing $a = x + iy$, $b = z + it$, we have

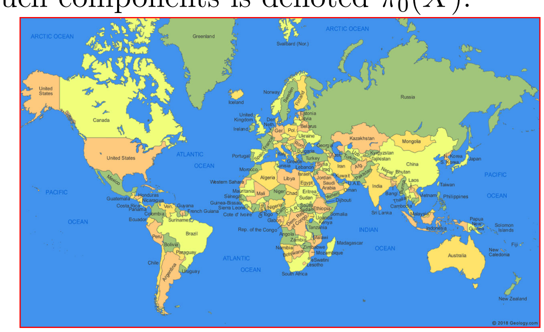
$$SU(2) = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 = 1\}.$$

Thus $SU(2)$ is a 3-dimensional real Lie group which as a manifold is the 3-dimensional sphere $S^3 \subset \mathbb{R}^4$.

6. Any countable group G with **discrete topology** (i.e., such that every set is open) is a (real and complex) Lie group.

2.4. **The connected component of** 1**.** Recall:

- A topological space X is **path-connected** if for any $P, Q \in X$ there is a continuous map $x : [0, 1] \to X$ such that $x(0) = P, x(1) = Q$.
- If X is any topological space, then for $P \in X$ we can define its **path-connected component** to be the set X_P of $Q \in X$ for which there is a continuous map $x : [0, 1] \to X$ such that $x(0) = P, x(1) = Q$ (such x is called **a path connecting** P **and** Q). Then X_P is the largest path-connected subset of X containing P. Clearly, the relation that Q belongs to X_P is an equivalence relation, which splits X into equivalence classes called **path connected components.** The set of such components is denoted $\pi_0(X)$.



• A topological space X is **connected** if the only subsets of X that are both open and closed are $\emptyset$ and X. For $P \in X$, the **connected component** of X is the union X^P of all connected subsets of X containing P, which is obviously connected itself (so it is the largest connected subset of X containing P). A path-connected space X is always connected but not vice versa; however, a connected *manifold* is path-connected (show it!), so for manifolds the notions of connected component and path-connected component coincide.

• If Y is a topological space, X is a set and $p : Y \to X$ is a surjective map (i.e., $X = Y/\sim$ is the quotient of Y by an equivalence relation) then X acquires a topology called the **quotient topology**, in which open sets are subsets $V \subset X$ such that $p^{-1}(V)$ is open.

Now let G be a real or complex Lie group, and G° the connected component of $1 \in G$. Then the connected component of any $g \in G$ is gG°.

Proposition 2.6. *(i) G° is a normal subgroup of G.*

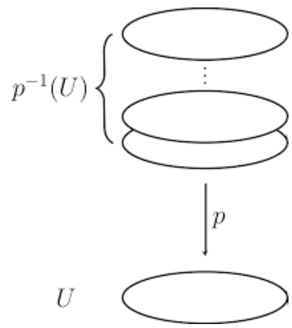
(ii) $\pi_0(G) = G/G^\circ$ with quotient topology is a discrete and countable group.

Proof. (i) Let $g \in G$, $a \in G^\circ$, and $x : [0,1] \to G$ be a path connecting 1 to a. Then gxg^{-1} is a path connecting 1 to gag^{-1}, so $gag^{-1} \in G^\circ$, hence G° is normal.

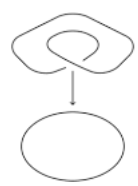
(ii) Since G is a manifold, for any $g \in G$, there is a neighborhood of g contained in $G_g = gG^\circ$. This implies that any coset of G° in G is open, hence G/G° is discrete. Also G/G° is countable since G has countable base. □

Thus we see that any Lie group is an extension of a discrete countable group by a connected Lie group. This essentially reduces studying Lie groups to studying connected Lie groups. In fact, one can further reduce to simply connected Lie groups, which is done in the next subsections.

2.5. **A crash course on coverings.** Now we need to review some more topology. Let X, Y be Hausdorff topological spaces, and $p : Y \to X$ a continuous map. Then p is called a **covering** if every point $x \in X$ has a neighborhood U such that $p^{-1}(U)$ is a union of disjoint open sets (called **sheets** of the covering) each of which is mapped homeomorphically onto U by p.



In other words, there exists a homeomorphism $h : U \times F \to p^{-1}(U)$ for some discrete space F with $(p \circ h)(u, f) = u$ for all $u \in U,\ f \in F$. I.e., informally speaking, a covering is a map that locally on X looks like the projection $X \times F \to X$ for some discrete F. It is clear that a covering of a manifold (C^k, real or complex analytic) is a manifold of the same type, and the covering map is regular.



Two paths $x_0, x_1 : [0,1] \to X$ such that $x_i(0) = P, x_i(1) = Q$ are said to be **homotopic** if there is a continuous map

$$x : [0,1] \times [0,1] \to X,$$

called a **homotopy** between x_0 and x_1, such that $x(t,0) = x_0(t)$ and $x(t,1) = x_1(t)$, $x(0,s) = P, x(1,s) = Q$.

`https://commons.wikimedia.org/wiki/File:Homotopy.gif#/media/File:HomotopySmall.gif`

For example, if $x(t)$ is a path and $g : [0,1] \to [0,1]$ is a change of parameter with $g(0) = 0,\ g(1) = 1$ then the paths $x_1(t) = x(t)$ and $x_2(t) = x(g(t))$ are clearly homotopic.

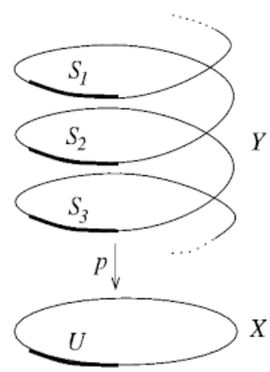
A path-connected Hausdorff space X is said to be **simply connected** if for any $P, Q \in X$ any paths $x_0, x_1 : [0,1] \to X$ such that $x_i(0) = P, x_i(1) = Q$ are homotopic.

Example 2.7. S^1 is not simply connected but S^n is simply connected for $n \geq 2$.

It is easy to show that any covering has a **homotopy lifting property**: if $b \in X$ and $\widetilde{b} \in p^{-1}(b) \subset Y$ then any path γ starting at b admits a unique lift to a path $\widetilde{\gamma}$ starting at $\widetilde{b}$, i.e., $p(\widetilde{\gamma}) = \gamma$. Moreover, if γ_1, γ_2 are homotopic paths on X then $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are homotopic on Y (in particular, have the same endpoint). Thus, if Z is a simply connected space with a point z then any continuous map $f : Z \to X$ with $f(z) = b$ lifts to a unique continuous map $\widetilde{f} : Z \to Y$ satisfying $\widetilde{f}(z) = \widetilde{b}$; i.e., $p \circ \widetilde{f} = f$. Namely, to compute $\widetilde{f}(w)$, pick a path β from z to w, let $\gamma = f(\beta)$ and consider the path $\widetilde{\gamma}$. Then the endpoint of $\widetilde{\gamma}$ is $\widetilde{f}(w)$, and it does not depend on the choice of β.

If Z, X are manifolds (of any regularity type), Z is simply connected, and $f : Z \to X$ is a regular map then the lift $\widetilde{f} : Z \to Y$ is also regular. Indeed, if we introduce local coordinates on Y using the homeomorphism between sheets of the covering and their images then $\widetilde{f}$ and f will be locally expressed by the same functions.

A covering $p : Y \to X$ of a path-connected space X is called **universal** if Y is simply connected.



If X is a sufficiently nice space, e.g., a manifold, its universal covering can be constructed as follows. Fix $b \in X$ and let $\widetilde{X}_b$ be the set of homotopy classes of paths on X starting at b. We have a natural map $p : \widetilde{X}_b \to X$, $p(\gamma) = \gamma(1)$. If $U \subset X$ is a small enough neighborhood of a point $x \in X$ then U is simply connected, so we have a natural identification $h : U \times F \to p^{-1}(U)$ with $(p \circ h)(u, f) = u$, where $F = p^{-1}(x)$ is the set of homotopy classes of paths from b to x; namely, $h(u, f)$ is the concatenation of f with any path connecting x with u inside U. The topologies on all such $p^{-1}(U)$ induced by these identifications glue together into a topology on Y, and the map $p : Y \to X$ is then a covering. Moreover, the homotopy lifting property implies that Y is simply connected, so this covering is universal.

It is easy to see that a universal covering $p : Y \to X$ covers any path-connected covering $p' : Y' \to X$, i.e., there is a covering $q : Y \to Y'$ such that $p = p' \circ q$; this is why it is called universal. Therefore a universal covering is unique up to an isomorphism (indeed, if Y, Y' are universal then we have coverings $q_1 : Y \to Y'$ and $q_2 : Y' \to Y$ and $q_1 \circ q_2 = q_2 \circ q_1 = \mathrm{Id}$).

Example 2.8. 1. The map $z \mapsto z^n$ defines an n-sheeted covering $S^1 \to S^1$.

2. The map $x \to e^{ix}$ defines the universal covering $\mathbb{R} \to S^1$.

Now denote by $\pi_1(X, x)$ the set of homotopy classes of *closed* paths on X, starting and ending at x. Then $\pi_1(X, x)$ is a group under concatenation of paths (concatenation is associative since the paths $a(bc)$ and $(ab)c$ differ only by parametrization and hence homotopic). This group is called the **fundamental group** of X relative to the point x. This group acts on the fiber $p^{-1}(x)$ for every covering $p : Y \to X$ (by lifting $\gamma \in \pi_1(X, x)$ to Y), which is called the action by **deck transformations**. This action is transitive iff Y is path-connected and moreover free iff Y is universal.

Finally, the group $\pi_1(X, x)$ does not depend on x up to an isomorphism. More precisely, conjugation by any path from x_1 to x_2 defines an isomorphism $\pi_1(X, x_1) \to \pi_1(X, x_2)$ (although two non-homotopic paths may define different isomorphisms if π_1 is non-abelian).

Example 2.9. 1. $\pi_1(S^1) = \mathbb{Z}$.

2. $\pi_1(\mathbb{C} \setminus \{z_1, ..., z_n\}) = F_n$ is a free group in n generators.

3. We have a 2-sheeted universal covering $S^n \to \mathbb{RP}^n$ (real projective space) for $n \geq 2$. Thus $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$ for $n \geq 2$.

Exercise 2.10. Make sure you can fill all the details in this subsection!

2.6. **Coverings of Lie groups.** Let G be a connected (real or complex) Lie group and $\widetilde{G} = \widetilde{G}_1$ be the universal covering of G, consisting of homotopy classes of paths $x : [0, 1] \to G$ with $x(0) = 1$. Then $\widetilde{G}$ is a group via $(x \cdot y)(t) = x(t)y(t)$, and also a manifold.

Proposition 2.11. *(i) $\widetilde{G}$ is a simply connected Lie group. The covering $p : \widetilde{G} \to G$ is a homomorphism of Lie groups.*

(ii) $\mathrm{Ker}(p)$ *is a central subgroup of $\widetilde{G}$ naturally isomorphic to $\pi_1(G) = \pi_1(G, 1)$. Thus, $\widetilde{G}$ is a central extension of G by $\pi_1(G)$. In particular, $\pi_1(G)$ is abelian.*

Proof. (i) We only need to show that $\widetilde{G}$ is a Lie group, i.e., that the multiplication map $\widetilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ is regular. But $\widetilde{G} \times \widetilde{G}$ is simply connected, and $\widetilde{m}$ is a lifting of the map

$$m' := m \circ (p \times p) : \widetilde{G} \times \widetilde{G} \to G \times G \to G,$$

so it is regular. In other words, $\widetilde{m}$ is regular since in local coordinates it is defined by the same functions as m.

(ii) is a homework problem. □

Remark 2.12. The same argument shows that more generally, the fundamental group of any path-connected topological group is abelian.

Example 2.13. 1. The map $z \mapsto z^n$ defines an n-sheeted covering of Lie groups $S^1 \to S^1$.

2. The map $x \to e^{ix}$ defines the universal covering of Lie groups $\mathbb{R} \to S^1$.

3. Consider the action of $SU(2)$ on trace zero Hermitian 2 by 2 matrices by conjugation. This action preserves the positive inner product $(A, B) = \text{Tr}(AB)$ and has determinant 1, so lands in $SO(3)$, and we'll see that the homomorphism $SU(2) \to SO(3)$ is surjective, with kernel ± 1. We will see it's a universal covering map (as $SU(2) = S^3$ is simply connected). Thus $\pi_1(SO(3)) = \mathbb{Z}/2$ (in fact, we see that $SO(3) \cong \mathbb{RP}^3$ as a manifold). This is demonstrated by the famous **Dirac belt trick**, which illustrates the notion of a **spinor**; namely, spinors are vectors in $\mathbb{C}^2$ acted upon by matrices from $SU(2)$. Here are some videos of the belt trick:

`https://www.youtube.com/watch?v=17Q0tJZcsnY`

`https://www.youtube.com/watch?v=Vfh21o-JW9Q`

2.7. **Lie subgroups.**

Definition 2.14. A **closed Lie subgroup** of a (real or complex) Lie group G is a subgroup which is also an embedded submanifold.

This terminology is justified by the following lemma.

Lemma 2.15. *A closed Lie subgroup of G is closed in G.*

Proof. Homework. □

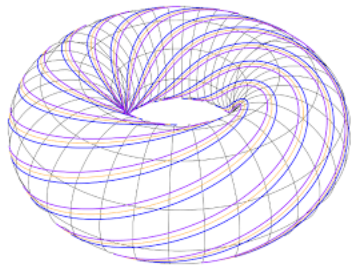
We also have

Theorem 2.16. *Any closed subgroup of a real Lie group G is a closed Lie subgroup.*

This theorem is rather nontrivial, and we will not prove it, but later will prove a weaker version which will suffice for our purposes.

Example 2.17. 1. $SL_n(K)$ is a closed Lie subgroup of $GL_n(K)$ for $K = \mathbb{R}, \mathbb{C}$. Indeed, the equation $\det A = 1$ defines a smooth hypersurface in the space of matrices (show it!).

2. Let $\phi : \mathbb{R} \to S^1 \times S^1$ be the irrational torus winding given by the formula $\phi(x) = (e^{ix}, e^{ix\sqrt{2}})$.



Then $\phi(\mathbb{R})$ is a subgroup of $S^1 \times S^1$ but not a closed Lie subgroup, since it is not an embedded submanifold: although ϕ is an immersion, the map $\phi^{-1} : \phi(\mathbb{R}) \to \mathbb{R}$ is not continuous.

2.8. **Generation of connected Lie groups by a neighborhood of 1.**

Proposition 2.18. *(i) If G is a connected Lie group and U a neighborhood of 1 in G then U generates G.*

(ii) If $f : G \to K$ is a homomorphism of Lie groups, K is connected, and $df_1 : T_1G \to T_1K$ is surjective, then f is surjective.

Proof. (i) Let H be the subgroup of G generated by U. Then H is open in G since $H = \cup_{h\in H} hU$. Thus H is an embedded submanifold of G, hence a closed Lie subgroup. Thus by Lemma 2.15 $H \subset G$ is closed. So $H = G$ since G is connected.

(ii) Since df_1 is surjective, by the implicit function theorem $f(G)$ contains some neghborhood of 1 in K. Thus it contains the whole K by (i). $\square$

3. Lecture 3

3.1. **Homogeneous spaces.**

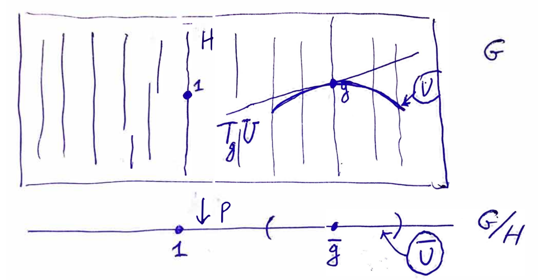
Recall that a regular map of manifolds $p : Y \to X$ is a said to be a **locally trivial fibration** (or **fiber bundle**) with **base** X and **total space** Y if every point $x \in X$ has a neighborhood U such that there is a manifold F (called the **fiber** of p at x) and a diffeomorphism $h : U \times F \cong p^{-1}(U)$ with $(p \circ h)(u, f) = u$. In other words, locally p looks like the projection $X \times F \to X$ (the trivial fiber bundle with fiber F over X), but not necessarily globally so. This generalizes the notion of a covering, in which case F is 0-dimensional (discrete).

Theorem 3.1. *(i) Let G be a Lie group of dimension n and $H \subset G$ a closed Lie subgroup of dimension k. Then the* **homogeneous space** *G/H has a natural structure of an $n - k$-dimensional manifold, and the map $p : G \to G/H$ is a locally trivial fibration with fiber H.*

(ii) If moreover H is normal in G then G/H is a Lie group.

(iii) We have a natural isomorphism $T_1(G/H) \cong T_1G/T_1H$.

Proof. Let $\bar{g} \in G/H$ and $g \in p^{-1}(\bar{g})$. Then $gH \subset G$ is an embedded submanifold (image of H under left translation by g). Pick a sufficiently small transversal submanifold U passing through g (i.e., $T_gG = T_g(gH) \oplus T_gU$).



By the inverse function theorem, the set UH is open in G. Let $\overline{U}$ be the image of UH in G/H. Since $p^{-1}(\overline{U}) = UH$ is open, $\overline{U}$ is open in the quotient topology. Also it is clear that $p : U \to \overline{U}$ is a homeomorphism. This defines a local chart near $\bar{g} \in G/H$, and it is easy to check that transition maps between such charts are regular. So G/H acquires the structure of a manifold, which is easily checked to be independent on the choices we made. Also the multiplication map $U \times H \to UH$ is a diffeomorphism, which implies that $p : G \to G/H$ is a locally trivial fibration with fiber H. Finally, we have a surjective linear map $T_gG \to T_{\bar{g}}G/H$ whose kernel is $T_g(gH)$. So in particular for $g = 1$ we get $T_1(G/H) \cong T_1G/T_1H$. This proves all parts of the proposition. □

Corollary 3.2. *Let $H \subset G$ be a closed Lie subgroup.*

(i) If H is connected then the map $p_0 : \pi_0(G) \to \pi_0(G/H)$ is a bijection.

(ii) If also G is connected then there is an exact sequence[1]

$$\pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to 1.$$

Proof. This follows from the theory of covering spaces using that $p : G \to G/H$ is a fibration. □

Exercise 3.3. Fill in the details in the proof of Corollary 3.2.

Remark 3.4. The sequence in Corollary 3.2(ii) is the end portion of the infinite **long exact sequence of homotopy groups of a fibration**,

$$\pi_i(H) \to \pi_i(G) \to \pi_i(G/H) \to \pi_{i-1}(H) \to ...$$

[1]This means that the map $\pi_1(G) \to \pi_1(G/H)$ is surjective and its kernel coincides with the image of the map $\pi_1(H) \to \pi_1(G)$.

3.2. **Lie subgroups.** We will call the image of an injective immersion of manifolds **an immersed submanifold**; it has a manifold structure coming from the source of the immersion.

Definition 3.5. A **Lie subgroup** of a Lie group G is a subgroup H which is also an *immersed* submanifold (but need not be an *embedded* submanifold, nor a closed subset).

It is clear that in this case H is still a Lie group and the inclusion $H \hookrightarrow G$ is a homomorphism of Lie groups.

Example 3.6. 1. The winding of a torus in Example 2.17(2) realizes $\mathbb{R}$ as a Lie subgroup of $S^1 \times S^1$ which is not closed.

2. Any countable subgroup of G is a 0-dimensional Lie subgroup (i.e., $\mathbb{Q} \subset \mathbb{R}$), but not always a closed one.

Proposition 3.7. *Let $f : G \to K$ be a homomorphism of Lie groups. Then $H := \mathrm{Ker} f$ is a closed normal Lie subgroup in G and $\mathrm{Im} f$ is a Lie subgroup (not necessarily closed) in K if it is an embedded submanifold. In this case we have an isomorphism of Lie groups $G/H \cong \mathrm{Im} f$.*

We will prove this later.

3.3. **Actions and representations of Lie groups.** Let X be a manifold, G a Lie group, and $a : G \times X \to X$ a set-theoretical left action of G on X.

Definition 3.8. This action is called **regular** if the map a is regular.

From now on, by an action of G on X we will always mean a regular action.

Example 3.9. 1. $GL_n(\mathbb{R})$ and any its Lie subgroup acts on $\mathbb{R}^n$ by linear transformations, and $GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$.

2. $SO(3)$ acts on S^2 by rotations.

Definition 3.10. A (real analytic) **finite dimensional representation** of a *real* Lie group G is a linear action of G on a finite dimensional vector space V over $\mathbb{R}$ or $\mathbb{C}$. Similarly, a (complex analytic) finite dimensional representation of a *complex* Lie group G is a linear action of G on a finite dimensional vector space V over $\mathbb{C}$.

In other words, a representation is a homomorphism of Lie groups $\pi_V : G \to GL(V)$.

Definition 3.11. A **morphism of representations** (or **intertwining operator**) $A : V \to W$ is a linear map which commutes with the G-action, i.e., $A\pi_V(g) = \pi_W(g)A$, $g \in G$.

As usual, an **isomorphism of representations** is an invertible morphism. With these definitions, finite dimensional representations of G form a *category*.

Note also that we have the operations of dual and tensor product on representations. Namely, given a representation V of G, we can define its representation V^* on its dual space by

$$\pi_{V^*}(g) = \pi_V(g^{-1})^*,$$

and if W is another representation of G then we can define a representation of G on $V \otimes W$ (the tensor product of vector spaces) by

$$\pi_{V\otimes W}(g) = \pi_V(g) \otimes \pi_W(g).$$

Also if $V \subset W$ is a **subrepresentation** (i.e., a subspace invariant under G) then W/V is also a representation of G, called the **quotient representation**.

3.4. **Orbits and stabilizers.** As in ordinary group theory, if G acts on X and $x \in X$ then we can define the **orbit** $Gx \subset X$ of x as the set of gx, $g \in G$, and the **stabilizer**, or **isotropy group** $G_x \subset G$ to be the group of $g \in G$ such that $gx = x$.

Proposition 3.12. *(The orbit-stabilizer theorem for Lie group actions) The stabilizer $G_x \subset G$ is a closed Lie subgroup, and the natural map $G/G_x \to X$ is an injective immersion whose image is Gx.*

This will be proved later.

Corollary 3.13. *The orbit $Gx \subset X$ is an immersed submanifold, and we have a natural isomorphism $T_x(Gx) \cong T_1G/T_1G_x$. If Gx is an embedded submanifold then the map $G/G_x \to Gx$ is a diffeomorphism.*

Remark 3.14. Note that Gx need not be closed in X. E.g., let $\mathbb{C}^\times$ act on $\mathbb{C}$ by multiplication. The orbit of 1 is $\mathbb{C}^\times \subset \mathbb{C}$, which is not closed.

Example 3.15. Suppose that G acts on X transitively. Then we get that $X \cong G/G_x$ for any $x \in X$, i.e., X is a **homogeneous space**.

Corollary 3.16. *If G acts transitively on X then the map $p : G \to X$ given by $p(g) = gx$ is a locally trivial fibration with fiber G_x.*

Example 3.17. 1. $SO(3)$ acts transitively on S^2 by rotations, $G_x = S^1 = SO(2)$, so $S^2 = SO(3)/S^1$. Thus $SO(3) = \mathbb{RP}^3$ fibers over S^2 with fiber S^1.

2. $SU(2)$ acts on $S^2 = \mathbb{CP}^1$, and the stabilizer is $S^1 = U(1)$. Thus $SU(2)/S^1 = S^2$, and $SU(2) = S^3$ fibers over S^2 with fiber S^1 (the **Hopf fibration**). Here is a nice keyring model of the Hopf fibration:

http://homepages.wmich.edu/~drichter/hopffibration.htm

3. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $F_n(\mathbb{K})$ the set of flags $0 \subset V_1 \subset ... \subset V_n = \mathbb{K}^n$ ($\dim V_i = i$). Then $G = GL_n(\mathbb{K})$ acts transitively on $F_n(\mathbb{K})$ (check it!). Also let $P \in F_n(\mathbb{K})$ be the flag for which $V_i = \mathbb{K}^i$ is the subspace of vectors whose all coordinates but the first i are zero. Then G_P is the subgroup $B_n(\mathbb{K}) \subset GL_n(\mathbb{K})$ of invertible upper triangular matrices. Thus $F_n(\mathbb{K}) = GL_n(\mathbb{K})/B_n(\mathbb{K})$ is a homogeneous space of $GL_n(\mathbb{K})$, in particular, a $\mathbb{K}$-manifold. It is called the **flag manifold**.

3.5. **Left translation, right translation, and adjoint action.** Recall that a Lie group G acts on itself by left translations $L_g(x) = gx$ and right translations $R_{g^{-1}}(x) = xg^{-1}$ (note that both are left actions).

Definition 3.18. The **adjoint action** $\mathrm{Ad}_g : G \to G$ is the action $\mathrm{Ad}_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$; i.e., $\mathrm{Ad}_g(x) = gxg^{-1}$.

Note this is an action by (inner) automorphisms. Also since $\mathrm{Ad}_g(1) = 1$, we have a linear map $d_1\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g} = T_1G$. We will abuse notation and denote this map just by Ad_g. This defines a representation of G on $\mathfrak{g}$ called the **adjoint representation**.

3.6. **A crash course on vector bundles.** Let X be a real manifold. A **vector bundle** on X is, informally speaking, a (locally trivial) fiber bundle on X whose fibers are finite dimensional vector spaces. In other words, it is a family of vector spaces parametrized by $x \in X$ and varying regularly with x. More precisely, we have the following definition.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 3.19. A $\mathbb{K}$-**vector bundle** of rank n on X is a manifold E with a surjective regular map $p : E \to X$ and a $\mathbb{K}$-vector space structure on each fiber $p^{-1}(x)$ such that every $x \in X$ has a neighborhood U admitting a diffeomorphism $h : U \times \mathbb{K}^n \to p^{-1}(U)$ with the following properties:

(i) $(p \circ g_U)(u, v) = u$, and

(ii) the map $g_U : p^{-1}(u) \to u \times \mathbb{K}^n$ is $\mathbb{K}$-linear.

In other words, locally on X, E is isomorphic to $X \times \mathbb{K}^n$, but not necessarily globally so.

As for ordinary fiber bundles, E is called the **total space** and X the **base** of the bundle.

Note that even if X is a complex manifold and $\mathbb{K} = \mathbb{C}$, E need not be a complex manifold.

Definition 3.20. A complex vector bundle $p : E \to X$ on a complex manifold X is said to be **holomorphic** if E is a complex manifold and the diffeomorphisms g_U can be chosen holomorphic.

From now on, unless specified otherwise, all complex vector bundles on complex manifolds we consider will be holomorphic.

It follows from the definition that if $p : E \to X$ is a vector bundle then X has an open cover $\{U_\alpha\}$ such that E trivializes on each U_α, i.e., there is a diffeomorphism $g_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ as above. In this case we have **clutching functions**

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{K})$$

(holomorphic if E is a holomorphic bundle), defined by the formula

$$(g_\alpha \circ g_\beta^{-1})(x, v) = (x, h_{\alpha\beta}(x)v)$$

which satisfy the **consistency conditions**

$$h_{\alpha\beta}(x) = h_{\beta\alpha}(x)^{-1}$$

and

$$h_{\alpha\beta}(x) \circ h_{\beta\gamma}(x) = h_{\alpha\gamma}(x)$$

for $x \in U_\alpha \cap U_\beta \cap U_\gamma$. Moreover, the bundle can be reconstructed from this data, starting from the disjoint union $\sqcup_\alpha U_\alpha \times \mathbb{K}^n$ and identifying (gluing) points according to

$$h_{\alpha\beta} : (x, v) \in U_\beta \times \mathbb{K}^n \sim (x, h_{\alpha\beta}(x)v) \in U_\alpha \times K^n.$$

The consistency conditions ensure that the relation $\sim$ is symmetric and transitive, so it is an equivalence relation, and we define E to be the space of equivalence classes with the quotient topology. Then E has a natural structure of a vector bundle on X.

This can also be used for constructing vector bundles. Namely, the above construction defines a $\mathbb{K}$-vector bundle on X once we are given a cover $\{U_\alpha\}$ on X and a collection of clutching functions

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{K})$$

satisfying the consistency conditions.

Remark 3.21. All this works more generally for non-linear fiber bundles if we drop the linearity conditions along fibers.

Example 3.22. 1. The **trivial bundle** $p : E = X \times \mathbb{K}^n \to X$, $p(x, v) = x$.

2. The **tangent bundle** is the vector bundle $p : TX \to X$ constructed as follows. For the open cover we take an atlas of charts (U_α, ϕ_α) with transition maps

$$\theta_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta),$$

and we set

$$h_{\alpha\beta}(x) := d_{\phi_\beta(x)}\theta_{\alpha\beta}.$$

(Check that these maps satisfy consistency conditions!)

Thus the tangent bundle TX is a vector bundle of rank $\dim X$ whose fiber $p^{-1}(x)$ is naturally the tangent space T_xX (indeed, the tangent vectors transform under coordinate changes exactly by multiplication by $h_{\alpha\beta}(x)$). In other words, it formalizes the idea of "the tangent space T_xX varying smoothly with $x \in X$".

Definition 3.23. A **section** of a map $p : E \to X$ is a map $s : X \to E$ such that $p \circ s = \mathrm{Id}_x$.

Example 3.24. If $p : X \times Y = E \to X$, $p(x, y) = x$ is the trivial bundle then a section $s : X \to E$ is given by $s(x) = (x, f(x))$ where $y = f(x)$ is a function $X \to Y$. So the notion of a section is a generalization of the notion of a function.

In particular, we may consider sections of a vector bundle $p : E \to X$ over an open set $U \subset X$. These sections form a vector space denoted $\Gamma(U, E)$.

Exercise 3.25. Show that a vector bundle $p : E \to X$ is trivial (i.e., globally isomorphic to $X \times \mathbb{K}^n$) if and only if it admits sections $s_1, ..., s_n$ which form a basis in every fiber $p^{-1}(x)$.

4. Lecture 4

4.1. Vector fields.

Definition 4.1. A **vector field** on X is a section of the tangent bundle TX.

Thus in local coordinates a vector field looks like

$$\mathbf{v} = \sum_i v_i \frac{\partial}{\partial x_i},$$

$v_i = v_i(\mathbf{x})$, and if $x_i \mapsto x_i'$ is a change of local coordinates then the expression for $\mathbf{v}$ in the new coordinates is

$$\mathbf{v} = \sum_i v_i' \frac{\partial}{\partial x_i'}$$

where

$$v_i' = \sum_j \frac{\partial x_i'}{\partial x_j} v_j,$$

i.e., the clutching function is the **Jacobi matrix** of the change of variable. Thus, every vector field $\mathbf{v}$ on X defines a derivation of the algebra $O(U)$ for every open set $U \subset X$ compatible with restriction maps $O(U) \to O(V)$ for $V \subset U$; in particular, a derivation $O_x \to O_x$

for all $x \in X$. Conversely, it is easy to see that such a collection of derivations gives rise to a vector field, so this is really the same thing.

A manifold X is called **parallelizable** if its tangent bundle is trivial. As explained above, this is equivalent to having a collection of vector fields $\mathbf{v}_1, ..., \mathbf{v}_n$ which form a basis in every tangent space (such a collection is called a **frame**). For example, the circle S^1 and hence the torus $S^1 \times S^1$ are parallelizable. On the other hand, the sphere S^2 is not parallelizable, since it does not even have a single nowhere vanishing vector field (the **Hairy Ball theorem**, or **Hedgehog theorem**). The same is true for any even-dimensional sphere S^{2m}, $m \geq 1$.

4.2. **Tensor fields, differential forms.** Since vector bundles are basically just smooth families of vector spaces varying over some base manifold X, we can do with them the same things we can do with vector spaces - duals, tensor products, symmetric and exterior powers, etc. E.g., the **contangent bundle** T^*X is dual to the tangent bundle TX.

More generally, we make the following definition.

Definition 4.2. A **tensor field** of rank (k, m) on a manifold X is a section of the tensor product $(TX)^{\otimes k} \otimes (T^*X)^{\otimes m}$.

For example, a tensor field of rank $(1, 0)$ is a vector field. Also, a skew-symmetric tensor field of rank $(0, m)$ is called a **differential** m**-form** on X. In other words, a differential m-form is a section of the vector bundle $\Lambda^m T^*X$.

For example, if $f \in O(X)$ then we have a differential 1-form df on X, the differential of f (indeed, recall that $d_x f : T_x X \to \mathbb{K}$). A general 1-form can therefore be written in local coordinates as

$$\omega = \sum_i a_i dx_i.$$

where $a_i = a_i(\mathbf{x})$. If coordinates are changed as $x_i \mapsto x_i'$, then in new coordinates

$$\omega = \sum_i a_i' dx_i'$$

where

$$a_i' = \sum_j \frac{\partial x_j}{\partial x_i'} a_j.$$

Thus the clutching function is the **inverse of the Jacobi matrix** of the change of variable. For instance,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

More generally, a differential m-form in local coordinates looks like

$$\omega = \sum_{1 \le i_1 < ... < i_m \le n} a_{i_1 ... i_m}(x) dx_{i_1} \wedge ... \wedge dx_{i_m}.$$

4.3. **Left and right invariant tensor fields on Lie groups.** Note that if a Lie group G acts on a manifold X, then it automatically acts on the tangent bundle TX and thus on vector and, more generally, tensor fields on X. In particular, G acts on tensor fields on itself by left and right translations, we will denote this action by L_g and R_g. We say that a tensor field T on G is **left invariant** if $L_g T = T$ for all g, and **right invariant** if $R_g T = T$ for all g.

Proposition 4.3. *(i) For any $\tau \in \mathfrak{g}^{\otimes k} \otimes \mathfrak{g}^{*\otimes m}$ there exists a unique left invariant tensor field τ_l and a unique right invariant tensor field τ_r whose value at 1 is τ. Thus, the spaces of such tensor fields are naturally isomorphic to $\mathfrak{g}^{\otimes k} \otimes \mathfrak{g}^{*\otimes m}$.*

(ii) τ_l is also right invariant iff τ_r is also left invariant iff τ is invariant under the adjoint representation Ad_g.

Proof. (i) Consider the tensor fields $\tau_l(g) := L_g \tau, \tau_r(g) := R_{g^{-1}} \tau$ (i.e., we "spread" $\tau \in T_1 G$ to other tangent spaces $T_g G$ by left/right translations). By construction, $R_{g^{-1}} \tau$ is right invariant, while $L_g \tau$ is left invariant, both with value τ at 1, and it is clear that these are unique.

(ii) is an exercise. □

Corollary 4.4. *A Lie group is parallelizable.*

Proof. Given a basis $e_1, ..., e_n$ of $\mathfrak{g} = T_1 G$, the vector fields $L_g e_1, ..., L_g e_n$ form a frame. □

Remark 4.5. In particular, S^1 and $SU(2) = S^3$ are parallelizable. It turns out that S^n for $n \ge 1$ is parallelizable if and only if $n = 1, 3, 7$ (a deep theorem). So spheres of other dimensions don't admit a Lie group structure (this can also be seen in a simpler way, and not very deep). The sphere S^7 does not admit one either, although it admits a weaker structure of a "homotopy Lie group", or H-space (arising from octonions) which suffices for parallelizability.

4.4. **Classical groups.** Roughly speaking, **classical groups** are groups arising from linear algebra. More precisely, classical groups are the following subgroups of the **general linear group** $GL_n(\mathbb{K})$: $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$ (the **special linear group**), $O_n(\mathbb{K})$, $SO_n(\mathbb{K})$, $Sp_{2n}(\mathbb{K})$, $O(p,q)$, $SO(p,q)$, $U(p,q)$, $SU(p,q)$, $Sp(2p,2q) := Sp_{2n}(\mathbb{C}) \cap U(2p,2q)$ for $p+q = n$ (and also some others we'll consider later).

Namely,

• The **orthogonal group** $O_n(\mathbb{K})$ is the group of matrices preserving the nondegenerate quadratic form in n variables, $Q = x_1^2 + ... + x_n^2$ (or, equivalently, the corresponding bilinear form $x_1y_1 + ... + x_ny_n$);

• The **symplectic group** $Sp_{2n}(\mathbb{K})$ is the group of matrices preserving a nondegenerate skew-symmetric form in $2n$ variables;

• The **pseudo-orthogonal group** $O(p, q)$, $p + q = n$ is the group of real matrices preserving a nondegenerate quadratic form of signature (p, q), $Q = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_n^2$ (or, equivalently, the corresponding bilinear form);

• The **pseudo-unitary group** $U(p, q)$, $p+q = n$ is the group of complex matrices preserving a nondegenerate Hermitian quadratic form of signature (p, q), $Q = |x_1|^2 + ... + |x_p|^2 - |x_{p+1}|^2 - ... - |x_n|^2$ (or, equivalently, the corresponding sesquilinear form);

• The **special pseudo-orthogonal, pseudo-unitary, and orthogonal groups** $SO(p, q) \subset O(p, q)$, $SU(p, q) \subset U(p, q)$, $SO_n \subset O_n$ are the subgroups of matrices of determinant 1.

Note that the groups don't change under switching p, q and that $(S)O_n(\mathbb{R}) = (S)O(n, 0)$; it is also denoted $(S)O(n)$. Also $(S)U(n, 0)$ is denoted by $(S)U(n)$.

Exercise 4.6. Show that the special (pseudo)orthogonal groups are index 2 subgroups of the (pseudo)orthogonal groups.

Let us show that they are all Lie groups. For this purpose we'll use the **exponential map** for matrices. Namely, recall from linear algebra that we have an analytic function $\exp : \mathfrak{gl}_n(\mathbb{K}) \to GL_n(\mathbb{K})$ given by the formula

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

and the matrix-valued analytic function log near $1 \in GL_n(\mathbb{K})$,

$$\log(A) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(A-1)^n}{n}.$$

Namely, this is well defined if the spectral radius of $A-1$ is < 1 (i.e., all eigenvalues are in the open unit disk). These maps have the following properties:

1. They are mutually inverse.
2. They are conjugation-invariant.
3. $d\exp_0 = d\log_1 = \mathrm{Id}$.
4. If $xy = yx$ then $\exp(x + y) = \exp(x)\exp(y)$. If $XY = YX$ then $\log(XY) = \log(X) + \log(Y)$ (for X, Y sufficiently close to 1).

5. For $x \in \mathfrak{gl}_n(\mathbb{K})$ the map $t \mapsto \exp(tx)$ is a homomorphism of Lie groups $\mathbb{K} \to GL_n(\mathbb{K})$.

6. $\det \exp(a) = \exp(\mathrm{tr}\ a)$, $\log(\det A) = \mathrm{tr}(\log A)$.

Now we can look at classical groups and see what happens to the equations defining them when we apply log.

1. $G = SL_n(\mathbb{K})$. We already did this example but let us re-do it by a different method. The group G is defined by the equation $\det A = 1$. So for A close to 1 $\log(\det A) = 0$, i.e., $\mathrm{tr} \log(A) = 0$. So $\log(A) \in \mathfrak{sl}_n(\mathbb{K}) = \mathfrak{g}$, the space of matrices with trace 0. This defines a local chart near $1 \in G$, showing that G is a manifold, hence a Lie group (namely, local charts near other points are obtained by translation).

2. $G = O_n(\mathbb{K})$. The equation is $A^T = A^{-1}$, thus $\log(A)^T = -\log(A)$, so $\log(A) \in \mathfrak{so}_n(\mathbb{K}) = \mathfrak{g}$, the space of skew-symmetric matrices.

3. $G = U(n)$. The equation is $\overline{A}^T = A^{-1}$, thus $\overline{\log(A)}^T = -\log(A)$, so $\log(A) \in \mathfrak{u}_n = \mathfrak{g}$, the space of skew-Hermitian matrices.

Exercise 4.7. Do the same for all classical groups listed above.

We obtain

Proposition 4.8. *Every classical group G from the above list is a Lie group, with $\mathfrak{g} = T_1 G \subset \mathfrak{gl}_n(\mathbb{K})$. Moreover, if $\mathfrak{u} \subset \mathfrak{gl}_n(\mathbb{K})$ is a small neighborhood of* 0 *and $U = \exp(\mathfrak{u})$ then* exp *and* log *define mutually inverse diffeomorphisms between $\mathfrak{u} \cap \mathfrak{g}$ and $U \cap G$.*

Exercise 4.9. Which of these groups are complex Lie groups?

Exercise 4.10. Use this proposition to compute the dimensions of classical groups: $\dim SL_n = n^2 - 1$, $\dim O_n = n(n-1)/2$, $\dim Sp_{2n} = n(2n+1)$, $\dim SU_n = n^2 - 1$, etc. (Note that for complex groups we give the dimension over $\mathbb{C}$).

5. Lecture 5

5.1. **Quaternions.** An important role in the theory of Lie groups is played by the **algebra of quaternions**, which is the only noncommutative finite dimensional division algebra over $\mathbb{R}$, discovered in the 19th century by W. R. Hamilton.

Definition 5.1. The **algebra of quaternions** is the $\mathbb{R}$-algebra with basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

This algebra is associative but not commutative.

Given a quaternion

$$\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},\ a, b, c, d \in \mathbb{R},$$

we define the **conjugate quaternion** by the formula

$$\overline{\mathbf{q}} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

Thus

$$\mathbf{q}\overline{\mathbf{q}} = |\mathbf{q}|^2 = a^2 + b^2 + c^2 + d^2 \in \mathbb{R},$$

where $|\mathbf{q}|$ is the length of $\mathbf{q}$ as a vector in $\mathbb{R}^4$. So if $\mathbf{q} \neq 0$ then it is invertible and

$$\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}.$$

Thus $\mathbb{H}$ is a **division algebra** (i.e., a skew-field). One can show that the only finite dimensional associative division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.

In particular, we can do linear algebra over $\mathbb{H}$ in almost the same way as we do over ordinary fields. Namely, every (left or right) module over $\mathbb{H}$ is free and has a basis; such a module is called a (left or right) **quaternionic vector space**. In particular, any (say, right) quaternionic vector space of dimension n (i.e., with basis of n elements) is isomorphic to $\mathbb{H}^n$. Moreover, $\mathbb{H}$-linear maps between such spaces are given by left multiplication by quaternionic matrices. Finally, it is easy to see that Gaussian elimination works the same way as over ordinary fields; in particular, every invertible square matrix over $\mathbb{H}$ is a product of elementary matrices of the form $1 + (\mathbf{q} - 1)E_{ii}$ and $1 + \mathbf{q}E_{ij}$, $i < j$, where $\mathbf{q} \in \mathbb{H}$ is nonzero.

Also it is easy to show that

$$\overline{q_1 q_2} = \overline{q_2 q_1},\ \ |\mathbf{q}_1\mathbf{q}_2| = |\mathbf{q}_1| \cdot |\mathbf{q}_2|$$

(check this!). So quaternions are similar to complex numbers, except they are non-commutative. Finally, note that $\mathbb{H}$ contains a copy of $\mathbb{C}$ spanned by $1, \mathbf{i}$; however, this does not make $\mathbb{H}$ as $\mathbb{C}$-algebra since $\mathbf{i}$ is not a central element.

Proposition 5.2. *The group of unit quaternions* $\{\mathbf{q} \in \mathbb{H} : |\mathbf{q}| = 1\}$ *under multiplication is isomorphic to* $SU(2)$ *as a Lie group.*

Proof. We can realize $\mathbb{H}$ as $\mathbb{C}^2$, where $\mathbb{C} \subset \mathbb{H}$ is spanned by $1, \mathbf{i}$; namely, $(z_1, z_2) \mapsto z_1 + \mathbf{j}z_2$. Then left multiplication by quaternions on $\mathbb{H} = \mathbb{C}^2$ commutes with right multiplication by $\mathbb{C}$, i.e., is $\mathbb{C}$-linear. So it is given by complex 2 by 2 matrices. It is easy to compute that the corresponding matrix is

$$z_1 + z_2\mathbf{j} \mapsto \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix},$$

and we showed before that such matrices (with $|z_1|^2 + |z_2|^2 = 1$) are exactly the matrices from $SU(2)$. □

This is another way to see that $SU(2) \cong S^3$ as a manifold (since the set of unit quaternions is manifestly S^3).

Corollary 5.3. *The map* $\mathbf{q} \mapsto (\frac{\mathbf{q}}{|\mathbf{q}|}, |\mathbf{q}|)$ *is an isomorphism of Lie groups* $\mathbb{H}^\times \cong SU(2) \times \mathbb{R}_{>0}$.

This is the quaternionic analog of the trigonometric form of complex numbers, except the "phase" factor $\frac{\mathbf{q}}{|\mathbf{q}|}$ is now not in S^1 but in $S^3 = SU(2)$.

5.2. **More classical groups.** Now we can define a new classical group $GL_n(\mathbb{H})$, a real Lie group of dimension $4n^2$, called the **quaternionic general linear group**. For example, as we just showed, $GL_1(\mathbb{H}) = \mathbb{H}^\times \cong SU(2) \times \mathbb{R}_{>0}$.

For $A \in GL_n(\mathbb{H})$, let $\det A$ be the determinant of A as a linear operator on $\mathbb{C}^{2n} = \mathbb{H}^n$.

Lemma 5.4. *We have* $\det A > 0$.

Proof. For $n = 1$, $A = \mathbf{q} \in \mathbb{H}^\times$ and $\det \mathbf{q} = |\mathbf{q}|^2 > 0$. It follows that $\det(1+(\mathbf{q}-1)E_{ii}) = |\mathbf{q}|^2 > 0$. Also it is easy to see that $\det(1+\mathbf{q}E_{ij}) = 1$ for $i \neq j$. It then follows by Gaussian elimination that for any A we have $\det(A) > 0$. □

Let $SL_n(\mathbb{H}) \subset GL_n(\mathbb{H})$ be the subgroup of matrices A with $\det A = 1$, called the **quaternionic special linear group**.

Exercise 5.5. Show that this is a normal subgroup, and $GL_n(\mathbb{H}) \cong SL_n(\mathbb{H}) \times \mathbb{R}_{>0}$.

Thus $SL_n(\mathbb{H})$ is a real Lie group of dimension $4n^2 - 1$.

We can also define groups of quaternionic matrices preserving various sesquilinear forms. Namely, let $V \cong \mathbb{H}^n$ be a right quaternionic vector space.

Definition 5.6. A **sesquilinear form** on V is a biadditive function $(,) : V \times V \to \mathbb{H}$ such that

$$(\mathbf{x}\alpha, \mathbf{y}\beta) = \overline{\alpha}(\mathbf{x}, \mathbf{y})\beta,\ \mathbf{x}, \mathbf{y} \in V,\ \alpha, \beta \in \mathbb{H}.$$

Such a form is called **Hermitian** if $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ and **skew-Hermitian** if $(\mathbf{x}, \mathbf{y}) = -\overline{(\mathbf{y}, \mathbf{x})}$.

Note that the order of factors is important here!

Proposition 5.7. *(i) Every nondegenerate Hermitian form on* V *in some basis takes the form*

$$(\mathbf{x}, \mathbf{y}) = \overline{x_1}y_1 + ... + \overline{x_p}y_p - \overline{x_{p+1}}y_{p+1} - ... - \overline{x_n}y_n$$

for a unique pair (p, q) with $p + q = n$.

(ii) Every nondegenerate skew-Hermitian form on V in some basis takes the form

$$(\mathbf{x}, \mathbf{y}) = \overline{x_1}\mathbf{j}y_1 + ...\overline{x_n}\mathbf{j}y_n.$$

Exercise 5.8. Prove this proposition.

In (i), the pair (p, q) is called the **signature** of the quaternionic Hermitian form.

Exercise 5.9. Show that a nondegenerate quaternionic Hermitian form of signature (p, q) can be written as

$$(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}, \mathbf{y}) + \mathbf{j}B_2(\mathbf{x}, \mathbf{y}),$$

with B_1, B_2 taking values in $\mathbb{C} = \mathbb{R} + \mathbb{R}\mathbf{i} \subset \mathbb{H}$, where B_1 is a usual nondegenerate Hermitian form of signature $(2p, 2q)$ and B_2 is a nondegenerate skew-symmetric bilinear form on V as a ($2n$-dimensional) $\mathbb{C}$-vector space. Show that $B_2(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}\mathbf{j}, \mathbf{y})$. Deduce that any complex linear transformation preserving B_1 and B_2 is $\mathbb{H}$-linear.

Thus the group of symmetries of a nondegenerate quaternionic Hermitian form of signature (p, q) is $Sp(2p, 2q) = Sp_{2n}(\mathbb{C}) \cap U(2p, 2q)$. It is called the **quaternionic pseudo-unitary group**.

One also sometimes uses the notation $U(p, q, \mathbb{R}) = O(p, q)$, $U(p, q, \mathbb{C}) = U(p, q)$, $U(p, q, \mathbb{H}) = Sp(2p, 2q)$, and $U(n, 0, \mathbb{K}) = U(n, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Exercise 5.10. Show that a nondegenerate quaternionic skew-Hermitian form can be written as

$$(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}, \mathbf{y}) + \mathbf{j}B_2(\mathbf{x}, \mathbf{y}),$$

with B_1, B_2 taking values in $\mathbb{C} = \mathbb{R} + \mathbb{R}\mathbf{i} \subset \mathbb{H}$, where B_1 is an ordinary skew-Hermitian form, while B_2 is a symmetric bilinear form (both nondegenerate). Show that $B_2(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}\mathbf{j}, \mathbf{y})$. Deduce that any complex linear transformation preserving B_1 and B_2 is $\mathbb{H}$-linear. Also show that the signature of the Hermitian form iB_1 is necessarily (n, n).

Thus the group of symmetries of a nondegenerate quaternionic skew-Hermitian form is $O_{2n}(\mathbb{C}) \cap U(n, n)$. This group is denoted by $O^*(2n)$ and called the **quaternionic orthogonal group**. There is also the subgroup $SO^*(2n) \subset O^*(2n)$ of matrices of determinant 1 (having index 2).

All of these groups are Lie groups, which is shown similarly to the above, using the exponential map.

Exercise 5.11. Compute the dimensions of all classical groups introduced above.

5.3. **Exponential map.** Let G be a real Lie group, $\mathfrak{g} = T_1 G$.

Proposition 5.12. *Let $x \in \mathfrak{g}$. There is a unique morphism of Lie groups $\gamma = \gamma_x : \mathbb{R} \to G$ such that $\gamma'(0) = x$.*

Proof. For such a morphism we should have

$$\gamma(t+s) = \gamma(t)\gamma(s),\ t, s \in \mathbb{R},$$

so differentiating by s at $s = 0$, we get[2]

$$\gamma'(t) = \gamma(t)x.$$

Thus $\gamma(t)$ is a solution of the ODE defined by the left-invariant vector field $\mathbf{L}_x$ corresponding to $x \in \mathfrak{g}$ with initial condition $\gamma(0) = 1$. By the existence and uniqueness theorem for solutions of ODE, this equation has a unique solution with this initial condition defined for $|t| < \varepsilon$ for some $\varepsilon > 0$. Moreover, if $|s| + |t| < \varepsilon$, both $\gamma_1(t) := \gamma(s+t)$ and $\gamma_2(t) := \gamma(s)\gamma(t)$ satisfy this differential equation with initial condition $\gamma_1(0) = \gamma_2(0) = \gamma(s)$, so $\gamma_1 = \gamma_2$. Thus

$$\gamma(s+t) = \gamma(s)\gamma(t),\ |s| + |t| < \varepsilon;$$

hence $\gamma(t)x = x\gamma(t)$ for $|t| < \varepsilon$.

We claim that the solution $\gamma(t)$ extends to all values of $t \in \mathbb{R}$. Indeed, let us prove that it extends to $|t| < 2^n\varepsilon$ for all $n \geq 0$ by induction in n. The base of induction ($n = 0$) is already known, so we only need to justify the induction step from $n-1$ to n. Given t with $|t| < 2^n\varepsilon$, we define

$$\gamma(t) := \gamma(\tfrac{t}{2})^2.$$

This agrees with the previously defined solution for $|t| < 2^{n-1}\varepsilon$, and we have

$$\gamma'(t) = \tfrac{1}{2}(\gamma'(\tfrac{t}{2})\gamma(\tfrac{t}{2}) + \gamma(\tfrac{t}{2})\gamma'(\tfrac{t}{2})) = \tfrac{1}{2}\gamma(\tfrac{t}{2})x\gamma(\tfrac{t}{2}) + \tfrac{1}{2}\gamma(\tfrac{t}{2})^2 x = \gamma(\tfrac{t}{2})^2 x = \gamma(t)x,$$

as desired.

Thus, we have a regular map $\gamma : \mathbb{R} \to G$ with $\gamma(s+t) = \gamma(s)\gamma(t)$ and $\gamma'(0) = x$, which is unique by the uniqueness of solutions of ODE. □

Definition 5.13. The **exponential map** $\exp : \mathfrak{g} \to G$ is defined by the formula $\exp(x) = \gamma_x(1)$.

Thus $\gamma_x(t) = \exp(tx)$. So we have

[2]For brevity for $g \in G$, $x \in \mathfrak{g}$ we denote $L_g x$ by gx and $R_g x$ by xg.

Proposition 5.14. *The flow defined by the right-invariant vector field* $\mathbf{R}_x$ *is given by* $g \mapsto \exp(tx)g$*, and the flow defined by the left-invariant vector field* $\mathbf{L}_x$ *is given by* $g \mapsto g\exp(tx)$.

Example 5.15. 1. Let $G = \mathbb{K}^n$. Then $\exp(x) = x$.

2. Let $G = GL_n(\mathbb{K})$ or its Lie subgroup. Then $\gamma_x(t)$ satisfies the matrix differential equation

$$\gamma'(t) = \gamma(t)x$$

with $\gamma(0) = 1$, so

$$\gamma_x(t) = e^{tx},$$

the matrix exponential. For example, if $n = 1$, this is the usual exponential function.

The following theorem describes the basic properties of the exponential map. Let G be a real or complex Lie group.

Theorem 5.16. *(i)* $\exp : \mathfrak{g} \to G$ *is a regular map which is a diffeomorphism of a neighborhood of* $0 \in \mathfrak{g}$ *onto a neighborhood of* $1 \in G$*, with* $\exp(0) = 1$, $\exp'(0) = \mathrm{Id}_{\mathfrak{g}}$.

(ii) $\exp((s+t)x) = \exp(sx)\exp(tx)$ *for* $x \in \mathfrak{g}$, $s, t \in \mathbb{K}$.

(iii) For any morphism of Lie groups $\phi : G \to K$ *and* $x \in T_1G$ *we have*

$$\phi(\exp(x)) = \exp(\phi_* x);$$

i.e., the exponential map commutes with morphisms.

(iv) For any $g \in G$, $x \in \mathfrak{g}$*, we have*

$$g\exp(x)g^{-1} = \exp(\mathrm{Ad}_g x).$$

Proof. (i) The regularity of exp follows from the fact that if a differential equation depends regularly on parameters then so do its solutions. Also $\gamma_0(t) = 1$ so $\exp(0) = 1$. We have $\exp'(0)x = \frac{d}{dt}\exp(tx)|_{t=0} = x$, so $\exp'(0) = \mathrm{Id}$. By the inverse function theorem this implies that exp is a diffeomorphism near the origin.

(ii) Holds since $\exp(tx) = \gamma_x(t)$.

(iii) Both $\phi(\exp(tx))$ and $\exp(\phi_*(tx))$ satisfy the equation $\gamma'(t) = \gamma(t)\phi_*(x)$ with the same initial conditions.

(iv) is a special case of (iii) with $\phi : G \to G$, $\phi(h) = ghg^{-1}$. ☐

Thus exp has an inverse $\log : U \to \mathfrak{g}$ defined on a neighborhood U of $1 \in G$ with $\log(1) = 0$. This map is called the **logarithm**. For $GL_n(\mathbb{K})$ and its Lie subgroups it coincides with the matrix logarithm. The logarithm map defines a canonical coordinate chart on G near 1, so a choice of a basis of $\mathfrak{g}$ gives a local coordinate system.

Proposition 5.17. *Let G be a connected Lie group and $\phi : G \to K$ a morphism of Lie groups. Then ϕ is completely determined by $\phi_* : T_1G \to T_1K$.*

Proof. We have $\phi(\exp(x)) = \exp(\phi_*(x))$, so since exp is a diffeomorphism near 0, ϕ is determined by ϕ_* on a neighborhood of $1 \in G$. This completely determines ϕ since this neighborhood generates G. □

6. Lecture 6

6.1. **The commutator.** In general (say, for $G = GL_n(\mathbb{K})$, $n \geq 2$), $\exp(x + y) \neq \exp(x)\exp(y)$. So let us consider the map

$$(x, y) \mapsto \mu(x, y) = \log(\exp(x)\exp(y))$$

which maps $U \times U \to \mathfrak{g}$, where $U \subset \mathfrak{g}$ is a neighborhood of 0. This map expresses the product in G in the coordinate chart coming from the logarithm map. We have $\mu(x, 0) = \mu(0, x) = x$ and $\mu_*(x, y) = x+y$, so

$$\mu(x, y) = x + y + \tfrac{1}{2}\mu_2(x, y) + ...$$

where $\mu_2 = d^2\mu_{(0,0)}$ is the quadratic part and ... are higher terms. Moreover, $\mu_2(x, 0) = \mu_2(0, y) = 0$, hence μ_2 is a bilinear map $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$. It is easy to see that $\mu(x, -x) = 0$, hence μ_2 is skew-symmetric.

Definition 6.1. The map μ_2 is called the **commutator** and denoted by $x, y \mapsto [x, y]$.

Thus we have

$$\exp(x)\exp(y) = \exp(x + y + \tfrac{1}{2}[x, y] + ...).$$

Example 6.2. Let $G = GL_n(\mathbb{K})$. Then

$$\exp(x)\exp(y) = (1+x+\tfrac{x^2}{2}+...)(1+y+\tfrac{y^2}{2}+...) = 1+x+y+\tfrac{x^2}{2}+xy+\tfrac{y^2}{2}+... =$$

$$1 + (x + y) + \tfrac{(x+y)^2}{2} + \tfrac{xy-yx}{2} + ... = \exp(x + y + \tfrac{xy-yx}{2} + ...)$$

Thus

$$[x, y] = xy - yx.$$

This justifies the term "commutator": it measures the failure of x and y to commute.

Corollary 6.3. *If $G \subset GL_n(\mathbb{K})$ is a Lie subgroup then $\mathfrak{g} = T_1G \subset \mathfrak{gl}_n(\mathbb{K})$ is closed under the commutator $[x, y] = xy-yx$, which coincides with the commutator of G.*

For $x \in \mathfrak{g}$ define the linear map $\mathrm{ad}x : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}x(y) = [x, y].$$

Proposition 6.4. *(i) Let G, K be Lie groups and $\phi : G \to K$ a morphism of Lie groups. Then $\phi_* : T_1G \to T_1K$ preserves the commutator:*

$$\phi_*([x, y]) = [\phi_*(x), \phi_*(y)].$$

(ii) The adjoint action preserves the commutator.
(iii) We have

$$\exp(x)\exp(y)\exp(x)^{-1}\exp(y)^{-1} = \exp([x, y] + ...)$$

where ... denotes cubic and higher terms.
(iv) Let $X(t), Y(s)$ be parametrized curves on G such that $X(0) = Y(0) = 1$, $X'(0) = x, Y'(0) = y$. Then we have

$$[x, y] = \lim_{s,t\to 0} \frac{\log(X(t)Y(s)X(t)^{-1}Y(s)^{-1})}{ts}.$$

In particular,

$$[x, y] = \lim_{s,t\to 0} \frac{\log(\exp(tx)\exp(sy)\exp(tx)^{-1}\exp(sy)^{-1})}{ts}$$

and

$$[x, y] = \frac{d}{dt}|_{t=0}\mathrm{Ad}_{X(t)}(y).$$

Thus $\mathrm{ad} = \mathrm{Ad}_*$, *the differential of* Ad *at* $1 \in G$.
(v) If G is commutative (=abelian) then $[x, y] = 0$ for all x, y.

Proof. (i) Follows since ϕ commutes with the exponential map.
(ii) Follows from (i) by setting $\phi = \mathrm{Ad}_g$.
(iii) Modulo cubic and higher terms we have

$$\log(\exp(x)\exp(y)) = \log(\exp(y)\exp(x)) + [x, y] + ...,$$

which implies the statement by exponentiation.
(iv) Let $\log X(t) = x(t)$, $\log Y(s) = y(s)$. Then by (iii) we have

$$\log(X(t)Y(s)X(t)^{-1}Y(s)^{-1}) =$$

$$\log(\exp(x(t))\exp(y(s))\exp(x(t))^{-1}\exp(y(s))^{-1}) = ts([x, y]+o(1)),\ t, s \to 0.$$

This implies the first two statements. The last statement follows by taking the limit in s first, then in t.
(v) follows from (iii). □

Motivated by part (v), a Lie algebra $\mathfrak{g}$ is said to be **commutative** or **abelian** if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

6.2. **The Jacobi identity.** The matrix commutator $[x, y] = xy - yx$ obviously satisfies the identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

called the **Jacobi identity**. Thus it is satisfied for any Lie subgroup of $GL_n(\mathbb{K})$.

Proposition 6.5. *The Jacobi identity holds for any Lie group G.*

Proof. Let $\mathfrak{g} = T_1 G$. By the above proposition

$$\mathrm{ad}x = \frac{d}{dt}|_{t=0}\mathrm{Ad}_{\exp(tx)}.$$

Also, the Jacobi identity is equivalent to $\mathrm{ad}x$ being a derivation of the commutator:

$$\mathrm{ad}x([y, z]) = [\mathrm{ad}x(y), z] + [y, \mathrm{ad}x(z)].$$

To show that it is indeed a derivation, let $g(t) = \exp(tx)$, then

$$\mathrm{Ad}_{g(t)}([y, z]) = [\mathrm{Ad}_{g(t)}(y), \mathrm{Ad}_{g(t)}(z)].$$

The desired identity is then obtained by differentiating this equality by t at $t = 0$ and using the Leibniz rule. □

Corollary 6.6. *We have* $\mathrm{ad}[x, y] = [\mathrm{ad}x, \mathrm{ad}y]$.

Proof. This is also equivalent to the Jacobi identity. □

Proposition 6.7. *For $x \in \mathfrak{g}$ one has* $\exp(\mathrm{ad}x) = \mathrm{Ad}_{\exp(x)} \in GL(\mathfrak{g})$.

Proof. We will show that $\exp(t\mathrm{ad}x) = \mathrm{Ad}_{\exp(tx)}$ for $t \in \mathbb{R}$. Let $\gamma_1(t) = \exp(t\mathrm{ad}x)$ and $\gamma_2(t) = \mathrm{Ad}_{\exp(tx)}$. Then γ_1, γ_2 both satisfy the differential equation $\gamma'(t) = \gamma(t)\mathrm{ad}x$ and equal 1 at $t = 0$. Thus $\gamma_1 = \gamma_2$. □

7. Lecture 7

7.1. **Lie algebras.**

Definition 7.1. A **Lie algebra** over a field $\mathbf{k}$ is a vector space $\mathfrak{g}$ over $\mathbf{k}$ equipped with bilinear operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **commutator** or **(Lie) bracket** which satisfies the following identities:

(i) $[x, x] = 0$ for all $x \in \mathfrak{g}$;
(ii) the Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

Remark 7.2. If $\mathbf{k}$ has characteristic $\neq 2$ then the condition $[x, x] = 0$ is equivalent to skew-symmetry $[x, y] = -[y, x]$, but in characteristic 2 it is stronger.

Example 7.3. Any subspace of $\mathfrak{gl}_n(\mathbf{k})$ closed under $[x, y] := xy - yx$ is a Lie algebra.

The notion of a **morphism of Lie algebras** is defined in an obvious way.

Example 7.4. The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a morphism of Lie algebras.

Thus we have

Theorem 7.5. *If G is a $\mathbb{K}$-Lie group (for $\mathbb{K} = \mathbb{R}, \mathbb{C}$) then $\mathfrak{g} := T_1G$ has a natural structure of a Lie algebra over $\mathbb{K}$. Moreover, if $\phi : G \to K$ is a morphism of Lie groups then $\phi_* : T_1G \to T_1K$ is a morphism of Lie algebras.*

We will denote the Lie algebra $\mathfrak{g} = T_1G$ by $\mathrm{Lie}G$ and call it the **Lie algebra of** G. We see that the assignment $G \mapsto \mathrm{Lie}G$ is a functor from the category of Lie groups to the category of Lie algebras. Thus we have a map $\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$, which is injective if G is connected.

7.2. **Lie subalgebras and ideals.** A **Lie subalgebra** of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ closed under the commutator. It is called a **Lie ideal** if moreover $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Proposition 7.6. *Let $H \subset G$ be a Lie subgroup. Then:*
(i) $\mathrm{Lie}H \subset \mathrm{Lie}G$ *is a Lie subalgebra;*
(ii) If H is normal then $\mathrm{Lie}H$ *is a Lie ideal in* $\mathrm{Lie}G$*;*
(iii) If G, H are connected and $\mathrm{Lie}H \subset \mathrm{LieG}$ *is a Lie ideal then H is normal in G.*

Proof. (i) If $x, y \in \mathfrak{h}$ then $\exp(tx), \exp(sy) \in H$, so

$$[x, y] = \lim_{t,s\to 0} \frac{\log(\exp(tx)\exp(sy)\exp(-tx)\exp(-sy))}{ts} \in \mathfrak{h}.$$

(ii) We have $ghg^{-1} \in H$ for $g \in G$ and $h \in H$. Thus, taking $h = \exp(sy)$, $y \in \mathfrak{h}$ and taking the derivative in s at zero we get $\mathrm{Ad}_g(y) \in \mathfrak{h}$. Now taking $g = \exp(tx)$, $x \in \mathfrak{g}$ and taking the derivative in t at zero, we get $[x, y] \in \mathfrak{h}$, i.e., $\mathfrak{h}$ is a Lie ideal.

(iii) If $x \in \mathfrak{g}$, $y \in \mathfrak{h}$ are small then

$$\exp(x)\exp(y)\exp(x)^{-1} =$$

$$\exp(\mathrm{Ad}_{\exp(x)}y) = \exp(\exp(\mathrm{ad}x)y) = \exp(\sum_{n=0}^{\infty} \tfrac{(\mathrm{ad}x)^n y}{n!}) \in H$$

since $\sum_{n=0}^{\infty} \frac{(\mathrm{ad}x)^n y}{n!} \in \mathfrak{h}$. So G acting on itself by conjugation maps a small neighborhood of 1 in H into H (as G is generated by its neighborhood of 1, since it is connected). But H is also connected, so is generated by its neighborhood of 1. Hence H is normal. $\square$

7.3. **The Lie algebra of vector fields.** Recall that a vector field on a manifold X is a compatible family of derivations $\mathbf{v} : O(U) \to O(U)$ for open subsets $U \subset X$.

Proposition 7.7. *If* $\mathbf{v}, \mathbf{w}$ *are derivations of an algebra* A *then so is* $[\mathbf{v}, \mathbf{w}] := \mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v}$.

Proof. We have

$$(\mathbf{vw} - \mathbf{wv})(ab) = \mathbf{v}(\mathbf{w}(a)b + a\mathbf{w}(b)) - \mathbf{w}(\mathbf{v}(a)b + a\mathbf{v}(b)) =$$
$$\mathbf{vw}(a)b + \mathbf{w}(a)\mathbf{v}(b) + \mathbf{v}(a)\mathbf{w}(b) + a\mathbf{vw}(b)$$
$$-\mathbf{wv}(a)b - \mathbf{v}(a)\mathbf{w}(b) - \mathbf{w}(a)\mathbf{v}(b) - a\mathbf{wv}(b) =$$
$$(\mathbf{vw} - \mathbf{wv})(a)b + a(\mathbf{vw} - \mathbf{wv})(b).$$

□

Thus, the space $\mathrm{Vect}(X)$ of vector fields on X is a Lie algebra under the operation

$$\mathbf{v}, \mathbf{w} \mapsto [\mathbf{v}, \mathbf{w}],$$

called the **Lie bracket of vector fields**. [3]

In local coordinates we have

$$\mathbf{v} = \sum_i v_i \frac{\partial}{\partial x_i}, \ \mathbf{w} = \sum w_j \frac{\partial}{\partial x_j},$$

so

$$[\mathbf{v}, \mathbf{w}] = \sum_i \left(\sum_j (v_j \tfrac{\partial w_i}{\partial x_j} - w_j \tfrac{\partial v_i}{\partial x_j}) \right) \tfrac{\partial}{\partial x_i}.$$

This implies that if vector fields $\mathbf{v}, \mathbf{w}$ are tangent to a k-dimensional submanifold $Y \subset X$ then so is their Lie bracket $[\mathbf{v}, \mathbf{w}]$. Indeed, in local coordinates Y is given by equations $x_{k+1} = ... = x_n = 0$, and in such coordinates a vector field is tangent to Y iff it does not contain terms with $\frac{\partial}{\partial x_j}$ for $j > k$.

Exercise 7.8. Let $U \subset \mathbb{R}^n$ be an open subset, $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(U)$ and g_t, h_t be the associated flows, defined in a neighborhood of every point of U for small t. Show that for any $\mathbf{x} \in U$

$$\lim_{t,s \to 0} \frac{g_t h_s g_t^{-1} h_s^{-1}(\mathbf{x}) - \mathbf{x}}{ts} = [\mathbf{v}, \mathbf{w}](\mathbf{x}).$$

Now let G be a Lie group and $\mathrm{Vect}_L(G), \mathrm{Vect}_R(G) \subset \mathrm{Vect}(G)$ be the subspaces of left and right invariant vector fields.

[3]Note that this Lie algebra is infinite dimensional for all real manifolds and many (but not all) complex manifolds of positive dimension.

Proposition 7.9. $\mathrm{Vect}_L(G), \mathrm{Vect}_R(G) \subset \mathrm{Vect}(G)$ *are Lie subalgebras which are both canonically isomorphic to* $\mathfrak{g} = \mathrm{Lie}G$.

Proof. The first statement is obvious, so we prove only the second statement. Let $\mathbf{x}, \mathbf{y} \in \mathrm{Vect}_L(G)$. Then $\mathbf{x} = \mathbf{L}_x$, $\mathbf{y} = \mathbf{L}_y$ for $x = \mathbf{x}(1), y = \mathbf{y}(1) \in \mathfrak{g}$, where $\mathbf{L}_z$ denotes the vector field on G obtained by right translations of $z \in \mathfrak{g}$. Then $[\mathbf{L}_x, \mathbf{L}_y] = \mathbf{L}_z$, where $z = [\mathbf{L}_x, \mathbf{L}_y](1)$. So let us compute z.

Let f be a regular function on a neighborhood of $1 \in G$. We have shown that for $u \in \mathfrak{g}$

$$(\mathbf{L}_u f)(g) = \tfrac{d}{dt}|_{t=0} f(g \exp(tu)).$$

Thus,

$$z(f) = x(\mathbf{L}_y f) - y(\mathbf{L}_x f) = x(\tfrac{d}{ds}|_{s=0} f(\bullet \exp(sy))) - y(\tfrac{d}{dt}|_{t=0} f(\bullet \exp(tx))) =$$

$$\tfrac{d}{dt}|_{t=0} \tfrac{d}{ds}|_{s=0} f(\exp(tx)\exp(sy)) - \tfrac{d}{ds}|_{s=0} \tfrac{d}{dt}|_{t=0} f(\exp(sy)\exp(tx)) =$$

$$\tfrac{\partial^2}{\partial t \partial s}|_{t=s=0}(F(tx + sy + \tfrac{1}{2}ts[x,y] + ...) - F(tx + sy - \tfrac{1}{2}ts[x,y] + ...)),$$

where $F(u) := f(\exp(u))$. It is easy to see by using Taylor expansion that this expression equals to $[x, y](f)$. Thus $z = [x, y]$, i.e., the map $\mathfrak{g} \to \mathrm{Vect}_L(G)$ given by $x \mapsto \mathbf{L}_x$ is a Lie algebra isomorphism. Similarly, the map $\mathfrak{g} \to \mathrm{Vect}_R(G)$ given by $x \mapsto -\mathbf{R}_x$ is a Lie algebra isomorphism, as claimed. □

7.4. **Proofs of Theorem 2.16, Proposition 3.12, Proposition 3.7.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and X be a manifold with an action $a : G \times X \to X$. Then for any $z \in \mathfrak{g}$ we have a vector field $a_*(z)$ on X given by

$$(a_*(z)f)(x) = \frac{d}{dt}|_{t=0} f(\exp(-tz)x),$$

where $t \in \mathbb{R}$, $f \in O(U)$ for some open set $U \subset X$ and $x \in U$.

Proposition 7.10. *The map* a_* *is linear and we have*

$$a_*([z, w]) = [a_*(z), a_*(w)].$$

In other words, the map $a_* : \mathfrak{g} \to \mathrm{Vect}(X)$ *is a homomorphism of Lie algebras.*

Proof. Exercise. □

This motivates the following definition.

Definition 7.11. An action of a Lie algebra $\mathfrak{g}$ on a manifold X is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{Vect}(X)$.

Thus an action of a Lie group G on X induces an action of the Lie algebra $\mathfrak{g} = \text{Lie}G$ on X.

Now let $x \in X$. Then we have a linear map $a_{*x} : \mathfrak{g} \to T_xX$ given by $a_{*x}(z) := a_*(z)(x)$

Theorem 7.12. *(i) The stabilizer G_x is a closed subgroup of G with Lie algebra*

$$\mathfrak{g}_x := \text{Ker}(a_{*x}).$$

(ii) The map $G/G_x \to X$ given by $g \mapsto gx$ is an immersion. So the orbit Gx is an immersed submanifold of X, and

$$T_x(Gx) \cong \text{Im}(a_{*x}) \cong \mathfrak{g}/\mathfrak{g}_x.$$

Part (i) of Theorem 7.12 is the promised weaker version of Theorem 2.16 sufficient for our purposes. Also, part (ii) implies Proposition 3.12.

Proof. (i) It is clear that G_x is closed in G, but we need to show it is a Lie subgroup and compute its Lie algebra.[4] It suffices to show that for some neighborhood U of 1 in G, $U \cap G_x$ is a (closed) submanifold of U such that $T_1(U \cap G_x) = \mathfrak{g}_x$.

Note that $\mathfrak{g}_x \subset \mathfrak{g}$ is a Lie subalgebra, since the commutator of vector fields vanishing at x also vanishes at x (by the formula for commutator in local coordinates). Also, for any $z \in \mathfrak{g}_x$, $\exp(tz)x$ is a solution of the ODE $\gamma'(t) = a_{*\gamma(t)}(z)$ with initial condition $\gamma(0) = x$, and $\gamma(t) = x$ is such a solution, so by uniqueness of ODE solutions $\exp(tz)x = x$, this $\exp(tz) \in G_x$.

Now choose a complement $\mathfrak{u}$ of $\mathfrak{g}_x$ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{u}$. Then $a_{*x} : \mathfrak{u} \to T_xX$ is injective. By the implicit function theorem, the map $\mathfrak{u} \to X$ given by $u \mapsto \exp(u)x$ is injective for small u, so $\exp(u) \in G_x$ for small $u \in \mathfrak{u}$ if and only if $u = 0$.

But in a small neighborhood U of 1 in G, any element g can be uniquely written as $g = \exp(u)\exp(z)$, where $u \in \mathfrak{u}$ and $z \in \mathfrak{g}_x$. So we see that $g \in G_x$ iff $u = 0$, i.e., $\log(g) \in \mathfrak{g}_x$. This shows that $U \cap G_x$ coincides with $U \cap \exp(\mathfrak{g}_x)$, as desired.

(ii) The same proof shows that we have an isomorphism $T_1(G/G_x) \cong \mathfrak{g}/\mathfrak{g}_x = \mathfrak{u}$, so the injectivity of $a_{*x} : \mathfrak{u} \to T_xX$ implies that the map $G/G_x \to X$ given by $g \mapsto gx$ is an immersion, as claimed. □

Corollary 7.13. *(Proposition 3.7) Let $\phi : G \to K$ be a morphism of Lie groups and $\phi_* : \text{Lie}G \to \text{Lie}K$ be the corresponding morphism of Lie algebras. Then $H := \text{Ker}(\phi)$ is a closed normal Lie subgroup with Lie algebra $\mathfrak{h} := \text{Ker}(\phi_*)$, and the map $\overline{\phi} : G/H \to K$ is an immersion.*

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[4]Although we claimed in Theorem 2.16 that a closed subgroup of a Lie group is always a Lie subgroup, we did not prove it, so we need to prove it in this case.

Moreover, if $\mathrm{Im}\overline{\phi}$ *is a submanifold of* K *then it is a closed Lie subgroup, and we have an isomorphism of Lie groups* $\overline{\phi} : G/H \cong \mathrm{Im}\overline{\phi}$.

Proof. Apply Theorem 7.12 to the action of G on $X = K$ via $g \circ k = \phi(g)k$, and take $x = 1$. □

Corollary 7.14. *Let* V *be a finite dimensional representation of a Lie group* G*, and* $v \in V$*. Then the stabilizer* G_v *is a closed Lie subgroup of* G *with Lie algebra* $\mathfrak{g}_v := \{z \in \mathfrak{g} : zv = 0\}$.

Example 7.15. Let A be a finite dimensional algebra (not necessarily associative, e.g. a Lie algebra). Then the group $G = \mathrm{Aut}(A) \subset GL(A)$ is a closed Lie subgroup with Lie algebra $\mathrm{Der}(A) \subset \mathrm{End}(A)$ of derivations of A, i.e., linear maps $d : A \to A$ such that

$$d(ab) = d(a)b + ad(b).$$

Indeed, consider the action of of $GL(A)$ on $\mathrm{Hom}(A \otimes A, A)$. Then $G = G_\mu$ where $\mu : A \otimes A \to A$ is the multiplication map. Also, if $g_t(ab) = g_t(a)g_t(b)$ and $d = \frac{d}{dt}|_{t=0}g_t$ then

$$d(ab) = d(a)b + ad(b)$$

and conversely, if d is a derivation then $g_t := \exp(td)$ is an automorphism.

7.5. **The center of G and $\mathfrak{g}$.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $Z = Z(G)$ the center of G, i.e. the set of $z \in G$ such that $zg = gz$ for all $g \in G$. Also let $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ be the set of $x \in \mathfrak{g}$ such that $[x, y] = 0$ for all $y \in \mathfrak{g}$; it is called the **center** of $\mathfrak{g}$.

Proposition 7.16. *If* G *is connected then* Z *is a closed (normal, commutative) Lie subgroup of* G *with Lie algebra* $\mathfrak{z}$.

Proof. Since G is connected, an element $g \in G$ belongs to Z iff it commutes with $\exp(tu)$ for all $u \in \mathfrak{g}$, i.e., iff $\mathrm{Ad}_g(u) = u$. Thus $Z = \mathrm{Ker}(\mathrm{Ad})$, where $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the adjoint representation. Thus by Proposition 3.7, $Z \subset G$ is a closed Lie subgroup with Lie algebra $\mathrm{Ker}(\mathrm{ad})$, as claimed. □

Remark 7.17. In general (when G is not necessarily connected), it can be shown that G/G_0 acts on $\mathfrak{z}$, and Z is a closed Lie subgroup with Lie algebra $\mathfrak{g}^{G/G^\circ}$ (the subspace of invariant vectors).

Definition 7.18. The group $G/Z(G)$ is called the **adjoint group** of G.

It is clear that $G/Z(G)$ is naturally isomorphic to the image of the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$, which motivates the terminology.

7.6. **The Baker-Campbell-Hausdorff formula.** We have defined the commutator $[x, y]$ on $\mathfrak{g} = \mathrm{Lie}G$ as the quadratic part of $\mu(x, y) = \log(\exp(x)\exp(y))$. So one may wonder if taking higher order terms in the Taylor explansion of $\mu(x, y)$,

$$\mu(x, y) = \sum_{n=1}^{\infty} \frac{\mu_n(x, y)}{n!}$$

would yield new operations on $\mathfrak{g}$. It turns out, however, that all these operations express via the commutator. Namely, we have

Theorem 7.19. *For each $n \geq 1$, $\mu_n(x, y)$ may be written as a $\mathbb{Q}$-Lie polynomial of x, y (i.e., a $\mathbb{Q}$-linear combination of Lie monomials, obtained by taking successive commutators of x, y), which is universal (i.e., independent on G).*

Example 7.20.

$$\mu_3(x, y) = \tfrac{1}{2}([x, [x, y]] + [y, [y, x]]).$$

Thus

$$\mu(x, y) = x + y + \tfrac{1}{2}[x, y] + \tfrac{1}{12}([x, [x, y]] + [y, [y, x]]) + ...$$

We will discuss the proof of this theorem and the exact formula for $\mu_n(x, y)$ (called the Baker-Campbell-Hausdorff formula) later.

8. Lecture 8

8.1. **The fundamental theorems of Lie theory.**

Theorem 8.1. *(First fundamental theorem of Lie theory) For a Lie group G, there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}G$, given by $\mathfrak{h} = \mathrm{Lie}H$.*

Theorem 8.2. *(Second fundamental theorem of Lie theory) If G and K are Lie groups with G simply connected then the map*

$$\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$$

given by $\phi \mapsto \phi_*$ is a bijection.

Theorem 8.3. *(Third fundamental theorem of Lie theory) Any finite dimensional Lie algebra is the Lie algebra of a Lie group.*

These theorems hold for real as well as complex Lie groups. Thus we have

Corollary 8.4. *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the assignment $G \mapsto \mathrm{Lie}G$ is an equivalence between the category of simply connected $\mathbb{K}$-Lie groups and the category of finite dimensional $\mathbb{K}$-Lie algebras. Moreover, any connected Lie group K has the form G/Γ where G is simply connected and $\Gamma \subset G$ is a discrete central subgroup.*

Proof. The second fundamental theorem says that the functor $G \mapsto \mathrm{Lie}G$ is fully faithful, and the third fundamental theorem says that it is essentially surjective. Thus it is an equivalence of categories. The last statement was a homework problem (G is the universal covering of K). □

We will discuss proofs later.

8.2. **Complexification of real Lie groups and real forms of complex Lie groups.** Let $\mathfrak{g}$ be a real Lie algebra. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex Lie algebra. We say that $\mathfrak{g}_{\mathbb{C}}$ is the **complexification** of $\mathfrak{g}$, and $\mathfrak{g}$ is a **real form** of $\mathfrak{g}_{\mathbb{C}}$.

Note that two non-isomorphic real Lie algebras can have isomorphic complexifications; in other words, the same complex Lie algebra can have different real forms. For example,

$$\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}_n(\mathbb{R})_{\mathbb{C}} \cong \mathfrak{gl}_n(\mathbb{C})$$

while for $n > 1$,

$$\mathfrak{u}(n) \ncong \mathfrak{gl}_n(\mathbb{R}),$$

since in the first algebra any element x with nilpotent $\mathrm{ad}x$ must be zero, while in the second one it must not.

Definition 8.5. Let G be a connected complex Lie group and $K \subset G$ a real Lie subgroup such that $\mathrm{Lie}K$ is a real form of $\mathrm{Lie}G$ (i.e., the natural map $\mathrm{Lie}K \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{Lie}G$ is an isomorphism). Then K is called a **real form** of G.

Example 8.6. Both $U(n)$ and $GL_n(\mathbb{R})$ are real forms of $GL_n(\mathbb{C})$ (but $GL_n(\mathbb{R})$ is not connected).

Let K be a simply connected real Lie group, and $\mathfrak{g} = \mathrm{Lie}K \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathrm{Lie}K$. By the third fundamental theorem of Lie theory, there exists a unique simply connected complex Lie group G such that $\mathrm{Lie}G = \mathfrak{g}$. It is called the **complexification** of K. We have a natural homomoprhism $K \to G$ coming from the homomorphism $\mathrm{Lie}K \to$ LieG (coming from the second fundamental theorem), but it need not be injective; e.g. it is not for K being the universal covering of $SL_2(\mathbb{R})$.

8.3. **Distributions and the Frobenius theorem.** The proofs of the fundamental theorems of Lie theory are based on the notion of an integrable distribution in differential geometry, and the Frobenius theorem about such distributions.

Definition 8.7. A k-dimensional distribution on a manifold X is a rank k subbundle $D \subset TX$.

This means that in every tangent space T_xX we fix a k-dimensional subspace D_x which varies regularly with x. In other words, on some neighborhood $U \subset X$ of every $x \in X$, D is spanned by vector fields $\mathbf{v}_1, ..., \mathbf{v}_k$ linearly independent at every point of U.

Definition 8.8. A distribution D is **integrable** if every point $x \in X$ has a neighborhood U and local coordinates $x_1, ..., x_n$ on U such that D is defined at every point of U by the equations $dx_{k+1} = ... = dx_n = 0$, i.e., it is spanned by vector fields $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, ..., k$.

This is equivalent to saying that every point x of X is contained in an **integral submanifold** for D, i.e., an immersed submanifold $S = S_x \subset X$ such that for any $y \in S$ the tangent space $T_yS \subset T_yX$ coincides with D_y. Namely, S_x is the set of all points of $y \in X$ that can be connected to x by a smooth curve $\gamma : [0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$ and $\gamma'(t) \in D_{\gamma(t)}$ for all $t \in [0,1]$ (show it!)

For this reason an integrable distribution is also called a **foliation** and the integral submanifolds S_x are called the **sheets of the foliation**. The manifold X falls into a disjoint union of such sheets. But note that the sheets need not be closed (i.e., think of the irrational torus winding!)

Example 8.9. A 1-dimensional distribution is the same thing as a **direction field.** It is always integrable, as follows from the existence theorem for ODE, and its integral submanifolds are called **integral curves**. They are geometric realizations of solutions of the corresponding ODE.

However, for $k \geq 2$ a distribution is not always integrable.

Theorem 8.10. *(The Frobenius theorem) A distribution D is integrable if and only if for every two vector fields* $\mathbf{v}, \mathbf{w}$ *contained in D, their commutator* $[\mathbf{v}, \mathbf{w}]$ *is also contained in D.*

Example 8.11. Let $\mathbf{v} = \partial_x$, $\mathbf{w} = x\partial_y + \partial_z$ in $\mathbb{R}^3$, and D be the 2-dimensional distribution spanned by $\mathbf{v}, \mathbf{w}$. Then $[\mathbf{v}, \mathbf{w}] = \partial_y \notin D$. So D is not integrable.

Proof. If D is integrable, a vector field is contained in D iff it is tangent to integral submanifolds of D. But the commutator of two vector fields tangent to a submanifold is itself tangent to this submanifold. This establishes the "only if" part.

It remains to prove the "if " part. The proof is by induction in the rank k of D. The base case $k = 0$ is trivial, so it suffices to establish the inductive step. The question is local, so we may work in a neighborhood U of $P \in X$. Suppose that $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathrm{Vect}(U)$ is a basis of D in U (on every tangent space). By local existence and uniqueness of solutions of ODE, in some local coordinates $x_1, ..., x_n = z$, the vector field $\mathbf{v}_k$ equals ∂_z. By subtracting from $\mathbf{v}_i, i < k$ a multiple of $\mathbf{v}_k$ we can make sure that $\mathbf{v}_i$ has no ∂_z-component. Then

$$\mathbf{v}_i = \sum_{j=1}^{n-1} a_{ij}(x_1, ..., x_{n-1}, z)\partial_{x_j}.$$

Thus we have

$$[\partial_z, \mathbf{v}_i] = \sum_{m=1}^{n-1} b_{im}(x_1, ..., x_{n-1}, z)\mathbf{v}_m.$$

Thus, setting $A = (a_{ij}(x_1, ..., x_{n-1}, z)), B = (b_{im}(x_1, ..., x_{n-1}, z))$, we have

$$\partial_z A = BA.$$

Let A_0 be the solution of this linear ODE with $A_0(x_1, ..., x_{n-1}, 0) = 1$. Then $A = A_0 C$, where $C = C(x_1, ..., x_{n-1})$ does not depend on z. So we have a basis of D given by

$$\mathbf{w}_i = \sum_j c_{ij}(x_1, ..., x_{n-1})\partial_{x_j}.$$

Thus there is a neighborhood U of P which can be represented as $U = (-a, a) \times U'$, where $\dim U' = n - 1$, so that $D = \mathbb{R} \oplus D'$, where D' is a $k - 1$-dimensional distribution on U'. It is clear that for any two vector fields $\mathbf{v}, \mathbf{w}$ on U' contained in D', so is $[\mathbf{v}, \mathbf{w}]$. Hence D' is integrable by the induction assumption. Therefore, so is D, justifying the inductive step.

□

8.4. **Proofs of the fundamental theorems.** We start by proving the first fundamental theorem.

Proposition 8.12. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then there is a unique (not necessarily closed) connected Lie subgroup H with Lie algebra $\mathfrak{h}$.*

Proof. The proof of existence of H is based on the Frobenius theorem.

Define the distribution D on G by left-translating $\mathfrak{h} \subset \mathfrak{g} = T_1G$, i.e., $D_g = L_g\mathfrak{h}$. So any vector field contained in D is of the form

$$\mathbf{v} = \sum f_i \mathbf{L}_{a_i},$$

where a_i is a basis of $\mathfrak{h}$ and f_i are regular functions. Now if

$$\mathbf{w} = \sum g_j \mathbf{L}_{a_j}$$

is another such field then

$$[\mathbf{v}, \mathbf{w}] = \sum_{i,j} (f_i \mathbf{L}_{a_i}(g_j)\mathbf{L}_{a_j} - g_j L_{a_j}(f_i)\mathbf{L}_{a_i} + f_i g_j [\mathbf{L}_{a_i}, \mathbf{L}_{a_j}]).$$

But $[a_i, a_j] = \sum_k c_{ij}^k a_k$, so

$$[\mathbf{L}_{a_i}, \mathbf{L}_{a_j}] = \sum_k c_{ij}^k \mathbf{L}_{a_k}.$$

Thus if $\mathbf{v}, \mathbf{w}$ are contained in D then so is $[\mathbf{v}, \mathbf{w}]$. Hence by the Frobenius theorem, D is integrable.

Now consider the integral (embedded) submanifold H of D going through $1 \in G$. We claim that H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$. Indeed, it suffices to show that H is a subgroup of G. But this is clear since H is the collection of elements of G of the form

$$g = \exp(a_1) ... \exp(a_m),$$

where $a_i \in \mathfrak{h}$.

Moreover, H is unique since it has to be generated by the image of the exponential map $\exp : \mathfrak{h} \to G$. □

Now we prove the second fundamental theorem.

Proposition 8.13. *The map* $\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$ *is a bijection if* G *is simply connected.*

Proof. We know this map is injective so we only need to establish surjectivity. For any morphism $\psi : \mathrm{Lie}G \to \mathrm{Lie}K$, consider the morphism

$$\theta = (\mathrm{id}, \psi) : \mathrm{Lie}G \to \mathrm{Lie}(G \times K) = \mathrm{Lie}G \oplus \mathrm{Lie}K$$

The previous proposition implies that there is a connected Lie subgroup $H \subset G \times K$ whose Lie algebra is $\mathrm{Im}\theta$. We have a projection homomorphisms $p_1 : H \to G$, $p_2 : H \to K$, and $(p_1)_* = \mathrm{id}$, so p_1 is a covering. Since G is simply connected, p_1 is an isomorphism, so we can define $\phi := p_2 \circ p_1^{-1} : G \to K$, and it is easy to see that $\psi = \phi_*$. □

Finally, let us discuss a proof of the third fundamental theorem, stating that any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the Lie algebra of a Lie group. We will deduce it from the following purely algebraic *Ado's theorem*.

Theorem 8.14. *(Ado) Any finite dimensional Lie algebra over $\mathbb{K}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.*

Ado's theorem in fact holds over any ground field, but it is rather nontrivial and we won't prove it here. But it immediately implies Theorem 8.3. Indeed, using Theorem 8.1, Ado's theorem it implies the following stronger statement:

Theorem 8.15. *Any finite dimensional $\mathbb{K}$-Lie algebra is the Lie algebra of a Lie subgroup of $\mathfrak{gl}_n(\mathbb{K})$ for some n.*

This implies

Corollary 8.16. *Any simply connected Lie group is the universal covering of a linear Lie group, i.e., of a Lie subgroup of $GL_n(\mathbb{K})$.*

However, it is not true that any Lie group is isomorphic to a Lie subgroup of $GL_n(\mathbb{K})$.

Exercise 8.17. Let G be the universal cover of $SL_2(\mathbb{R})$. Show that G is not isomorphic to a Lie subgroup of $GL_n(\mathbb{R})$ for any n and that moreover, the only quotients of G that are such subgroups are $SL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$.

Hint: use the representation theory of the Lie algebra $\mathfrak{sl}_2$.

9. Lecture 9

9.1. **Representations.**

We have previously defined (finite dimensional) representations of Lie groups and (iso)morphisms between them. We can do the same for Lie algebras:

Definition 9.1. A representation of a Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ (or a $\mathfrak{g}$-module) is a vector space V over $\mathbf{k}$ equipped with a homomorphism of Lie algebras $\rho = \rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$. A morphism of representations $A : V \to W$ (also called an intertwining operator) is a linear map which commutes with the $\mathfrak{g}$-action: $A\rho_V(b) = \rho_W(b)A$ for $b \in \mathfrak{g}$. Such A is an isomorphism if it is an isomorphism of vector spaces.

The fundamental theorems of Lie theory imply:

Corollary 9.2. *Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}G$.*

(i) Any finite dimensional representation $\rho : G \to GL(V)$ gives rise to a Lie algebra representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$, and any morphism of G-representations is also a morphism of $\mathfrak{g}$-representations.

(ii) If G is connected then any morphism of $\mathfrak{g}$-representations is a morphism of G-representations.

(iii) If G is simply connected then the assignment $\rho \mapsto \rho_*$ is an equivalence of categories $\operatorname{Rep} G \to \operatorname{Rep} \mathfrak{g}$ between the corresponding categories of finite dimensional representations. In particular, any finite dimensional representation of $\mathfrak{g}$ can be uniquely exponentiated to G.

Exercise 9.3. Let $\mathfrak{g}$ be a complex Lie algebra. Show that $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \mathfrak{g}$. Deduce that if G is a simply connected complex Lie group then $\operatorname{Rep}_{\mathbb{R}} G \cong \operatorname{Rep}(\mathfrak{g} \oplus \mathfrak{g})$, where $\operatorname{Rep}_{\mathbb{R}} G$ is the category of finite dimensional representations of G regarded as a real Lie group.

Example 9.4. 1. The trivial representation: $\rho(g) = 1, g \in G$, $\rho_*(x) = 0$, $x \in \mathfrak{g}$.

2. The adjoint representation: $\rho(g) = \operatorname{Ad}_g, \rho(x) = \operatorname{ad} x$.

As usual, a **subrepresentation** of a representation V is a subspace $W \subset V$ invariant under a G-action (resp. $\mathfrak{g}$-action). In this case the quotient space V/W has a natural structure of a representation. The notion of direct sum of representations is defined in an obvious way. Also we have the notion of a dual representation:

$$\rho_{V^*}(g) = \rho_V(g^{-1})^*, g \in G;\ \rho_{V^*}(x) = -\rho_V(x)^*, x \in \mathfrak{g},$$

and tensor product:

$$\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g),\ \rho_{V\otimes W}(x) = \rho_V(x) \otimes 1_W + 1_V \otimes \rho_W(x).$$

Thus we have the notion of symmetric and exterior powers $S^m V, \wedge^m V$ of a representation V. Also for representations V, W, $\operatorname{Hom}(V, W)$ is a representation via

$$g \circ A = \rho_W(g) A \rho_V(g^{-1}),\ x \circ A = \rho_W(x) A - A \rho_V(x),$$

so if V is finite dimensional then $\operatorname{Hom}(V, W) \cong V^* \otimes W$. Finally, for every representation V we have the notion of invariants:

$$V^G = \{v \in V; gv = v\ \forall g \in G\},\ V^{\mathfrak{g}} = \{v \in V : xv = 0\ \forall x \in \mathfrak{g}\}.$$

Thus $V^G \subset V^{\mathfrak{g}}$ and $V^G = V^{\mathfrak{g}}$ for connected G (in general, $V^G = (V^{\mathfrak{g}})^{G/G^\circ}$). Also $\operatorname{Hom}(V, W)^G \cong \operatorname{Hom}_G(V, W)$ and $\operatorname{Hom}(V, W)^{\mathfrak{g}} = \operatorname{Hom}_{\mathfrak{g}}(V, W)$, the spaces of intertwining operators. Note that in all cases the formula for Lie algebras is determined by the formula for groups by the requirement that these definitions should be consistent with the assignment $\rho \mapsto \rho_*$.

Definition 9.5. A representation $V \neq 0$ of G or $\mathfrak{g}$ is **irreducible** if any subrepresentation $W \subset V$ is either 0 or V and is **indecomposable** if for any decomposition $V \cong V_1 \oplus V_2$, we have $V_1 = 0$ or $V_2 = 0$.

It is clear that any finite dimensional representation is isomorphic to a direct sum of indecomposable representations (in fact, uniquely so by the *Krull-Schmidt theorem*). However, not any V is a direct sum of irreducible representations, e.g. $\rho : \mathbb{C} \to GL_2(\mathbb{C})$, $\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

Definition 9.6. A representation V is called **completely reducible** if it is isomorphic to a direct sum of irreducible representations.

Some of the main problems of representation theory are:
1) Classify irreducible representations;
2) If V is a completely reducible representation, find its decomposition into irreducibles.
3) For which G are all representations completely reducible?

Example 9.7. Let V be a finite dimensional $\mathbb{C}$-representation of $\mathfrak{g}$ or G and $A : V \to V$ be a homomorphism of representations (e.g., defined by a central element). Then we have a decomposition of representations $V = \oplus_\lambda V(\lambda)$, where $V(\lambda)$ is the generalized eigenspace of V with eigenvalue λ.

Example 9.8. Let V be the vector representation of $GL(V)$. Then V is irreducible, and more generally so are $S^m V, \wedge^n V$ (show it!). Thus $V \otimes V$ is completely reducible: $V \otimes V \cong S^2 V \oplus \wedge^2 V$.

9.2. **Schur's lemma.**

Lemma 9.9. *(Schur's lemma) Let V, W be irreducible finite dimensional complex representations of G or $\mathfrak{g}$. Then $\mathrm{Hom}_{G,\mathfrak{g}}(V, W) = 0$ if V, W are not isomorphic, and $\mathrm{Hom}_{G,\mathfrak{g}}(V, V) = \mathbb{C}$.*

Proof. Let $A : V \to W$ be a nonzero morphism of representations. Then $\mathrm{Im}(A) \subset W$ is a nonzero subrepresentation, hence $\mathrm{Im}(A) = W$. Also $\mathrm{Ker}(A) \subset V$ is a proper subrepresentation, so $\mathrm{Ker}(A) = 0$. Thus A is an isomorphism, i.e., we may assume that $W = V$. In this case, let λ be an eigenvalue of A. Then $A - \lambda I : V \to V$ is a morphism of representations but not an isomorphism, hence it must be zero, so $A = \lambda I$. □

Corollary 9.10. *The center of $G, \mathfrak{g}$ acts on an irreducible representation by a scalar. In particular, if G or $\mathfrak{g}$ is abelian then its every irreducible representation is 1-dimensional.*

Example 9.11. Irreducible representations of $\mathbb{R}$ are χ_s given by $\chi_s(a) = \exp(sa)$, $s \in \mathbb{C}$. Irreducible representations of $\mathbb{R}^\times = \mathbb{R}_{>0} \times \mathbb{Z}/2$ are $\chi_{s,+}(a) = |a|^s$, $\chi_{s,-}(a) = |a|^s \mathrm{sign}(a)$. Irreducible representations of S^1 are $\chi_n(z) = z^n$, $n \in \mathbb{Z}$. Irreducible representations of the real group $\mathbb{C}^* = \mathbb{R}_{>0} \times S^1$ are $\chi_{s,n}(z) = |z|^s (z/|z|)^n$, $s \in \mathbb{C}$, $n \in \mathbb{Z}$.

Corollary 9.12. *Let $V = \oplus_i n_i V_i, W = \oplus_i m_i V_i$ be completely reducible complex representations of G or $\mathfrak{g}$. Then $\mathrm{Hom}_{G,\mathfrak{g}}(V, W) = \oplus_i \mathrm{Mat}_{m_i,n_i}(\mathbb{C})$.*

9.3. **Unitary representations.** A finite dimensional representation V of G is said to be **unitary** if it is equipped with a positive definite Hermitian inner product $B(,)$ invariant under G, i.e., $B(gv, gw) = B(v, w)$ for $v, w \in V$, $g \in G$.

Proposition 9.13. *Any unitary representation is completely reducible.*

Proof. If $W \subset V$ is a subrepresentation of a unitary representation V then let $W^\perp$ be its orthogonal complement under B. Then $W^\perp$ is also a subrepresentation since B is invariant, and $V = W \oplus W^\perp$ since B is positive definite.

Now we can prove complete reducibility of V by induction in $\dim V$. The base $\dim V = 1$ is clear so let us make the inductive step. Pick an irreducible $W \subset V$. Then $V = W \oplus W^\perp$, and $W^\perp$ is a unitary representation of dimension smaller than $\dim V$, so is completely reducible by the induction assumption. □

Proposition 9.14. *Any finite dimensional complex representation V of a finite group G is unitary. Moreover, if V is irreducible, the unitary structure is unique up to a positive factor.*

Proof. Let B be any positive definite inner product on V. Let

$$\widehat{B}(v, w) := \sum_{g \in G} B(gv, gw).$$

Then $\widehat{B}$ is positive definite and invariant, so V is unitary.

If V is irreducible and B_1, B_2 are two unitary structures on V then $B_1(v, w) = B_2(Av, w)$ for some homomorphism $A : V \to V$. Thus by Schur's lemma $A = \lambda I$, and $\lambda > 0$ since B_1, B_2 are positive definite. □

Corollary 9.15. *Every finite dimensional complex representation of a finite group G is completely reducible.*

9.4. **Representations of $\mathfrak{sl}_2$.** The Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ has basis $e = E_{12}$, $f = E_{21}$, $h = E_{11} - E_{22}$ with commutator

$$[e, f] = h, \ [h, e] = 2e, \ [h, f] = -2f.$$

Since 2-by-2 matrices act on variables x, y, they also act on the space $V = \mathbb{C}[x, y]$ of polynomials in x, y. Namely, this action is given by the formula

$$e = x\partial_y,\ f = y\partial_x, h = x\partial_x - y\partial_y.$$

This infinite-dimensional representation has the form $V = \oplus_{n\ge 0} V_n$, where V_n is the space of polynomials of degree n. The space V_n is invariant under e, f, h, so it is an $n+1$-dimensional representation of $\mathfrak{sl}_2$. It has basis $v_{pq} = x^p y^q$, such that

$$hv_{pq} = (p-q)v_{pq},\ ev_{pq} = qv_{p+1,q-1},\ fv_{pq} = pv_{p-1,q+1}.$$

Thus V_0 is the trivial representation, and V_1 is the tautological representation by 2-by-2 matrices. Also it is easy to see that V_2 is the adjoint representation.

Theorem 9.16. *(i) V_n is irreducible.*

(ii) If $V \neq 0$ is a finite dimensional representation of $\mathfrak{sl}_2$ then $e|_V$ and $f|_V$ are nilpotent, so $U := \mathrm{Ker}(e) \neq 0$. Moreover, h preserves U and acts diagonalizably on it, with nonnegative integer eigenvalues.

(iii) Any irreducible finite dimensional representation V of $\mathfrak{sl}_2$ is isomoprhic to V_n for some n.

(iv) Any finite dimensional representation V of $\mathfrak{sl}_2$ is completely reducible.

Proof. (i) Let $W \subset V_n$ be a nonzero subrepresentation. Since it is h-invariant, it must be spanned by vectors $v_{p,n-p}$ for a nonempty subset $S \subset [0, n]$. Since W is e-invariant and f-invariant, if $m \in S$ then so are $m+1, m-1$ (if they are in $[0, n]$). Thus $S = [0, n]$ and $W = V_n$.

(ii) Let V be a finite dimensional representation of $\mathfrak{sl}_2$. We can write V as a direct sum of generalized eigenspaces of h: $V = \oplus_\lambda V(\lambda)$. Since $he = e(h+2)$, $hf = f(h-2)$, we have $e : V(\lambda) \to V(\lambda+2)$, $f : V(\lambda) \to V(\lambda-2)$. Thus $e|_V$, $f|_V$ are nilpotent, so $U \neq 0$.

If $v \in U$ then $e(hv) = (h-2)ev = 0$, so $hv \in U$, i.e., U is h-invariant.

Given $v \in U$, consider the vector $v_m := e^m f^m v$. We have

(1)
$$ef^m v = fef^{m-1}v + hf^{m-1}v = fef^{m-1}v + f^{m-1}(h-2(m-1))v = ...$$
$$= f^{m-1}m(h-m+1)v.$$

Thus

$$v_m = e^{m-1}f^{m-1}m(h-m+1)v.$$

Hence

$$v_m = m!h(h-1)...(h-m+1)v.$$

But for large enough m, $v_m = 0$, since f is nilpotent, so

$$h(h-1)...(h-m+1)v = 0.$$

Thus h acts diagonalizably on U with nonnegative integer eigenvalues.

(iii) Let $v \in U$ be an eigenvector of h, i.e., $hv = \lambda v$. Let $w_m = f^m v$. Then

$$fw_m = w_{m+1}, hw_m = (\lambda - 2m)w_m.$$

Also, it follows from (1) that

$$ew_m = m(\lambda - m + 1)w_{m-1}.$$

Thus if $w_m \neq 0$ and $\lambda \neq m$ then $w_{m+1} \neq 0$. Also the nonzero vectors w_m are linearly independent since they have different eigenvalues of h. Thus $\lambda = n$ must be a nonnegative integer (as also follows from (ii)), and $w_{n+1} = 0$. So V, being irreducible, has a basis w_m, $m = 0, ..., n$. Now it is easy to see that $V \cong V_n$, via the assignment

$$w_m \mapsto n(n-1)...(n-m+1)x^m y^{n-m}.$$

(iv) Consider the **Casimir operator**

$$C = 2fe + \frac{h^2}{2} + h.$$

It is easy to check that $[C, e] = [C, f] = [C, h] = 0$, so $C : V \to V$ is a homomorphism. Thus $C|_{V_n} = \frac{n(n+2)}{2}$ (it is a scalar by Schur's lemma, and acts with such eigenvalue on $v_{0n} \in V_n$); note that these are different for different n. For a general representation, we have $V = \oplus_c V_c$, the direct sum of generalized eigenspaces of C.

Assume V is indecomposable. We prove that V is completely reducible by induction on $\dim V$. The base $\dim V = 1$ is obvious, so we only need to justify the inductive step. Since V is indecomposable, C has a single eigenvalue c on V. Let $W \subset V$ be an irreducible subrepresentation. By (iii), $W = V_n$ for some n, so $c = \frac{n(n+2)}{2}$. Moreover, V/W has smaller dimension, so is completely reducible, i.e., $V/W \cong (m-1)V_n$ for some m (again by (iii)). Thus $\dim V = m(n+1)$ and $V(n)$ has dimension m, with h acting on it by n, by (ii). Let $u_1, ..., u_m$ be a basis of $V(n)$. As in (iii), we define subrepresentations $W_i \subset V$ generated by u_i. It is easy to see that the natural morphism $W_1 \oplus ... \oplus W_m \to V$ is injective. Hence it is an isomorphism, i.e., V is completely reducible. □

Corollary 9.17. *(The Jacobson-Morozov lemma for $GL(V)$) Let V be a finite dimensional complex vector space and $N : V \to V$ be a nilpotent operator. Then there is a unique up to isomorphism action of $\mathfrak{sl}_2$ on V for which e acts by N.*

Proof. This follows from the above theorem and the Jordan normal form theorem for operators on V. □

For a representation V define its **character** by

$$\chi_V(z) = \mathrm{Tr}_V(z^h) = \sum_m \dim V(m) z^m.$$

Thus

$$\chi_{V_n}(z) = z^n + z^{n-2} + ... + z^{-n} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}.$$

Since these functions are linearly independent, we see that a finite dimensional representation of $\mathfrak{sl}_2$ is determined by its character. Also it is easy to see that

$$\chi_{V\oplus W} = \chi_V + \chi_W, \chi_{V\otimes W} = \chi_V \chi_W.$$

Theorem 9.18. *(The Clebsch-Gordan rule) We have*

$$V_m \otimes V_n \cong \oplus_{i=0}^{\min(m,n)} V_{|m-n|+2i}.$$

Proof. It suffices to note that we have the corresponding character identity:

$$\chi_{V_m}\chi_{V_n} = \sum_{i=0}^{\min(m,n)} \chi_{V_{|m-n|+2i}}.$$

□

Exercise 9.19. Show that V_n has an invariant nondegenerate inner product (i.e., such that $(av, w) + (v, aw) = 0$ for $a \in \mathfrak{sl}_2$, $v, w \in V_n$) which is symmetric for even n and skew-symmetric for odd n. In particular, $V_n^* \cong V_n$.

10. Lecture 10

10.1. **The universal enveloping algebra.** Let V be a vector space over a field $\mathbf{k}$. Recall that the **tensor algebra** of V is the $\mathbb{Z}$-graded associative algebra $TV := \oplus_{n\geq 0} V^{\otimes n}$ (with $\deg(V^{\otimes n}) = n$), with multiplication given by $a \cdot b = a \otimes b$ for $a \in V^{\otimes m}$ and $b \in V^{\otimes n}$. If $\{x_i\}$ is a basis of V then TV is just the free algebra with generators x_i (i.e., without any relations). Its basis consists of various words in the letters x_i.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{k}$.

Definition 10.1. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is the quotient of $T\mathfrak{g}$ by the ideal I generated by the elements $xy - yx - [x, y]$, $x, y \in \mathfrak{g}$.

Recall that any associative algebra A is also a Lie algebra with operation $[a, b] := ab - ba$. The following proposition follows immediately from the definition of $U(\mathfrak{g})$.

Proposition 10.2. *(i) Let $J \subset T\mathfrak{g}$ be an ideal, and $\rho : \mathfrak{g} \to T\mathfrak{g}/J$ the natural linear map. Then ρ is a homomorphism of Lie algebras if and only if $J \supset I$, so that $T\mathfrak{g}/J$ is a quotient of $T\mathfrak{g}/I = U(\mathfrak{g})$. In other words, $U(\mathfrak{g})$ is the largest quotient of $T\mathfrak{g}$ for which ρ is a homomorphism of Lie algebras.*

(ii) Let A be any associative algebra over $\mathbf{k}$. Then the map

$$\mathrm{Hom}_{\mathrm{associative}}(U(\mathfrak{g}), A) \to \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A)$$

given by $\phi \mapsto \phi \circ \rho$ is a bijection.

Part (ii) of this proposition implies that any Lie algebra map $\psi : \mathfrak{g} \to A$ can be uniquely extended to an associative algebra map $\phi : U(\mathfrak{g}) \to A$ so that $\psi = \phi \circ \rho$. This is the universal property of $U(\mathfrak{g})$ which justifies the term "universal enveloping algebra".

In particular, it follows that a representation of $\mathfrak{g}$ on a vector space V is the same thing as an algebra map $U(\mathfrak{g}) \to \mathrm{End}(V)$ (i.e., a representation of $U(\mathfrak{g})$ on V). Thus, to understand the representation theory of $\mathfrak{g}$, it is helpful to understand the structure of $U(\mathfrak{g})$; for example, every central element $C \in U(\mathfrak{g})$ gives rise to a morphism of representations $V \to V$.

In terms of the basis $\{x_i\}$ of $\mathfrak{g}$, we can write the bracket as

$$[x_i, x_j] = \sum_k c_{ij}^k x_k,$$

where $c_{ij}^k \in \mathbf{k}$ are the **structure constants**. Then the algebra $U(\mathfrak{g})$ can be described as the quotient of the free algebra $\mathbf{k}\langle\{x_i\}\rangle$ by the relations

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k.$$

Example 10.3. 1. If $\mathfrak{g}$ is abelian (i.e., $c_{ij}^k = 0$) then $U(\mathfrak{g}) = S\mathfrak{g} = \mathbf{k}[\{x_i\}]$ is the symmetric algebra of $\mathfrak{g}$, $S\mathfrak{g} = \oplus_{n \geq 0} S^n \mathfrak{g}$, which in terms of the basis is the polynomial algebra in x_i.

2. $U(\mathfrak{sl}_2(\mathbf{k}))$ is generated by e, f, h with defining relations

$$he - eh = 2e, \; hf - fh = -2f, \; ef - fe = h.$$

Recall that $\mathfrak{g}$ acts on $T\mathfrak{g}$ by derivations via the adjoint action. Moreover, using the Jacobi identity, we have

$$\mathrm{ad} z(xy - yx - [x, y]) = [z, x]y + x[z, y] - [z, y]x - y[z, x] - [z, [x, y]] =$$

$$([z,x]y - y[z,x] - [[z,x],y]) + (x[z,y] - [z,y]x - [x,[z,y]]).$$

Thus $\mathrm{ad}z(I) \subset I$, and hence the action of $\mathfrak{g}$ on $T\mathfrak{g}$ descends to its action on $U(\mathfrak{g})$ by derivations (also called the adjoint action). It is easy to see that these derivations are in fact inner:

$$\mathrm{ad}z(a) = za - az$$

for $a \in U(\mathfrak{g})$. Indeed, it suffices to note that this holds for $a \in \mathfrak{g}$ by the definition of $U(\mathfrak{g})$.

Thus we get

Proposition 10.4. *The center $Z(U(\mathfrak{g}))$ of $U(\mathfrak{g})$ coincides with the subalgebra of invariants $U(\mathfrak{g})^{\mathrm{ad}\mathfrak{g}}$.*

Example 10.5. The Casimir operator $C = 2fe + \frac{h^2}{2} + h$ which we used to study representations of $\mathfrak{g} = \mathfrak{sl}_2$ is in fact a central element of $U(\mathfrak{g})$.

10.2. **Graded and filtered algebras.** Recall that a $\mathbb{Z}_{\geq 0}$-**filtered** algebra is an algebra A equipped with a filtration

$$0 = F_{-1}A \subset F_0A \subset F_1A \subset ... \subset F_nA \subset ...$$

such that $1 \in F_0A$, $\cup_{n\geq 0} F_nA = A$ and $F_iA \cdot F_jA \subset F_{i+j}A$. In particular, if A is generated by $\{x_i\}$ then a filtration on A can be obtained by declaring x_i to be of degree 1; i.e., $F_nA = (F_1A)^n$ is the span of all words in x_i of degree $\leq n$.

If $A = \oplus_{i\geq 0} A_i$ is $\mathbb{Z}_{\geq 0}$-graded then we can define a filtration on A by setting $F_nA := \oplus_{i=0}^n A_i$; however, not any filtered algebra is obtained in this way, and having a filtration is a weaker condition than having a grading. Still, if A is a filtered algebra, we can define its **associated graded algebra** $\mathrm{gr}(A) := \oplus_{n\geq 0}\mathrm{gr}_n(A)$, where $\mathrm{gr}_n(A) := F_nA/F_{n-1}A$. The multiplication in $\mathrm{gr}(A)$ is given by the "leading terms" of multiplication in A: for $a \in F_iA$, $b \in F_jA$, pick their representatives $\widetilde{a} \in F_iA$, $\widetilde{b} \in F_jA$ and let ab be the projection of $\widetilde{a}\widetilde{b}$ to $\mathrm{gr}_{i+j}(A)$.

Proposition 10.6. *If $\mathrm{gr}(A)$ is a domain (has no zero divisors) then so is A.*

Proof. Exercise. □

11. Lecture 11

11.1. **The Poincaré-Birkhoff-Witt theorem.** Now let us come back to the case of the universal enveloping algebra. The grading on $T\mathfrak{g}$ does not descend to $U(\mathfrak{g})$, in general, since the relation $xy - yx = [x,y]$ is not homogeneous: the right hand side has degree 2 while the left hand side has degree 1. So we can only define a filtration on $U(\mathfrak{g})$ by setting

$\deg(\mathfrak{g}) = 1$. Thus $F_nU(\mathfrak{g})$ is the image of $\oplus_{i=0}^n \mathfrak{g}^{\otimes i}$. Note that since $xy - yx = [x,y]$ for $x \in \mathfrak{g}$, we have $[F_iU(\mathfrak{g}), F_jU(\mathfrak{g})] \subset F_{i+j-1}U(\mathfrak{g})$. Thus, $\mathrm{gr}(U(\mathfrak{g}))$ is commutative; in other words, we have a surjective algebra homomorpism $\phi : S\mathfrak{g} \to \mathrm{gr}U(\mathfrak{g})$.

Theorem 11.1. *(Poincaré-Birkhoff-Witt theorem) The homomorphism ϕ is an isomorphism.*

We will prove this theorem later. Now let us discuss its reformulation in terms of a basis and corollaries.

Given a basis $\{x_i\}$ of $\mathfrak{g}$, fix an ordering on this basis and consider ordered monomials $\prod_i x_i^{n_i}$, where the product is ordered according to the ordering of the basis. The statement that ϕ is surjective is equivalent to saying that ordered monomials span $U(\mathfrak{g})$. This is also easy to see directly: any monomial can be ordered using the commutation relations at the cost of an error of lower degree, so we can write any monomial as a linear combination of ordered ones. Thus the PBW theorem can be formulated as follows:

Theorem 11.2. *The ordered monomials are linearly independent, hence form a basis of $U(\mathfrak{g})$.*

Corollary 11.3. *The map $\rho : \mathfrak{g} \to U(\mathfrak{g})$ is injective.*

Remark 11.4. Let $\mathfrak{g}$ be a vector space equipped with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then one can define the algebra $U(\mathfrak{g})$ as above. However, if the map $\rho : \mathfrak{g} \to U(\mathfrak{g})$ is injective then we clearly must have $[x,x] = 0$ and Jacobi identity for $x \in \mathfrak{g}$, i.e., $\mathfrak{g}$ has to be a Lie algebra. Thus the PBW theorem fails without the axioms of a Lie algebra.

Corollary 11.5. *Let $\mathfrak{g}_i$ be Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ as a vector space. Then the multiplication map in any order $\otimes_i U(\mathfrak{g}_i) \to U(\mathfrak{g})$ in any order is a linear isomorphism.*

Proof. The corollary follows immediately from the PBW theorem by choosing a basis of each $\mathfrak{g}_i$. □

Remark 11.6. This corollary applies to the case of infinitely many $\mathfrak{g}_i$ if we understand the tensor product accordingly: the span of tensor products of elements of $U(\mathfrak{g}_i)$ where almost all of these elements are equal to 1. Note that if $\dim \mathfrak{g}_i = 1$, this recovers the PBW theorem itself, so this is in fact a generalization of the PBW theorem.

Let $\mathrm{char}(\mathbf{k}) = 0$. Define the **symmetrization map** $\sigma : S\mathfrak{g} \to U(\mathfrak{g})$ given by

$$\sigma(y_1 \otimes ... \otimes y_n) = \frac{1}{n!} \sum_{s \in S_n} y_{s(1)} ... y_{s(n)}.$$

It is easy to see that this map commutes with the adjoint action of $\mathfrak{g}$.

Corollary 11.7. *σ is an isomorphism.*

Proof. It is easy to see that grσ (the induced map on the associated graded algebra) coincides with ϕ. □

Corollary 11.8. *The map σ defines an isomorphism between the center $Z(U(\mathfrak{g}))$ and the algebra $(S\mathfrak{g})^{\mathrm{ad}\mathfrak{g}}$.*

In the case when $\mathfrak{g} = \mathrm{Lie}G$ for a connected Lie group G, we thus obtain an isomorphism of the center of $U(\mathfrak{g})$ with $(S\mathfrak{g})^{\mathrm{Ad}G}$.

Example 11.9. Let $\mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{so}_3$. Then $\mathfrak{g}$ has a basis x, y, z with $[x, y] = z$, $[y, z] = x$, $[z, x] = y$, and $G = SO(3)$ acts on these elements by ordinary rotations of the 3-dimensional space. So the only G invariant polynomials of x, y, z are polynomials of $r^2 = x^2 + y^2 + z^2$. Thus we get that $Z(U(\mathfrak{g})) = \mathbb{C}[x^2 + y^2 + z^2]$. In terms of e, f, h, we have

$$x^2 + y^2 + z^2 = -4fe - h^2 - 2h = -2C,$$

where C is the Casimir element.

11.2. **Proof of the PBW theorem.**

Lemma 11.10. *There exists a unique linear map $\phi : T\mathfrak{g} \to S\mathfrak{g}$ such that*

(i) for an **ordered** *monomial $X := x_{i_1}...x_{i_m} \in \mathfrak{g}^{\otimes m}$ one has $\phi(X) = X$;*

(ii) one has $\phi(I) = 0$; in other words, ϕ descends to a linear map $\overline{\phi} : U(\mathfrak{g}) \to S\mathfrak{g}$.

Remark 11.11. The map ϕ is not canonical and depends on the choice of the basis x_i of $\mathfrak{g}$.

Note that the lemma immediately implies the PBW theorem, since the images of ordered monomials under ϕ are linearly independent in $S\mathfrak{g}$, implying that these monomials themselves are linearly independent in $U(\mathfrak{g})$.

Proof. It is clear that ϕ is unique if exists since ordered monomials span $U(\mathfrak{g})$. We will construct ϕ by defining it inductively on $F_nT\mathfrak{g}$ for $n \geq 0$.

Suppose ϕ is already defined on $F_{n-1}T\mathfrak{g}$ and let us extend it to $F_nT\mathfrak{g} = F_{n-1}T\mathfrak{g} \oplus \mathfrak{g}^{\otimes n}$. So we should define ϕ on $\mathfrak{g}^{\otimes n}$. Since ϕ is already defined on ordered monomials X (by $\phi(X) = X$), we need to extend this definition to all monomials.

Namely, let X be an ordered monomial of degree n, and let us define ϕ on monomials of the form $s(X)$ for $s \in S_n$, where

$$s(y_1...y_n) := y_{s(1)}...y_{s(n)}.$$

To this end, fix a decomposition D of s into a product of transpositions of neighbors:

$$s = s_{j_r}...s_{j_1},$$

and define $\phi(s(X))$ by the formula

$$\phi(s(X)) := X + \Phi_D(s, X),$$

where

$$\Phi_D(s, X) := \sum_{m=0}^{r-1} \phi([,]_{j_{m+1}}(s_{j_m}...s_{j_1}(X))),$$

and

$$[,]_j(y_1...y_j y_{j+1}...y_n) := y_1...[y_j, y_{j+1}]...y_n.$$

We need to show that $\phi(s(X))$ is well defined, i.e., $\Phi_D(s, X)$ does not really depend on the choice of D and s but only on $s(X)$. We first show that $\Phi_D(s, X)$ is independent on D.

To this end, recall that the symmetric group S_n is generated by $s_j, 1 \leq j \leq n-1$ with defining relations

$$s_j^2 = 1; \; s_j s_k = s_k s_j, |j-k| \geq 2; \; s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}.$$

Thus any two decompositions of s into a product of transpositions of neighbors can be related by a sequence of applications of these relations somewhere inside the decomposition.

Now, the first relation does not change the outcome by the identity $[x, y] = -[y, x]$.

For the second relation, suppose that $j < k$ and we have two decompositions D_1, D_2 of s given by $s = ps_j s_k q$ and $s = ps_k s_j q$, where q is a product of m transpositions of neighbors. Let $q(X) = YabZcdT$ where $a, b, c, d \in \mathfrak{g}$ stand in positions $j, j+1, k, k+1$. Let $\Phi_1 := \Phi_{D_1}(s, X)$, $\Phi_2 := \Phi_{D_2}(s, X)$. Then the sums defining Φ_1 and Φ_2 differ only in the m-th and $m+1$-th term, so we get

$$\Phi_1 - \Phi_2 =$$

$$\phi(YabZ[c, d]T) + \phi(Y[a, b]ZdcT) - \phi(Y[a, b]ZcdT) - \phi(YbaZ[c, d]T),$$

which equals zero by the induction assumption.

For the third relation, suppose that we have two decompositions D_1, D_2 of s given by $s = ps_j s_{j+1} s_j q$ and $s = ps_{j+1} s_j s_{j+1} q$, where q is a product of k transpositions of neighbors. Let $q(X) = YabcZ$ where $a, b, c \in \mathfrak{g}$ stand in positions $j, j+1, j+2$. Let $\Phi_1 := \Phi_{D_1}(s, X)$,

$\Phi_2 := \Phi_{D_2}(s, X)$. Then the sums defining Φ_1 and Φ_2 differ only in the k-th, $k+1$-th, and $k+2$-th terms, so we get

$$\Phi_1 - \Phi_2 =$$

$$(\phi(Y[a,b]cZ) + \phi(Yb[a,c]Z) + \phi(Y[b,c]aZ)) -$$

$$(\phi(Ya[b,c]Z) + \phi(Y[a,c]bZ) + \phi(Yc[a,b]Z)).$$

So the Jacobi identity

$$[[b,c],a] + [b,[a,c]] + [[a,b],c] = 0$$

combined with property (ii) in degree $n-1$ implies that $\Phi_1 - \Phi_2 = 0$, i.e., $\Phi_1 = \Phi_2$, as claimed. Thus we will denote $\Phi_D(s, X)$ just by $\Phi(s, X)$.

It remains to show that $\Phi(s, X)$ does not depend on the choice of s and only depends on $s(X)$. Let $X = x_{i_1}...x_{i_n}$; then $s(X) = s'(X)$ if and only if $s = s't$, where t is the product of transpositions s_k for which $i_k = i_{k+1}$. Thus, it suffices to show that $\Phi(s, X) = \Phi(ss_k, X)$ for such k. But this follows from the the fact that $[x, x] = 0$.

Now, it follows from the construction of ϕ that for any monomial X of degree n (not necessarily ordered), $\phi(s_j(X)) = \phi(X) + \phi([,]_j(X))$. Thus ϕ satisfies property (ii) in degree n. The lemma is proved. □

12. Lecture 12

12.1. **Ideals and commutant.** Let $\mathfrak{g}$ be a Lie algebra. Recall that an ideal in $\mathfrak{g}$ is a subspace $\mathfrak{h}$ such that $[\mathfrak{g}, \mathfrak{h}] = \mathfrak{h}$. If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal then $\mathfrak{g}/\mathfrak{h}$ has a natural structure of a Lie algebra. Moreover, if $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomoprphism of Lie algebras then $\mathrm{Ker}\phi$ is an ideal in $\mathfrak{g}_1$, $\mathrm{Im}\phi$ is a Lie subalgebra in $\mathfrak{g}_2$, and ϕ induces an isomorphism $\mathfrak{g}_1/\mathrm{Ker}\phi \cong \mathrm{Im}\phi$ (check it!).

Lemma 12.1. *If $I_1, I_2 \subset \mathfrak{g}$ are ideals then so are $I_1 \cap I_2, I_1 + I_2, [I_1, I_2]$.*

Proof. Exercise. □

Definition 12.2. The commutant of $\mathfrak{g}$ is the ideal $[\mathfrak{g}, \mathfrak{g}]$.

Lemma 12.3. *The quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian; moreover, if $I \subset \mathfrak{g}$ is an ideal such that $\mathfrak{g}/I$ is abelian then $I \supset [\mathfrak{g}, \mathfrak{g}]$.*

Proof. Exercise. □

Example 12.4. The commutant of $\mathfrak{gl}_n(\mathbf{k})$ is $\mathfrak{sl}_n(\mathbf{k})$ (check it!).

Exercise 12.5. Prove that if G is a connected Lie group with Lie algebra $\mathfrak{g}$ then $[G, G]$ is a Lie subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$.

12.2. **Solvable Lie algebras.** For a Lie algebra $\mathfrak{g}$ define its **derived series** recursively by the formulas $D^0(\mathfrak{g}) = \mathfrak{g}$, $D^{n+1}(\mathfrak{g}) = [D^n(\mathfrak{g}), D^n(\mathfrak{g})]$. This is a descending sequence of ideals in $\mathfrak{g}$.

Definition 12.6. A Lie algebra $\mathfrak{g}$ is said to be **solvable** if $D^n(\mathfrak{g}) = 0$ for some n.

Proposition 12.7. *The following conditions on $\mathfrak{g}$ are equivalent:*

(i) $\mathfrak{g}$ is solvable;

(ii) There exists a sequence of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset ... \supset \mathfrak{g}_m = 0$ such that $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

Proof. It is clear that (i) implies (ii), since we can take $\mathfrak{g}_i = D^i\mathfrak{g}$. Conversely, by induction we see that $D^i\mathfrak{g} \subset \mathfrak{g}_i$, as desired. □

Proposition 12.8. *(i) Any Lie subalgebra or quotient of a solvable Lie algebra is solvable.*

(ii) If $I \subset \mathfrak{g}$ is an ideal and $I, \mathfrak{g}/I$ are solvable then $\mathfrak{g}$ is solvable.

Proof. Exercise. □

12.3. **Nilpotent Lie algebras.** For a Lie algebra $\mathfrak{g}$ define its **lower central series** recursively by the formulas $D_0(\mathfrak{g}) = \mathfrak{g}$, $D_{n+1}(\mathfrak{g}) = [\mathfrak{g}, D_n(\mathfrak{g})]$. This is a descending sequence of ideals in $\mathfrak{g}$.

Definition 12.9. A Lie algebra $\mathfrak{g}$ is said to be **nilpotent** if $D_n(\mathfrak{g}) = 0$ for some n.

Proposition 12.10. *The following conditions on $\mathfrak{g}$ are equivalent:*

(i) $\mathfrak{g}$ is nilpotent;

(ii) There exists a sequence of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset ... \supset \mathfrak{g}_m = 0$ such that $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$.

Proof. It is clear that (i) implies (ii), since we can take $\mathfrak{g}_i = D_i\mathfrak{g}$. Conversely, by induction we see that $D_i\mathfrak{g} \subset \mathfrak{g}_i$, as desired. □

Remark 12.11. Any nilpotent Lie algebra is solvable since $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ implies $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$, hence $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

Proposition 12.12. *Any Lie subalgebra or quotient of a nilpotent Lie algebra is nilpotent.*

Proof. Exercise. □

Example 12.13. (i) The Lie algebra of upper triangular matrices of size n is solvable, but it is not nilpotent for $n \geq 2$.

(ii) The Lie algebra of strictly upper triangular matrices is nilpotent.

(iii) The Lie algebra of all matrices of size $n \geq 2$ is not solvable.

13. Lecture 13

13.1. **Lie's theorem.**

Theorem 13.1. *(Lie) Let* $\mathbf{k}$ *be an algebraically closed field of characteristic zero, and* $\mathfrak{g}$ *a finite dimensional solvable Lie algebra over* $\mathbf{k}$*. Then any irreducible finite dimensional representation of* $\mathfrak{g}$ *is 1-dimensional.*

Proof. Let V be a finite dimensional representation of $\mathfrak{g}$. It suffices to show that V contains a common eigenvector of $\mathfrak{g}$. The proof is by induction in $\dim \mathfrak{g}$. The base is trivial so let us justify the induction step. Since $\mathfrak{g}$ is solvable, $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$, so fix a subspace $\mathfrak{h} \subset \mathfrak{g}$ of codimension 1 containing $[\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, hence solvable. Thus by the induction assumption, there is a nonzero common eigenvector $v \in V$ for $\mathfrak{h}$, i.e., there is a linear functional $\lambda : \mathfrak{h} \to \mathbf{k}$ such that $av = \lambda(a)v$ for all $a \in \mathfrak{h}$.

Let $x \in \mathfrak{g}$ be an element not belonging to $\mathfrak{h}$ and W be the subspace of V spanned by $v, xv, x^2v, ...$ We claim that W is a subrepresentation of V. Indeed, it suffices to show that W is $\mathfrak{h}$-invariant, i.e., $ax^nv \in W$ for $a \in \mathfrak{h}$. But we have

$$ax^nv = xax^{n-1}v + [a, x]x^{n-1}v. \tag{2}$$

Therefore, it follows by induction in n that ax^nv is a linear combination of $v, xv, ..., x^nv$.

Let n be the smallest integer such that x^nv is a linear combination of x^iv with $i < n$. Then $v_i := x^{i-1}v$ for $i = 1, ..., n$ is a basis of W and $\dim W = n$. As shown above, any element $a \in \mathfrak{h}$ acts this basis by an upper triangular matrix with all diagonal entries equal $\lambda(a)$. Thus $\mathrm{tr}(a|_W) = n\lambda(a)$.

On the other hand, if $a \in [\mathfrak{g}, \mathfrak{g}]$ then $\mathrm{tr}(a|_W) = 0$, thus $n\lambda(a) = 0$ in $\mathbf{k}$. Since $\mathrm{char}(\mathbf{k}) = 0$, this implies that $\lambda(a) = 0$.

Thus it follows from (2) by induction in n that for every $a \in \mathfrak{h}$, we have

$$ax^nv = \lambda(a)x^nv.$$

In particular, $[\mathfrak{g}, \mathfrak{g}]$ acts by zero on W, hence W is a representation of the abelian Lie algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Now the statement follows since every finite dimensional representation of an abelian Lie algebra has a common eigenvector. □

Remark 13.2. Lie's theorem does not hold in characteristic $p > 0$. Indeed, let $\mathfrak{g}$ be the Lie algebra with basis x, y and $[x, y] = y$, and let V be the space with basis $v_0, ..., v_{p-1}$ and action of $\mathfrak{g}$ given by

$$xv_i = iv_i,\ yv_i = v_{i+1},$$

where $i+1$ is taken modulo p. It is easy to see that V is irreducible.

Here is an another formulation of Lie's theorem:

Corollary 13.3. *Every finite dimensional representation V of a finite dimensional solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbf{k}$ of characteristic zero has a basis in which all elements of $\mathfrak{g}$ act by upper triangular matrices. In other words, there is a sequence of subrepresentations $0 = V_0 \subset V_1 \subset ... \subset V_n = V$ such that $\dim(V_{k+1}/V_k) = 1$.*

In the case $\dim \mathfrak{g} = 1$, this recovers the well known theorem in linear algebra that any linear operator on a finite dimensional $\mathbf{k}$-vector space is upper triangular in some basis (which is actually true in any characteristic).

Proof. The proof is by induction in $\dim V$ (where the base is obvious). By Lie's theorem, there is a common eigenvector $v_0 \in V$ for $\mathfrak{g}$. Let $V' := V/\mathbf{k}v_0$. Then by the induction assumption V' has a basis $v_1', ..., v_n'$ in which $\mathfrak{g}$ acts by upper triangular matrices. Let $v_1, ..., v_n$ be any lifts of $v_1', ..., v_n'$ to V. Then $v_0, v_1, ..., v_n$ is a basis of V in which $\mathfrak{g}$ acts by upper triangular matrices. $\square$

Corollary 13.4. *Over an algebraically closed field of characteristic zero, the following hold.*

(i) A solvable finite dimensional Lie algebra $\mathfrak{g}$ admits a sequence of ideals $0 = I_0 \subset I_1 \subset ... \subset I_n = \mathfrak{g}$ such that $\dim(I_{k+1}/I_k) = 1$.

(ii) A finite dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof. (i) Apply Corollary 13.3 to the adjoint representation of $\mathfrak{g}$.

(ii) If $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent then it is solvable and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, so $\mathfrak{g}$ is solvable. Conversely, if $\mathfrak{g}$ is solvable then by Corollary 13.3 elements of $[\mathfrak{g}, \mathfrak{g}]$ act on $\mathfrak{g}$, hence on $[\mathfrak{g}, \mathfrak{g}]$ by strictly upper triangular matrices, which implies the statement. $\square$

Example 13.5. Let $\mathfrak{g}, V$ be as in Remark 13.2 and $\mathfrak{h} = \mathfrak{g} \ltimes V$ be the semidirect product, i.e. $\mathfrak{h} = \mathfrak{g} \oplus V$ as a space with

$$[(g_1, v_1), (g_2, v_2)] = ([g_1, g_2], g_1 v_2 - g_2 v_1).$$

Then $\mathfrak{h}$ is a counterexample to Corollary 13.4 both (i) and (ii) in characteristic $p > 0$.

13.2. **Engel's theorem.**

Theorem 13.6. *Let $V \neq 0$ be a finite dimensional vector space over any field $\mathbf{k}$, and $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra consisting of nilpotent operators. Then there exists a nonzero vector $v \in V$ such that $\mathfrak{g}v = 0$.*

Proof. The proof is by induction on the dimension of $\mathfrak{g}$ (note the structure of the proof is similar to that for Lie's theorem). The base case $\mathfrak{g} = 0$ is trivial and we assume the dimension of $\mathfrak{g}$ is positive.

First we find an ideal $\mathfrak{h}$ of codimension one in $\mathfrak{g}$. Let $\mathfrak{h}$ be a maximal (proper) subalgebra of $\mathfrak{g}$, which exists by finite-dimensionality of $\mathfrak{g}$. We claim that $\mathfrak{h} \subset \mathfrak{g}$ is an ideal and has codimension one.

Indeed, for each $x \in \mathfrak{h}$, $\mathrm{ad}x$ induces a linear map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$, and this induced map is nilpotent (in fact, $\mathrm{ad}x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent). Thus, by the inductive hypothesis, there exists a nonzero element $\bar{a}$ in $\mathfrak{g}/\mathfrak{h}$ such that $\mathrm{ad}x \cdot \bar{a} = 0$ for each $x \in \mathfrak{h}$. Let a be a lift of $\bar{a}$ to $\mathfrak{g}$. Then $[x, a] \in \mathfrak{h}$ for all $x \in \mathfrak{g}$. Let $\mathfrak{h}'$ be the span of $\mathfrak{h}$ and a. Then $\mathfrak{h}' \subset \mathfrak{g}$ is a Lie subalgebra in which $\mathfrak{h}$ is an ideal. Hence, by maximality, $\mathfrak{h} = \mathfrak{g}$. This proves the claim.

Now let $W = V^{\mathfrak{h}} \subset V$. By the inductive hypothesis, $W \neq 0$. Also for $w \in W$ and $x \in \mathfrak{h}$, we have $xaw = axw + [x, a]w = 0$, so $aw \in W$. Thus W is a $\mathfrak{g}$-subrepresentation of V.

Now take $w \neq 0$ in W. Let k be the smallest positive integer such that $a^k w = 0$; it exists since a acts nilpotently on V. Let $v = a^{k-1} w \in W$. Then $v \neq 0$ but $\mathfrak{h}v = av = 0$, so $\mathfrak{g}v = 0$, as desired. □

Definition 13.7. An element $x \in \mathfrak{g}$ is said to be **nilpotent** if the operator $\mathrm{ad}x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent.

Corollary 13.8. *(Engel's theorem) A finite dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if every element $x \in \mathfrak{g}$ is nilpotent.*

Proof. The "only if" direction is easy. To prove the "if" direction, note that by Theorem 13.6, in some basis v_i of $\mathfrak{g}$ all elements $\mathrm{ad}x$ act by strictly upper triangular matrices. Let I_m be the subspace of $\mathfrak{g}$ spanned by the vectors $v_1, ..., v_m$. Then $I_m \subset I_{m+1}$ and $[\mathfrak{g}, I_{m+1}] \subset I_m$, hence $\mathfrak{g}$ is nilpotent. □

14. Lecture 14

14.1. Semisimple and reductive Lie algebras, the radical.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbf{k}$.

Proposition 14.1. $\mathfrak{g}$ *contains the largest solvable ideal which contains all solvable ideals of* $\mathfrak{g}$.

Definition 14.2. This ideal is called **the radical** of $\mathfrak{g}$ and denoted $\mathrm{rad}(\mathfrak{g})$.

Proof. Let I, J be solvable ideals of $\mathfrak{g}$. Then $I + J \subset \mathfrak{g}$ is an ideal, and $(I + J)/I = J/(I \cap J)$ is solvable, so $I + J$ is solvable. Thus the sum of finitely many solvable ideals is solvable. Hence the sum of all solvable ideals in $\mathfrak{g}$ is a solvable ideal, as desired. $\square$

Definition 14.3. (i) $\mathfrak{g}$ is called **semisimple** if $\mathrm{rad}(\mathfrak{g}) = 0$, i.e., $\mathfrak{g}$ does not contain nonzero solvable ideals.

(ii) A non-abelian $\mathfrak{g}$ is called **simple** if it contains no ideals other than $0, \mathfrak{g}$. In other words, a non-abelian $\mathfrak{g}$ is simple if its adjoint representation is irreducible (=simple).

Thus if $\mathfrak{g}$ is both solvable and semisimple then $\mathfrak{g} = 0$.

Proposition 14.4. *(i) We have* $\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h}) = \mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h})$*. In particular, the direct sum of semisimple Lie algebras is semisimple.*

(ii) A simple Lie algebra is semisimple. Thus a direct sum of simple Lie algebras is semisimple.

Proof. (i) The images of $\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h})$ in $\mathfrak{g}$ and in $\mathfrak{h}$ are solvable, hence contained in $\mathrm{rad}(\mathfrak{g})$, respectively $\mathrm{rad}(\mathfrak{h})$. Thus $\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h}) \subset \mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h})$. But $\mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h})$ is a solvable ideal in $\mathfrak{g} \oplus \mathfrak{h}$, so $\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h}) = \mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h})$.

(ii) The only nonzero ideal in $\mathfrak{g}$ is $\mathfrak{g}$, and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ since $\mathfrak{g}$ is not abelian. Hence $\mathfrak{g}$ is not solvable. Thus $\mathfrak{g}$ is semisimple. $\square$

Example 14.5. The Lie algebra $\mathfrak{sl}_2(\mathbf{k})$ is simple if $\mathrm{char}(\mathbf{k}) \neq 2$. Likewise, $\mathfrak{so}_3(\mathbf{k})$ is simple.

Proposition 14.6. *The Lie algebra* $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ *is semisimple. Thus any* $\mathfrak{g}$ *can be included in an exact sequence*

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}} \to 0,$$

where $\mathrm{rad}(\mathfrak{g})$ *is a solvable ideal and* $\mathfrak{g}_{\mathrm{ss}}$ *is semisimple. Moreover, if* $\mathfrak{h} \subset \mathfrak{g}$ *is a solvable ideal such that* $\mathfrak{g}/\mathfrak{h}$ *is semisimple then* $\mathfrak{h} = \mathrm{rad}(\mathfrak{g})$*.*

Proof. Let $I \subset \mathfrak{g}_{\mathrm{ss}}$ be a solvable ideal, and let $\widetilde{I}$ be its preimage in $\mathfrak{g}$. Then $\widetilde{I}$ is a solvable ideal in $\mathfrak{g}$. Thus $\widetilde{I} = \mathrm{rad}(\mathfrak{g})$ and $I = 0$. $\square$

In fact, in characteristic zero there is a stronger statement, which says that this extension splits:

Theorem 14.7. *(Levi decomposition) If* $\mathrm{char}(\mathbf{k}) = \mathbf{0}$ *then we have* $\mathfrak{g} \cong \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}$*, where* $\mathfrak{g}_{\mathrm{ss}} \subset \mathfrak{g}$ *is a semisimple subalgebra (but not necessarily an ideal); i.e.,* $\mathfrak{g}$ *is isomorphic to the semidirect product* $\mathfrak{g}_{\mathrm{ss}} \ltimes \mathrm{rad}(\mathfrak{g})$*. In other words, the projection* $p : \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}}$ *admits an (in general, non-unique) splitting* $q : \mathfrak{g}_{\mathrm{ss}} \to \mathfrak{g}$*, i.e., a Lie algebra map such that* $p \circ q = \mathrm{Id}$*.*

This theorem will be proved later.

Example 14.8. Let G be the group of motions of the Euclidean space $\mathbb{R}^3$ (generated by rotations and translations). Then $G = SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$, so $\mathfrak{g} = \mathrm{Lie}G = \mathfrak{so}_3(\mathbb{R}) \ltimes \mathbb{R}^3$, hence $\mathrm{rad}(\mathfrak{g}) = \mathbb{R}^3$ (abelian Lie algebra) and $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{so}_3(\mathbb{R})$.

Proposition 14.9. *Let* $\mathrm{char}(\mathbf{k}) = 0$*,* $\mathbf{k}$ *algebraically closed, and* V *be an irreducible representation of* $\mathfrak{g}$*. Then* $\mathrm{rad}(\mathfrak{g})$ *acts on* V *by scalars, and* $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ *by zero.*

Proof. By Lie's theorem, there is a nonzero vector $v \in V$ and $\lambda \in \mathrm{rad}(\mathfrak{g})^*$ such that $av = \lambda(a)v$ for $a \in \mathrm{rad}(\mathfrak{g})$. Let $x \in \mathfrak{g}$ and $\mathfrak{g}_x \subset \mathfrak{g}$ be the Lie subalgebra spanned by $\mathrm{rad}(\mathfrak{g})$ and x. Similarly to the proof of Lie's theorem, by induction in n one shows that for $a \in \mathrm{rad}(\mathfrak{g})$

$$ax^n v = \lambda(a)x^n v + \sum_{i=1}^{n} c_i x^{n-i} v, \ c_i \in \mathbf{k}.$$

Let W be the span of $x^n v$ for $n \geq 0$. It follows that W is a $\mathfrak{g}_x$-subrepresentation of V on which $a \in \mathrm{rad}(\mathfrak{g})$ has the only eigenvalue $\lambda(a)$. Thus for $x \in \mathfrak{g}$, $a \in \mathrm{rad}(\mathfrak{g})$, we have $\lambda([x, a]) = 0$ (as $\mathrm{tr}([x, a]|_W) = \lambda(a) \dim W = 0$). So

$$axv = xav + [a, x]v = \lambda(a)xv + \lambda([a, x])v = \lambda(a)xv.$$

Thus the λ-eigenspace V_λ of V is a $\mathfrak{g}$-subrepresentation of V, which implies that $V_\lambda = V$ since V is irreducible. ☐

Definition 14.10. $\mathfrak{g}$ is called **reductive** if $\mathrm{rad}(\mathfrak{g})$ coincides with the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$.

In other words, $\mathfrak{g}$ is reductive if $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = 0$.

The Levi decomposition theorem implies that a reductive Lie algebra in characteristic zero is a direct sum of a semisimple Lie algebra and an abelian Lie algebra (its center).

14.2. **Invariant inner products.** Let B be a bilinear form on a Lie algebra $\mathfrak{g}$. Recall that B is invariant if $B([x, y], z) = B(x, [y, z])$ for any $x, y, z \in \mathfrak{g}$.

Example 14.11. If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional representation of $\mathfrak{g}$ then the form

$$B_V(x, y) := \mathrm{Tr}(\rho(x)\rho(y))$$

is an invariant symmetric bilinear form on $\mathfrak{g}$. Indeed, the symmetry is obvious and

$$B_V([x, y], z) = B_V(x, [y, z]) = \mathrm{Tr}|_V(\rho(x)\rho(y)\rho(z) - \rho(x)\rho(z)\rho(y)).$$

Proposition 14.12. *If B is a symmetric invariant bilinear form on $\mathfrak{g}$ and $I \subset \mathfrak{g}$ is an ideal then the orthogonal complement $I^\perp \subset \mathfrak{g}$ is also an ideal. In particular, $\mathfrak{g}^\perp = \mathrm{Ker}(B)$ is an ideal in $\mathfrak{g}$.*

Proof. Exercise. □

Proposition 14.13. *If B_V is nondegenerate for some V then $\mathfrak{g}$ is reductive.*

Proof. Let $V_1, ..., V_n$ be the simple composition factors of V; i.e., V has a filtration by subrepresentations such that $F^{i+1}V/F^iV = V_i$, $F^0V = 0$ and $F^nV = V$. Then $B_V(x, y) = \sum_i B_{V_i}(x, y)$. Now, if $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ then $x|_{V_i} = 0$, so $B_{V_i}(x, y) = 0$ for all $y \in \mathfrak{g}$, hence $B_V(x, y) = 0$. □

Example 14.14. It is clear that if $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{k})$ and $V = \mathbf{k}^n$ then the form B_V is nondegenerate, as $B_V(E_{ij}, E_{kl}) = \delta_{il}\delta_{jk}$. Thus $\mathfrak{g}$ is reductive. Also if n is not divisible by the characteristic of $\mathbf{k}$ then $\mathfrak{sl}_n(\mathbf{k})$ is semisimple, since it is orthogonal to scalars under B_V (hence reductive), and has trivial center. In fact, it is easy to show that in this case $\mathfrak{sl}_n(\mathbf{k})$ is a simple Lie algebra (another way to see that it is semisimple).

In fact, we have the following proposition.

Proposition 14.15. *All classical Lie algebras over $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ are reductive.*

Proof. Let $\mathfrak{g}$ be a classical Lie algebra and V its standard matrix representation. It is easy to check that the form B_V on $\mathfrak{g}$ is nondegenerate, which implies that $\mathfrak{g}$ is reductive. □

For example, the Lie algebras $\mathfrak{so}_n(\mathbb{K})$, $\mathfrak{sp}_{2n}(\mathbb{K})$, $\mathfrak{su}(p, q)$ have trivial center and therefore are semisimple.

14.3. **The Killing form and the Cartan criterion.**

Definition 14.16. The Killing form of a Lie algebra $\mathfrak{g}$ is the form $B_\mathfrak{g}(x, y) = \mathrm{tr}(\mathrm{ad}x\mathrm{ad}y)$.

The Killing form is denoted by $K_\mathfrak{g}(x, y)$ or shortly by $K(x, y)$.

Theorem 14.17. *(Cartan criterion of solvability) A Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero is solvable if and only if $[\mathfrak{g}, \mathfrak{g}] \subset \mathrm{Ker}(K)$.*

Theorem 14.18. *(Cartan criterion of semisimplicity) A Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero is semisimple if and only if its Killing form is nondegenerate.*

14.4. **Jordan decomposition.** To prove the Cartan criteria, we will use the Jordan decomposition of a square matrix. Let us recall it.

Proposition 14.19. *A square matrix $A \in \mathfrak{gl}_N(\mathbf{k})$ over a field $\mathbf{k}$ of characteristic zero can be uniquely written as $A_s + A_n$, where $A_s \in \mathfrak{gl}_N(\mathbf{k})$ is semisimple (i.e. diagonalizes over the algebraic closure of $\mathbf{k}$) and $A_n \in \mathfrak{gl}_N(\mathbf{k})$ is nilpotent in such a way that $A_sA_n = A_nA_s$. Moreover, $A_s = P(A)$ for some $P \in \mathbf{k}[x]$.*

Proof. By the Chinese remainder theorem, there exists a polynomial $P \in \overline{\mathbf{k}}[x]$ such that for every eigenvalue λ of A we have $P(x) = \lambda$ modulo $(x-\lambda)^N$, i.e.,

$$P(x) - \lambda = (x-\lambda)^N Q_\lambda(x)$$

for some polynomial Q_λ. Then on the generalized eigenspace $V(\lambda)$ for A, we have

$$P(A) - \lambda = (A-\lambda)^N Q_\lambda(A) = 0,$$

so $A_s := P(A)$ is semisimple and $A_n = A - P(A)$ is nilpotent, with $A_nA_s = A_sA_n$. If $A = A_s' + A_n'$ is another such decomposition then A_s', A_n' commute with A, hence with A_s and A_n. Also we have

$$A_s - A_s' = A_n' - A_n.$$

Thus this matrix is both semisimple and nilpotent, so it is zero. Finally, since A_s, A_n are unique, they are invariant under the Galois group of $\overline{\mathbf{k}}$ over $\mathbf{k}$ and therefore have entries in $\mathbf{k}$. □

Remark 14.20. 1. If $\mathbf{k}$ is algebraically closed, then A admits a basis in which it is upper triangular, and A_s is the diagonal part while A_n is the off-diagonal part of A.

2. Proposition 14.19 holds with the same proof in characteristic p if the field $\mathbf{k}$ is perfect, i.e., the Frobenius map $x \to x^p$ is surjective on $\mathbf{k}$. However, if $\mathbf{k}$ is not perfect, the proof fails: the fact that A_s, A_n are Galois invariant does not imply that their entries are in $\mathbf{k}$. Also the statement fails: if $\mathbf{k} = \mathbb{F}_p(t)$ and $Ae_i = e_{i+1}$ for $i = 1, .., p-1$ while $Ae_p = te_1$ then A has only one eigenvalue $t^{1/p}$, so $A_s = t^{1/p} \cdot \mathrm{Id}$, i.e., does not have entries in $\mathbf{k}$.

14.5. **Proofs of the Cartan solvability criterion.** It is clear that $\mathfrak{g}$ is solvable if and only if so is $\mathfrak{g} \otimes_{\mathbf{k}} \overline{\mathbf{k}}$, so we may assume that $\mathbf{k}$ is algebraically closed.

For the "only if" part, note that by Lie's theorem, $\mathfrak{g}$ has a basis in which the operators $\mathrm{ad}x$, $x \in \mathfrak{g}$, are upper triangular. Then $[\mathfrak{g},\mathfrak{g}]$ acts in this basis by strictly upper triangular matrices, so $K(x,y) = 0$ for $x \in [\mathfrak{g},\mathfrak{g}]$ and $y \in \mathfrak{g}$.

To prove the "if" part, let us prove the following lemma.

Lemma 14.21. *Let* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *be a Lie subalgebra such that for any* $x \in [\mathfrak{g}, \mathfrak{g}]$ *and* $y \in \mathfrak{g}$ *we have* $\mathrm{tr}(xy) = 0$*. Then* $\mathfrak{g}$ *is solvable.*

Proof. Let $x \in [\mathfrak{g}, \mathfrak{g}]$. Let $\lambda_i, i = 1, ..., m$, be the distinct eigenvalues of x. Let $E \subset \mathbf{k}$ be a $\mathbb{Q}$-span of λ_i. Let $b : E \to \mathbb{Q}$ be a linear functional. There exists an interpolation polynomial $Q \in \mathbf{k}[t]$ such that $Q(\lambda_i - \lambda_j) = b(\lambda_i - \lambda_j) = b(\lambda_i) - b(\lambda_j)$ for all i, j.

By Proposition 14.19, we can write x as $x = x_s + x_n$. Then the operator $\mathrm{ad}x_s$ is diagonalizable with eigenvalues $\lambda_i - \lambda_j$. So

$$Q(\mathrm{ad}x_s) = \mathrm{ad}b,$$

where $b : V \to V$ is the operator acting by $b(\lambda_j)$ on the generalized λ_j-eigenspace of x.

Also we have

$$\mathrm{ad}x = \mathrm{ad}x_s + \mathrm{ad}x_n$$

a sum of commuting semisimple and nilpotent operators. Thus

$$\mathrm{ad}x_s = (\mathrm{ad}x)_s = P(\mathrm{ad}x),$$

and $P(0) = 0$ since 0 is an eigenvalue of $\mathrm{ad}x$. Thus

$$\mathrm{ad}b = R(\mathrm{ad}x),$$

where $R(t) = Q(P(t))$ and $R(0) = 0$.

Let $x = \sum_j [y_j, z_j]$, $y_j, z_j \in \mathfrak{g}$, and d_j be the dimension of the generalized λ_j-eigenspace of x. Then

$$\sum_j d_j b(\lambda_j)\lambda_j = \mathrm{tr}(bx) =$$

$$\mathrm{tr}(\sum_j b[y_j, z_j]) = \mathrm{tr}(\sum_j [b, y_j]z_j) = \mathrm{tr}(\sum_j R(\mathrm{ad}x)(y_j)z_j).$$

Since $R(0) = 0$, we have $R(\mathrm{ad}x)(y_j) \in [\mathfrak{g}, \mathfrak{g}]$, so by assumption we get

$$\sum_j d_j b(\lambda_j)\lambda_j = 0.$$

Applying b, we get $\sum_j d_j b(\lambda_j)^2 = 0$. Thus $b(\lambda_j) = 0$ for all j. Hence $b = 0$, so $E = 0$.

Thus, the only eigenvalue of x is 0, i.e., x is nilpotent. But then by Engel's theorem, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Thus $\mathfrak{g}$ is solvable. Thus proves the lemma. ☐

Now the "if" part of the Cartan solvability criterion follows easily by applying Lemma 14.21 to $V = \mathfrak{g}$ and replacing $\mathfrak{g}$ by the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$.

14.6. **Proof of the Cartan semisimplicity criterion.** Assume that $\mathfrak{g}$ is semisimple, and let $I = \mathrm{Ker}(K_{\mathfrak{g}})$, and ideal in $\mathfrak{g}$. Then $K_I = (K_{\mathfrak{g}})|_I = 0$. Thus by Cartan's criterion I is solvable. Hence $I = 0$.

Conversely, suppose $K_{\mathfrak{g}}$ is nondegenerate. Then $\mathfrak{g}$ is reductive. Moreover, the center of $\mathfrak{g}$ is contained in the kernel of $K_{\mathfrak{g}}$, so it must be trivial. Thus $\mathfrak{g}$ is semisimple.

15. Lecture 15

15.1. **Properties of semisimple Lie algebras.**

Proposition 15.1. *Let* $\mathrm{char}\mathbf{k} = 0$ *and* $\mathfrak{g}$ *be a finite dimensional Lie algebra over* $\mathbf{k}$. *Then* $\mathfrak{g}$ *is semisimple iff* $\mathfrak{g} \otimes_{\mathbf{k}} \overline{\mathbf{k}}$ *is semisimple.*

Proof. Immediately follows from Cartan's criterion of semisimplicity. Here is another proof (of the nontrivial direction): if $\mathfrak{g}$ is semisimple and I is a nonzero solvable ideal in $\mathfrak{g} \otimes_{\mathbf{k}} \overline{\mathbf{k}}$ then it has a finite Galois orbit $I_1, ..., I_n$ and $I_1 + ... + I_n$ is a Galois invariant solvable ideal, so it comes from a solvable ideal in $\mathfrak{g}$. □

Remark 15.2. This theorem fails if we replace the word "semisimple" by "simple": e.g., if $\mathfrak{g}$ is a simple complex Lie algebra regarded as a real Lie algebra then $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \mathfrak{g}$ is semisimple but not simple.

Theorem 15.3. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* $I \subset \mathfrak{g}$ *an ideal. Then there is an ideal* $J \subset \mathfrak{g}$ *such that* $\mathfrak{g} = I \oplus J$.

Proof. Let $I^\perp$ be the orthogonal complement with respect to the Killing form, an ideal in $\mathfrak{g}$. Consider the intersection $I \cap I^\perp$. It is an ideal in $\mathfrak{g}$ with the zero Killing form. Thus, by the Cartan criterion, it is solvable. By definition of a semisimple Lie algebra, this means that $I \cap I^\perp = 0$, so we may take $J = I^\perp$. □

We will see below that J is in fact unique and must equal $I^\perp$.

Corollary 15.4. *A Lie algebra* $\mathfrak{g}$ *is semisimple iff it is a direct sum of simple Lie algebras.*

Proof. We have already shown that a direct sum of simple Lie algebras is semisimple. The opposite direction easily follows by induction from the previous theorem. □

Corollary 15.5. *If* $\mathfrak{g}$ *is a semisimple Lie algebra, then* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Proof. For a simple Lie algebra it is clear because $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$ which cannot be zero (otherwise, $\mathfrak{g}$ would be abelian). So the result follows since a semisimple Lie algebra is a direct sum of simple ones. □

Proposition 15.6. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ be a semisimple Lie algebra, with $\mathfrak{g}_i$ being simple. Then any ideal I in $\mathfrak{g}$ is of the form $I = \oplus_{i\in S}\mathfrak{g}_i$ for some subset $S \subset \{1, ..., k\}$.*

Proof. The proof goes by induction in k. Let $p_k : \mathfrak{g} \to \mathfrak{g}_k$ be the projection. Consider $p_k(I) \subset \mathfrak{g}_k$. Since $\mathfrak{g}_k$ is simple, either $p_k(I) = 0$, in which case $I \subset \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_{k-1}$ and we can use the induction assumption, or $p_k(I) = \mathfrak{g}_k$. Then $[\mathfrak{g}_k, I] = [\mathfrak{g}_k, p_k(I)] = \mathfrak{g}_k$. Since I is an ideal, $I \supset \mathfrak{g}_k$, so $I = I' \oplus \mathfrak{g}_k$ for some subspace $I' \subset \mathfrak{g}_1 \oplus \quad \oplus \mathfrak{g}_{k-1}$. It is immediate that then I' is an ideal in $\mathfrak{g}_1 \oplus \quad \oplus \mathfrak{g}_{k-1}$ and the result again follows from the induction assumption. □

Corollary 15.7. *Any ideal in a semisimple Lie algebra is semisimple. Also, any quotient of a semisimple Lie algebra is semisimple.*

Let $\mathrm{Der}\mathfrak{g}$ be the Lie algebra of derivations of a Lie algebra $\mathfrak{g}$. We have a homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}\mathfrak{g}$ whose kernel is the center $\mathfrak{z}(\mathfrak{g})$. Thus if $\mathfrak{g}$ has trivial center (e.g., is semisimple) then the map ad is injective and identifies $\mathfrak{g}$ with a Lie subalgebra of $\mathrm{Der}\mathfrak{g}$. Moreover, or $d \in \mathrm{Der}\mathfrak{g}$ and $x \in \mathfrak{g}$, we have

$$[d, \mathrm{ad}x](y) = d[x, y] - [x, dy] = [dx, y] = \mathrm{ad}(dx)(y).$$

Thus $\mathfrak{g} \subset \mathrm{Der}\mathfrak{g}$ is an ideal.

Proposition 15.8. *If $\mathfrak{g}$ is semisimple then $\mathfrak{g} = \mathrm{Der}\mathfrak{g}$.*

Proof. Consider the invariant symmetric bilinear form

$$K(a, b) = \mathrm{tr}|_{\mathfrak{g}}(ab)$$

on $\mathrm{Der}\mathfrak{g}$. This is an extension of the Killing form of $\mathfrak{g}$ to $\mathrm{Der}\mathfrak{g}$, so its restriction to $\mathfrak{g}$ is nondegenerate. Let $I = \mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ in $\mathrm{Der}\mathfrak{g}$ under K. It follows that I is an ideal, $I\cap\mathfrak{g} = 0$, and $I \oplus \mathfrak{g} = \mathrm{Der}\mathfrak{g}$. Since both I and $\mathfrak{g}$ are ideals, we have $[\mathfrak{g}, I] = 0$. Thus for $d \in I$ and $x \in \mathfrak{g}$, $[d, \mathrm{ad}x] = \mathrm{ad}(dx) = 0$, so dx belongs to the center of $\mathfrak{g}$. Thus $dx = 0$, i.e., $d = 0$. It follows that $I = 0$, as claimed. □

Corollary 15.9. *Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra, and $G = \mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$. Then G is a Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$. Thus G acts on $\mathfrak{g}$ by the adjoint action.*

Proof. It is easy to show that for any finite dimensional real or complex Lie algebra $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g})$, so the statement follows. □

16. Lecture 16

16.1. **Extensions of Lie algebra representations.** Let $\mathfrak{g}$ be a Lie algebra and U, W be representations of $\mathfrak{g}$. We would like to classify all representations V which fit into a short exact sequence

$$0 \to U \to V \to W \to 0,$$

i.e., $U \subset V$ is a subrepresentation such that the surjection $p : V \to W$ has kernel U and thus defines an isomorphism $V/U \cong W$. In other words, V is endowed with a 2-step filtration with $F_0V = U$ and $F_1V = V$ such that $F_1V/F_0V = W$, so $\mathrm{gr}(V) = U \oplus W$. To do so, pick a splitting of this sequence as a sequence of vector spaces, i.e. an injection $i : W \to V$ (not a homomorphism of representations, in general) such that $p \circ i = \mathrm{Id}_W$. This defines a linear isomorphism $\widetilde{i} : U \oplus W \to V$ given by $(u, w) \mapsto u + i(w)$, which allows us to rewrite the action of $\mathfrak{g}$ on V as an action on $U \oplus W$. Since $\widetilde{i}$ is not in general a morphism of representations, this action is given by

$$\rho(x)(u, w) = (xu + a(x)w, xw)$$

where $a : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{k}}(W, U)$ is a linear map, and $\widetilde{i}$ is a morphism of representations iff $a = 0$.

What are the conditions on a to give rise to a representation? We compute:

$$\rho([x, y])(u, w) = ([x, y]u + a([x, y])w, [x, y]w),$$

$$[\rho(x), \rho(y)](u, w) = ([x, y]u + ([x, a(y)] + [a(x), y])w, [x, y]w).$$

Thus the condition to give a representation is the Leibniz rule

$$a([x, y]) = [x, a(y)] + [a(x), y] = [x, a(y)] - [y, a(x)].$$

In general, if E is a representation of $\mathfrak{g}$ then a linear function $a : \mathfrak{g} \to E$ such that

$$a([x, y]) = x \circ a(y) - y \circ a(x)$$

is called a $\mathbf{1-}$**cocycle** of $\mathfrak{g}$ with values in E. The space of 1-cocycles is denoted by $Z^1(\mathfrak{g}, E)$.

Example 16.1. We have $Z^1(\mathfrak{g}, \mathbf{k}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ and $Z^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Der}\mathfrak{g}$.

Thus we see that in our setting $a : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{k}}(W, U)$ defines a representation if and only if $a \in Z^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U))$. Denote the representation V attached to such a by V_a. Then we have a natural short exact sequence

$$0 \to U \to V_a \to W \to 0.$$

It may, however, happen that some $a \neq 0$ defines a trivial extension $V \cong U \oplus W$, i.e., $V_a \cong V_0$, and more generally $V_a \cong V_b$ for $a \neq b$.

Let us determine when this happens. More precisely, let us look for isomorphisms $f : V_a \to V_b$ preserving the structure of the short exact sequences, i.e., such that $\mathrm{gr}(f) = \mathrm{Id}$. Then

$$f(u, w) = (u + Aw, w)$$

where $A : W \to U$ is a linear map. Then we have

$$xf(u, w) = x(u + Aw, w) = (xu + xAw + b(x)w, xw)$$

and

$$fx(u, w) = f(xu + a(x)w, xw) = (xu + a(x)w + Axw, xw),$$

so we get that $xf = fx$ iff

$$[x, A] = a(x) - b(x).$$

In particular, setting $b = 0$, we see that V is a trivial extension if and only if $a(x) = [x, A]$ for some A.

More generally, if E is a $\mathfrak{g}$-representation, the linear function $a : \mathfrak{g} \to E$ given by $a(x) = xv$ for some $v \in E$ is called the **1-coboundary** of v, and one writes $a = dv$. The space of 1-coboundaries is denoted by $B^1(\mathfrak{g}, E)$; it is easy to see that it is a subspace of $Z^1(\mathfrak{g}, E)$, i.e., a 1-coboundary is always a 1-cocycle. Thus in our setting $f : V_a \to V_b$ is an isomorphism of representations iff

$$a - b = dA,$$

i.e., there is an isomorphism $f : V_a \cong V_b$ with $\mathrm{gr}(f) = \mathrm{Id}$ if and only if $a = b$ in the quotient space

$$\mathrm{Ext}^1(W, U) := Z^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U))/B^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U)).$$

The notation is justified by the fact that this space parametrizes extensions of W by U. More precisely, every extension V gives rise to a class $[V] \in \mathrm{Ext}^1(W, U)$, and the extension is trivial iff $[V] = 0$.

More generally, for a $\mathfrak{g}$-module E the space

$$H^1(\mathfrak{g}, E) := Z^1(\mathfrak{g}, E)/B^1(\mathfrak{g}, E)$$

is called the **first cohomology** of $\mathfrak{g}$ with coefficients in E. Thus,

$$\mathrm{Ext}^1(W, U) = H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U)).$$

Lemma 16.2. *A short exact sequence* $0 \to U \to V \to W \to 0$ *gives rise to an exact sequence*

$$0 \to U^{\mathfrak{g}} \to V^{\mathfrak{g}} \to W^{\mathfrak{g}} \to H^1(\mathfrak{g}, U) \to H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, W).$$

Proof. First of all we need to define the connecting map

$$\beta : W^{\mathfrak{g}} \to H^1(\mathfrak{g}, U).$$

To do so, for $w \in W^{\mathfrak{g}}$ pick a lift $\widetilde{w}$ of w to V and let $\widetilde{\beta}(x) = x\widetilde{w}$ for $x \in \mathfrak{g}$. Since w is $\mathfrak{g}$-invariant, this takes values in U, and is clearly a 1-cocycle, i.e., $\widetilde{\beta} \in Z^1(\mathfrak{g}, U)$. Let $\beta(w)$ be the class of $\widetilde{\beta}$ in $H^1(\mathfrak{g}, U)$. Then $\beta(w)$ is well defined (i.e., independent on the choice of $\widetilde{w}$) since any two lifts of $\widetilde{w}_1, \widetilde{w}_2$ of w differ by an element $u \in U$, hence the corresponding difference $\widetilde{\beta}_1 - \widetilde{\beta}_2$ is the coboundary du of u.

The exactness of the sequence can now be checked directly. Namely, it is easy to see that the composition of any two consecutive arrows is zero. So we just need to show that the kernel of each map is the image of the previous one. Let us do so in the two most complicated terms (the rest is fairly obvious).

The term $W^{\mathfrak{g}}$. Suppose $w \in W^{\mathfrak{g}}$ and $\beta(w) = 0$. Then $x \mapsto x\widetilde{w}$ is a coboundary, i.e., there exists $u \in U$ such that $x\widetilde{w} = xu$. So replacing $\widetilde{w}$ by $\widetilde{w} - u$, we may assume that $x\widetilde{w} = 0$, i.e., $\widetilde{w} \in V^{\mathfrak{g}}$.

The term $H^1(\mathfrak{g}, U)$. Suppose $a \in H^1(\mathfrak{g}, U)$ and its image in $H^1(\mathfrak{g}, V)$ is zero. Then $a(x) = xv$ for some $v \in V$. So $a = \beta(w)$, where w is the projection of v to W. □

16.2. **Complete reducibility of representations of semisimple Lie algebras.** We have shown above that for a semisimple $\mathfrak{g}$ over a field of characteristic zero, $H^1(\mathfrak{g}, \mathbf{k}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* = 0$, and $H^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Der}\mathfrak{g}/\mathfrak{g} = 0$. In fact, these are special cases of a more general theorem.

Theorem 16.3. *If $\mathfrak{g}$ is semisimple in characteristic zero then for every finite dimensional representation V of $\mathfrak{g}$ we have $H^1(\mathfrak{g}, V) = 0$.*

The proof of Theorem 16.3 is in the next subsection.

Theorem 16.4. *Every finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ over a field of characteristic zero is completely reducible, i.e., isomorphic to a direct sum of irreducible representations.*

Proof. Theorem 16.3 implies that for any finite dimensional representations W, U of $\mathfrak{g}$ one has $\mathrm{Ext}^1(W, U) = 0$. Thus any short exact sequence

$$0 \to U \to V \to W \to 0$$

splits, which implies the statement. □

16.3. **Proof of Theorem 16.3.** We will use the following lemma, which holds over any field.

Lemma 16.5. *Let E be a representation of a Lie algebra $\mathfrak{g}$ and $C \in U(\mathfrak{g})$ be a central element which acts by 0 on the trivial representation of $\mathfrak{g}$ and by some scalar $\lambda \neq 0$ on E. Then $H^1(\mathfrak{g}, E) = 0$.*

Proof. We have seen that $H^1(\mathfrak{g}, E) = \mathrm{Ext}^1(\mathbf{k}, E)$, so our job is to show that any extension

$$0 \to E \to V \to \mathbf{k} \to 0$$

splits. Let $p : V \to \mathbf{k}$ be the projection. We claim that there exists a unique vector $v \in V$ such that $p(v) = 1$ and $Cv = 0$. Indeed, pick some $w \in V$ with $p(w) = 1$. Then $Cw \in E$, so set $v = w - \lambda^{-1}Cw$. Since $C^2 w = \lambda Cw$, we have $Cv = 0$. Also if v' is another such vector then $v - v' \in E$ so $C(v - v') = \lambda(v - v') = 0$, hence $v = v'$.

Thus $\mathbf{k}v \subset V$ is a $\mathfrak{g}$-invariant complement to E (as C is central), which implies the statement. ☐

The key lemma used in the proof is the following.

Lemma 16.6. *Let $\mathfrak{g}$ be semisimple in characteristic zero and V be a nontrivial finite dimensional irreducible $\mathfrak{g}$-module. Then there is a central element $C \in U(\mathfrak{g})$ such that $C|_{\mathbf{k}} = 0$ and $C|_V \neq 0$.*

Proof. Consider the invariant symmetric bilinear form on $\mathfrak{g}$

$$B_V(x, y) = \mathrm{tr}|_V(xy).$$

We claim that $B_V \neq 0$. Indeed, let $\bar{\mathfrak{g}} \subset \mathfrak{gl}(V)$ be the image of $\mathfrak{g}$, then $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] = \bar{\mathfrak{g}}$ since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Thus by a result proved above, if $B_V = 0$ then $\bar{\mathfrak{g}}$ is solvable, so, being the quotient of a semisimple Lie algebra $\mathfrak{g}$, it must be zero, hence V is trivial, a contradiction.

Let $I = \mathrm{Ker}(B_V)$. Then $I \subset \mathfrak{g}$ is an ideal, so $\mathfrak{g} = I \oplus \mathfrak{g}'$ for some semisimple Lie algebra $\mathfrak{g}'$, and B_V is nondegenerate on $\mathfrak{g}'$. Let x_i be a basis of $\mathfrak{g}'$ and x^i the dual basis, and define the Casimir element $C = \sum_i x_i x^i$. It is easy to show that C is independent on the choice of the basis. Also C is central: for $y \in \mathfrak{g}$,

$$[y, C] = \sum_i ([y, x_i]x^i + x_i[y, x^i]) = 0$$

since

$$\sum_i ([y, x_i] \otimes x^i + x_i \otimes [y, x^i]) = 0.$$

Moreover, $\mathrm{tr}_V(C) = \sum_i B_V(x_i, x^i) = \dim \mathfrak{g}'$, so $C|_V = \frac{\dim \mathfrak{g}'}{\dim V} \neq 0$. Also it is clear that $C|_{\mathbf{k}} = 0$, so the lemma follows. ☐

Corollary 16.7. *For any irreducible finite dimensional representation V of a semisimple Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero, we have $H^1(\mathfrak{g}, V) = 0$.*

Proof. If V is nontrivial, this follows from Lemmas 16.5 and 16.6. On the other hand, if $V = \mathbf{k}$ then $H^1(\mathfrak{g}, V) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^* = 0$. □

Now we can prove Theorem 16.3. By Lemma 16.2 (the last portion of the exact sequence), it suffices to prove the theorem for irreducible V, which is guaranteed by Corollary 16.7.

Corollary 16.8. *A reductive Lie algebra $\mathfrak{g}$ in characteristic zero is uniquely a direct sum of a semisimple and abelian Lie algebra.*

Proof. Consider the adjoint representation of $\mathfrak{g}$. It is a representation of $\mathfrak{g}' = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, which fits into a short exact sequence

$$0 \to \mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}' \to 0.$$

By complete reducibility, this sequence splits, i.e. we have a decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$ as a direct sum of ideals, and it is clearly unique. □

16.4. **Semisimple elements.** Let x be an element of a Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbf{k}$. Let $\mathfrak{g}_\lambda \subset \mathfrak{g}$ be the generalized eigenspace of $\mathrm{ad}x$ with eigenvalue λ. Then $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_\lambda$.

Lemma 16.9. *We have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.*

Proof. Let $y \in \mathfrak{g}_\lambda, z \in \mathfrak{g}_\mu$. We have

$$(\mathrm{ad}x-\lambda-\mu)^N([y,z]) = \sum_{p+q+r+s=N} (-1)^{r+s}\frac{N!}{p!q!r!s!}\lambda^r\mu^s[(\mathrm{ad}x)^p(y), (\mathrm{ad}x)^q(z)] =$$

$$\sum_{k+\ell=N} \frac{N!}{k!\ell!}[(\mathrm{ad}x-\lambda)^k(y), (\mathrm{ad}x-\mu)^\ell(z)].$$

Thus if $(\mathrm{ad}x-\lambda)^N(y) = 0$ and $(\mathrm{ad}x-\mu)^M(z) = 0$ then

$$(\mathrm{ad}x-\lambda-\mu)^{m+n}([y,z]) = 0,$$

so $[y,z] \in \mathfrak{g}_{\lambda+\mu}$. □

Definition 16.10. An element x of a Lie algebra $\mathfrak{g}$ is called **semisimple** if the operator $\mathrm{ad}x$ is semisimple and **nilpotent** if this operator is nilpotent.

It is clear that any element which is both semisimple and nilpotent is central, so for a semisimple Lie algebra it must be zero. Note also that for $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$ this coincides with the usual definition.

Proposition 16.11. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then every element $x \in \mathfrak{g}$ has a unique decomposition as $x = x_s + x_n$, where x_s is semisimple and x_n is nilpotent. Moreover, if $y \in \mathfrak{g}$ and $[x,y] = 0$ then $[x_s, y] = 0$.*

Proof. Recall that $\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ via the adjoint representation. So we can consider the Jordan decomposition $x = x_s + x_n$, with $x_s, x_n \in \mathfrak{gl}(\mathfrak{g})$. We have $x_s(y) = \lambda y$ for $y \in \mathfrak{g}_\lambda$. Thus $y \mapsto x_s(y)$ is a derivation of $\mathfrak{g}$ by Lemma 16.9. But we know that every derivation of $\mathfrak{g}$ is inner, which implies that $x_s \in \mathfrak{g}$. It is clear that x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$. Also if $[x, y] = 0$ then $\mathrm{ad}y$ preserves $\mathfrak{g}_\lambda$ for all λ, hence $[x_s, y] = 0$ as linear operators on $\mathfrak{g}$ and thus as elements of $\mathfrak{g}$. This also implies that the decomposition is unique since if $x = x_s' + x_n'$ then $x_s - x_s' = x_n' - x_n$ is both semisimple and nilpotent, hence zero. □

Corollary 16.12. *Any nonzero semisimple Lie algebra $\mathfrak{g}$ contains nonzero semisimple elements.*

Proof. Otherwise, by Proposition 16.11, every element $x \in \mathfrak{g}$ is nilpotent, which by Engel's theorem would imply that $\mathfrak{g}$ is nilpotent, hence solvable, hence zero. □

16.5. **Toral subalgebras.**

Definition 16.13. An abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a **toral subalgebra** if it consists of semisimple elements.

Proposition 16.14. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a toral subalgebra, and B a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g., the Killing form).*

(i) We have a decomposition $\mathfrak{g} = \oplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the subspace of $x \in \mathfrak{g}$ such that for $h \in \mathfrak{h}$ we have $[h, x] = \alpha(h)x$, and $\mathfrak{g}_0 \supset \mathfrak{h}$.

(ii) We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(iii) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under B.

(iv) B restricts to a nondegenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbf{k}$.

Proof. (i) is just the joint eigenspace decomposition for $\mathfrak{h}$ acting in $\mathfrak{g}$. (ii) is a very easy special case of Lemma 16.9. (iii) and (iv) follow from the fact that B is nondegenerate and invariant. □

Corollary 16.15. *(i) The Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is reductive.*

(ii) if $x \in \mathfrak{g}_0$ then $x_s, x_n \in \mathfrak{g}_0$.

Proof. (i) This follows by Cartan's criterion from the fact that for $x, y \in \mathfrak{g}_0$, the form $\mathrm{tr}|_{\mathfrak{g}}(xy)$ is nondegenerate. (Proposition 16.14 (iv) for the Killing form of $\mathfrak{g}$).

(ii) We have $[h, x] = 0$ for $h \in \mathfrak{h}$, so $[h, x_s] = 0$, hence $x_s \in \mathfrak{g}_0$. □

17. Lecture 17

17.1. **Cartan subalgebras.**

Definition 17.1. A Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ is a toral subalgebra $\mathfrak{h} \in \mathfrak{g}$ such that $\mathfrak{g}_0 = \mathfrak{h}$.

Example 17.2. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$. Then the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of diagonal matrices is a Cartan subalgebra.

It is clear that any Cartan subalgebra is a maximal toral subalgebra of $\mathfrak{g}$. The following theorem, stating the converse, shows that Cartan subalgebras exist.

Theorem 17.3. *Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra.*

Proof. Let $x \in \mathfrak{g}_0$, then, as shown above, $x_s \in \mathfrak{g}_0$, so $x_s \in \mathfrak{h}$ by maximality of $\mathfrak{h}$. Thus $\mathrm{ad}x|_{\mathfrak{g}_0} = \mathrm{ad}x_n|_{\mathfrak{g}_0}$ is nilpotent. So by Engel's theorem $\mathfrak{g}_0$ is nilpotent. But it is also reductive, hence abelian.

Now let us show that every $x \in \mathfrak{g}_0$ which is nilpotent in $\mathfrak{g}$ must be zero. Indeed, in this case, for any $y \in \mathfrak{g}_0$, the operator $\mathrm{ad}x \cdot \mathrm{ad}y : \mathfrak{g} \to \mathfrak{g}$ is nilpotent, so $\mathrm{tr}|_{\mathfrak{g}}(\mathrm{ad}x \cdot \mathrm{ad}y) = 0$. But this form is nondegenerate on $\mathfrak{g}_0$, which implies that $x = 0$.

Thus for any $x \in \mathfrak{g}_0$, $x_n = 0$, so $x = x_s$ is semisimple. Hence $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{h}$ is a Cartan subalgebra. □

We will later show that all Cartan subalgebras of $\mathfrak{g}$ are conjugate under $\mathrm{Aut}(\mathfrak{g})$, in particular they all have the same dimension, which is called the **rank** of $\mathfrak{g}$.

17.2. **Root decomposition.**

Proposition 17.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and B a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g., the Killing form).*

(i) We have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the subspace of $x \in \mathfrak{g}$ such that for $h \in \mathfrak{h}$ we have $[h, x] = \alpha(h)x$, and R is the (finite) set of $\alpha \in \mathfrak{h}^*$, $\alpha \neq 0$, such that $\mathfrak{g}_\alpha \neq 0$.

(ii) We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(iii) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under B.

(iv) B restricts to a nondegenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbf{k}$.

Proof. This immediately follows from Theorem 16.14. □

Definition 17.5. The set R is called the **root system** of $\mathfrak{g}$ and its elements are called **roots**.

Proposition 17.6. *Let $\mathfrak{g}_1, ..., \mathfrak{g}_n$ be simple Lie algebras and let $\mathfrak{g} = \oplus_i \mathfrak{g}_i$.*

(i) Let $\mathfrak{h}_i \subset \mathfrak{g}_i$ be Cartan subalgebras of $\mathfrak{g}_i$ and $R_i \subset \mathfrak{h}$ the corresponding root systems of $\mathfrak{g}_i$. Then $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ is a Cartan subalgebra in $\mathfrak{g}$ and the corresponding root system R is the disjoint union of R_i.

(ii) Each Cartan subalgebra in $\mathfrak{g}$ must have the form $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ where $\mathfrak{h}_i \subset \mathfrak{g}_i$ is a Cartan subalgebra in $\mathfrak{g}_i$.

Proof. (i) is obvious. To prove (ii), given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, let $\mathfrak{h}_i$ be the projections of $\mathfrak{h}$ to $\mathfrak{g}_i$. It is easy to see that $\mathfrak{h}_i \subset \mathfrak{g}_i$ are Cartan subalgebras. Also $\mathfrak{h} \subset \oplus_i \mathfrak{h}_i$ and the latter is toral, which implies that $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ since $\mathfrak{h}$ is a Cartan subalgebra. □

Example 17.7. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$. Then the subspace of diagonal matrices $\mathfrak{h}$ is a Cartan subalgebra, which can be identified as the space of vectors $\mathbf{x} = (x_1, ..., x_n)$ such that $\sum x_i = 0$. Let e_i be the linear functionals on this space given by $e_i(\mathbf{x}) = x_i$. We have $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{i \neq j} \mathbf{k} E_{ij}$ and $[\mathbf{x}, E_{ij}] = (x_i - x_j)E_{ij}$. Thus the root system R consists of vectors $e_i - e_j \in \mathfrak{h}^*$ for $i \neq j$ (so there are $n(n-1)$ roots).

Now let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let $(,)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, for example the Killing form. Since the restriction of $(,)$ to $\mathfrak{h}$ is nondegenerate, it defines an isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ given by $h \mapsto (h, ?)$. The inverse of this isomorphism will be denoted by $\alpha \mapsto H_\alpha$. We also have the inverse form on $\mathfrak{h}^*$ which we also will denote by $(,)$; it is given by $(\alpha, \beta) := \alpha(H_\beta) = (H_\alpha, H_\beta)$.

Lemma 17.8. *For any $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ we have*

$$[e, f] = (e, f)H_\alpha.$$

Proof. We have $[e, f] \in \mathfrak{h}$ so it is enough to show that the inner product of both sides with any $h \in \mathfrak{h}$ is the same. We have

$$([e, f], h) = (e, [f, h]) = \alpha(h)(e, f) = ((e, f)H_\alpha, h),$$

as desired. □

Lemma 17.9. *(i) If α is a root then $(\alpha, \alpha) \neq 0$.*

(ii) Let $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ be such that $(e, f) = \frac{2}{(\alpha,\alpha)}$, and let $h_\alpha := \frac{2H_\alpha}{(\alpha,\alpha)}$. Then e, f, h_α satisfy the commutation relations of the Lie algebra $\mathfrak{sl}_2$.

(iii) h_α is independent on the choice of $(,)$.

Proof. (i) Pick $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ with $(e, f) \neq 0$. Let $h := [e, f] = (e, f)H_\alpha$ (by Lemma 17.8) and consider the Lie algebra $\mathfrak{a}$ generated by e, f, h. Then we see that

$$[h, e] = \alpha(h)e = (\alpha, \alpha)(e, f)e, \ [h, f] = -\alpha(h)f = (\alpha, \alpha)(e, f)f,$$

Thus if $(\alpha, \alpha) = 0$ then $\mathfrak{a}$ is a solvable Lie algebra. By Lie's theorem, we can choose a basis in $\mathfrak{g}$ such that operators $\mathrm{ad}e$, $\mathrm{ad}f$, $\mathrm{ad}h$ are upper triangular. Since $h = [e, f]$, $\mathrm{ad}h$ will be strictly upper-triangular and thus nilpotent. But since $h \in \mathfrak{h}$, it is also semisimple. Thus, $\mathrm{ad}h = 0$, so $h = 0$ as $\mathfrak{g}$ is semisimple. On the other hand, $h = (e, f)H_\alpha \neq 0$. This contradiction proves the first part of the theorem.

(ii) This follows immediately from the formulas in the proof of (i).

(iii) It's enough to check the statement for a simple Lie algebra, and in this case this is easy since $(,)$ is unique up to scaling. ☐

The Lie subalgebra of $\mathfrak{g}$ spanned by e, f, h_α, which we've shown to be isomorphic to $\mathfrak{sl}_2(\mathbf{k})$, will be denoted by $\mathfrak{sl}_2(\mathbf{k})_\alpha$ (we will see that $\mathfrak{g}_\alpha$ are 1-dimensional so it is independent on the choices).

Proposition 17.10. *Let* $\mathfrak{a}_\alpha = \mathbf{k}H_\alpha \oplus \oplus_{k\neq 0}\mathfrak{g}_{k\alpha} \subset \mathfrak{g}$. *Then* $\mathfrak{a}_\alpha$ *is a Lie subalgebra of* $\mathfrak{g}$.

Proof. This follows from the fact that for $e \in \mathfrak{g}_{k\alpha}, f \in \mathfrak{g}_{-k\alpha}$ we have $[e, f] = (e, f)H_{k\alpha} = k(e, f)H_\alpha$. ☐

Corollary 17.11. *(i) The space* $\mathfrak{g}_\alpha$ *are 1-dimensional for each root* α *of* $\mathfrak{g}$.

(ii) If α *is a root of* $\mathfrak{g}$ *and* $k \geq 2$ *is an integer then* $k\alpha$ *is not a root of* $\mathfrak{g}$.

Proof. For a root α the Lie algebra $\mathfrak{a}_\alpha$ contains $\mathfrak{sl}_2(\mathbf{k})_\alpha$, so it is a finite dimensional representation this Lie algebra. Also the kernel of h_α on this representation is spanned by h_α, hence 1-dimensional, and eigenvalues of h_α are even integers since $\alpha(h_\alpha) = 2$. Thus by the representation theory of $\mathfrak{sl}_2$, this representation is irreducible, i.e., eigenspaces of h_α (which are $\mathfrak{g}_{k\alpha}$ and $\mathbf{k}H_\alpha$) are 1-dimensional. Therefore the map $[e, ?] : \mathfrak{g}_\alpha \to \mathfrak{g}_{2\alpha}$ is zero (as $\mathfrak{g}_\alpha$ is spanned by e). So again by representation theory of $\mathfrak{sl}_2$ we have $\mathfrak{g}_{k\alpha} = 0$ for $|k| \geq 2$. ☐

Theorem 17.12. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra with Cartan subalgebra* $\mathfrak{h}$ *and root decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha\in R}\mathfrak{g}_\alpha$. *Let* $(,)$ *be a nondegenerate symmetric invariant bilinear form on* $\mathfrak{g}$.

(i) R *spans* $\mathfrak{h}^*$ *as a vector space, and elements* h_α, $\alpha \in R$ *span* $\mathfrak{h}$ *as a vector space.*

(ii) For any two roots α, β, *the number* $a_{\alpha,\beta} := \beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ *is an integer.*

(iii) For $\alpha \in R$, *define the* **reflection operator** $s_\alpha : \mathfrak{h} \to \mathfrak{h}$ *by*

$$s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha = \lambda - 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

Then for any roots α, β, $s_\alpha(\beta)$ is also a root.

(iv) For roots $\alpha, \beta \neq \pm\alpha$, the subspace $V_{\alpha,\beta} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}_{\beta+k\alpha} \subset \mathfrak{g}$ is an is an irreducible representation of $\mathfrak{sl}_2(\mathbf{k})_\alpha$.

Proof. (i) Suppose $h \in \mathfrak{h}$ is such that $\alpha(h) = 0$ for all roots α. Then $\mathrm{ad}h = 0$, hence $h = 0$ as $\mathfrak{g}$ is semisimple. This implies both statements.

(ii) $a_{\alpha,\beta}$ is the eigenvalue of h_α on e_β, hence an integer by the representation theory of $\mathfrak{sl}_2$.

(iii) Let $x \in \mathfrak{g}_\beta$ be nonzero. If $\beta(h_\alpha) \geq 0$ then let $y = f_\alpha^{\beta(h_\alpha)}x$. If $\beta(h_\alpha) \leq 0$ then let $y = e_\alpha^{-\beta(h_\alpha)}x$. Then by representation theory of $\mathfrak{sl}_2$, $y \neq 0$. We also have $[h, y] = s_\alpha(\beta)(h)y$. This implies the statement.

(iv) It is clear that $V_{\alpha,\beta}$ is a representation. Also all weight spaces of $V_{\alpha,\beta}$ are 1-dimensional, and its weights are either all odd or all even. This implies that it is irreducible. □

Corollary 17.13. *Let $\mathfrak{h}_\mathbb{R}$ be the $\mathbb{R}$-span of h_α. Then $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$ and the restriction of the Killing form to $\mathfrak{h}_\mathbb{R}$ is real-valued and positive definite.*

Proof. It follows from the previous theorem that the eigenvalues of $\mathrm{ad}h$, $h \in \mathfrak{h}_\mathbb{R}$, are real. So $\mathfrak{h}_\mathbb{R} \cap i\mathfrak{h}_\mathbb{R} = 0$, which implies the first statement. Now, $K(h,h) = \sum_i \lambda_i^2$ where λ_i are the eigenvalues of $\mathrm{ad}h$ (which are not all zero if $h \neq 0$). Thus $K(h,h) > 0$ if $h \neq 0$. □

18. Lecture 18

18.1. **Regular elements.** In this section we will discuss another way of constructing Cartan subalgebras. First consider an example.

Example 18.1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $x \in \mathfrak{g}$ be a diagonal matrix with distinct eigenvalues. Then the centralizer $\mathfrak{h} = C(x)$ is the space of all diagonal matrices of trace 0, which is a Cartan subalgebra. Thus the same applies to any diagonalizable matrix, i.e., a generic matrix (one for which the discriminant of the characteristic polynomial is nonzero).

So we may hope that if we take a generic element x in a semisimple Lie algebra then its centralizer is a Cartan subalgebra. But for that we have to define what we mean by generic.

Definition 18.2. The **nullity** $n(x)$ of an element $x \in \mathfrak{g}$ is the multiplicity of the eigenvalue 0 for the operator $\mathrm{ad}x$ (i.e., the dimension of the generalized 0-eigenspace). The **rank** $\mathrm{rank}(\mathfrak{g})$ of $\mathfrak{g}$ is the minimal value of $n(x)$. An element x is **regular** if $n(x) = \mathrm{rank}(\mathfrak{g})$.

Example 18.3. It is easy to check that for $\mathfrak{g} = \mathfrak{sl}_n$, x is regular if and only if its eigenvalues are distinct.

We will need the following auxiliary lemma.

Lemma 18.4. *Let $P(z_1, ..., z_n)$ be a nonzero complex polynomial, and $U \subset \mathbb{C}^n$ be the set of points $(z_1, ..., z_n) \in \mathbb{C}^n$ such that $P(z_1, ..., z_n) \neq 0$. Then U is path connected, dense and open.*

Proof. It is clear that U is open, since it is the preimage of the open set $\mathbb{C}^\times \subset \mathbb{C}$ under a continuous map. It is also dense, as its complement, the hypersurface $P = 0$, cannot contain a ball. Finally, to see that it is path connected, take $\mathbf{x}, \mathbf{y} \in U$, and consider the polynomial $Q(t) := P((1-t)\mathbf{x} + t\mathbf{y})$. It has only finitely many zeros, hence the entire line $\mathbf{z} = (1-t)\mathbf{x}+t\mathbf{y}$ except finitely many points is contained in U. Clearly, $\mathbf{x}$ and $\mathbf{y}$ can be connected by a path inside this line. $\square$

Lemma 18.5. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then the set $\mathfrak{g}^{\text{reg}}$ of regular elements is connected, dense and open in $\mathfrak{g}$.*

Proof. Consider the characteristic polynomial $P_x(t)$ of $\text{ad}x$. We have

$$P_x(t) = t^{\text{rank}(\mathfrak{g})}(t^m + a_{m-1}(x)t^{m-1} + ... + a_0(x)),$$

where $m = \dim \mathfrak{g} - \text{rank}\mathfrak{g}$ and a_i are some polynomials of x, with $a_0 \neq 0$. Then x is regular if and only if $a_0(x) \neq 0$. This implies the statement by Lemma 18.4. $\square$

Proposition 18.6. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Then*

(i) $\dim \mathfrak{h} = \text{rank}(\mathfrak{g})$*; and*

(ii) the set $\mathfrak{h}^{\text{reg}} := \mathfrak{h} \cap \mathfrak{g}^{\text{reg}}$ coincides with the set $V := \{h \in \mathfrak{h} : \alpha(h) \neq 0 \ \forall \alpha \in R\}$. In particular, $\mathfrak{h}^{\text{reg}}$ is open and dense in $\mathfrak{h}$.

Proof. (i) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ (we know it exists, e.g. we can take G to be the connected component of the identity in $\text{Aut}(\mathfrak{g})$). Let $\phi : G \times V \to \mathfrak{g}$ be the map defined by $\phi(g, x) := \text{Ad}g \cdot x$. Let us compute the differential $\phi_* : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g}$ at the point $(1, x)$ for $x \in \mathfrak{h}$. We obtain

$$\phi_*(y, h) = [y, x] + h.$$

The kernel of this map is identified with the set of $y \in \mathfrak{g}$ such that $[y, x] \in \mathfrak{h}$. But then $K([y, x], z) = K(y, [x, z]) = 0$ for all $z \in \mathfrak{h}$, so $[y, x] = 0$. Thus $\text{Ker}\phi_* = C(x)$.

Now let $x \in V$. Then $C(x) = \mathfrak{h}$. Thus ϕ_* is surjective by dimension count, hence ϕ is a submersion at $(1, x)$. This means that $U := \text{Im}\phi$ contains x together with its neighborhood in $\mathfrak{g}$. Hence the same holds for $\text{Ad}g \cdot x$, which implies that U is open. Since $\mathfrak{g}^{\text{reg}}$ is open and dense, we see that $U \cap \mathfrak{g}^{\text{reg}} \neq \emptyset$. But

$$n(\text{Ad}g \cdot x) = n(x) = \dim C(x) = \dim \mathfrak{h}.$$

for $x \in V$. This implies that $\mathrm{rank}\mathfrak{g} = \dim \mathfrak{h}$.

(ii) It is clear that for $x \in \mathfrak{h}$, we have

$$n(x) = \dim \mathrm{Ker}(\mathrm{ad}x) = \dim \mathfrak{h} + \#\{\alpha \in R : \alpha(x) = 0\}.$$

This implies the statement. □

18.2. **Conjugacy of Cartan subalgebras.**

Theorem 18.7. *(i) Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $x \in \mathfrak{g}$ a regular semisimple element (which exists by Proposition 18.6). Then the centralizer $C(x)$ is a Cartan subalgebra of $\mathfrak{g}$.*

(ii) Any Cartan subalgebra is of this form.

Proof. Consider the eigenspace decomposition of $\mathrm{ad}x$: $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_\lambda$. Since $\mathbb{C}x$ is a toral subalgebra, the Lie algebra $\mathfrak{g}_0 = C(x)$ is reductive, with $\dim(\mathfrak{g}_0) = \mathrm{rank}\mathfrak{g}$.

We claim that $\mathfrak{g}_0$ is also nilpotent. By Engel's theorem, to establish this, it suffices to show that the restriction of $\mathrm{ad}y$ to $\mathfrak{g}_0$ is nilpotent for $y \in \mathfrak{g}_0$. But $\mathrm{ad}(x + ty) = \mathrm{ad}x + t\mathrm{ad}y$ is invertible on $\mathfrak{g}/\mathfrak{g}_0$ for small t, since it is so for $t = 0$ and the set of invertible matrices is open. Thus $\mathrm{ad}(x + ty)$ must be nilpotent on $\mathfrak{g}_0$, as the multiplicity of the eigenvalue 0 for this operator must be (at least) $\mathrm{rank}\mathfrak{g} = \dim \mathfrak{g}_0$. But $\mathrm{ad}(x + ty) = t\mathrm{ad}y$ on $\mathfrak{g}_0$, which implies that $\mathrm{ad}y$ is nilpotent on $\mathfrak{g}_0$, as desired.

Thus $\mathfrak{g}_0$ is abelian. Moreover, for $y, z \in \mathfrak{g}_0$ the operator $\mathrm{ad}y_n \cdot \mathrm{ad}z$ is nilpotent on $\mathfrak{g}$ (as the product of two commuting operators one of which is nilpotent), so $K_\mathfrak{g}(y_n, z) = 0$, which implies that $y_n = 0$, as $K_\mathfrak{g}$ restricts to a nondegenerate form on $\mathfrak{g}_0$ and z is arbitrary. It follows that any $y \in \mathfrak{g}_0$ is semisimple, so $\mathfrak{g}_0$ is a toral subalgebra. Moreover, it is maximal since any element commuting with x is in $\mathfrak{g}_0$. Thus $\mathfrak{g}_0$ is a Cartan subalgebra.

(ii) Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Then as proved above, it contains a regular element x, which is automatically semisimple. Then $\mathfrak{h} = C(x)$. □

Corollary 18.8. *(i) Any regular element $x \in \mathfrak{g}$ is semisimple.*

(ii) Such x is contained in a unique Cartan subalgebra, namely $\mathfrak{h}_x = C(x)$.

Proof. (i) It is clear that if x is regular then so is x_s. Since $x \in C(x_s)$ and as shown above $C(x_s)$ is a Cartan subalgebra, it follows that x is semisimple.

(ii) Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra containing x. Then $\mathfrak{h} \supset \mathfrak{h}_x$, thus by dimension count $\mathfrak{h} = \mathfrak{h}_x$. □

Remark 18.9. In the literature regular elements are usually defined by a weaker condition, as elements $x \in \mathfrak{g}$ for which the **ordinary** (not generalized) 0-eigenspace of $\mathrm{ad}x$ (i.e., the centralizer $C(x)$ of x) has dimension $\mathrm{rank}\mathfrak{g}$. Such elements don't have to be semisimple, e.g. the nilpotent Jordan block in $\mathfrak{sl}_n$ is regular in this sense. In this case, elements with $n(x) = \mathrm{rank}\mathfrak{g}$ (i.e., ones we call regular) are called **regular semisimple**, as they are the elements that are both regular in this sense and semisimple.

Theorem 18.10. *Any two Cartan subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$ are conjugate. I.e., if $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$ are two Cartan subalgebras and G a connected Lie group with Lie algebra $\mathfrak{g}$ then there exists an element $g \in G$ such that $\mathrm{Ad}g \cdot \mathfrak{h}_1 = \mathfrak{h}_2$.*

Proof. As we have shown, every element $x \in \mathfrak{g}^{\mathrm{reg}}$ is contained in a unique Cartan subalgebra $\mathfrak{h}_x$. Introduce an equivalence relation on $\mathfrak{g}^{\mathrm{reg}}$ by setting $x \sim y$ if $\mathfrak{h}_x$ is conjugate to $\mathfrak{h}_y$. It is clear that if $x, y \in \mathfrak{h}$ are regular elements in a Cartan subalgebra $\mathfrak{h}$ then $\mathfrak{h}_x = \mathfrak{h}_y = \mathfrak{h}$, so for any $g \in G$, $\mathrm{Ad}g \cdot x \sim y$, and any element equivalent to y has this form. So as shown above the equivalence class U_y of y is open. However, we have seen that $\mathfrak{g}^{\mathrm{reg}}$ is connected. Thus there is only one equivalence class. Hence any two Cartan subalgebras of the form $\mathfrak{h}_x$ for regular x are conjugate. This implies the result, since we know that any Cartan subalgebra is of the form $\mathfrak{h}_x$. □

Remark 18.11. The same results and proofs apply over any algebraically closed field $\mathbf{k}$ of characteristic zero if we use the Zariski topology instead of the usual topology of $\mathbb{C}^n$ when working with the notions of a connected, open and dense set.

18.3. **Root systems of classical Lie algebras.**

Example 18.12. Let $\mathfrak{g}$ be the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbf{k})$. Thus $\mathfrak{g}$ consists of square matrices A of size $2n$ such that

$$AJ + JA^T = 0$$

where $J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$, with blocks being of size n. So writing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get $A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$, where b, c are symmetric. A Cartan subalgebra $\mathfrak{h}$ is then spanned by matrices A such that $a = \mathrm{diag}(x_1, ..., x_n)$ and $b = c = 0$. So $\mathfrak{h} \cong \mathbf{k}^n$. In this case we have roots coming from the a-part, which are simply the roots $e_i - e_j$ of $\mathfrak{gl}_n \subset \mathfrak{sp}_{2n}$ (defined by the condition that $b = c = 0$) and also the roots coming form the b-part, which are $e_i + e_j$ (including $i = j$, when we get $2e_i$), and the c-part, which gives the negatives of these roots, $-e_i - e_j$, including $-2e_i$.

This is the **root system of type C_n.**

Example 18.13. Let $\mathfrak{g}$ be the orthogonal Lie algebra $\mathfrak{so}_{2n}(\mathbf{k})$, preserving the quadratic form $Q = x_1x_{n+1} + ... + x_nx_{2n}$. Then the story is almost the same. The Lie algebra $\mathfrak{g}$ consists of square matrices A of size $2n$ such that

$$AJ + JA^T = 0$$

where $J = \begin{pmatrix} 0 & \mathbf{1} \\ +\mathbf{1} & 0 \end{pmatrix}$, with blocks being of size n. So writing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get $A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$, where b, c are now skew-symmetric. A Cartan subalgebra $\mathfrak{h}$ is again spanned by matrices A such that $a = \mathrm{diag}(x_1, ..., x_n)$ and $b = c = 0$. So $\mathfrak{h} \cong \mathbf{k}^n$. In this case we again have roots coming from the a-part, which are simply the roots $e_i - e_j$ of $\mathfrak{gl}_n \subset \mathfrak{so}_{2n}$ (defined by the condition that $b = c = 0$) and also the roots coming form the b-part, which are $e_i + e_j$ (but now excluding $i = j$, so only for $i \neq j$), and the c-part, which gives the negatives of these roots, $-e_i - e_j$, $i \neq j$.

This is the **root system of type D_n.**

Example 18.14. Let $\mathfrak{g}$ be the orthogonal Lie algebra $\mathfrak{so}_{2n+1}(\mathbf{k})$, preserving the quadratic form $Q = x_0^2 + x_1x_{n+1} + ... + x_nx_{2n}$. Then the Lie algebra $\mathfrak{g}$ consists of square matrices A of size $2n + 1$ such that

$$AJ + JA^T = 0$$

where

$$J = \begin{pmatrix} \mathbf{1}_1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_n \\ 0 & \mathbf{1}_n & 0 \end{pmatrix},$$

So writing

$$A = \begin{pmatrix} p & u & v \\ w & a & b \\ z & c & d \end{pmatrix},$$

we get

$$A = \begin{pmatrix} 0 & u & -u \\ w & a & b \\ -w & c & -a^T \end{pmatrix},$$

where b, c are skew-symmetric. A Cartan subalgebra $\mathfrak{h}$ is spanned by matrices A such that $a = \mathrm{diag}(x_1, ..., x_n)$ and $b = c = 0$, $u = w = 0$. So $\mathfrak{h} \cong \mathbf{k}^n$. In this case we again have roots coming from the a-part, which are simply the roots $e_i - e_j$ of $\mathfrak{gl}_n \subset \mathfrak{so}_{2n+1}$ (defined by the condition that $b = c = 0$, $u = w = 0$) and also the roots coming form the b-part, which are $e_i + e_j$, $i \neq j$, and the c-part, which gives the negatives of these roots, $-e_i - e_j$, $i \neq j$. But we also have the roots coming from the w-part, which are e_i, and from the u part, which are $-e_i$.

This is the **root system of type B_n.**

19. Lecture 19

19.1. **Abstract root systems.** Let $E \cong \mathbb{R}^r$ be a Euclidean space with a positive inner product.

Definition 19.1. An **abstract root system** is a finite set $R \subset E \setminus 0$ satisfying the following axioms:

(R1) R spans E;

(R2) For all $\alpha, \beta \in R$ the number $n_{\alpha\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer;

(R3) If $\beta \in R$ then $s_\alpha(\beta) := \beta - n_{\alpha\beta}\alpha \in R$.

Elements of R are called **roots**. The number $r = \dim E$ is called the **rank** of R.

In particular, taking $\beta = \alpha$ in R3 yields that R is centrally symmetric, i.e., $R = -R$. Also note that s_α is the reflection with respect to the hyperplane $(\alpha, x) = 0$, so R3 just says that R is invariant under such reflections.

Note also that if $R \subset E$ is a root system, $\bar{E} \subset E$ a subspace, and $R' = R \cap E'$ then R' is also a root system inside $E' = \mathrm{Span}(R') \subset \bar{E}$.

For a root α the corresponding **coroot** $\alpha^\vee \in E^*$ is defined by the formula $\alpha^\vee(x) = \frac{2(\alpha,x)}{(\alpha,\alpha)}$. Thus $\alpha^\vee(\alpha) = 2$, $n_{\alpha\beta} = \alpha^\vee(\beta)$ and $s_\alpha(\beta) = \beta - \alpha^\vee(\beta)\alpha$.

Definition 19.2. A root system R is **reduced** if for $\alpha, c\alpha \in R$, we have $c = \pm 1$.

In previous lectures we proved the following result:

Theorem 19.3. *If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra then the corresponding set of roots R is a reduced root system, and $\alpha^\vee = h_\alpha$.*

Example 19.4. 1. The root system of $\mathfrak{sl}_n$ is called A_{n-1}. In this case, as we have seen, the roots are $e_i - e_j$, and $s_{e_i - e_j} = (ij)$, the transposition of the i-th and j-th coordinates.

2. The subset $\{1, 2, -1, -2\}$ of $\mathbb{R}$ is a root system which is not reduced.

Definition 19.5. Let $R_1 \subset E_1, R_2 \subset E_2$ be root systems. An **isomorphism of root systems** $\phi : R_1 \to R_2$ is an isomorphism $\phi : E_1 \to E_2$ which maps R_1 to R_2 and preserves the number $n_{\alpha\beta}$.

So an isomorphism does not have to preserve the inner product, e.g. it may rescale it.

Definition 19.6. The **Weyl group** of a root system R is the group of automorphisms of E generated by s_α.

Proposition 19.7. *W is a finite subgroup of $O(E)$ which preserves R.*

Proof. Since s_α are orthogonal reflections, $W \subset O(E)$. By R3, s_α preserves R. By $R1$ an element of W is determined by its action on R, hence W is finite. $\square$

Example 19.8. For the root system A_{n-1}, $W = S_n$, the symmetric group. Note that for $n \geq 3$, the automorphism $x \mapsto -x$ of R is not in W, so W is, in general, a proper subgroup of $\mathrm{Aut}(R)$.

19.2. **Root systems of rank 2.** If α, β are linearly independent roots in R and $E' \subset E$ is spanned by α, β then $R' = R \cap E'$ is a root system in E' of rank 2. So to classify reduced root systems, it is important to classify reduced root systems of rank 2 first.

Theorem 19.9. *Let R be a reduced root system and $\alpha, \beta \in R$ be two linearly independent roots with $|\alpha| \geq |\beta|$. Let ϕ be the angle between α and β. Then we have one of the following possibilities:*
(1) $\phi = \pi/2$, $n_{\alpha\beta} = n_{\beta\alpha} = 0$;
(2a) $\phi = 2\pi/3$, $|\alpha|^2 = |\beta|^2$, $n_{\alpha\beta} = n_{\beta\alpha} = -1$;
(2b) $\phi = \pi/3$, $|\alpha|^2 = |\beta|^2$, $n_{\alpha\beta} = n_{\beta\alpha} = 1$;
(3a) $\phi = 3\pi/4$, $|\alpha|^2 = 2|\beta|^2$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -2$;
(3b) $\phi = \pi/4$, $|\alpha|^2 = 2|\beta|^2$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 2$;
(4a) $\phi = 5\pi/6$, $|\alpha|^2 = 3|\beta|^2$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -3$;
(4b) $\phi = 5\pi/6$, $|\alpha|^2 = 3|\beta|^2$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 3$.

Proof. We have $(\alpha, \beta) = 2|\alpha| \cdot |\beta| \cos\phi$, so $n_{\alpha\beta} = 2\frac{|\beta|}{|\alpha|}\cos\phi$. Thus $n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\phi$. Thus this number can only take values $0, 1, 2, 3$ (as it is an integer by R2) and $\frac{n_{\alpha\beta}}{n_{\beta\alpha}} = \frac{|\alpha|^2}{|\beta|^2}$ if $n_{\alpha\beta} \neq 0$. The rest is obtained by analysis of each case. $\square$

In fact, all these possibilities are realized. Namely, we have root systems $A_1 \times A_1$, A_2, $B_2 = C_2$ (the root system of the Lie algebras $\mathfrak{sp}_4$ and $\mathfrak{so}_5$, which are in fact isomorphic, consisting of the vertices and midpoints of edges of a square), and G_2, generated by α, β with $(\alpha, \alpha) = 6$, $(\beta, \beta) = 2$, $(\alpha, \beta) = -3$, and roots being $\pm\alpha, \pm\beta, \pm(\alpha+\beta)$, $\pm(\alpha + 2\beta)$, $\pm(\alpha + 3\beta)$, $\pm(2\alpha + 3\beta)$.

Theorem 19.10. *Any reduced rank 2 root system R is of the form $A_1 \times A_1$, A_2, B_2 or G_2.*

Proof. Pick independent roots $\alpha, \beta \in R$ such that the angle ϕ is as large as possible. Then $\phi \geq \pi/2$ (otherwise can replace α with $-\alpha$), so we are in one of the cases $1, 2a, 3a, 4a$. Now the statement follows by inspection of each case, giving $A_1 \times A_1$, A_2, B_2 and G_2 respectively. □

Corollary 19.11. *If $\alpha, \beta \in R$ are independent roots with $(\alpha, \beta) < 0$ then $\alpha + \beta \in R$.*

Proof. This is easy to see from the classification of rank 2 root systems. □

19.3. **Positive and simple roots.** Let R be a reduced root system and $t \in E^*$ be such that $t(\alpha) \neq 0$ for any $\alpha \in R$. We say that a root is **positive** (with respect to t) if $t(\alpha) > 0$ and **negative** if $t(\alpha) < 0$. The set of positive roots is denoted by R_+ and of negative ones by R_-, so $R_+ = -R_-$ and $R = R_+ \cup R_-$ (disjoint union). This decomposition is called a **polarization** of R; it depends on the choice of t.

Example 19.12. Let R be of type A_{n-1}. Then for $t = (t_1, ..., t_n)$ we have $t(\alpha) \neq 0$ for all α iff $t_i \neq t_j$ for any i, j. E.g. suppose $t_1 > t_2 > ... > t_n$, then we have $e_i - e_j \in R_+$ iff $i < j$. We see that polarizations are in bijection with permutations in S_n, i.e., with elements of the Weyl group, which acts simply transitively on them. We will see that this is, in fact, the case for any reduced root system.

Definition 19.13. A root $\alpha \in R_+$ is **simple** if it is not a sum of two other positive roots.

Lemma 19.14. *Every positive root is a sum of simple roots.*

Proof. If α is not simple then $\alpha = \beta + \gamma$ where $\beta, \gamma \in R_+$. We have $t(\alpha) = t(\beta) + t(\gamma)$, so $t(\beta), t(\gamma) < t(\alpha)$. If β or γ is not simple, we can continue this process, and it will terminate since t has finitely many values on R. □

Lemma 19.15. *If $\alpha, \beta \in R_+$ are simple roots then $(\alpha, \beta) \leq 0$.*

Proof. Assume $(\alpha, \beta) > 0$. Then $(-\alpha, \beta) < 0$ so by Lemma 19.11 $\gamma := \beta - \alpha$ is a root. If γ is positive then $\beta = \alpha + \gamma$ is not simple. If γ is negative then $-\gamma$ is positive so $\alpha = \beta + (-\gamma)$ is not simple. □

Theorem 19.16. *The set $\Pi \subset R_+$ of simple roots is a basis of E.*

Proof. We will use the following linear algebra lemma:

Lemma 19.17. *Let v_i be vectors in a Euclidean space E such that $(v_i, v_j) \leq 0$ when $i \neq j$ and $t(v_i) > 0$ for some $t \in E^*$. Then v_i are linearly independent.*

Proof. Suppose we have a nontrivial relation

$$\sum_{i \in I} c_i v_i = \sum_{i \in J} c_i v_i$$

where I, J are disjoint and $c_i > 0$ (clearly, every nontrivial relation can be written in this form). Evaluating t on this relation, we deduce that both sides are nonzero. Now let us compute the square of the left hand side:

$$0 < |\sum_{i \in I} c_i v_i|^2 = (\sum_{i \in I} c_i v_i, \sum_{j \in J} c_j v_j) \leq 0.$$

This is a contradiction. □

Now the result follows from Lemma 19.15 and Lemma 19.17. □

Thus the set Π of simple roots has r elements: $\Pi = (\alpha_1, ..., \alpha_r)$.

Example 19.18. For A_{n-1} with polarization given by t with decreasing coordinates, $\alpha_i := e_i - e_{i+1}$ are simple roots.

Corollary 19.19. *Any root $\alpha \in R$ can be uniquely written as $\alpha = \sum_{i=1}^r n_i \alpha_i$, where $n_i \in \mathbb{Z}$. If α is positive then $n_i \geq 0$ and if α is negative then $n_i \leq 0$.*

For a positive root α, its **height** $h(\alpha)$ is the number $\sum n_i$. So simple roots are the roots of height 1, and the height of $e_i - e_j$ in $R = A_{n-1}$ is $j - i$.

20. Lecture 20

20.1. **Dual root system.** For a root system R, the set $R^\vee \subset E^*$ of $\alpha^\vee$ for all $\alpha \in R$ is also a root system, such that $(R^\vee)^\vee = R$. It is called the **dual root system** to R. For example, B_n is dual to C_n, while A_{n-1}, D_n and G_2 are self-dual.

Moreover, it is easy to see that any polarization of R gives rise to a polarization of $R^\vee$ (using the image $t^\vee$ of t under the isomorphism $E \to E^*$ induced by the inner product), and the corresponding system $\Pi^\vee$ of simple roots consists of $\alpha_i^\vee$ for $\alpha_i \in \Pi$.

20.2. **Root and weight lattices.** Recall that a **lattice** in a real vector space E is a subgroup $Q \subset E$ generated by a basis of E. Of course, every lattice is isomorphic to $\mathbb{Z}^n \subset \mathbb{R}^n$. Also recall that for a lattice $Q \subset E$ the **dual lattice** $Q^* \subset E^*$ is the set of $f \in E^*$ such that $f(v) \in \mathbb{Z}$ for all $v \in Q$. If Q is generated by a basis e_i of E then Q^* is generated by the dual basis e_i^*.

In particular, for a root system R we can define the **root lattice** $Q \subset E$, which is generated by the simple roots α_i with respect to some polarization of R. Since Q is also generated by all roots in R, it is independent on the choice of the polarization. Similarly, we can define the **coroot lattice** $Q^\vee \subset E^*$ generated by $\alpha^\vee, \alpha \in R$, which is just the root lattice of $R^\vee$.

Also we define the **weight lattice** $P \subset E$ to be the dual lattice to $Q^\vee$: $P = (Q^\vee)^*$, and the **coweight lattice** $P^\vee \subset E^*$ to be the dual lattice to Q: $P^\vee = Q^*$, so $P^\vee$ is the weight lattice of $R^\vee$. Thus

$$P = \{\lambda \in E : (\lambda, \alpha^\vee) \in \mathbb{Z}\, \forall \alpha \in R\},\ P^\vee = \{\lambda \in E^* : (\lambda, \alpha) \in \mathbb{Z}\, \forall \alpha \in R\}.$$

Since for $\alpha, \beta \in R$ we have $(\alpha^\vee, \beta) = n_{\alpha\beta} \in \mathbb{Z}$, we have $Q \subset P$, $Q^\vee \subset P^\vee$.

Given a system of simple roots $\Pi = \{\alpha_1, ..., \alpha_r\}$, we define **fundamental coweights** $\omega_i^\vee$ to be the dual basis to α_i and **fundamental weights** ω_i to be the dual basis to $\alpha_i^\vee$: $(\omega_i, \alpha_j^\vee) = (\omega_i^\vee, \alpha_j) = \delta_{ij}$. Thus P is generated by ω_i and $P^\vee$ by $\omega_i^\vee$.

Example 20.1. Let R be of type A_1. Then $(\alpha, \alpha^\vee) = 2$ for the unique positive root α, so $\omega = \frac{1}{2}\alpha$, thus $P/Q = \mathbb{Z}/2$. More generally, if R is of type A_{n-1} and we identify $Q \cong Q^\vee, P \cong P^\vee$, then P becomes the set of $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ such that $\sum_i \lambda_i = 0$ and $\lambda_i - \lambda_j = \mathbb{Z}$. So we have a homomorphism $\phi : P \to \mathbb{R}/\mathbb{Z}$ given by $\phi(\lambda) = \lambda_i \bmod \mathbb{Z}$ (for any i). Since $\sum_i \lambda_i = 0$, we have $\phi : P \to \mathbb{Z}/n$, and $\mathrm{Ker}\phi = Q$ (integer vectors with sum zero). Also it is easy to see that ϕ is surjective (we may take $\lambda_i = \frac{k}{n}$ for $i \neq n$ and $\lambda_n = \frac{k}{n} - k$, then $\phi(\lambda) = \frac{k}{n}$). Thus $P/Q \cong \mathbb{Z}/n$.

20.3. **Weyl chambers.** Suppose we have two polarizations of a root system R defined by $t, t' \in E$, and Π, Π' are the corresponding systems of simple roots. Are Π, Π' equivalent in a suitable sense? The answer turns out to be yes. To show this, we will need the notion of a Weyl chamber.

Note that the polarization defined by t depends only on the signs of (t, α), so does not change when t is continuously deformed without crossing the hyperplanes $(t, \alpha) = 0$. This motivates the following definition:

Definition 20.2. A **Weyl chamber** is a connected component of the complement of the root hyperplanes $(x, \alpha) = 0$ in E ($\alpha \in R$).

Thus a Weyl chamber is defined by a system of strict homogeneous linear inequalities $\pm(\alpha, x) = 0$, $\alpha \in R$. More precisely, the set of solutions of such a system is either empty of a Weyl chamber.

Thus the polarization defined by t depends only on the Weyl chamber containing t.

The following lemma is geometrically obvious.

Lemma 20.3. *(i) The closure $\overline{C}$ of a Weyl chamber C is a convex cone.*

(ii) The boundary of $\overline{C}$ is a union of codimension 1 faces F_i which are convex cones inside one of the root hyperplanes defined inside it by a system of non-strict homogeneous linear inequalities.

The root hyperplanes containing the faces F_i are called the **walls** of C.

We have seen above that every Weyl chamber defines a polarization of R. Conversely, every polarization defines the corresponding **positive Weyl chamber** C_+ defined by the conditions $(\alpha, x) > 0$ for $\alpha \in R_+$ (this set is nonempty since it contains t, hence is a Weyl chamber). Thus C_+ is the set of vectors of the form $\sum c_i \omega_i$ with $c_i > 0$. So C_+ has r faces $L_{\alpha_1}, ..., L_{\alpha_r}$.

The following lemma is left as an exercise:

Lemma 20.4. *These assignments are mutually inverse bijections between polarizations of R and Weyl chambers.*

Since the Weyl group W permutes the roots, it acts on the set of Weyl chambers.

Theorem 20.5. *W acts transitively on the set of Weyl chambers.*

Proof. Let us say that Weyl chambers C, C' are **adjacent** if they share a common face $F \subset L_\alpha$. In this case it is easy to see that $s_\alpha(C) = C'$. Now given any Weyl chambers C, C', pick generic $t \in C, t' \in C'$ and connect them with a straight segment. This will define a sequence of Weyl chambers visited by this segment: $C_0 = C, C_1, ..., C_m = C'$, and C_i, C_{i+1} are adjacent for each i. So C_i, C_{i+1} lie in the same W-orbit. Hence so do C, C'. □

Corollary 20.6. *Every Weyl chamber has r walls.*

Proof. This follows since it is true for the positive Weyl chamber and the Weyl group acts transitively on the Weyl chambers. □

Corollary 20.7. *Any two polarizations of R are related by the action of an element $w \in W$. Thus if Π, Π' are systems of simple roots corresponding to two polarizations then there is $w \in W$ such that $w(\Pi) = \Pi'$.*

20.4. **Simple reflections.** Given a polarization of R and the corresponding system of simple roots $\Pi = \{\alpha_1, ..., \alpha_r\}$, the **simple reflections** are the reflections s_{α_i}, denoted by s_i.

Lemma 20.8. *For every Weyl chamber C there exist $i_1, ..., i_m$ such that $s_{i_1}...s_{i_m}(C_+)$.*

Proof. Pick $t \in C, t_+ \in C_+$ generically and connect them with a straight segment as before. Let m be the number of chamber walls crossed by this segment. The proof is by induction in m (with obvious base). Let C' be the chamber entered by our segment from C and L_α the wall separating C, C', so that $C = s_\alpha(C')$. By the induction assumption $C' = u(C_+)$, where $u = s_{i_1}....s_{i_{m-1}}$. So $L_\alpha = u(L_{\alpha_j})$ for some j. Thus $s_\alpha = us_ju^{-1}$. Thus $C = s_\alpha(C') = s_\alpha u(C_+) = us_j(C_+)$, so we get the result with $i_m = j$. □

Corollary 20.9. *(i) The simple reflections s_i generate W;*
(ii) $W(\Pi) = R$.

Proof. (i) This follows since for any root α, the hyperplane L_α is a wall of some Weyl chamber, so s_α is a product of s_i.
(ii) Follows from (i). □

Thus R can be reconstructed from Π as $W(\Pi)$, where W is the subgroup of $O(E)$ generated by s_i.

Example 20.10. For root system A_{n-1} part (i) says that any element of S_n is a product of transpositions of neighbors.

21. Lecture 21

21.1. **Length of an element of W.** Let us say that a root hyperplane L_α **separates** two Weyl chambers C, C' if they lie on different sides of L_α.

Definition 21.1. The **length** $l(w)$ of $w \in W$ is the number of root hyperplanes separating C_+ and $w(C_+)$.

We have $t \in C_+, w(t) \in w(C_+)$, so $l(w)$ is the number of roots α such that $(t, \alpha) > 0$ but $(w(t), \alpha) = (t, w^{-1}\alpha) < 0$. Note that if α is a root satisfying this condition then $\beta = -w^{-1}\alpha$ satisfies the conditions $(t, \beta) > 0$, $(t, w\beta) < 0$. Thus $l(w) = l(w^{-1})$ and $l(w)$ is the number of positive roots which are mapped by w to negative roots. Note also that the notion of length depends on the polarization of R (as it refers to the positive chamber C_+ defined using the polarization).

Example 21.2. Let s_i be a simple reflection. Then $s_i(C_+)$ is adjacent to C_+, with the only separating hyperplane being L_{α_i}. Thus $l(s_i) = 1$. It follows that the only positive root mapped by s_i to a negative root is α_i (namely, $s_i(\alpha_i) = -\alpha_i$), and thus s_i permutes $R_+ \setminus \{\alpha_i\}$.

Proposition 21.3. *Let $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$. Then $(\rho, \alpha_i^\vee) = 1$ for all i. Thus $\rho = \sum_i \omega_i$.*

The weight ρ plays an important role in representation theory, as it occurs in the character formula for representations of semisimple Lie algebras.

Proof. We have $\rho = \frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha\in R_+,\alpha\neq\alpha_i}\alpha$. Since s_i permutes $R_+\setminus\{\alpha_i\}$, we get $s_i\rho = \rho - \alpha_i$. But for any λ, $s_i\lambda = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$. This implies the statement. □

Theorem 21.4. *Let $w = s_{i_1}...s_{i_l}$ be a representation of $w \in W$ as a product of simple reflections that has minimal possible length. Then $l = l(w)$.*

Proof. As before, define a chain of Weyl chambers $C_k = s_{i_1}...s_{i_k}(C_+)$, so that $C_0 = C_+$ and $C_k = w(C_+)$. We have seen that C_k and C_{k-1} are adjacent. So there is a zigzag path from C_+ to $w(C_+)$ that intersects at most l root hyperplanes (namely, the segment from C_{k-1} to C_k intersects only one hyperplane). Thus $l(w) \leq l$. On the other hand, pick generic points in C_+ and $w(C_+)$ and connect them with a straight segment. This segment intersects every separating root hyperplane exactly once, so produces an expression of w as a product of $l(w)$ simple reflections. This implies the statement. □

An expression $w = s_{i_1}...s_{i_l}$ is called **reduced** if $l = l(w)$.

Proposition 21.5. *The Weyl group W acts simply transitively on Weyl chambers.*

Proof. We have already shown before that the action is transitive, so we just have to show that if $w(C_+) = C_+$ then $w = 1$. But in this case $l(w) = 0$, so w has to be a product of zero simple reflections, i.e., indeed $w = 1$. □

Thus we see that $\overline{C}_+$ is a *fundamental domain* of the action of W on E. Moreover, one can show that $E/W = \overline{C}_+$, i.e., every W-orbit on E has a unique representative in C_+.

Corollary 21.6. *Let $C_- = -C_+$ be the* **negative Weyl chamber**. *Then there exists a unique $w_0 \in W$ such that $w_0(C_+) = C_-$. We have $l(w_0) = |R_+|$ and for any $w \neq w_0$, $l(w) < l(w_0)$. Also $w_0^2 = 1$.*

Proof. Exercise. □

The element w_0 is therefore called the **longest element** of W.

Example 21.7. For the root system A_{n-1} the element w_0 is the order reversing involution: $w_0(1, 2, ..., n) = (n, ..., 2, 1)$.

21.2. **Cartan matrices and Dynkin diagrams.** Our goal now is to classify reduced root systems. We have shown that classifying root systems is equivalent to classifying sets Π of simple roots. So we need to classify such sets Π. Before doing so, note that we have a nice notion of **direct product** of root systems.

Namely, let $R_1 \subset E_1$ and $R_2 \subset E_2$ be two root systems. Let $E = E_1 \oplus E_2$ (orthogonal decomposition) and $R = R_1 \sqcup R_2$ (with $R_1 \perp R_2$). If $t_1 \in E_1, t_2 \in E_2$ define polarizations of R_1, R_2 with systems of simple roots Π_1, Π_2 then $t = t_1 + t_2$ defines a polarization of R with $\Pi = \Pi_1 \sqcup \Pi_2$ (with $\Pi_1 \perp \Pi_2$ and $\Pi_i = \Pi \cap R_i$).

Definition 21.8. A root system R is **irreducible** if it cannot be written (nontrivially) in this way.

Lemma 21.9. *If R is a root system with system of simple roots $\Pi = \Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$ then $R = R_1 \sqcup R_2$ where R_i is the root system generated by Π_i.*

Proof. If $\alpha \in \Pi_1, \beta \in \Pi_2$ then $s_\alpha(\beta) = \beta$, $s_\beta(\alpha) = \alpha$ and s_α and s_β commute. So if W_i is the group generated by $s_\alpha, \alpha \in \Pi_i$ then $W = W_1 \times W_2$, with W_1 acting trivially on Π_2 and W_2 on Π_1. Thus

$$R = W(\Pi) = W(\Pi_1 \sqcup \Pi_2) = W_1(\Pi_1) \sqcup W_2(\Pi_2) = R_1 \sqcup R_2.$$

□

Proposition 21.10. *Any root system is uniquely a union of irreducible ones.*

Proof. The decomposition is given by the maximal decomposition of Π into mutually orthogonal systems of simple roots. □

Thus it suffices to classify irreducible root systems.

As noted above, a root system is determined by pairwise inner products of positive roots. However, it is more convenient to encode them by the **Cartan matrix** A defined by

$$a_{ij} = n_{\alpha_j \alpha_i} = (\alpha_i^\vee, \alpha_j).$$

The following properties of the Cartan matrix follow immediately from the above results:

Proposition 21.11. *(i)* $a_{ii} = 2$;

(ii) a_{ij} *is a nonpositive integer;*

(iii) for any $i \neq j$, $a_{ij}a_{ij} = 4\cos^2\phi \in \{0, 1, 2, 3\}$, *where* ϕ *is the angle between* α_i *and* α_j*;*

(iv) Let $d_i = |\alpha_i|^2$. *Then the matrix* $d_i a_{ij}$ *is symmetric and positive definite.*

We will see later that conversely, any such matrix defines a root system.

Example 21.12. 1. Type A_{n-1}: $a_{ii} = 2, a_{i,i+1} = a_{i+1,i} = -1$, $a_{ij} = 0$ otherwise.

2. Type B_n: $a_{ii} = 2$, $a_{i,i+1} = a_{i+1,i} = -1$ except that $a_{n,n-1} = -2$.

3. Type C_n: transposed to B_n.

4. Type D_n: same as B_n but $a_{n-1,n-2} = a_{n,n-2} = a_{n-2,n} = a_{n-2,n-1} = -1$, $a_{n,n-1} = a_{n-1,n} = 0$.

5. Type G_2: $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

It is convenient to encode such matrices by **Dynkin diagrams:**

- Indices i are vertices;
- Vertices i and j are connected by $a_{ij}a_{ji}$ lines;
- If $a_{ij} \neq a_{ji}$, i.e., $|\alpha_i|^2 \neq |\alpha_j|^2$, then the arrow on the lines goes from long root to short root ("less than" sign).

It is clear that such diagram completely determines the Cartan matrix (if we fix the labeling of vertices), and vice versa. Also it is clear that the root system is irreducible if and only if its Dynkin diagram is connected.

Proposition 21.13. *The Cartan matrix determines the root system uniquely.*

Proof. We may assume the Dynkin diagram is connected. The Cartan matrix determines, for any pair of simple roots, the angle between them (which is right or obtuse) and the ratio of their lengths if they are not orthogonal. By the classification of rank 2 root systems, this determines the inner product on simple roots, up to scaling, which implies the statement. □

22. Lecture 22

22.1. **Classification of Dynkin diagrams.** The following theorem gives a complete classification of irreducible root systems.

Theorem 22.1. *(i) Connected Dynkin diagrams are classified by the list given in the first picture at* `https://en.wikipedia.org/wiki/Dynkin_diagram`*; i.e., they are A_n, B_n, C_n, D_n, G_2 which we have already met, along with four more: F_4, E_6, E_7, E_8.*

(ii) Every matrix satisfying the conditions of Proposition 21.11 is a Cartan matrix of some root system.

The proof of this theorem is rather long but direct. It consists of several steps. The first step is construction of the remaining root systems F_4, E_6, E_7, E_8.

22.2. **The root system F_4.**

Definition 22.2. The root system F_4 is the union of the root system $B_4 \subset \mathbb{R}^4$ with the vectors

$$\left(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\right) = \sum_{i=1}^{4}(\pm\tfrac{1}{2}e_i),$$

for all choices of signs.

Thus besides the roots of B_4, which are $\pm e_i \pm e_j$ (24 of them, squared length 2) and $\pm e_i$ (8 of them, squared length 1), we have the 16 new roots $\sum_{i=1}^{4}(\pm\frac{1}{2}e_i)$ (squared length 1); this gives a total of 48.

Exercise 22.3. Check that this is an irreducible root system.

To give a polarization of the F_4 root system, pick $t = (t_1, t_2, t_3, t_4)$ with $t_1 \gg t_2 \gg t_3 \gg t_4$.

Exercise 22.4. Check that for this polarization, the simple positive roots are, $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\alpha_2 = e_4$, $\alpha_3 = e_3 - e_4$, $\alpha_4 = e_2 - e_3$. Thus $\alpha_1^\vee = e_1 - e_2 - e_3 - e_4$, $\alpha_2^\vee = 2e_4$, $\alpha_3^\vee = e_3 - e_4$, $\alpha_4^\vee = e_2 - e_3$. So $a_{12} = -1, a_{23} = -2, a_{34} = -1$, $a_{21} = -1, a_{32} = -1$, $a_{43} = -1$, which gives the Dynkin diagram of F_4.

22.3. **The root system E_8.**

Definition 22.5. The root system E_8 is the union of the root system $D_8 \subset \mathbb{R}^8$ with the vectors $\sum_{i=1}^{8}(\pm\frac{1}{2}e_i)$, for all choices of signs with even number of minuses.

Thus besides the roots of D_8, $\pm e_i \pm e_j$ (112 of them), we have 128 new roots $\sum_{i=1}^{8}(\pm\frac{1}{2}e_i)$. So in total we have 240 roots. All roots have squared length 2.

Exercise 22.6. Show that it is an irreducible root system.

To give a polarization of the E_8 root system, pick t so that $t_i \gg t_{i+1}$.

Exercise 22.7. Check that for this polarization, the simple positive roots are, $\alpha_1 = \frac{1}{2}(e_1 + e_8 - \sum_{i=2}^{7} e_i)$, $\alpha_2 = e_7 + e_8$ and $\alpha_i = e_{10-i} - e_{11-i}$ for $3 \le i \le 8$. Thus the roots $\alpha_2, ..., \alpha_8$ generate the root system D_7, while $a_{13} = -1$ and $a_{1i} = 0$ for all $i \neq 1, 3$. This recovers the Dynkin diagram E_8.

22.4. **The root system E_7.**

Definition 22.8. The root system E_7 is the subsystem of E_8 generated by $\alpha_1, ..., \alpha_7$.

Note that these roots (unlike $\alpha_8 = e_2 - e_3$) satisfy the equation $x_1 + x_2 = 0$. Thus E_7 is the intersection of E_8 with this subspace. So it includes the roots $\pm e_i \pm e_j$ with $3 \le i, j \le 8$ distinct (60 roots), $\pm(e_1 - e_2)$ (2 roots) and $\sum_{i=1}^{8}(\pm\frac{1}{2}e_i)$ with even number of minuses and the opposite signs for e_1 and e_2 (64 roots). Altogether we get 126 roots.

22.5. **The root system E_6.**

Definition 22.9. The root system E_6 is the subsystem of E_8 and E_7 generated by $\alpha_1, ..., \alpha_6$.

Note that these roots (unlike $\alpha_8 = e_2 - e_3$ and $\alpha_7 = e_3 - e_4$) satisfy the equations $x_1 + x_2 = 0, x_2 + x_3 = 0$. Thus E_6 is the intersection of E_8 with this subspace. So it includes the roots $\pm e_i \pm e_j$ with $4 \le i, j \le 8$ distinct (40 roots), and $\sum_{i=1}^{8}(\pm\frac{1}{2}e_i)$ with even number of minuses and the opposite signs for e_1 and e_2 and for e_2 and e_3 (32 roots). Altogether we get 72 roots.

22.6. **Proof of the classification theorem.** Now that we have shown that there exist root systems attached to all Cartan matrices, it remains to classify Cartan matrices (or Dynkin diagrams), i.e. show that there are no others than those we have considered. For this purpose we consider Dynkin diagrams as graphs with certain kind of special edges (with one, two or three lines and a possible orientation). Note first that any subgraph of a Dynkin diagram must itself be a Dynkin diagram, since a principal submatrix of a positive definite symmetric matrix is itself positive definite. On the other hand, consider **untwisted and twisted affine Dynkin diagrams** depicted on the first picture at `https://en.wikipedia.org/wiki/Affine_Lie_algebra`. These are not Dynkin diagrams since the corresponding matrix A is degenerate, hence not positive definite.

Exercise 22.10. Prove this by showing that in each case there exists a nonzero vector v such that $Av = 0$. For example, in the simply laced case (only simple edges), this amounts to finding a labeling of the vertices by nonzero numbers such that the sum of labels of the neighbors to each vertex is twice the label of that vertex, and in the non-simply laced case it's a weighted version of that.

Thus they cannot occur inside a Dynkin diagram.

We conclude that a Dynkin diagram is a tree. Indeed, it cannot have a loop with simple edges, since this is the affine diagram $\widetilde{A}_{n-1}$, which has a null vector $(1, ..., 1)$. If there is a loop with non-simple edges, this is even worse - this vector will have a negative inner product with itself.

Further, it cannot have vertices with more than four simple edges coming out since it cannot have a subdiagram $\widetilde{D}_4$ (and for non-simple edges it is even worse, as before). Thus all the vertices of our tree are i-valent for $i \leq 3$.

Also we cannot have a subdiagram $\widetilde{D}_n$, $n \geq 5$, which implies that there is at most one trivalent vertex.

Further, if there is a triple edge then the diagram is G_2. There is no way to attach any edge to the G_2 diagram because $D_4^{(3)}$ and $\widetilde{G}_2$ are forbidden.

Next, if there is a trivalent vertex then there cannot be a non-simple edge anywhere in the diagram (as we have forbidden affine diagrams $A_{2k-1}^{(2)}, \widetilde{B}_n$). So in this case the diagram is simply laced, so it must be on our list (D_n, E_6, E_7, E_8) since it cannot contain affine diagrams $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$.

It remains to consider chain-shaped diagrams. They can't contain two double edges (affine diagrams $A_{2k}^{(2)}, D_{k+1}^{(2)}, \widetilde{C}_n$). Thus if the double edge is at the end, we can only get B_n and C_n.

Finally, if the double edge is in the middle, we can't have affine subdiagram $\widetilde{F}_4$ and $E_6^{(2)}$, so our diagram must be F_4. The theorem is proved.

Remark 22.11. Note that we have **exceptional isomorphisms** $D_2 = A_1 \times A_1$, $D_3 = A_3$, $B_2 = C_2$. Otherwise the listed root systems are distinct.

22.7. **Simply laced and non-simply laced diagrams.** As we already mentioned, a Dynkin diagram (or the corresponding root system) is called **simply laced** if all the edges are simple, i.e. $a_{ij} = 0, -1$ for $i \neq j$. This is equivalent to the Cartan matrix being symmetric, or to all roots having the same length. The connected simply-laced diagrams are A_n, D_n, E_6, E_7, E_8. The remaining diagrams B_n, C_n, F_4, G_2 are not simply laced, but they contain roots of only two squared lengths, whose ratio is 2 for double edge (B_n, C_n, F_4) and 3 for triple edge (G_2). The roots of the bigger length are called **long** and of the smaller length are called **short**.

It is easy to see that long and short roots form a root subsystem of the same rank (but not necessarily irreducible). For instance, in G_2 both form a root subsystem of type A_2, and in B_2 both are $A_1 \times A_1$. In B_3 long roots form D_3 and short ones form $A_1 \times A_1 \times A_1$.

23. Lecture 23

23.1. **Serre relations.** Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. We would like to show that any reduced root system gives rise to a semisimple Lie algebra over $\mathbf{k}$, and moreover a unique one. To this end, it suffices to show that any reduced *irreducible* root system gives rise to a unique (finite dimensional) *simple* Lie algebra.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbf{k}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and root system $R \subset \mathfrak{h}^*$ (which is thus reduced and irreducible). Fix a polarization of R with the set of simple roots $\Pi = (\alpha_1, ..., \alpha_r)$, and let $A = (a_{ij})$ be the Cartan matrix of R. We have a decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{n}_\pm := \oplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$ are the Lie subalgebras spanned by positive, respectively negative root vectors. Pick elements $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ so that $e_i, f_i, h_i = [e_i, f_i]$ form an $\mathfrak{sl}_2$-triple.

Theorem 23.1. *(Serre relations) (i) The elements e_i, f_i, h_i, $i = 1, ..., r$ generate $\mathfrak{g}$.*

(ii) These elements satisfy the following relations:

$$[h_i, h_j] = 0,\ [h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j,\ [e_i, f_j] = \delta_{ij} h_i,$$
$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0,\ (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0,\ i \neq j.$$

The last two sets of relations are called **Serre relations**. Note that if $a_{ij} = 0$ then the Serre relation just says that $[e_i, e_j] = 0$.

Proof. (i) We know that h_i form a basis of $\mathfrak{h}$, so it suffices to show that e_i generate $\mathfrak{n}_+$ and f_i generate $\mathfrak{n}_-$. We only prove the first statement, the second being the same for the opposite polarization.

Let $\mathfrak{n}_+' \subset \mathfrak{n}_+$ be the Lie subalgebra generated by e_i. It is clear that $\mathfrak{n}_+' = \oplus_{\alpha \in R_+'} \mathfrak{g}_\alpha$ where $R_+' \subset R_+$. Assume the contrary, that $R_+' \neq R_+$. Pick $\alpha \in R_+ \setminus R_+'$ with the smallest height (it is not a simple root). Then $\mathfrak{g}_{\alpha - \alpha_i} \subset \mathfrak{n}_+'$, so $[e_i, \mathfrak{g}_{\alpha - \alpha_i}] = 0$. Let $x \in \mathfrak{g}_{-\alpha}$ be a nonzero element. We have

$$([x, e_i], y) = (x, [e_i, y]) = 0$$

for any $y \in \mathfrak{g}_{\alpha - \alpha_i}$. Thus $[x, e_i] = 0$ for all i, which implies, by the representation theory of $\mathfrak{sl}_2$, that $(\alpha, \alpha_i^\vee) \leq 0$ for all i, hence $(\alpha, \alpha_i) \leq 0$

for all i. This would imply that $(\alpha, \alpha) \leq 0$, a contradiction. This proves (i).

(ii) All the relations except the Serre relations follow from the definition and properties of root systems. So only the Serre relations require proof. We prove only the relation involving f_i, the other one being the same for the opposite polarization. Consider the $(\mathfrak{sl}_2)_i$-submodule M_{ij} of $\mathfrak{g}$ generated by f_j. It is finite dimensional and we have $[h_i, f_j] = -a_{ij} f_j$, $[f_i, f_j] = 0$. Thus we must have $M_{ij} \cong V_{-a_{ij}}$. Hence $(\mathrm{ad} f_i)^{-a_{ij}+1} f_j = 0$. □

23.2. **Free Lie algebras.** Let $x_1, .., x_m$ be letters, and $\mathbf{k}$ a field. Then we can define the **free Lie algebra** $FL_m(\mathbf{k})$ generated by $x_1, ..., x_m$ (infinite dimensional for $m \geq 1$), which is spanned by all possible iterated commutators of $x_1, ..., x_m$ modulo the relations $[x, x] = 0$ and the Jacobi identity. This is a Lie algebra graded by positive integers (with finite dimensional graded pieces) whose universal enveloping algebra is just the free associative algebra $\mathbf{k}\langle x_1, ..., x_m\rangle$ with basis formed by all words in the alphabet $x_1, ..., x_m$.

For example, FL_2 is generated by x, y with $FL_2[1]$ having basis x, y, $FL_2[2]$ having basis $[x, y]$, $FL_2[3]$ having basis $[x, [x, y]]$, $[y, [x, y]]$, etc. Similarly, FL_3 is generated by x, y, z with $FL_3[1]$ having basis x, y, z, $FL_3[2]$ having basis $[x, y], [x, z], [y, z]$, $FL_3[3]$ having basis $[x, [x, y]]$, $[y, [x, y]]$, $[y, [y, z]]$, $[z, [y, z]]$, $[x, [x, z]$, $[z, [x, z]]$, $[x, [y, z]]$, $[y, [z, x]]$ (note that $[z, [x, y]]$ expresses in terms of the last two using the Jacobi identity).

The free Lie algebra has a universal property: for any Lie algebra $\mathfrak{g}$ over $\mathbf{k}$, $\mathrm{Hom}(FL_m(\mathbf{k}), \mathfrak{g}) = \mathfrak{g}^m$, i.e. a homomorphism from $FL_m(\mathbf{k})$ to $\mathfrak{g}$ is completely determined by the images of $x_1, ..., x_m$, which can be chosen arbitrarily.

Free Lie algebras deserve a much more detailed discussion (in particular, it requires a little bit of work to show that they are really well defined), which we will try to make in the spring semester.

23.3. **The Serre presentation for semisimple Lie algebras.** Now for any reduced root system R let $\mathfrak{g}(R)$ be the Lie algebra generated by $e_i, f_i, h_i, i = 1, ..., r$, with **defining relations** being the relations of Theorem 23.1. Precisely, this means that $\mathfrak{g}(R)$ is the quotient of the free Lie algebra $FL_{3r}(\mathbf{k})$ with generators e_i, f_i, h_i modulo the Lie ideal generated by the differences of the left and right hand sides of these relations.

Theorem 23.2. *(Serre) (i) The Lie subalgebra $\mathfrak{n}_+$ of $\mathfrak{g}(R)$ generated by e_i has the Serre relations $(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0$ as the defining relations.*

Similarly, the Lie subalgebra $\mathfrak{n}_-$ of $\mathfrak{g}(R)$ generated by f_i has the Serre relations $(\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0$ as the defining relations. In particular, $e_i, f_i \neq 0$ in $\mathfrak{g}(R)$ and thus h_i are linearly independent.

(ii) $\mathfrak{g}(R)$ is a sum of finite dimensional modules over every simple root subalgebra $(\mathfrak{sl}_2)_i = (e_i, f_i, h_i)$.

(iii) $\mathfrak{g}(R)$ is finite dimensional.

(iv) $\mathfrak{g}(R)$ is semisimple and has root system R.

Proof. It is easy to see that $\mathfrak{g}(R_1 \sqcup R_2) = \mathfrak{g}(R_1) \oplus \mathfrak{g}(R_2)$, so it suffices to prove the theorem for irreducible root systems.

(i) Consider the (in general, infinite dimensional) Lie algebra $\widetilde{\mathfrak{g}(R)}$ generated by e_i, f_i, h_i with the defining relations of Theorem 23.1 without the Serre relations. Then the relations imply that

$$\widetilde{\mathfrak{g}(R)} = \widetilde{\mathfrak{n}_+} \oplus \mathfrak{h}' \oplus \widetilde{\mathfrak{n}_-},$$

where $\widetilde{\mathfrak{n}_+}$ is generated by e_i, $\widetilde{\mathfrak{n}_-}$ is generated by f_i, and $\mathfrak{h}'$ is spanned by h_i (indeed, any commutator can be simplified to have only e_i, only f_i, or only a single h_i).

Lemma 23.3. *The Lie algebra $\widetilde{\mathfrak{n}_+}$ is free on the generators e_i and $\widetilde{\mathfrak{n}_-}$ is free on the generators f_i.*

Proof. We prove only the first statement, the second being the same for the opposite polarization. Let $\mathfrak{h}' \supset R$ be a vector space with basis h_i', $i = 1, ..., r$ and consider the Lie algebra $\mathfrak{a} := FL_r \rtimes \mathfrak{h}'$, where FL_r is freely generated by $f_1', ..., f_r'$ and

$$[h_i', f_j'] = -a_{ij} f_j', \ [h_i', h_j'] = 0.$$

Consider the universal enveloping algebra

$$U = U(\mathfrak{a}) = \mathbf{k}\langle f_1, ..., f_r\rangle \rtimes \mathbf{k}[h_1', ..., h_r'].$$

Then we can define an action of $\widetilde{\mathfrak{g}(R)}$ on the space U as follows: for $P \in \mathbf{k}[h_1', ..., h_r']$ and w a word in f_i' of weight $-\alpha$, we set

$$h_i(w \otimes P) = w \otimes (h_i' - \alpha(h_i))P, \ f_i(w \otimes P) = f_i' w \otimes P,$$

$$e_i(f_{j_1} ... f_{j_s} \otimes P) = \sum_{k: j_k = i} f_{j_1}' .... \widehat{f_{j_k}'} ... f_{j_s}' \otimes (h_{j_k}' - (\alpha_{j_{k+1}} + ... + \alpha_{j_s})(h_i'))P$$

(where the hat means that the corresponding factor is omitted). It is easy to check that this indeed defines an action, i.e., the relations of $\widetilde{\mathfrak{g}(R)}$ are satisfied (check it!). Thus we have a linear map $\mathfrak{g}(R) \to U$ given by $x \mapsto x(1)$. By the PBW theorem, the restriction of this map to $\widetilde{\mathfrak{n}_+}$ is a map $\phi : \widetilde{\mathfrak{n}}_+ \to FL_r$ which sends every iterated commutator of f_i to itself. This implies that ϕ is an isomorphism, i.e., $\widetilde{\mathfrak{n}_+}$ is free. $\square$

Now consider the element $S_{ij}^+ := (\mathrm{ad} e_i)^{1-a_{ij}} e_j$ in $\widetilde{\mathfrak{n}_+}$ and $S_{ij}^- := (\mathrm{ad} f_i)^{1-a_{ij}} f_j$ in $\widetilde{\mathfrak{n}_-}$. It is easy to check that $[f_k, S_{ij}] = 0$ (this follows easily from the representation theory of $\mathfrak{sl}_2$ -check it!). Therefore, setting I_+ to be the ideal in the Lie algebra $\widetilde{\mathfrak{n}_+}$ generated by S_{ij}^+, and I_- to be the ideal in the Lie algebra $\widetilde{\mathfrak{n}_-}$ generated by S_{ij}^-, we see that the ideal of Serre relations in $\mathfrak{g}(R)$ is $I_+ \oplus I_-$. Lemma 23.3 now implies (i).

(ii) The Serre relations imply that e_j generates the representation $V_{-a_{ij}}$ of $(\mathfrak{sl}_2)_i$ for $j \neq i$, and so does f_j. Also any element of $\mathfrak{h}$ generates V_0 or V_2 or the sum of the two, and e_i, f_i generate V_2. This implies (ii) since $\mathfrak{g}(R)$ is generated by e_i, f_i, h_i, and if x generates a representation X of $(\mathfrak{sl}_2)_i$ and y generates a representation Y then $[x, y]$ generates a quotient of $X \otimes Y$.

(iii) We have $\mathfrak{g}(R) = \oplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ are the subspaces of $\mathfrak{g}(R)$ of weight α, and $\mathfrak{g}_0 = \mathfrak{h}$. Let Q_+ be the $\mathbb{Z}_+$-span of α_i. Then $\mathfrak{g}_\alpha$ is zero unless $\alpha \in Q_+$ or $-\alpha \in Q_+$, and is finite dimensional for any α.

We will now show that if $\mathfrak{g}_\alpha \neq 0$ then $\alpha \in R$ or $\alpha = 0$, which implies (iii). It suffices to consider $\alpha \in Q_+$. We prove the statement by induction in the height $\mathrm{ht}(\alpha) = \sum_i k_i$ where $\alpha = \sum_i k_i \alpha_i$. The base case (height 1) is obvious, so we only need to justify the inductive step. We have $(\alpha, \omega_i^\vee) = k_i \geq 0$ for all i. If there is only one i with $k_i \geq 0$ then the statement is clear since $\mathfrak{g}_{m\alpha_i} = 0$ if $m \geq 2$. (as $\mathfrak{n}_+$ is generated by e_i). So assume that there are at least two such indices i. Since $(\alpha, \alpha) > 0$, there exists i such that $(\alpha, \alpha_i^\vee) > 0$. By the representation theory of $\mathfrak{sl}_2$, $\mathfrak{g}_{s_i\alpha} \neq 0$. Clearly, $s_i\alpha = \alpha - (\alpha, \alpha_i^\vee)\alpha_i \notin -Q_+$ (since $k_j > 0$ for at least two indices j), so $s_i\alpha \in Q_+$ but has height smaller than α (as $(\alpha, \alpha_i^\vee) > 0$). So by the induction assumption $s_i\alpha \in R$, which implies $\alpha \in R$. This proves (iii).

(iv) We see that $\mathfrak{g}(R) = \mathfrak{h} \oplus \oplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ are 1-dimensional. Let I be a nonzero ideal in $\mathfrak{g}$. Then $I \supset \mathfrak{g}_\alpha$ for some $\alpha \neq 0$. Also, by representation theory of $\mathfrak{sl}_2$, $I_\beta \neq 0$ implies $I_{w\beta} \neq 0$ for all $w \in W$. Thus $I_{\alpha_i} \neq 0$ for some i, i.e., $e_i \in I$. Hence $h_i, f_i \in I$. Now let J be the set of indices j for which $e_j, f_j, h_j \in I$ (or, equivalently, just $e_j \in I$); we have shown it is nonempty. Since $[h_j, e_k] = a_{jk} e_k$, we find that if $j \in J$ and $a_{jk} \neq 0$ (i.e., k is connected to j in the Dynkin diagram) then $k \in J$. Since the Dynkin diagram is connected, $J = [1, ..., r]$ and $I = \mathfrak{g}$. Thus $\mathfrak{g}$ is simple and clearly has root system R. This proves (iv) and completes the proof of the theorem. □

Corollary 23.4. *Isomorphism classes of simple Lie algebras over* $\mathbf{k}$ *are in bijection with Dynkin diagrams* A_n, $n \geq 1$, B_n, $n \geq 2$, C_n, $n \geq 3$, D_n, $n \geq 4$, E_6, E_7, E_8, F_4 *and* G_2.

24. Lecture 24

24.1. **Representations of semisimple Lie algebras.** We will now develop representation theory of complex semisimple Lie algebras. The representation theory of semisimple Lie algebras over algebraically closed field of characteristic zero is completely parallel, so we will stick to the complex case. So all representations will be over $\mathbb{C}$. We will mostly be interested in finite dimensional representations; as we know, they can be exponentiated to holomorphic representations of the corresponding simply connected complex Lie group group G, which defines a bijection between isomorphism classes of such representations of $\mathfrak{g}$ and G.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Recall that every finite dimensional representation of $\mathfrak{g}$ is completely reducible, so to classify finite dimensional representations it suffices to classify irreducible representations.

As in the simplest case of $\mathfrak{sl}_2$, a crucial tool is the decomposition of the representation in a direct sum of eigenspaces of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Definition 24.1. Let $\lambda \in \mathfrak{h}^*$, and V a representation of $\mathfrak{g}$ (possibly infinite dimensional). Then a vector $v \in V$ is said to have **weight** λ if $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$. The subspace of such vectors is denoted by $V[\lambda]$. If $V[\lambda] \neq 0$, we say that λ is a weight of V, and the set of weights of V is denoted by $P(V)$.

It is easy to see that $\mathfrak{g}_\alpha V[\lambda] \subset V[\lambda + \alpha]$.

Let $V' \subset V$ be the span of all weight vectors in V. Then it is clear that $V' = \oplus_{\lambda \in \mathfrak{h}} V[\lambda]$.

Definition 24.2. We say that V **has a weight decomposition** (with respect to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$) if $V' = V$, i.e., if $V = \oplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$.

Note that not every representation of $\mathfrak{g}$ has a weight decomposition (e.g., for $V = U(\mathfrak{g})$ with $\mathfrak{g}$ acting by left multiplication all weight subspaces are zero).

Proposition 24.3. *Any finite dimensional representation V of $\mathfrak{g}$ has a weight decomposition. Moreover, all weights of V are* **integral***, i.e., $P(V)$ is a finite subset of the weight lattice $P \subset \mathfrak{h}^*$ of $\mathfrak{g}$.*

Proof. For each $i = 1, ..., r$, V is a finite dimensional representation of the root subalgebra $(\mathfrak{sl}_2)_i$, so its element h_i acts semisimply on V. Thus $\mathfrak{h}$ acts semisimply on V, hence V has a weight decomposition. Also eigenvalues of h_i are integers, so for any $\lambda \in P(V)$ we have $\lambda(h_i) = (\lambda, \alpha_i^\vee) \in \mathbb{Z}$, hence $\lambda \in P$. $\square$

Definition 24.4. A vector v in $V[\lambda]$ is called a **highest weight vector of weight** λ if $e_i v = 0$ for all i, i.e., if $\mathfrak{n}_+ v = 0$. A representation V of $\mathfrak{g}$ is a **highest weight representation with highest weight** λ if it is generated by such a nonzero vector.

Proposition 24.5. *Any finite dimensional representation $V \neq 0$ contains a nonzero highest weight vector of some weight λ. Thus every irreducible finite dimensional representation of $\mathfrak{g}$ is a highest weight representation.*

Proof. Note that $P(V)$ is a finite set. Pick $\lambda \in P(V)$ so that $(\lambda, \rho^\vee)$ is maximal. Then $\lambda + \alpha_i \notin P(V)$ for any i, since $(\lambda + \alpha_i, \rho^\vee) = (\lambda, \rho^\vee) + 1$. Hence for any nonzero $v \in V[\lambda]$ (which exists as $\lambda \in P(V)$) we have $e_i v = 0$.

The second statement follows since an irreducible representation is generated by each its nonzero vector. ☐

24.2. **Verma modules.** Even though we are mostly interested in finite dimensional representations of $\mathfrak{g}$, it is useful to consider some infinite dimensional representations, which are called **Verma modules**.

The Verma module M_λ is defined as "the largest highest weight representation with highest weight λ". Namely, it is generated by a single highest weight vector v_λ with **defining relations** $hv = \lambda(h)v$ for $h \in \mathfrak{h}$ and $e_i v = 0$. More formally, we make the following definition.

Definition 24.6. Let $I_\lambda \in U(\mathfrak{g})$ be the left ideal generated by the elements $h - \lambda(h), h \in \mathfrak{h}$ and e_i, $i = 1, ..., r$. Then the **Verma module** M_λ is the quotient $U(\mathfrak{g})/I_\lambda$.

In this realization, the highest weight vector v_λ is just the class of the unit 1 of $U(\mathfrak{g})$.

Proposition 24.7. *The map $\phi : U(\mathfrak{n}_-) \to M_\lambda$ given by $\phi(x) = xv_\lambda$ is an isomorphism of left $U(\mathfrak{n}_-)$-modules.*

Proof. By the PBW theorem, the multiplication map

$$\xi : U(\mathfrak{n}_-) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+) \to U(\mathfrak{g})$$

is a linear isomorphism. It is easy to see that $\xi^{-1}(I_\lambda) = U(\mathfrak{n}_-) \otimes K_\lambda$, where

$$K_\lambda := \sum_i U(\mathfrak{h} \oplus \mathfrak{n}_+)(h_i - \lambda(h_i)) + \sum_i U(\mathfrak{h} \oplus \mathfrak{n}_+) e_i$$

is the kernel of the homomorphism $\chi_\lambda : U(\mathfrak{h} \oplus \mathfrak{n}_+) \to \mathbb{C}$ given by $\chi_\lambda(h) = \lambda(h)$, $h \in \mathfrak{h}$, $\chi_\lambda(e_i) = 0$. Thus, we have a natural isomorphism of left $U(\mathfrak{n}_-)$-modules

$$U(\mathfrak{n}_-) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+)/K_\lambda \to M_\lambda,$$

as claimed. □

Remark 24.8. The definition of M_λ means that it is the **induced module** $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h} \oplus \mathfrak{n}_+$ on which it acts via χ_λ.

Corollary 24.9. *M_λ has a weight decomposition with $P(M_\lambda) = \lambda - Q_+$, $\dim M_\lambda[\lambda] = 1$, and weight subspaces of M_λ are finite dimensional.*

Proposition 24.10. *(i) If V is a representation of $\mathfrak{g}$ and $v \in V$ is a vector such that $hv = \lambda(h)v$ for $h \in h$ and $e_iv = 0$ then there is a unique homomorphism $\eta : M_\lambda \to V$ such that $\eta(v_\lambda) = v$. In particular, if V is generated by such $v \neq 0$ (i.e., V is a highest weight representation with highest weight vector v) then V is a quotient of M_λ.*

(ii) Every highest weight representation has a weight decomposition into finite dimensional weight subspaces.

Proof. (i) Uniqueness follows from the fact that v_λ generates M_λ. To construct η, note that we have a natural map of $\mathfrak{g}$-modules $\widetilde{\eta} : U(\mathfrak{g}) \to V$ given by $\widetilde{\eta}(x) = xv$. Moreover, $\widetilde{\eta}|_{I_\lambda} = 0$ thanks to the relations satisfied by v, so $\widetilde{\eta}$ descends to a map $\eta : U(\mathfrak{g})/I_\lambda = M_\lambda \to V$. Moreover, if V is generated by v then this map is surjective, as desired.

(ii) This follows from (i) since a quotient of any representation with a weight decomposition must itself have a weight decomposition. □

Corollary 24.11. *Every highest weight representation V has a unique highest weight generator, up to scaling.*

Proof. Suppose v, w are two highest weight generators of V of weights λ, μ. If $\lambda = \mu$ then they are proportional since $\dim V[\lambda] \leq \dim M_\lambda[\lambda] = 1$, as V is a quotient of M_λ. On the other hand, if $\lambda \neq \mu$, then we can assume without loss of generality that $\lambda - \mu \notin Q_+$ (otherwise switch λ, μ). Then $\mu \notin \lambda - Q_+$, hence $\mu \notin P(V)$, a contradiction. □

Proposition 24.12. *For every $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has a unique irreducible quotient L_λ. Moreover, L_λ is a quotient of every highest weight $\mathfrak{g}$-module V with highest weight λ.*

Proof. Let $Y \subset M_\lambda$ be a proper submodule. Then Y has a weight decomposition, and cannot contain a nonzero multiple of v_λ (as otherwise $Y = M_\lambda$), so $P(Y) \subset (\lambda - Q_+) \setminus \{\lambda\}$. Now let J_λ be the sum of all proper submodules $Y \subset M_\lambda$. Then $P(J_\lambda) \subset (\lambda - Q_+) \setminus \{\lambda\}$, so J_λ is also a proper submodule of M_λ (the maximal one). Thus, $L_\lambda := M_\lambda/J_\lambda$

is an irreducible highest weight module with highest weight λ. Moreover, if V is any nonzero quotient of M_λ then the kernel K of the map $M_\lambda \to V$ is a proper submodule, hence contained in J_λ. Thus the surjective map $M_\lambda \to L_\lambda$ descends to a surjective map $V \to L_\lambda$. The kernel of this map is a proper submodule of V, hence zero if V is irreducible. Thus in the latter case $V \cong L_\lambda$. □

Corollary 24.13. *Irreducible highest weight $\mathfrak{g}$-modules are classified by their highest weight $\lambda \in \mathfrak{h}^*$, via the bijection $\lambda \mapsto L_\lambda$.*

24.3. **Finite dimensional modules.** Since every finite dimensional irreducible $\mathfrak{g}$-module is highest weight, it is of the form L_λ for λ belonging to some subset $P_F \subset P$, the set of weights λ such that L_λ is finite dimensional. So to obtain a final classification of finite dimensional irreducible representations of $\mathfrak{g}$, we should determine the subset P_F.

Let $P_+ \subset P$ be the intersection of P with the closure of the dominant Weyl chamber C_+; i.e., P_+ is the set of nonnegative integer linear combinations of the fundamental weights ω_i. In other words, P_+ is the set of $\lambda \in P$ such that $(\lambda, \alpha^\vee) \in \mathbb{Z}_+$. Weights belonging to P_+ are called **dominant integral**.

Proposition 24.14. *We have $P_F \subset P_+$.*

Proof. The vector v_λ is highest weight for $(\mathfrak{sl}_2)_i$ with highest weight $\lambda(h_i) = (\lambda, \alpha_i^\vee)$. This must be a nonnegative integer for the corresponding $\mathfrak{sl}_2$-module to be finite dimensional. □

Lemma 24.15. *If $\lambda \in P_+$ then in L_λ, we have $f_i^{\lambda(h_i)+1} v_\lambda = 0$.*

Proof. By the representation theory of $\mathfrak{sl}_2$, we have $e_i f_i^{\lambda(h_i)+1} v_\lambda = 0$. Also it is obvious that $e_j f_i^{\lambda(h_i)+1} v_\lambda = 0$ for $j \neq i$. Thus, $w := f_i^{\lambda(h_i)+1} v_\lambda$ is a highest weight vector in L_λ. So w cannot be a generator (as the highest weight generator is unique up to scaling). Thus w generates a proper submodule in L_λ, which must be zero since L_λ is irreducible. □

Lemma 24.16. *Let V be a $\mathfrak{g}$-module with weight decomposition into finite dimensional weight subspaces. If V is a sum of finite dimensional $(\mathfrak{sl}_2)_i$-modules for each $i = 1, ..., r$, then for each $\lambda \in P$ and $w \in W$, $\dim V[\lambda] = \dim V[w\lambda]$. In particular, $P(V)$ is W-invariant.*

Proof. It suffices to prove the statement for $w = s_i$, and in fact to prove that $\dim V[\lambda] \le \dim V[s_i\lambda]$ (as $s_i^2 = 1$).

If $(\lambda, \alpha_i^\vee) = m \ge 0$ then consider the operator $f_i^m : V[\lambda] \to V[s_i\lambda]$. We claim that this operator is injective, which implies the desired inequality. Indeed, let $v \in V[\lambda]$ be a nonzero vector and E be the representation of $(\mathfrak{sl}_2)_i$ generated by v. Then E is finite dimensional, and $v \in E[m]$, so by representation theory of $\mathfrak{sl}_2$, $f_i^m v \neq 0$, as claimed.

Similarly, if $(\lambda, \alpha_i^\vee) = -m \leq 0$ then the operator $e_i^m : V[\lambda] \to V[s_i\lambda]$ is injective. This proves the lemma. □

Now we are ready to state the main classification theorem.

Theorem 24.17. *For any $\lambda \in P_+$, L_λ is finite dimensional; i.e., $P_F = P_+$. Thus finite dimensional irreducible representations of $\mathfrak{g}$ are classified, up to an isomorphism, by their highest weight $\lambda \in P_+$, via the bijection $\lambda \mapsto L_\lambda$. Moreover, for any $\mu \in P$ and $w \in W$,* $\dim L_\lambda[\mu] = \dim L_\lambda[w\mu]$.

Proof. Since $f_i^{\lambda(h_i)+1} v_\lambda = 0$, we see that v_λ generates the irreducible finite dimensional $(\mathfrak{sl}_2)_i$-module of highest weight $\lambda(h_i)$. Also, every nonzero element of $\mathfrak{g}$ generates a finite dimensional $(\mathfrak{sl}_2)_i$-module. Hence every vector in L_λ generates a finite dimensional $(\mathfrak{sl}_2)_i$-module. Thus by Lemma 24.16, $P(L_\lambda)$ is W-invariant.

Now let $\mu \in P(L_\lambda) \cap P_+$. Then $\mu = \lambda - \beta$, $\beta \in Q_+$, so $(\mu, \rho^\vee) = (\lambda, \rho^\vee) - (\beta, \rho^\vee) \leq (\lambda, \rho^\vee)$. So if $\mu = \sum m_i \omega_i$, $m_i \in \mathbb{Z}_+$ then $\sum_i m_i(\omega_i, \rho^\vee) \leq (\lambda, \rho^\vee)$. Since $(\omega_i, \rho^\vee) \geq \frac{1}{2}$, this implies that $\mu \in P(L_\lambda) \cap P_+$ is finite. But we know that $WP_+ = P$, hence $W(P(L_\lambda) \cap P_+) = P(L_\lambda)$, as $P(L_\lambda)$ is W-invariant. It follows that $P(L_\lambda)$ is finite, hence L_λ is finite dimensional. □

25. Lecture 25

25.1. **Characters.** Let V be a finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$. Recall that the action of $\mathfrak{g}$ on V can be exponentiated to the action of the corresponding simply connected complex Lie group G. Recall also that the **character** of a finite dimensional representation V of any group G is the function

$$\chi_V(g) = \mathrm{Tr}|_V(g).$$

Let us compute this character in our case. To this end, let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $h \in \mathfrak{h}$, and let us compute $\chi_V(e^h)$. Note that this completely determines χ_V since it determines $\chi_V(e^x)$ for any semisimple element $x \in \mathfrak{g}$, and semisimple elements form a dense open set in $\mathfrak{g}$ (complement of zeros of some polynomial). So elements of the form e^x as above form a dense open set at least in some neighborhood of 1 in G, and an analytic function is determined by its values on any nonempty open set.

We know that V has a weight decomposition: $V = \oplus_{\mu\in P} V[\mu]$. Thus we have

$$\chi_V(e^h) = \sum_{\mu\in P} \dim V[\mu] e^{\mu(h)}.$$

Consider the group algebra $\mathbb{Z}[P]$. It sits naturally inside the algebra of analytic functions on $\mathfrak{h}$ via $\lambda \mapsto e^\lambda$, where $e^\lambda(h) := e^{\lambda(h)}$, and we see that $\chi_V \in \mathbb{Z}[P]$, namely

$$\chi_V = \sum_{\mu\in P} \dim V[\mu] e^{\mu}.$$

We will call the element χ_V the **character** of V.

25.2. **Category $\mathcal{O}$.** Note that the above definition of character is a purely formal algebraic definition, i.e., χ_V is simply the generating function of dimensions of weight subspaces of V. So it makes sense for any (possibly infinite dimensional) representation V with a weight decomposition into finite dimensional weight subspaces, except we may obtain an infinite sum. More precisely, we make the following definition.

Definition 25.1. The category $\mathcal{O}$ is the category of representations V of $\mathfrak{g}$ with weight decomposition into finite dimensional weight spaces $V = \oplus_{\mu\in P} V[\mu]$, such that $P(V)$ is contained in the union of sets $\lambda^i - Q_+$ for a finite collection of weights $\lambda^1, ..., \lambda^N \in P$ (depending on V).

For example, any highest weight module belongs to $\mathcal{O}$.

Let R be the ring of series $a := \sum_{\mu\in P} a_\mu e^\mu$ ($a_\mu \in \mathbb{Z}$) such that the set $P(a)$ of μ with $a_\mu \neq 0$ is contained in the union of sets $\lambda^i - Q_+$ for a finite collection of weights $\lambda^1, ..., \lambda^N \in P$. Then for every $V \in \mathcal{O}$ we can define the character $\chi_V \in R$. Moreover, it is easy to see that if

$$0 \to X \to Y \to Z \to 0$$

is a short exact sequence in $\mathcal{O}$ then $\chi_Y = \chi_X + \chi_Z$, and that for any $V, U \in \mathcal{O}$ we have $V \otimes U \in \mathcal{O}$ and $\chi_{V\otimes U} = \chi_V \chi_U$.

Example 25.2. Let $V = M_\lambda$ be the Verma module. Recall that as a vector space $M_\lambda = U(\mathfrak{n}_-) = \otimes_{\alpha\in R_+} \mathbb{C}[e_{-\alpha}]$ (using the PBW theorem). Thus

$$\sum_\mu U(\mathfrak{n}_-)[\mu] e^\mu = \frac{1}{\prod_{\alpha\in R_+}(1 - e^{-\alpha})}.$$

Hence

$$\chi_{M_\lambda} = \frac{e^\lambda}{\prod_{\alpha\in R_+}(1 - e^{-\alpha})}.$$

It is convenient to rewrite this formula as follows:

$$\chi_{M_\lambda} = \frac{e^{\lambda+\rho}}{\Delta},\ \Delta := \prod_{\alpha\in R_+}(e^{\alpha/2} - e^{-\alpha/2}).$$

The polynomial Δ is called the **Weyl denominator**.

Note that we have a homomorphism $\varepsilon : W \to \mathbb{Z}/2$ given by $w \to \det(w|_{\mathfrak{h}})$, i.e. $w \mapsto (-1)^{l(w)}$; it is defined on simple reflections by $s_i \mapsto -1$. This homomorphism is called the **sign character**. For example, for type A_{n-1} this is the sign of a permutation in S_n. We will say that an element of $f \in \mathbb{C}[P]$ is anti-invariant under W is $w(f) = (-1)^{l(w)} f$ for all $w \in W$.

Proposition 25.3. *The Weyl denominator Δ is anti-invariant under W.*

Proof. Since s_i permutes positive roots not equal to α_i and send α_i to $-\alpha_i$, it follows that $s_i\Delta = -\Delta$. □

25.3. **Weyl character formula.**

Theorem 25.4. *(Weyl character formula) For any $\lambda \in P_+$ the character $\chi_\lambda := \chi_{L_\lambda}$ is given by*

$$\chi_{L_\lambda} = \frac{\sum_{w\in W}(-1)^{l(w)} e^{w(\lambda+\rho)}}{\Delta}.$$

The proof of this theorem is in the next subsection.

Corollary 25.5. *(Weyl denominator formula) One has*

$$\Delta = \sum_{w\in W}(-1)^{l(w)} e^{w\rho}.$$

Proof. This follows from the Weyl character formula by setting $\lambda = 0$ (as $L_0 = \mathbb{C}$ is the trivial representation). □

25.4. **Proof of the Weyl character formula.** Consider the product $\Delta\chi_\lambda \in \mathbb{Z}[P]$. We know that χ_λ is W-invariant, so this product is W-antiinvariant. Thus,

$$\Delta\chi_\lambda = \sum c_\mu e^\mu,$$

where $c_{w\mu} = (-1)^{l(w)} c_\mu$. Moreover, $c_\mu = 0$ unless $\mu \in \lambda + \rho - Q_+$, and $c_{\lambda+\rho} = 1$. Thus to prove the Weyl character formula, we need to show that $c_\mu = 0$ if $\mu \in P_+ \cap (\lambda + \rho - Q_+)$ and $\mu \neq \lambda + \rho$.

To this end, we will construct the above decomposition $\Delta\chi_\lambda$ using representation theory, so that the vanishing property we need is apparent from the construction.

First recall that we have the Casimir element C of $U(\mathfrak{g})$ given by the formula $C = \sum a_i^2$ for an orthonormal basis $a_i \in \mathfrak{g}$. This element is central, so acts by a scalar on every highest weight (in particular, finite dimensional irreducible) representation. We can write C in the form

$$C = \sum_j x_j^2 + \sum_{\alpha \in R_+} (e_{-\alpha} e_\alpha + e_\alpha e_{-\alpha}),$$

for an orthonormal basis x_j of $\mathfrak{h}$. Since $[e_\alpha, e_{-\alpha}] = h_\alpha$, we find that

$$C = \sum_j x_j^2 + 2 \sum_{\alpha \in R_+} e_{-\alpha} e_\alpha + \sum_{\alpha \in R_+} h_\alpha.$$

Thus we get

Lemma 25.6. *If V is a highest weight representation with highest weight λ then $C|_V = (\lambda, \lambda + 2\rho) = |\lambda + \rho|^2 - |\rho|^2$.*

Now we will define a sequence of modules $K(b)$ from category $\mathcal{O}$ parametrized by some binary strings b. This is done inductively. We set $K(\emptyset) = L_\lambda$. Now suppose $K(b)$ is already defined. If $K(b) = 0$, we do not define $K(b')$ for any longer string b' starting with b. Otherwise, pick a nonzero vector $v_b \in K(b)$, of some weight $\nu(b) \in \lambda - Q_+$ such that the height of $\lambda - \nu(b)$ is minimal possible. Then v_b is a highest weight vector, and we can consider the corresponding the homomorphism $\xi_b : M_{\nu_b} \to K(b)$. Let $K(b1), K(b0)$ be the kernel and cokernel of ξ_b. We have

$$\chi_{K(b1)} - \chi_{M_{\nu(b)}} + \chi_{K(b)} - \chi_{K(b0)} = 0.$$

Thus we have

$$\chi_{K(b)} = \chi_{M_{\nu(b)}} - \chi_{K(b1)} + \chi_{K(b0)}.$$

It is clear that for every b and μ, there is b' starting with b such that $K_{b'}[\mu] = 0$. So iterating this formula starting with $b = \emptyset$, we will get

$$\chi_\lambda = \sum_b (-1)^{\Sigma(b)} \chi_{M_{\nu(b)}}$$

where Σ is the sum of digits of b. So

$$\Delta \chi_\lambda = \sum_b (-1)^{\Sigma(b)} e^{\nu(b)+\rho}.$$

Also note that by induction in the length of b we can conclude that the eigenvalue of C on $M_{\nu(b)}$ is $|\lambda + \rho|^2 - |\rho|^2$, which implies that

$$|\nu(b) + \rho|^2 = |\lambda + \rho|^2$$

for all b.

So it remains to show that if $\mu = \lambda + \rho - \beta \in P_+$ with $\beta \in Q_+$ and $\beta \neq 0$ then $|\mu|^2 < |\lambda + \rho|^2$. Indeed,

$$|\lambda + \rho|^2 - |\mu|^2 = |\lambda + \rho|^2 - |\lambda - \beta + \rho|^2 =$$
$$2(\lambda + \rho, \beta) - |\beta|^2 > (\lambda + \rho, \beta) - |\beta|^2 = (\lambda + \rho - \beta, \beta) \geq 0.$$

This completes the proof of the Weyl character formula.

18.755: Lie Groups and Lie algebras II

26. Lecture 1

This lecture starts the second half of the course "Lie groups and Lie algebras". We begin by reviewing the material covered in the previous semester.

26.1. **Generalities on Lie groups.** A real (complex) Lie group is a real (complex) analytic manifold G with a group structure such that the group law is a regular (i.e., analytic) map $G \times G \to G$. A complex Lie group can be regarded as a real one by forgetting the complex structure. A homomorphism of Lie groups is a regular group homomorphism between Lie groups. Examples of real Lie groups: $\mathbb{R}^n$, $U(n)$, $SU(n)$, $GL_n(\mathbb{R})$, $O(p,q)$, $Sp_{2n}(\mathbb{R})$. Examples of complex Lie groups: $\mathbb{C}^n$, $GL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$.

The connected component of the identity $G^\circ \subset G$ is a normal subgroup, and G/G° is a discrete countable group.

A connected Lie group G has a universal covering $\widetilde{G}$ which is a simply connected Lie group, and we have a surjective homomorphism $\widetilde{G} \to G$ whose kernel is a discrete central subgroup $Z \subset \widetilde{G}$ isomorphic to $\pi_1(G)$. In particular, $\pi_1(G)$ is abelian.

A Lie subgroup of a Lie group G is a subgroup $H \subset G$ which is also an immersed submanifold (i.e. H is a Lie group and the map $H \to G$ is regular with injective differential everywhere). A Lie subgroup $H \subset G$ is closed if it is moreover an embedded submanifold (i.e., is locally closed); equivalently, if H is a closed subset in G. For example, the irrational torus winding is a Lie subgroup but not a closed one. Example of a closed Lie subgroup: $O_n(\mathbb{K}) \subset GL_n(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In fact, any closed subgroup of a Lie group is a closed Lie subgroup, but we did not prove that.

Any connected Lie group is generated by any neighborhood of 1.

If $H \subset G$ is a closed Lie subgroup then G/H is a manifold with a transitive action of G, called a homogeneous G-space. If moreover H is normal then G/H is a Lie group. Also if a Lie group G acts transitively on a manifold X (i.e., the action map $G \times X \to X$ is regular) then $X = G/G_x$, where G_x is the stabilizer of a point $x \in X$, which is a closed Lie subgroup of G. More generally, for any regular action of G on X (not necessarily transitive), the orbit $Gx \subset X$ is a manifold (as $Gx = G/G_x$) which is an immersed submanifold of X.

A linear action of a Lie group G on a finite dimensional vector space V is called a representation of G. We will usually consider complex representations (even for real Lie groups). Thus an n-dimensional representation of G is just a homomorphism $G \to GL_n(\mathbb{C})$. The usual notions of representation theory apply verbatim to representations of Lie groups - homomorphisms (intertwining operators), subrepresentations, quotient representations, irreducible and indecomposable representations, dual representation, tensor product. Schur's lemma: if V, W are irreducible non-isomorphic complex representations of G then every homomorphism $V \to W$ is zero and every homomorphism $V \to V$ is a scalar. Example: the adjoint representation of G on $T_1G = \mathfrak{g}$ is the differential at 1 of the action of G on itself by conjugation. A unitary representation is one preserving a positive definite Hermitian form, i.e., given by a homomorphism $G \to U(n)$. Such representations are completely reducible (direct sums of irreducible representations). This implies that any finite dimensional complex representation of a finite group is completely reducible.

26.2. **Lie algebras.** A Lie group G acts on itself by right translations. Therefore, every element $a \in \mathfrak{g} = T_1G$ gives rise to a unique left invariant vector field $\mathbf{L}_a$ on G such that $\mathbf{L}_a|_1 = a$; namely, $\mathbf{L}_a$ is obtained by left translations of a: i.e., $\mathbf{L}_a|g = ga$.

Recall that a vector field on a manifold X may be viewed as a coherent system of derivations of the algebras of regular functions on open subsets $U \subset X$. So we may consider the operator $[\mathbf{L}_a, \mathbf{L}_b] = \mathbf{L}_a\mathbf{L}_b - \mathbf{L}_b\mathbf{L}_a$. It is easy to check that this is a left-invariant derivation, so we have

$$[\mathbf{L}_a, \mathbf{L}_b] = \mathbf{L}_{[a,b]},$$

where $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map. The map $[,]$ is called the commutator or Lie bracket. It satisfies the skew symmetry

$$[a, a] = 0$$

(hence $[a, b] = -[b, a]$) and Jacobi identit

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

A vector space $\mathfrak{g}$ over any field with a bilinear operation $[,]$ satisfying these properties is called a Lie algebra. So in our case $\mathfrak{g} = T_1G$ is a Lie algebra, which is called the Lie algebra of G, denoted Lie(G).

A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ closed under the commutator. Such subalgebra is called an ideal if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. In the latter case $\mathfrak{g}/\mathfrak{h}$ is naturally a Lie algebra. If $H \subset G$ is a Lie subgroup then Lie$H$ $\subset$ Lie$G$ is a Lie subalgebra, which is an ideal if H is a normal subgroup, and in the latter case Lie(G/H) = Lie(G)/Lie(H).

An example of an ideal in $\mathfrak{g}$ is its center $\mathfrak{z}(\mathfrak{g})$, the set of elements $z \in \mathfrak{g}$ such that $[z, x] = 0$ for all $x \in \mathfrak{g}$.

Example: $\mathrm{Lie}(GL_n(\mathbb{K})) = \mathfrak{gl}_n(\mathbb{K})$, the Lie algebra of all matrices with operation $[a, b] = ab - ba$. $\mathrm{Lie}(O_n(\mathbb{K})) = \mathfrak{so}_n(\mathbb{K})$, the Lie subalgebra of skew-symmetric matrices with the same operation.

A finite dimensional representation of a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}_n(\mathbb{K})$. The general notions and results of group representation theory apply to Lie algebras with obvious modifications.

26.3. **Exponential map.** Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}G$. Given $a \in \mathfrak{g}$, we can define the 1-parameter subgroup $g(t) = \exp(ta)$, $t \in \mathbb{K}$, which by definition is the solution of the differential equation $g'(t) = g(t)a$, i.e., $g'(t) = \mathbf{L}_a|_{g(t)}$, with initial condition $g(0) = 1$. The image of the map $t \mapsto \exp(ta)$ for $a \neq 0$ is a 1-dimensional Lie subgroup of G (which may not be closed). In particular, setting $t = 1$, we obtain a regular map $\exp : \mathfrak{g} \to G$. For example, for $G = GL_n(\mathbb{K})$ it is the usual matrix exponential, hence the notation.

The differential of exp at 1 is the identity, so it is locally invertible near $1 \in G$ (even though not globally so, in general). The inverse is called the logarithm and denoted $\log : U \to \mathfrak{g}$ where $U \subset G$ is a neighborhood of 1. For $G = GL_n(\mathbb{K})$ this is the usual logarithm for matrices. We have

$$\log(\exp(a)\exp(b)) = a + b + \frac{1}{2}[a, b] + \text{ cubic and higher terms.}$$

One also has

$$\log(\exp(a)\exp(b)\exp(-a)\exp(-b)) = [a, b] + \text{ cubic and higher terms.}$$

Thus $[a, b]$ is twice the quadratic part of $\log(\exp(a)\exp(b))$, and the leading part of $\log(\exp(a)\exp(b)\exp(-a)\exp(-b))$ near 0, which gives another definition of the Lie bracket $[a, b]$. This shows that the Lie bracket measures the failure of G to be commutative; in particular, if G is commutative then $[,] = 0$, and the converse holds for connected groups.

26.4. **Fundamental theorems of Lie theory.** The general theory of Lie groups rests on the following three fundamental theorems, which hold for both real and complex Lie groups.

Theorem 26.1. *(First fundamental theorem of Lie theory) For a Lie group G, there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}G$, given by $\mathfrak{h} = \mathrm{Lie}H$.*

Theorem 26.2. *(Second fundamental theorem of Lie theory) If G and K are Lie groups with G simply connected then the map*

$$\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$$

given by $\phi \mapsto \phi_*$ is a bijection.

Theorem 26.3. *(Third fundamental theorem of Lie theory) Any finite dimensional Lie algebra is the Lie algebra of a Lie group.*

We proved the first two theorems but not the third one.
These theorem imply

Corollary 26.4. *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the assignment $G \mapsto \mathrm{Lie}G$ is an equivalence between the category of simply connected $\mathbb{K}$-Lie groups and the category of finite dimensional $\mathbb{K}$-Lie algebras. Moreover, any connected Lie group K has the form G/Γ where G is simply connected and $\Gamma \subset G$ is a discrete central subgroup.*

26.5. **Representations of $\mathfrak{sl}_2$.** The Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ of the Lie group $SL_2(\mathbb{C})$ has basis e, f, h with

$$[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f.$$

Its irreducible representations (or, equivalently, irreducible representations of the Lie group $SL_2(\mathbb{C})$) are the $n+1$-dimensional representations V_n, $n \geq 0$, realized on homogeneous polynomials of degree n in two variables x, y. The operator $h : V_n \to V_n$ is diagonalizable with eigenvalues (called weights) $n, n-2, ..., -n$, so V_n is called the irreducible representation with highest weight n. Any finite dimensional representation is a direct sum of irreducible ones. Moreover, we have the Clebsch-Gordan rule

$$V_n \otimes V_m = \oplus_{i=1}^{\min(m,n)} V_{|m-n|+2i-1}.$$

26.6. **Universal enveloping algebra.** If $\mathfrak{g}$ is a Lie algebra then its universal enveloping algebra is the algebra freely generated by $\mathfrak{g}$ (i.e., the tensor algebra $T\mathfrak{g} = \oplus_{n\geq 0}\mathfrak{g}^{\otimes n}$) modulo the relations

$$x \otimes y - y \otimes x = [x, y], x, y \in \mathfrak{g}.$$

Then representations of $\mathfrak{g}$ is the same thing as representations of $U(\mathfrak{g})$. It is easy to see that if $x_i, i \in I$ is a basis of $\mathfrak{g}$ then elements $\prod_i x_i^{n_i}$ (ordered products under some total ordering on I) span $U(\mathfrak{g})$. The nontrivial Poincaré-Birkhoff-Witt theorem (which we proved) states that in fact these elements are linearly independent, i.e., form a basis of $U(\mathfrak{g})$.

26.7. **Solvable and nilpotent Lie algebras.** For a Lie algebra $\mathfrak{g}$ let $D(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}]$. A Lie algebra $\mathfrak{g}$ is solvable if $D^n(\mathfrak{g}) = 0$ for some n. Also define $L_i(\mathfrak{g})$ recursively by $L_1(\mathfrak{g}) = \mathfrak{g}$ and $L_i(\mathfrak{g}) := [\mathfrak{g}, L_{i-1}(\mathfrak{g})]$. A Lie algebra $\mathfrak{g}$ is called nilpotent if $L_n(\mathfrak{g}) = 0$ for some n. It is easy to see that a nilpotent Lie algebra is solvable. Example: the Lie algebra of strictly upper triangular matrices is nilpotent and the Lie algebra of all upper triangular matrices is solvable but not nilpotent (for matrices of size ≥ 2).

We proved the following two important theorems.

Theorem 26.5. *(Lie) If $\mathfrak{g}$ is a finite dimensional solvable Lie algebra over an algebraically closed field of characteristic zero then every irreducible representation of $\mathfrak{g}$ is 1-dimensional. Thus every finite dimensional representation of $\mathfrak{g}$ has a basis in which all elements of $\mathfrak{g}$ act by upper triangular matrices.*

Theorem 26.6. *(Engel) A finite dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if every element $x \in \mathfrak{g}$ is nilpotent, i.e. the operator $\mathrm{ad}_x = [x, ?] : \mathfrak{g} \to \mathfrak{g}$ is nilpotent.*

26.8. **Semisimple and reductive Lie algebras, the radical.** From now on we consider finite dimensional Lie algebras over an algebraically closed field of characteristic zero, e.g. $\mathbb{C}$.

The sum of all solvable ideals in a Lie algebra $\mathfrak{g}$ is itself its (largest) solvable ideal, called the radical of $\mathfrak{g}$ and denoted $\mathrm{rad}(\mathfrak{g})$. We call $\mathfrak{g}$ semisimple if $\mathrm{rad}(\mathfrak{g}) = 0$. So for any $\mathfrak{g}$ the Lie algebra $\mathfrak{g}_{ss} := \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple. A semisimple Lie algebra is the same thing as a direct sum of simple Lie algebras, i.e. ones that have no ideals but $0, \mathfrak{g}$. Example: $\mathfrak{sl}_n$ is a simple Lie algebra, so is $\mathfrak{so}_n$ for $n \geq 3, n \neq 4$. For $n = 4$ we have $\mathfrak{so}_4 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, so it is semisimple but not simple.

A semisimple Lie algebra $\mathfrak{g}$ is the Lie algebra of the connected Lie group $G = \mathrm{Aut}(\mathfrak{g})^\circ$.

We proved the following important result on the structure of finite dimensional Lie algebras.

Theorem 26.7. *(Levi decomposition) We have $\mathfrak{g} \cong \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}$, where $\mathfrak{g}_{\mathrm{ss}} \subset \mathfrak{g}$ is a semisimple subalgebra (but not necessarily an ideal); i.e., $\mathfrak{g}$ is isomorphic to the semidirect product $\mathfrak{g}_{\mathrm{ss}} \ltimes \mathrm{rad}(\mathfrak{g})$. In other words, the projection $p : \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}}$ admits an (in general, non-unique) splitting $q : \mathfrak{g}_{\mathrm{ss}} \to \mathfrak{g}$, i.e., a Lie algebra map such that $p \circ q = \mathrm{Id}$.*

A reductive Lie algebra $\mathfrak{g}$ is one where the radical $\mathrm{rad}(\mathfrak{g})$ coincides with the center of $\mathfrak{g}$. A reductive Lie algebra is the same thing as a direct sum of a semisimple Lie algebra and an abelian Lie algebra.

26.9. **The Killing form and the Cartan criterion.**

Definition 26.8. The Killing form of a Lie algebra $\mathfrak{g}$ is the form $B_{\mathfrak{g}}(x, y) = \mathrm{tr}(\mathrm{ad}x\mathrm{ad}y)$.

The Killing form is denoted by $K_{\mathfrak{g}}(x, y)$ or shortly by $K(x, y)$. It is an invariant symmetric bilinear form, i.e., $K([x, y], z) = K(x, [y, z])$. We proved the following two important theorems.

Theorem 26.9. *(Cartan criterion of solvability) A Lie algebra $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}] \subset \mathrm{Ker}(K)$.*

Theorem 26.10. *(Cartan criterion of semisimplicity) A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate.*

More generally, for a representation V of $\mathfrak{g}$ with representation map $\rho : \mathfrak{g} \to \mathrm{End}V$ we can consider the invariant form

$$B_V(x, y) = \mathrm{Tr}(\rho(x)\rho(y)).$$

If this form is nondegenerate for some V then $\mathfrak{g}$ is reductive.

26.10. **Complete reducibility of representations.**

Theorem 26.11. *Every finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ over a field of characteristic zero is completely reducible, i.e., isomorphic to a direct sum of irreducible representations.*

27. Lecture 2

27.1. **Jordan decomposition, Cartan subalgebras.** Let $\mathfrak{g}$ be a semisimple Lie algebra. An element $x \in \mathfrak{g}$ is semisimple if the operator $\mathrm{ad}x$ is semisimple and nilpotent if this operator is nilpotent. The Jordan decomposition theorem: any element $x \in \mathfrak{g}$ admits a unique decomposition $x = x_s + x_n$ where x_s is semisimple, x_n is nilpotent and $[x_s, x_n] = 0$.

A Cartan subalgebra in $\mathfrak{g}$ is a maximal commutative subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting of semisimple elements.

Theorem 27.1. *All Cartan subalgebras of $\mathfrak{g}$ are conjugate by the action of the corresponding connected Lie group G.*

In particular, all of them have the same dimension, called the rank of $\mathfrak{g}$.

Example: a Cartan subalgebra of $\mathfrak{sl}_n$ is the subalgebra $\mathfrak{h}$ of diagonal matrices of trace zero. So its rank is $n - 1$.

27.2. **Root decomposition.**

Proposition 27.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and B a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g., the Killing form).*

(i) We have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the subspace of $x \in \mathfrak{g}$ such that for $h \in \mathfrak{h}$ we have $[h, x] = \alpha(h)x$, and R is the (finite) set of $\alpha \in \mathfrak{h}^*$, $\alpha \neq 0$, such that $\mathfrak{g}_\alpha \neq 0$.

(ii) We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(iii) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under B.

(iv) B restricts to a nondegenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$.

(v) $\dim \mathfrak{g}_\alpha = 1$.

Definition 27.3. The set R is called the **root system** of $\mathfrak{g}$ and its elements are called **roots**.

We showed that the roots span a real subspace $E = \mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$ whose complexification is $\mathfrak{h}^*$, and the restriction of the inverse Killing form to this subspace is positive definite.

27.3. **Abstract root systems.** Let $E \cong \mathbb{R}^r$ be a Euclidean space with a positive inner product.

Definition 27.4. An **abstract root system** is a finite set $R \subset E \setminus 0$ satisfying the following axioms:

(R1) R spans E;

(R2) For all $\alpha, \beta \in R$ the number $n_{\alpha\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer;

(R3) If $\beta \in R$ then $s_\alpha(\beta) := \beta - n_{\alpha\beta}\alpha \in R$.

Elements of R are called **roots**. The number $r = \dim E$ is called the **rank** of R.

Definition 27.5. A root system R is **reduced** if for $\alpha, c\alpha \in R$, we have $c = \pm 1$. It is called **irreducible** if it cannot be written as a direct product of two root systems.

The set of roots $R \subset \mathfrak{h}^*$ of a semisimple Lie algebra $\mathfrak{g}$ is a reduced root system, which is irreducible if and only if $\mathfrak{g}$ is simple.

27.4. **The Weyl group.**

Definition 27.6. The **Weyl group** of a root system R is the group of automorphisms of E generated by s_α.

Proposition 27.7. *W is a finite subgroup of $O(E)$ which preserves R.*

27.5. **Positive and simple roots.** Let R be a reduced root system and $t \in E^*$ be such that $t(\alpha) \neq 0$ for any $\alpha \in R$. We say that a root is **positive** (with respect to t) if $t(\alpha) > 0$ and **negative** if $t(\alpha) < 0$. The set of positive roots is denoted by R_+ and of negative ones by R_-, so $R_+ = -R_-$ and $R = R_+ \cup R_-$ (disjoint union). This decomposition is called a **polarization** of R; it depends on the choice of t.

The Weyl group W acts simply transitively on polarizations, so there is only one up to symmetry.

Definition 27.8. A root $\alpha \in R_+$ is **simple** if it is not a sum of two other positive roots.

Lemma 27.9. *Every positive root is a sum of simple roots.*

Lemma 27.10. *If $\alpha, \beta \in R_+$ are simple roots then $(\alpha, \beta) \leq 0$.*

Theorem 27.11. *The set $\Pi \subset R_+$ of simple roots is a basis of E.*

Corollary 27.12. *Any root $\alpha \in R$ can be uniquely written as $\alpha = \sum_{i=1}^r n_i \alpha_i$, where $n_i \in \mathbb{Z}$. If α is positive then $n_i \geq 0$ and if α is negative then $n_i \leq 0$.*

27.6. **Classification of reduced irreducible root systems.** For example, the roots of $\mathfrak{sl}_n$ are $e_i - e_j$ inside the space E of vectors in $\mathbb{R}^n$ with zero sum of coordinates, and $W = S_n$ is the symmetric group. Other classical root systems are B_n of $\mathfrak{so}_{2n+1}$, C_n of $\mathfrak{sp}_{2n}$ and D_n of $\mathfrak{so}_{2n}$. There are also 5 exceptional root systems, G_2, F_4, E_6, E_7, E_8.

We have shown that this exhausts all reduced irreducible root systems, and discussed how these root systems are encoded by Dynkin diagrams. Such a diagram is attached to the Cartan matrix of R, an r by r matrix (where r is the rank of R) with entries $a_{ij} = (\alpha_i^\vee, \alpha_j)$, which are integers. We have $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, $a_{ij} = 0$ iff $a_{ji} = 0$, $a_{ij}a_{ji} \in [0, 3]$ for $i \neq j$. The Weyl group is generated by the simple reflections $s_i = s_{\alpha_i}$ with defining relations

$$s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1,$$

where $m_{ij} = 2, 3, 4, 6$ if $a_{ij}a_{ji} = 0, 1, 2, 3$.

27.7. **The dual root system, root and weight lattices.** Given a root system $R \subset E$ and a root $\alpha \in R$, define the corresponding **coroot** $\alpha^\vee \in E^*$ by $s_\alpha \alpha^\vee = -\alpha^\vee$ and $(\alpha, \alpha^\vee) = 2$. The coroots $\alpha^\vee$ span the **dual root system** $R^\vee \subset E^*$ with the same Weyl group. For example, the dual root system in B_n is C_n, while other reduced irreducible root systems are self-dual.

In particular, for a root system R we can define the **root lattice** $Q \subset E$, which is generated by the simple (or all) roots. Similarly, we can define the **coroot lattice** $Q^\vee \subset E^*$ generated by $\alpha^\vee, \alpha \in R$, which is just the root lattice of $R^\vee$.

Also we define the **weight lattice** $P \subset E$ to be the dual lattice to $Q^\vee$: $P = (Q^\vee)^*$, and the **coweight lattice** $P^\vee \subset E^*$ to be the dual lattice to Q: $P^\vee = Q^*$, so $P^\vee$ is the weight lattice of $R^\vee$. Thus

$$P = \{\lambda \in E : (\lambda, \alpha^\vee) \in \mathbb{Z}\, \forall \alpha \in R\},\ P^\vee = \{\lambda \in E^* : (\lambda, \alpha) \in \mathbb{Z}\, \forall \alpha \in R\}.$$

The **fundamental weights** of R (for some polarization) are elements $\omega_i \in P$ defined by $(\omega_i, \alpha_j^\vee) = \delta_{ij}$. A **dominant integral weight** is a weight of the form $\lambda = \sum_i n_i \omega_i$ where n_i are nondegative integers. The set of such weights is denoted by P_+.

27.8. **Serre relations.** It turns out that a semisimple Lie algebra can be uniquely reconstructed from its Cartan matrix (or Dynkin diagram), and in particular any reduced irreducible root system gives rise to a unique simple Lie algebra. This is done using the so called **Serre presentation**. Namely, let $\mathfrak{g}$ be a simple Lie algebra, $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$, normalized so that $h_i = [e_i, f_i] = \alpha_i^\vee \in \mathfrak{h}$.

Theorem 27.13. *(Serre relations) (i) The elements* e_i, f_i, h_i, $i = 1, ..., r$ *generate* $\mathfrak{g}$.

(ii) These elements satisfy the following relations:

$$[h_i, h_j] = 0,\ [h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j,\ [e_i, f_j] = \delta_{ij} h_i,$$
$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0,\ (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0,\ i \neq j.$$

(iii) For any reduced irreducible root system, these generators and relations determine a finite dimensional simple Lie algebra $\mathfrak{g}$.

27.9. **Representations of semisimple Lie algebras.** We started to develop representation theory of a semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\mathfrak{n}_\pm = \oplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$, and for $\lambda \in \mathfrak{h}^*$ define the **Verma module** M_λ generated by a vector v_λ with defining relation $h v_\lambda = \lambda(h) v_\lambda, h \in \mathfrak{h}$, and $e_i v_\lambda = 0$. This is a free module over $U(\mathfrak{n}_-)$ of rank 1. A highest weight module with highest weight λ is a quotient of M_λ. In particular, the smallest nonzero quotient is the irreducible highest weight module L_λ, infinite dimensional in general. But all finite dimensional irreducible modules are of this form. Thus the following theorem (along with the complete reducibility theorem) gives a complete classification of finite dimensional representations of $\mathfrak{g}$.

Theorem 27.14. L_λ *is finite dimensional if and only if* $\lambda = \sum n_i \omega_i$ *is a dominant integral weight. Moreover, in this case* L_λ *is the quotient of* M_λ *by the submodule generated by* $f_i^{n_i+1} v_\lambda$, $i = 1, ..., r$.

27.10. **Weyl character formula.** Every highest weight representation V (say, with highest weight $\lambda \in P$) has a weight space decomposition $V = \oplus_{\mu \in \mathfrak{h}^*} V[\mu]$, where $V[\mu]$ is the (finite dimensional) eigenspace of $\mathfrak{h}$ with eigenvalue μ. The formal character of V is then the series

$$\chi_V = \sum \dim V[\mu] e^{\mu},$$

an element of a certain completion of the group algebra $\mathbb{Z}[P]$. For example, if V is finite dimensional then $\chi_V \in \mathbb{C}[P]$ (no completion needed), i.e. it is a Laurent polynomial. The Weyl character formula is the fundamental result in representation theory giving the character of L_λ for $\lambda \in P_+$ (i.e., of finite dimensional modules).

For $w \in W$ let $\det(w)$ be the determinant of w on $\mathfrak{h}$ (it is ± 1, i.e., $\det(s_i) = -1$). Let $\rho = \sum_i \omega_i$.

Theorem 27.15. *(Weyl character formula) One has*

$$\chi_{L_\lambda} = \frac{\sum_{w \in W} \det(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}.$$

In particular, $L_0 = \mathbb{C}$, so its character is 1. This implies

Corollary 27.16. *(Weyl denominator formula)*

$$\prod_{\alpha \in R_+} (1 - e^{-\alpha}) = \sum_{w \in W} \det(w) e^{w\rho - \rho}.$$

For example, for $\mathfrak{sl}_n$ this reduces to the formula for the Vandermonde determinant.

Exercise 27.17. Let Q be the root lattice of a simple Lie algebra $\mathfrak{g}$, Q_+ its positive part. Define the **Kostant partition function** to be the function $p : Q_+ \to \mathbb{Z}_{\geq 0}$ which attaches to $\beta \in Q_+$ the number of ways to write β as a sum of positive roots of $\mathfrak{g}$ (where the order does not matter).

(i) Show that

$$\sum_{\beta \in Q_+} p(\beta) e^{-\beta} = \frac{1}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}.$$

(ii) Prove the **Kostant multiplicity formula**

$$\dim L_\lambda[\gamma] = \sum_{w \in W} \det(w) p(w(\lambda + \rho) - \rho - \gamma),$$

where we set $p(\beta) = 0$ if $\beta \notin Q_+$.

(iii) Compute $p(k_1\alpha_1 + k_2\alpha_2)$ for $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{sp}_4$.

(iv) Use (iii) to compute explicitly the weight multiplicities of the irreducible representations L_λ for $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{sp}_4$. (You should get a sum of 6, respectively 8 terms, not particularly appealing, but easily computable in each special case).

28. Lecture 3

28.1. **Weyl dimension formula.** Recall that the Weyl character formula can be written as a trace formula: for $h \in \mathfrak{h}$

$$\chi_{L_\lambda}(e^h) = \mathrm{Tr}|_{L_\lambda}(e^h) = \frac{\sum_{w\in W}\det(w)e^{(w(\lambda+\rho)-\rho,h)}}{\prod_{\alpha\in R_+}(1-e^{-(\alpha,h)})}.$$

The dimension of L_λ should be obtained from this formula when $h = 0$. However, we don't not immediately get the answer since this formula gives the character as a ratio of two trigonometric polynomials which both vanish at $h = 0$, giving an indeterminacy. We know the limit exists since the character is a trigonometric polynomial, but we need to compute it. This can be done as follows.

Let us restrict attention to $h = th_\rho$ where $t \in \mathbb{R}$ and $h_\rho \in \mathfrak{h}$ corresponds to $\rho \in \mathfrak{h}^*$ using the identification induced by the invariant form. We have

$$\chi_{L_\lambda}(e^{th_\rho}) = \frac{\sum_{w\in W}\det(w)e^{t(w(\lambda+\rho)-\rho,\rho)}}{\prod_{\alpha\in R_+}(1-e^{-t(\alpha,\rho)})}.$$

The key idea is that for this specialization the numerator can also be factored using the denominator formula, which will allow us to resolve the indeterminacy. Namely, we have

$$\chi_{L_\lambda}(e^{th_\rho}) = e^{t(\lambda,\rho)}\frac{\prod_{\alpha\in R_+}(1-e^{t(\alpha,\lambda+\rho)})}{\prod_{\alpha\in R_+}(1-e^{-t(\alpha,\rho)})}. \tag{3}$$

Now sending $t \to 0$, we obtain

Proposition 28.1. *We have*

$$\dim L_\lambda = \frac{\prod_{\alpha\in R_+}(\alpha,\lambda+\rho)}{\prod_{\alpha\in R_+}(\alpha,\rho)}.$$

Note that this number is an integer, but this is not obvious without its intertpretation as the dimension of a representation.

Formula (3) has a meaning even before taking the limit. Namely, the eigenvalues of the element $2h_\rho$ define a $\mathbb{Z}$-grading on the representation L_λ called the **principal grading**, and we obtain a product formula for the Poincaré polynomial of this grading.

28.2. **Tensor products of fundamental representations.** The following result shows that if we understand fundamental representations of a semisimple Lie algebra $\mathfrak{g}$, we can gain some insight into general finite dimensional representations.

Proposition 28.2. *Let $\lambda = \sum_{i=1}^r m_i\omega_i$ be a dominant integral weight for $\mathfrak{g}$. Consider the tensor product $T_\lambda := \otimes_i L_{\omega_i}^{\otimes m_i}$, and let $v := \otimes_i v_{\omega_i}^{\otimes m_i}$ be the tensor product of the highest weight vectors. Let V be the subrepresentation of T_λ generated by v. Then $V \cong L_\lambda$.*

Proof. We have $V = L_\lambda \oplus \bigoplus_{\mu\in(\lambda-Q_+)\cap P_+} N_{\lambda\mu}L_\mu$ where $N_{\lambda\mu}$ are positive integers. Let $C \in U(\mathfrak{g})$ be the Casimir element for $\mathfrak{g}$. Recall that $C|_{L_\mu} = (\mu, \mu + 2\rho)$. We have seen that for any $\mu \in (\lambda - Q_+) \cap P_+$ such that $\mu \neq \lambda$, we have $(\mu, \mu + 2\rho) < (\lambda, \lambda + 2\rho)$. Therefore, since $C|_V = (\lambda, \lambda + 2\rho)$, we see that $N_{\lambda\mu} = 0$ for $\mu \neq \lambda$. □

28.3. **Representations of $SL_n(\mathbb{C})$.** Let us now discuss more explicitly the representation theory of $SL_n(\mathbb{C})$. We will consider its finite dimensional complex analytic representations as a complex Lie group. We have shown that this is equivalent to considering finite dimensional representations of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. We have also seen that these are completely reducible and the irreducible representations are L_λ, where $\lambda = \sum_{i=1}^{n-1} k_i\omega_i$, ω_i are the fundamental weights, and $k_i \in \mathbb{Z}_{\geq 0}$.

First let us compute ω_i. Recall that the standard Cartan subalgebra $\mathfrak{h}$ is the space $\mathbb{C}_0^n$ of vectors in $\mathbb{C}^n$ with zero sum of coordinates (diagonal matrices with trace zero). So elements of $\mathfrak{h}^*$ can be viewed as vectors $(x_1, ..., x_n) \in \mathbb{C}^n$ modulo simultaneous shift of all coordinates by the same number (i.e., $\mathfrak{h}^* = \mathbb{C}^n/\mathbb{C}_{\mathrm{diag}}$).

Recall that the simple roots are $\alpha_i^\vee = \mathbf{e}_i - \mathbf{e}_{i+1}$. Thus ω_i are determined by the conditions

$$(\omega_i, \mathbf{e}_j - \mathbf{e}_{j+1}) = \delta_{ij}.$$

This means that $\omega_i = (1, ..., 1, 0, ..., 0)$ where there are i copies of 1. Thus a dominant integral weight λ has the form

$$\lambda = (m_1 + ... + m_{n-1}, m_2 + ... + m_{n-1}, ..., m_{n-1}, 0).$$

So dominant integral weights are parametrized by non-increasing sequences $\lambda_1 \geq ... \geq \lambda_{n-1}$ of nonnegative integers. This agrees with the representation theory of $SL_2(\mathbb{C})$ that we worked out before: in this case the sequence has just one term.

Let us now describe explicitly the fundamental representations L_{ω_i}. Consider first the representation $V = \mathbb{C}^n$ with the usual action of matrices. It is called the **vector representation** or the **tautological representation** (as every matrix goes to itself). It is irreducible and has a standard basis $v_1, ..., v_n$. To find its highest weight, we have to find the vector $v \neq 0$ such that $e_i v = 0$. As $e_i = E_{i,i+1}$, we have $v = v_1$. It is easy to see that $hv = \omega_1(h)v$, so we see that v has weight ω_1, hence $L_{\omega_1} = V$.

To construct L_{ω_m} for $m > 1$, consider the exterior power $\wedge^m V$. It is easy to show that it is irreducible (we did this last term). A basis of $\wedge^m V$ consists of wedges $v_{i_1} \wedge ... \wedge v_{i_m}$ where $i_1 < ... < i_m$. The highest weight vector is clearly $v_1 \wedge ... \wedge v_m$, and it has weight ω_m. Thus $L_{\omega_m} = \wedge^m V$.

Note that $\wedge^n V = \mathbb{C}$ (the trivial representation) since every matrix in $SL_n(\mathbb{C})$ acts by its determinant, which is 1, and $\wedge^m V = 0$ for $m > n$. Also $V^* \cong \wedge^{n-1} V$ since the wedge pairing $V \otimes \wedge^{n-1} V \to \wedge^n V = \mathbb{C}$ is invariant and nondegenerate. Similarly, $\wedge^m V^* \cong \wedge^{n-m} V$.

We now see that the irreducible representation L_λ for $\lambda = \sum m_i \omega_i$ is generated inside $\otimes_i (\wedge^i V)^{\otimes m_i}$ by the tensor product of the highest weight vectors.

Example 28.3. $L_{N\omega_1} = S^N V$.

28.4. **Representations of $GL_n(\mathbb{C})$.** Let us now explain how to extend these results to $GL_n(\mathbb{C})$. This is easy to do since $GL_n(\mathbb{C})$ is not very different from the direct product $\mathbb{C}^\times \times SL_n(\mathbb{C})$. Namely, $GL_n(\mathbb{C}) = (\mathbb{C}^\times \times SL_n(\mathbb{C}))/\mu_n$ where μ_n is the group of roots of unity of order n embedded as $z \mapsto (z^{-1}, z\mathbf{1}_n)$. Indeed, the corresponding covering $\mathbb{C}^\times \times SL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is given by $(z, A) \mapsto zA$. So it suffices to classify holomorphic representations of the complex Lie group $\mathbb{C}^\times \times SL_n(\mathbb{C})$.

For $n = 1$ this is just the problem of describing the holomorphic representations of $\mathbb{C}^\times$. This is easy. The Lie algebra is spanned by a single element h such that $e^{2\pi i h} = 1$. This element must act in a representation by an operator H such that $e^{2\pi i H} = 1$. It follows that H is diagonalizable with integer eigenvalues. Thus representations of $\mathbb{C}^\times$ are completely reducible, with irreducibles χ_N one-dimensional and labeled by integers $N \in \mathbb{Z}$, $\chi_N(z) = z^N$.

We have a similar answer for $\mathbb{C}^\times \times SL_n$: representations are completely reducible with irreducibles being $L_{\lambda,N} = \chi_N \otimes L_\lambda$. Moreover, the ones factoring through GL_n just have $N = nr + \sum_{i=1}^{n-1} \lambda_i$ for some integer r.

Recall that GL_n has reductive Lie algebra $\mathfrak{gl}_n$ with Cartan subalgebra $\mathfrak{h} = \mathbb{C}^n$. The highest weight of $L_{\lambda, nm_n + \sum_{i=1}^{n-1} \lambda_i}$ is easily computed and equals $(m_1 + ... + m_{n-1} + m_n, ..., m_{n-1} + m_n, m_n)$. Thus highest weights of finite dimensional representations are non-increasing sequences $(\lambda_1, ..., \lambda_n)$ of integers which don't have to be positive. The fundamental representations are still $L_{\omega_m} = \wedge^m V$, and the only difference with SL_n is that now the top exterior power $\wedge^n V$ is not trivial but rather is a 1-dimensional **determinant character** with highest weight $\omega_n = (1, ..., 1)$. The highest weight of a finite dimensional representation then has the form $\lambda = \sum_{i=1}^n m_i\omega_i$, where $m_i \geq 0$ for $i \neq n$, while m_n is an arbitrary integer. Consequently, L_λ is found inside $\otimes_i(\wedge^i V)^{\otimes m_i}$ as the representation generated by the product of highest weight vectors. Note that it makes sense to take $m_n < 0$, as for a one-dimensional representation and $k < 0$ it is natural to define $\chi^{\otimes k} := (\chi^*)^{\otimes -k}$.

The representations with $m_n \geq 0$ are especially important; it is easy to see that these are exactly the ones that occur inside $V^{\otimes N}$ for some N (check it!). These representations are called **polynomial** since their matrix coefficients are polynomial functions of the matrix entries x_{ij} of $X \in GL_n(\mathbb{C})$, and consequently they extend by continuity to representations of the semigroup $\mathrm{Mat}_n(\mathbb{C}) \supset GL_n(\mathbb{C})$. Note that any irreducible representation is a polynomial one tensored with a non-positive power of the determinant character $\wedge^n V$.

28.5. **Schur-Weyl duality.** Note that highest weights of polynomial representations are non-increasing sequences of nonnegative integers $(\lambda_1, ..., \lambda_n)$, i.e. **partitions** with $\leq n$ parts. Namely, they are partitions of $|\lambda| = \sum_i \lambda_i$, which is just the eigenvalue of $\mathbf{1}_n \in \mathfrak{gl}_n$ on L_λ and can also be defined as the number N such that L_λ occurs in $V^{\otimes N}$.

Thus we have

$$V^{\otimes N} = \oplus_{\lambda:|\lambda|=N} L_\lambda \otimes \pi_\lambda,$$

where $\pi_\lambda := \mathrm{Hom}_{GL_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$ are multiplicity spaces. Here the summation is over partitions of N, and $L_\lambda = 0$ if λ has more than n parts. To understand the spaces π_λ, note that the symmetric group S_N acts on $V^{\otimes N}$ and commutes with $GL_n(\mathbb{C})$, so it gets to act on each π_λ.

Let A be the image of $U(\mathfrak{gl}_n)$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes N})$, and B be the image there of $\mathbb{C}S_N$. The algebras A, B commute.

Theorem 28.4. *(Schur-Weyl duality) (i) The centralizer of A is B and vice versa.*

(ii) If λ has at most n parts then the representations π_λ of B (hence S_N) are irreducible and pairwise non-isomorphic.

(iii) If $\dim V \geq N$ then π_λ exhaust all irreducible representations of S_N.

Proof. We start with

Lemma 28.5. *If U is a $\mathbb{C}$-vector space then $S^N U$ is spanned by elements $x \otimes ... \otimes x$, $x \in U$.*

Proof. It suffices to consider the case when U is finite dimensional. Then the span of these vectors is a nonzero subrepresentation in the irreducible $GL(U)$-representation $S^N U$, which implies the statement. □

Lemma 28.6. *For any associative algebra R over $\mathbb{C}$, the algebra $S^N R$ is generated by elements*

$$\Delta_N(x) := x \otimes 1 \otimes ... \otimes 1 + 1 \otimes x \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes 1 \otimes x$$

for $x \in R$.

Proof. Let P_N be the Newton polynomial expressing $z_1 ... z_N$ via $p_k := \sum_{i=1}^N z_i^k$, $k = 1, ..., N$ (it exists and is unique by the fundamental theorem on symmetric functions). Then we have

$$x \otimes ... \otimes x = P_N(\Delta_N(x), ..., \Delta_N(x^N)).$$

Hence the lemma follows from Lemma 28.5. □

Let us now show that A is the centralizer $Z(B)$ of B. Note that $Z(B) = S^N(\mathrm{End}V)$. Thus the statement follows from Lemma 28.6.

We will now use the following easy but important lemma (which actually holds over any field).

Lemma 28.7. *(Double centralizer lemma) Let V be a finite dimensional vector space and $A, B \subset \mathrm{End}V$ be subalgebras such that B is isomorphic to a direct sum of matrix algebras and A is the centralizer of B. Then A is also isomorphic to a direct sum of matrix algebras, and moreover*

$$V = \oplus_{i=1}^n W_i \otimes U_i,$$

where W_i run through all irreducible A-modules and U_i through irreducible B-modules. In particular, A is the centralizer of B and we have a natural bijection between irreducible A-modules and irreducible B-modules which matches W_i and U_i.

Proof. We have $V = \oplus_{i=1}^n W_i \otimes U_i$ where is U_i run through irreducible representations of B and $W_i = \mathrm{Hom}_B(U_i, V) \neq 0$ are multiplicity spaces. Thus $A = \oplus_{i=1}^n \mathrm{End}W_i$ and $B = \oplus_{i=1}^n \mathrm{End}U_i$, which implies the statement. □

Since the algebra B is a direct sum of matrix algebras (by complete reducibility of representations of finite groups), Lemma 28.7 yields (i).[5]

To prove (ii), it suffices to note that if λ has $\leq n$ parts then L_λ occurs in $V^{\otimes N}$, so $\pi_\lambda \neq 0$. The rest follows from (i) and Lemma 28.7.

[5]This also gives another proof of the fact that A is a direct sum of matrix algebras, i.e. complete reducibility of $V^{\otimes N}$

(iii) If $\dim V \geq N$ then pick N linearly independent vectors $v_1, ..., v_N \in V$. It is easy to see that the map $\mathbb{C}S_N \to V^{\otimes N}$ defined by $s \mapsto s(v_1 \otimes ... \otimes v_N)$ is injective. Thus $B = \mathbb{C}S_N$. This implies the statement. $\square$

Remark 28.8. The algebra A is called the **Schur algebra**.

Thus we see that representations of S_N are labeled by partitions λ of N, and those that occur in $V^{\otimes n}$ correspond to the partitions that have $\leq \dim V$ parts. Moreover, we claim that this labeling of representations by partitions does not depend on $\dim V$. To show this, suppose λ has $\leq n$ parts and $V = \mathbb{C}^n$. We have the Schur-Weyl decomposition of $GL_{n+1}(\mathbb{C}) \times S_N$-modules

$$(V \oplus \mathbb{C})^{\otimes N} = \oplus_\mu L_\mu^{(n+1)} \otimes \pi_\mu^{(n+1)},$$

Let us restrict this sum to $GL_n(\mathbb{C}) \times S_N$, and consider what happens to the summand $L_\lambda^{(n+1)} \otimes \pi_\lambda^{(n+1)}$. The highest weight vector v in $L_\lambda^{(n+1)}$ tensored with any element w of $\pi_\lambda^{(n+1)}$ sits in $V^{\otimes N} \subset (V \oplus \mathbb{C})^{\otimes N}$, since the $n+1$-th component of its weight is zero. Hence $v \otimes w$ generates a copy of $L_\lambda^{(n)} \otimes \pi_\lambda^{(n)}$ as a $GL_n(\mathbb{C}) \times S_N$-module. This implies that $\pi_\lambda^{(n+1)} \cong \pi_\lambda^{(n)}$.

Exercise 28.9. Let $R = \mathbb{C}[x_1, ..., x_N, y_1, ..., y_N]^{S_N}$ (the algebra of invariant polynomials). Show that R is generated by the elements $Q_{rs} := \sum_{i=1}^N x_i^r y_i^s$ where $1 \leq r + s \leq N$.

29. Lecture 4

29.1. **Schur functors.**

Definition 29.1. For a partition λ of N we define the **Schur functor** S^λ on the category of complex vector spaces (or complex representations of any group or Lie algebra) by $S^\lambda V = \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes n})$.

Thus we have

$$V^{\otimes N} = \oplus_\lambda S^\lambda V \otimes \pi_\lambda,$$

and if λ has $\leq n$ parts and $V = \mathbb{C}^n$ then $S^\lambda V = L_\lambda$ as a representation of $GL(V) = GL_n(\mathbb{C})$.

Example 29.2. 1. We have $S^{(n)}V = S^n V$, $S^{(1^n)}V = \wedge^n V$.

2. We have

$$V \otimes V = \mathbb{C}_+ \otimes S^{(2)}V \oplus \mathbb{C}_- \otimes S^{(1,1)}V = S^2 V \oplus \wedge^2 V$$

where S_2 acts in the first summand trivially and in the second one by sign.

Consider now the decomposition of $V \otimes V \otimes V$. We have

$$V\otimes V\otimes V = \mathbb{C}_+\otimes S^{(3)}V\oplus\mathbb{C}^2\otimes S^{(2,1)}V\oplus\mathbb{C}_-\otimes S^{(1,1,1)}V = S^3V+\mathbb{C}^2\otimes S^{(2,1)}V\oplus\wedge^3V$$

Thus

$$S^2V \otimes V = S^3V \oplus S^{(2,1)}V,\ \wedge^2V \otimes V = \wedge^3V \oplus S^{(2,1)}V.$$

We conclude that $S^{(2,1)}V$ can be described as the space of tensors symmetric in the first two components whose full symmetrization is zero, or tensors antisymmetric on the first two components whose antisymmetrization is zero.

Exercise 29.3. 1. Let $V = \mathbb{C}^n$, $n \geq 4$. Decompose $V^{\otimes 4}$ as a direct sum of irreducible representations of $GL_n(\mathbb{C}) \times S_4$. Characterize the occurring Schur functors as spaces of tensors with certain symmetry properties, similarly to the above description of $S^{(2,1)}V$. Compute the decompositions of $V \otimes S^3V$, $V \otimes \wedge^3V$, $S^2V \otimes S^2V, S^2V \otimes \wedge^2V$ and $\wedge^2V \otimes \wedge^2V$ into Schur functors.

2. Decompose $V \otimes V^*$, $V \otimes V \otimes V^*$ into a direct sum of irreducible representations. Describe the algebra $\mathrm{End}_{GL_n(\mathbb{C})}(V \otimes V^* \otimes V^*)$.

Let us compute the dimension of $S^\lambda V$ when $\dim V = N$ and λ has k parts. We have $\rho = (N-1, N-2, ..., 1, 0)$ (for SL_N), so the Weyl dimension formula tells us that

$$\dim S^\lambda V = \prod_{1\leq i<j\leq N} \frac{\lambda_i - \lambda_j + j - i}{j-i} =$$

$$\prod_{1\leq i<j\leq k} \frac{\lambda_i - \lambda_j + j - i}{j-i} \prod_{1\leq i\leq k<j\leq N} \frac{\lambda_i + j - i}{j-i} =$$

$$\prod_{1\leq i<j\leq k} \frac{\lambda_i - \lambda_j + j - i}{j-i} \prod_{i=1}^{k} \frac{(N+1-i)...(N+\lambda_i-i)}{(k+1-i)...(k+\lambda_i-i)}.$$

We obtain

Proposition 29.4. $\dim S^\lambda V = P_\lambda(N)$ *where* P_λ *is a polynomial of degree* $|\lambda|$ *with rational coefficients and integer roots. Moreover, the roots of* P_λ *are all the integers in the interval* $[1-\lambda_1, k-1]$ *(occurring with multiplicities).*

Moreover, we see that $P_\lambda(N)$ is an integer-valued polynomial, i.e., it takes integer values at integer points (this is equivalent to being an integer linear combination of $\binom{N}{j}$).

Example 29.5.

$$P_{(n)}(N) = \dim S^n V = \binom{N+n-1}{n}, \ P_{(1^n)}(N) = \dim \wedge^n V = \binom{N}{n}.$$

Also

$$P_{(a,b)}(N) = (a-b+1)\frac{N...(N+a-1)\cdot(N-1)...(N+b-2)}{(a+1)!b!} =$$

$$\frac{a-b+1}{a+1}\binom{N+a-1}{a}\binom{N+b-2}{b}$$

E.g., $P_{(2,1)}(N) = \dim S^{(2,1)}V = \frac{N(N+1)(N-1)}{3}$. Also,

$$P_{(a,a)}(N) = \frac{1}{a+1}\binom{N+a-1}{a}\binom{N+a-2}{a} =$$

$$\frac{1}{N+a-1}\binom{N+a-1}{N-1}\binom{N+a-2}{N-2} = \mathrm{Nar}(N+a-1, N-1),$$

the **Narayana numbers**.

Exercise 29.6. Let g_q be the diagonal matrix with diagonal elements $1, q, q^2, ..., q^{N-1}$. Compute the trace of g_q in $S^\lambda V$ in the product form. Write the answer explicitly (as a polynomial in q) with positive coefficients in the case $|\lambda| \le 3$.

Exercise 29.7. Draw the weights of the representation $S^{(2,2)}\mathbb{C}^3$ of $SL(3)$ on the hexagonal lattice, and indicate their multiplicities.

29.2. **The fundamental theorem of invariant theory.** Suppose we have a finite dimensional vector space V and a collection of tensors $T_i \in V^{\otimes m_i} \otimes V^{*\otimes n_i}$, $i = 1, ..., k$. An important problem is to describe "coordinate free" invariants of such a collection of tensors, i.e., polynomials functions $F(T_1, ..., T_k)$ which are invariant under the action of $GL(V)$. How can we classify such functions? This sounds formidably hard in such generality, but turns out to be very easy using Schur-Weyl duality.

It suffices to study such functions that have homogeneity degree d_i with respect to each T_i. To do so, we will depict each T_i by a vertex with m_i incoming and n_i outgoing arrows. We should think of incoming arrows as V-components and outgoing ones as V^*-components. Let us draw d_i such vertices for each i. To construct an invariant, let us connect the arrows preserving orientation so that all the arrows are used (this will only be possible if the number of incoming arrows equals the number of outgoing ones; otherwise every invariant of the multidegree $(d_1, ..., d_k)$ will be zero). To the obtained graph Γ we can assign the

convolution of tensors, which gives an invariant function F_Γ of the correct multidegree.

Theorem 29.8. *The functions F_Γ for various Γ span the space of invariant functions.*

Proof. An invariant function may be viewed as an element of the space $\bigotimes_{i=1}^k (V^{*\otimes m_i} \otimes V^{\otimes n_i})^{\otimes d_i}$, which we may write as the space of linear maps $V^{\otimes M} \to V^{\otimes N}$, where $M = \sum d_i m_i$ is the number of incoming arrows and $N = \sum d_i n_i$ the number of outgoing arrows. If $M \neq N$, there are no nonzero invariant maps. Otherwise, by the Schur-Weyl duality, the space of such maps is spanned by maps defined by permutations. But any such permutation defines a graph Γ, so the corresponding invariant is just the convolution F_Γ, which implies the statement. $\square$

Remark 29.9. Note that this proof also implies that if $\dim V$ is large compared to m_i, n_i, d_i then the functions F_Γ for non-isomorphic graphs Γ are linearly independent, so they form a basis in the algebra of A of invariant functions.

Example 29.10. Assume that $m_i = n_i = 1$, i.e., $T_1, ..., T_k$ are just matrices with GL_n acting by conjugation. Then all graphs that we can get are unions of cycles, so Theorem 29.8 implies that the algebra $A_{k,n}$ of such invariants (where $n = \dim V$) is generated by traces of words

$$F_{j_1,...,j_r} = \mathrm{Tr}(T_{j_1}...T_{j_r}).$$

Moreover, these elements are "asymptotically algebraically independent", i.e. there is no nonzero polynomial of them that vanishes for all sizes of matrices n.

This implies that there are no universal polynomial identities for matrices of all sizes. Indeed, if $P(T_1, ..., T_k) = 0$ for square matrices $T_1, ..., T_k$ of any size n (where P is a fixed noncommutative polynomial) then adding another matrix T_{k+1}, we get $\mathrm{Tr}(P(T_1, ..., T_k)T_{k+1}) = 0$, which contradicts linear independence of $F_{j_1,...,j_r}$.

This is false, however, if the size of matrices is fixed; in this case there are plenty of polynomial identities for each matrix size. For example, for matrices of size 1 we have $[X, Y] = 0$ and for matrices of size 2 we have $[Z, [X, Y]^2] = 0$. For general n there is the Amitsur-Levitzki identity given in Exercise 29.11.

Exercise 29.11. Let $X_1, ..., X_{2n}$ be complex n by n matrices. Let $\Lambda = \wedge(\xi_1, ..., \xi_{2n})$ be the exterior algebra generated by ξ_i with relations $\xi_i\xi_j = -\xi_j\xi_i, \xi_i^2 = 0$. Let X be the matrix over Λ given by

$$X := X_1\xi_1 + ... + X_{2n}\xi_{2n}.$$

1. Let $Y = X^2$. Show that $Y \in \mathrm{Mat}_n(\Lambda_+)$ where Λ_+ is the commutative subalgebra of Λ spanned by the elements of even degrees. Compute Y^n.

2. Show that $\mathrm{Tr}(Y^k) = 0 \in \Lambda_+$ for $k = 1, ..., n$.

3. Deduce that $Y^n = 0$. This should yield the Amitsur-Levitzki identity

$$\sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma) X_{\sigma(1)} ... X_{\sigma(2n)} = 0.$$

4. Deduce the same identity over any commutative ring R.

29.3. **Schur polynomials and characters of representations of S_N.** Using Schur-Weyl duality and the character formula for representations of GL_n, we can obtain information about characters of the symmetric group. Namely, it follows from the Weyl character formula that the characters of representations of GL_n are given by the formula

$$s_\lambda(x_1, ..., x_n) = \frac{\sum_{s \in S_n} \det(s) x_{s(1)}^{\lambda_1 + N - 1} ... x_{s(n)}^{\lambda_n}}{\prod_{i<j}(x_i - x_j)} = \frac{\det(x_i^{\lambda_j + N - j})}{\prod_{i<j}(x_i - x_j)}.$$

These symmetric polynomials are called **Schur polynomials**. For example, the character of $S^m V$ is

$$s_{(m)}(x_1, ..., x_n) = \sum_{i_1, ..., i_m : i_1 + ... + i_m = n} x_1^{i_1} ... x_m^{i_m} = h_m(x_1, ..., x_m),$$

the m-th **complete symmetric function**, and the character of $\wedge^m V$ is

$$s_{(1^m)}(x_1, ..., x_n) = \sum_{1 \le j_1 < ... < j_m \le n} x_{j_1} ... x_{j_m} = e_m(x_1, ..., x_m),$$

the m-th **elementary symmetric function**.

Let us now compute the trace in $V^{\otimes N}$ of $x \otimes \sigma$, where $x = \mathrm{diag}(x_1, ..., x_n)$ is a diagonal matrix and $\sigma \in S_N$ a permutation. Let σ have m_i cycles of length i. Then we have

$$\mathrm{Tr}|_{V^{\otimes N}}(x \otimes \sigma) = \prod_i (x_1^i + ... + x_n^i)^{m_i}.$$

On the other hand, using Schur-Weyl duality, we get

$$\mathrm{Tr}|_{V^{\otimes N}}(x \otimes \sigma) = \sum_\lambda \chi_\lambda(\sigma) s_\lambda(x),$$

where $\chi_\lambda(\sigma) = \mathrm{Tr}|_{\pi_\lambda}(\sigma)$ is the character of the representation π_λ of S_N. Thus we have

$$\sum_\lambda \chi_\lambda(\sigma) s_\lambda(x) = \prod_i (x_1^i + ... + x_n^i)^{m_i}.$$

Multiplying this by the discriminant, we get

$$\sum_\lambda \chi_\lambda(\sigma)\det(x_i^{\lambda_j+N-j}) = \prod_{i<j}(x_i - x_j)\cdot\prod_i (x_1^i + ... + x_n^i)^{m_i}.$$

Thus we get

Theorem 29.12. *(Frobenius character formula) The character value $\chi_\lambda(\sigma)$ is the coefficient of $x_1^{\lambda_1+N-1}...x_N^{\lambda_N}$ in the polynomial*

$$\prod_{i<j}(x_i - x_j)\cdot\prod_i (x_1^i + ... + x_n^i)^{m_i}$$

Exercise 29.13. Let $V = \mathbb{C}^2$ be the 2-dimensional tautological representation of $GL_2(\mathbb{C})$. Decompose $V^{\otimes N}$ into a direct sum of irreducible representations of $GL_2(\mathbb{C}) \times S_N$ and compute the characters and dimensions of all the irreducible representations of GL_2 and S_N that occur.

Exercise 29.14. Compute characters and dimensions of irreducible representations $L_{a+b,b,0}$ of $SL_3(\mathbb{C})$, where $a, b \geq 0$. Compute the weight multiplicities and draw the weights on the hexagonal lattice for $a+b \leq 3$, indicating the multiplicities. What are the special features of the case $b = 0$?

Hint. The best way to do this exercise is to compute the characters recursively, using that $V \otimes L_{a+b,b,0} = L_{a+b+1,b,0} \oplus L_{a+b,b+1,0} \oplus L_{a+b-1,b-1,0}$ (if $a = 0$, the second summand drops out and if $b = 0$ then the third one drops out), by the "addable boxes" rule. This allows you to express the characters for $b+1$ in terms of the characters for b and $b-1$. And we know the characters of $L_{a,0,0}$ - they are the complete symmetric functions h_a.

30. Lecture 5

30.1. **Howe duality.** Howe duality is another instance when we have a double centralizer property. Consider two finite dimensional complex vector spaces V, W, and consider the symmetric power $S^n(V \otimes W)$ as a representation of $GL(V) \times GL(W)$.

Theorem 30.1. *(Howe duality) We have a decomposition*

$$S^n(V \otimes W) = \oplus_{\lambda:|\lambda|=n} S^\lambda V \otimes S^\lambda W.$$

Note that if λ has more parts than $\dim V$ or $\dim W$ then the corresponding summand is zero.

Proof. We have

$$S^n(V \otimes W) = ((V \otimes W)^{\otimes n})^{S_n} = (V^{\otimes n} \otimes W^{\otimes n})^{S_n}$$

So using the Schur-Weyl duality, we get

$$S^n(V \otimes W) = ((\oplus_{\lambda:|\lambda|=n} S^\lambda V \otimes \pi_\lambda) \otimes (\oplus_{\mu:|\mu|=n} S^\mu W \otimes \pi_\mu))^{S_n} =$$

$$\oplus_{\lambda,\mu:|\lambda|=|\mu|=n} S^\lambda V \otimes S^\mu W \otimes (\pi_\lambda \otimes \pi_\mu)^{S_n}.$$

But the character of π_λ is integer-valued, so $\pi_\lambda = \pi_\lambda^*$. Thus by Schur's lemma $(\pi_\lambda \otimes \pi_\mu)^{S_n} = \mathbb{C}^{\delta_{\lambda\mu}}$, and we get

$$S^n(V \otimes W) = \oplus_{\lambda:|\lambda|=n} S^\lambda V \otimes S^\lambda W,$$

as claimed. □

Note that we never used that V, W were finite dimensional, so the statement is valid for any complex vector spaces V, W.

Corollary 30.2. *(Cauchy identity) If $x = (x_1, ..., x_r)$ and $y = (y_1, ..., y_s)$ then one has*

$$\sum_\lambda s_\lambda(x) s_\lambda(y) z^{|\lambda|} = \prod_{i=1}^r \prod_{j=1}^s \frac{1}{1 - z x_i y_j}.$$

Proof.

Lemma 30.3. *(Molien formula). Let $A : V \to V$ be a linear operator on a finite dimensional vector space V. Denote by $S^n A$ the induced linear operator $A^{\otimes n}$ on $S^n V$. Then*

$$\sum_{n=0}^\infty \mathrm{Tr}(S^n A) z^n = \frac{1}{\det(1 - zA)}.$$

Proof. Let A have eigenvalues $x_1, ..., x_r$. Then the eigenvalues of $S^n A$ are all possible monomials in x_i of degree r. Thus $\mathrm{Tr}(S^n A)$ is the sum of these monomials, which is the complete symmetric function $h_n(x_1, ..., x_r)$. So

$$\sum_{n=0}^\infty \mathrm{Tr}(S^n A) z^n = \sum_{n \geq 0} h_n(x_1, ..., x_r) z^n = \prod_{i=1}^r \frac{1}{1 - z x_i} = \frac{1}{\det(1 - zA)}.$$

□

Now let $g \in GL(V)$ with eigenvalues $x_1, ..., x_r$ and $h \in GL(W)$ with eigenvalues of $y_1, ..., y_s$. Then by Howe duality

$$\mathrm{Tr}(S^n(g \otimes h)) = \sum_{\lambda:|\lambda|=n} s_\lambda(x) s_\lambda(y).$$

On the other hand, by Molien's formula

$$\sum_{n\geq 0}\mathrm{Tr}(S^n(g\otimes h))z^n=\frac{1}{\det(1-z(g\otimes h))}=\prod_{i,j}\frac{1}{1-zx_iy_j}.$$

Comparing the two formulas, we obtain the statement. □

31. Lecture 6

31.1. **Minuscule weights.** Let $\mathfrak{g}$ be a simple complex Lie algebra. Minuscule weights for $\mathfrak{g}$ are highest weights for which irreducible representations are especially simple.

Definition 31.1. A dominant integral weight ω for $\mathfrak{g}$ is called **minuscule** if $(\omega,\beta)\leq 1$ for all positive coroots β.

Equivalently, $|(\omega,\beta)|\leq 1$ for any coroot β.

Obviously, $\omega=0$ is minuscule, but there may exist other minuscule weights. For example, for $\mathfrak{g}=\mathfrak{sl}_n$, all fundamental weights are minuscule, since $(\omega_i,\mathbf{e}_j-\mathbf{e}_k)=0$ if $j,k\leq i$ or $j,k>i$ and $(\omega_i,\mathbf{e}_j-\mathbf{e}_k)=1$ if $j\leq i<k$.

It is easy to see that any minuscule weight $\omega\neq 0$ is fundamental. Indeed, we can have $(\omega,\alpha_i^\vee)=1$ only for one i, and for all other simple coroots this inner product must be zero. Otherwise we will have $(\omega,\theta^\vee)\geq 2$, where $\theta^\vee$ is the maximal coroot.

On the other hand, not all fundamental weights are minuscule. In fact, we will see that the simple Lie algebras of types G_2, F_4 and E_8 do not have any nonzero minuscule weights. To formulate a criterion for a fundamental weight to be minuscule, recall that $\theta^\vee=\sum_i m_i\alpha_i^\vee$, where $m_i=(\omega_i,\theta^\vee)$ are strictly positive integers.

Lemma 31.2. *A fundamental weight ω_i is minuscule if and only if $m_i=1$.*

Proof. The definition of minuscule means that $m_i\leq 1$. On the other hand, if $m_i=1$ then given a positive coroot $\beta=\sum n_j\alpha_j^\vee$, we have $n_j\leq m_j$, in particular $n_i\leq 1$, so ω_i is minuscule. □

Lemma 31.3. *Let $\omega\in Q$ and $|(\omega,\beta)|\leq 1$ for all coroots β. Then $\omega=0$.*

Proof. Assume the contrary. Choose a counterexample $\omega=\sum_i m_i\alpha_i$ so that $\sum_i|m_i|$ is minimal possible. We have

$$(\omega,\omega)=\sum_i m_i(\omega,\alpha_i)>0.$$

So there exists j such that m_j and $(\omega, \alpha_j^\vee)$ are nonzero and have the same sign. Replacing ω with $-\omega$ if needed, we may assume that both are positive, then $(\omega, \alpha_j^\vee) = 1$. Then $s_j\omega = \omega - \alpha_j = \sum_j m_i'\alpha_i$ where $m_j' = m_j - 1$ and $m_i' = m_i$ for all $i \neq j$ is another counterexample. But we have $\sum_i |m_i'| = \sum_i |m_i| - 1$, a contradiction. □

Why are minuscule weights interesting? It is because of the following result.

Proposition 31.4. *The following conditions on a dominant integral weight ω are equivalent:*
(1) ω is minuscule;
(2) all weights of the representation L_ω belong to the orbit $W\omega$;
(3) if λ is a dominant integral weight such that $\omega - \lambda \in Q_+$ then $\lambda = \omega$.

Proof. Let us prove that (1) implies (3). If $\omega = 0$, there is nothing to prove, since then $-\lambda \in Q_+$, so $(\lambda, \rho) \leq 0$, hence $\lambda = 0$. So suppose that $\omega = \omega_i$ is minuscule. We have $\omega_i - \lambda = \sum_k m_k\alpha_k$ with $m_k \geq 0$. If $m_k = 0$ for some $k \neq i$ then the problem reduces to smaller rank by deleting the vertex k from the Dynkin diagram. So we may assume $m_k > 0$ for all $k \neq i$. Let β be a positive coroot. Then

$$(\omega_i - \lambda, \beta) = (\omega_i, \beta) - (\lambda, \beta) \leq (\omega_i, \beta) \leq 1$$

and if $\alpha_i^\vee$ does not occur in β then it is ≤ 0. So in particular we have $(\omega_i - \lambda, \alpha_j^\vee) \leq 0$ if $j \neq i$. If also $(\omega_i - \lambda, \alpha_i^\vee) \leq 0$ then $(\omega_i - \lambda, \omega_i - \lambda) \leq 0$, so $\omega_i = \lambda$, as claimed. Thus we may assume that $(\omega_i - \lambda, \alpha_i^\vee) = 1$, i.e., $m_i > 0$, so $m_j > 0$ for all j. Thus, $(\omega_i - \lambda, \theta^\vee) \geq 1$ (as $\theta^\vee$ is a dominant coweight). Hence $(\lambda, \theta^\vee) \leq 0$, i.e., $\lambda = 0$, as $\theta^\vee$ contains all $\alpha_j^\vee$ with positive coefficients. Thus $\omega_i \in Q$. But this is impossible by Lemma 31.3.

To see that (3) implies (2), note that if μ is any weight of L_ω then for some $w \in W$ the weight $\lambda = w\mu$ is dominant and $\omega - \lambda \in Q_+$, so $\lambda = \omega$ and $\mu = w^{-1}\omega$.

Finally, we show that (2) implies (1). Assume (2) holds. If ω is not minuscule then there is a positive root α such that $(\omega, \alpha^\vee) > 1$, hence $2(\omega, \alpha) > (\alpha, \alpha)$. Then $\omega - \alpha$ is a weight of L_ω (the weight of the nonzero vector $f_\alpha v_\omega$), and it is not W-conjugate to ω, as

$$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega).$$

Let $w \in W$ be such that the weight $\lambda := w(\omega - \alpha)$ is dominant. This is also a weight of L_ω, so $\omega - \lambda \in Q_+$. Thus $\lambda = \omega$, a contradiction. □

This immediately implies

Corollary 31.5. *The character of L_ω with minuscule ω is*

$$\chi_\omega = \sum_{\gamma \in W\omega} e^\gamma.$$

Proposition 31.6. *$\omega \in P_+$ is minuscule if and only if the restriction of L_ω to any root $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ is the direct sum of 1-dimensional and 2-dimensional representations.*

Proof. Let ω be minuscule and $v \in L_\omega$ be a highest weight vector for $(\mathfrak{sl}_2)_\alpha$. Then $h_\alpha v = (\omega, \alpha^\vee)v$. Thus $h_\alpha v = 0$ or $h_\alpha v = v$, as claimed.

On the other hand, if ω is not minuscule then there is a positive root α such that $(\omega, \alpha^\vee) = m > 1$. So $h_\alpha v_\omega = m v_\omega$ and v_ω generates the irreducible $m+1$-dimensional representation of $(\mathfrak{sl}_2)_\alpha$. □

Corollary 31.7. *If ω is minuscule then for any dominant integral weight λ of $\mathfrak{g}$ we have*

$$L_\omega \otimes L_\lambda = \oplus_{\gamma \in W\omega} L_{\lambda+\gamma},$$

where if $\lambda + \gamma$ is not dominant then we agree that $L_{\lambda+\gamma} = 0$.

Proof. By the Weyl character formula and Corollary 31.5, the character of $L_\omega \otimes L_\lambda$ is

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\mu \in W\omega} \sum_{w \in W} \det(w) e^{w(\lambda+\rho)+\mu}}{\prod_{\alpha \in R_+}(e^{\alpha/2} - e^{-\alpha/2})} =$$

$$\frac{\sum_{\gamma \in W\omega} \sum_{w \in W} \det(w) e^{w(\lambda+\gamma+\rho)}}{\prod_{\alpha \in R_+}(e^{\alpha/2} - e^{-\alpha/2})}.$$

If $\lambda+\gamma \notin P_+$ then for some i we have $(\lambda+\gamma, \alpha_i^\vee) < 0$. But $(\gamma, \alpha_i^\vee) \geq -1$. So $(\lambda+\gamma, \alpha_i^\vee) = -1$ and thus $(\lambda+\gamma+\rho, \alpha_i^\vee) = 0$. So for such γ, for any $w \in W$ the summand for w cancels with the summand for ws_i. Thus we get

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \sum_{w \in W} \det(w) e^{w(\lambda+\gamma+\rho)}}{\prod_{\alpha \in R_+}(e^{\alpha/2} - e^{-\alpha/2})} = \sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \chi_{L_{\lambda+\gamma}}.$$

□

Example 31.8. 1. Let V be the vector representation of GL_n. Then for a partition λ, $V \otimes L_\lambda = \oplus_\mu L_\mu$, where μ runs over all partitions obtained by adding one **addable** box to the Young diagram of λ, i.e., such that it remains a Young diagram.

2. More generally, $\wedge^m V \otimes L_\lambda = \oplus_\mu L_\mu$, where we sum over partitions obtained by adding m addable boxes to different rows.

Exercise 31.9. Compute the decomposition of $\wedge^m V \otimes S^k V$, $\wedge^m V \otimes \wedge^k V$, $S^2(\wedge^m V)$, $\wedge^2(\wedge^m V)$ into irreducible representations of $GL(V)$.

Exercise 31.10. Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra, and V a finite dimensional representation of $\mathfrak{g}$. Given a homomorphism $\Phi : L_\lambda \to V \otimes L_\mu$, let $\langle \Phi \rangle := (\mathrm{Id} \otimes v_\mu^*, \Phi v_\lambda) \in V$, where v_λ is a highest weight vector of L_λ and v_μ^* the lowest weight vector of L_μ^*. In other words, we have

$$\Phi v_\lambda = \langle \Phi \rangle \otimes v_\mu + \text{lower terms}$$

where the lower terms have lower weight than μ in the second component.

(i) Show that $\langle \Phi \rangle$ has weight $\lambda - \mu$.

(ii) Show that $f_i^{(\lambda,\alpha_i^\vee)+1} \langle \Phi \rangle = 0$ for all i.

(iii) Let $V[\nu]_\lambda$ be the subspace of vectors $v \in V[\nu]$ of weight ν which satisfy the equalities $f_i^{(\lambda,\alpha_i^\vee)+1} v = 0$ for all i. Show that the map $\Phi \mapsto \langle \Phi \rangle$ defines an isomorphism of vector spaces $\mathrm{Hom}_\mathfrak{g}(L_\lambda, V \otimes L_\mu) \cong V[\lambda - \mu]_\lambda$.

Hint. Let M_λ be the Verma module with highest weight λ, and $\overline{M}_{-\mu}$ be the **lowest weight** Verma module with lowest weight $-\mu$, i.e., generated by a vector $v_{-\mu}$ with defining relations $hv_{-\mu} = -\mu(h)v_{-\mu}$ for $h \in \mathfrak{h}$ and $f_i v_{-\mu} = 0$. Show first that the map $\Phi \mapsto \langle \Phi \rangle$ defines an isomorphism $\mathrm{Hom}_\mathfrak{g}(M_\lambda, V \otimes \overline{M}^*_{-\mu}) \cong V[\lambda - \mu]$. Next, show that $\Phi : \mathrm{Hom}_\mathfrak{g}(M_\lambda, V \otimes \overline{M}^*_{-\mu})$ factors through L_λ iff $\langle \Phi \rangle \in V[\lambda - \mu]_\lambda$, i.e., $f_i^{(\lambda,\alpha_i^\vee)+1} \langle \Phi \rangle = 0$ (for this, use that $e_j f_i^{(\lambda,\alpha_i^\vee)+1} v_\lambda = 0$, and that the kernel of $M_\lambda \to L_\lambda$ is generated by the vectors $f_i^{(\lambda,\alpha_i^\vee)+1} v_\lambda$). This implies that the above map defines an isomorphism $\mathrm{Hom}_\mathfrak{g}(L_\lambda, V \otimes \overline{M}^*_{-\mu}) \cong V[\lambda - \mu]_\lambda$. Finally, show that every homomorphism $L_\lambda \to V \otimes \overline{M}^*_{-\mu}$ in fact lands in $V \otimes L_\mu \subset V \otimes \overline{M}^*_{-\mu}$.

(iv) Let V be the vector representation of $SL_n(\mathbb{C})$. Determine the weight subspaces of $S^m V$, and compute the decomposition of $S^m V \otimes L_\mu$ into irreducibles for all μ (use (iii)).

(v) For any $\mathfrak{g}$, compute the decomposition of $\mathfrak{g} \otimes L_\mu$, where $\mathfrak{g}$ is the adjoint representation of $\mathfrak{g}$ (again use (iii)).

In both (iv) and (v) you should express the answer in terms of the numbers k_i such that $\mu = \sum_i k_i \omega_i$ and the Cartan matrix entries.

Proposition 31.11. *Every coset in P/Q contains a unique minuscule weight. This gives a bijection between P/Q and minuscule weights. So* *the number of minuscule weights equals* $\det A$*, where* A *is the Cartan matrix.*

Proof. Let $C := a + Q \in P/Q$ be a coset, and consider the intersection $C \cap P_+$. Let $\omega \in C \cap P_+$ be an element with smallest $(\omega, \rho^\vee)$. If λ is a dominant weight of L_ω then $\lambda \in C \cap P_+$, so $(\lambda, \rho^\vee) \geq (\omega, \rho^\vee)$, hence $(\omega - \lambda, \rho^\vee) \leq 0$. But $\omega - \lambda \in Q_+$, so $\lambda = \omega$. Thus ω is minuscule. On the other hand, if $\omega_1, \omega_2 \in C$ are minuscule then by Lemma 31.3, there is a coroot β such that $(\omega_1 - \omega_2, \beta) \geq 2$. So $(\omega_1, \beta) = 1$ and $(\omega_2, \beta) = -1$. The first identity implies $\beta > 0$ and the second one $\beta < 0$, a contradiction. □

31.2. **Fundamental weights of classical Lie algebras.** Let us now determine the fundamental weights of classical Lie algebras of types B_n, C_n, D_n.

Type C_n**.** Then $\mathfrak{g} = \mathfrak{sp}_{2n}$. The roots are $\mathbf{e}_i \pm \mathbf{e}_j, 2\mathbf{e}_i$, the simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \alpha_n = 2\mathbf{e}_n$, so $\alpha_i^\vee = \alpha_i$ for $i \neq n$ and $\alpha_n^\vee = \mathbf{e}_n$. So $\omega_i = (1, ..., 1, 0, ..., 0)$ (i ones) for $1 \leq i \leq n$.

Type B_n Then $\mathfrak{g} = \mathfrak{so}_{2n+1}$, so we have the same story as for C_n except $\alpha_n = \mathbf{e}_n$ and $\alpha_n^\vee = 2\mathbf{e}_n$, so we have the same ω_i for $i < n$ but $\omega_n = (\frac{1}{2}, ..., \frac{1}{2})$.

Type D_n**.** Then $\mathfrak{g} = \mathfrak{so}_{2n}$, so the roots are $\mathbf{e}_i \pm \mathbf{e}_j$, the simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \alpha_{n-2} = \mathbf{e}_{n-2} - \mathbf{e}_{n-1}$, $\alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n$, $\alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n$. So $\omega_i = (1, ..., 1, 0, ..., 0)$ (i ones) for $i = 1, ..., n-2$, but $\omega_{n-1} = (\frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2})$, $\omega_n = (\frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2})$.

31.3. **Minuscule weights outside type** A**.** Proposition 31.11 immediately tells us how many minuscule weights we have. For type A we saw that all fundamental weights are minuscule. For G_2, F_4, E_8, $\det A = 1$, so the only minuscule weight is 0. For type B_n we have $\det A = 2$, so we should have one nonzero minuscule weight, and this is the weight $(\frac{1}{2}, ..., \frac{1}{2})$. The corresponding representation has dimension 2^n. It is called the **spinor representation**, denoted S.

For C_n we also have $\det A = 2$, so we again have a unique nonzero minuscule weight. Namely, it is the weight $(1, 0, ..., 0)$ (so the minuscule representation is the tautological representation of $\mathfrak{sp}_{2n}$, of dimension $2n$). For D_n we have $\det A = 4$, so we have three nontrivial minuscule representations, with highest weights $\omega_1, \omega_{n-1}, \omega_n$, of dimensions $2n, 2^{n-1}, 2^{n-1}$. The first one is the tautological representation and the remaining two are the **spinor representations** S_+, S_-.

For E_6 there are two nontrivial minuscule representations V, V^* of dimension 27. For E_7 there is just one of dimension 56.

32. Lecture 7

32.1. **Fundamental representations of classical Lie algebras.** Let us now consider fundamental representations of classical Lie algebras.

Type C_n. Since the fundamental weights for $\mathfrak{g} = \mathfrak{sp}_{2n}$ are $\omega_i = (1, ..., 1, 0, .., 0)$ (i ones), same as for $\mathfrak{gl}_n$, one may think that the fundamental representations are also "the same", i.e. $\wedge^i V$, where V is the $2n$-dimensional vector representation. Indeed, a Cartan subalgebra in $\mathfrak{g}$ is the space of matrices $\mathrm{diag}(a_1, ..., a_n, -a_1, ..., -a_n)$, so $L_{\omega_1} = V$, with highest weight vector e_1. However, the representation $\wedge^2 V$ is not irreducible, even though it has the correct highest weight ω_2. Indeed, we have $\wedge^2 V = \wedge_0^2 V \oplus \mathbb{C}$, where $\mathbb{C}$ is the trivial representation spanned by the inverse $B^{-1} = \sum_i e_{i+n} \wedge e_i$ of the invariant nondegenerate skew-symmetric form $B = \sum_i e_i^* \wedge e_{i+n}^* \in \wedge^2 V^*$ preserved by $\mathfrak{g}$, and $\wedge_0^2 V$ is the orthogonal complement of B.

It turns out that $\wedge_0^2 V$ is irreducible. (You can show it directly or using the Weyl dimension formula). Thus we have $L_{\omega_2} = \wedge_0^2 V$ (if $n \geq 2$).

So what happens for L_{ω_i} with any $i \geq 2$? To determine this, note that we have a homomorphism of representations $\iota_B : \wedge^{i+1} V \to \wedge^{i-1} V$, which is just the contraction with B (we agree that $\wedge^{-1} V = 0$). So we may consider the subrepresentation $\wedge_0^i V = \mathrm{Ker}(\iota_B|_{\wedge^i V}) \subset \wedge^i V$.

Exercise 32.1. (i) Let $m_B : \wedge^{i-1} V \to \wedge^{i+1} V$ be the operator defined by $m_B(u) := B^{-1} \wedge u$. Show that the operators m_B, ι_B generate a representation of the Lie algebra $\mathfrak{sl}_2$ on $\wedge V := \oplus_i \wedge^i V$ where they are proportional to the operators e, f, such that h acts on $\wedge^i V$ by multiplication by $i - n$.

(ii) Show that ι_B is injective when $i \geq n$ and surjective when $i \leq n$ (so an isomorphism for $i = n$).

(iii) Show that $\mathrm{Ker}(\iota_B|_{\wedge^j V})$ is irreducible for $j \leq n$, and is isomorphic to L_{ω_j}, where we agree that $\omega_0 = 0$. Deduce that

$$\wedge V = \oplus_{i=0}^n L_{\omega_i} \otimes L_{n-j}$$

as a representation of $\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$, where L_m is the $m+1$-dimensional irreducible representation of $\mathfrak{sl}_2$ of highest weight m.

(iv) Show that every irreducible representation of $\mathfrak{sp}_{2n}$ occurs in $V^{\otimes N}$ for some N.

Thus we see another instance of the double centralizer property.

Type B_n. We have $\mathfrak{g} = \mathfrak{so}_{2n+1}$, preserving the quadratic form $Q = \sum_{i=1}^n x_i x_{i+1} + x_{2n+1}^2$. A Cartan subalgebra consists of matrices $\mathrm{diag}(a_1, ..., a_n, -a_1, ..., -a_n, 0)$. So the representations $\wedge^i V$, $1 \leq i \leq n$, where V is the $2n+1$-dimensional vector representation, have highest weight $(1,...,1,0,...0)$ (i ones), which is ω_i if $i \le n-1$.

Exercise 32.2. Show that the representation $\wedge^i V$ is irreducible for $0 \le i \le n$.

Thus for $1 \le i \le n-1$ we have $\wedge^i V = L_{\omega_i}$. On the other hand, the representation $\wedge^n V$, even though irreducible, is not fundamental. Indeed, its highest weight is $(1,...,1) = 2\omega_n$, as $\omega_n = (\frac{1}{2},...,\frac{1}{2})$. In fact, we see that the representation L_{ω_n} does not occur in $V^{\otimes N}$ for any N, since coordinates of its highest weight are not integer. This representation is called the **spinor representation** S. Vectors in S are called **spinors**. The weights of S are Weyl group translates of ω_n, so they are $(\pm\frac{1}{2},...,\pm\frac{1}{2})$ for any choices of sign, so $\dim S = 2^n$, and the character of S is given by the formula

$$\chi_S(x_1,...,x_n) = (x_1^{\frac{1}{2}} + x_1^{-\frac{1}{2}})...(x_n^{\frac{1}{2}} + x_n^{-\frac{1}{2}}).$$

This is supposed to be the trace of $\mathrm{diag}(x_1,...,x_n,x_1^{-1},...,x_n^{-1},1) \in SO_{2n+1}(\mathbb{C})$, which does not make sense since the square roots on the right hand side are defined only up to sign. This shows that the spinor representation S **does not lift** to the group $SO_{2n+1}(\mathbb{C})$. Namely, the group $SO_{2n+1}(\mathbb{C})$ is not simply connected, and the representation S only lifts to the universal covering group $\widetilde{SO}_{2n+1}(\mathbb{C})$, which is called the **spin group**, and is denoted $\mathrm{Spin}_{2n+1}(\mathbb{C})$.

Example 32.3. Let $n=1$. Then $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$ and S is the 2-dimensional irreducible representation. We know that this representation does not lift to $SO_3(\mathbb{C})$ but only to its double cover $SL_2(\mathbb{C})$, which is simply connected (so $\pi_1(SO_3(\mathbb{C})) = \mathbb{Z}/2$, demonstrated by the famous **belt trick**). So we have $\mathrm{Spin}_3(\mathbb{C}) = SL_2(\mathbb{C})$. This is related to the spin phenomenon in quantum mechanics, hence the terminology.

Proposition 32.4. *For $n \ge 3$ we have $\pi_1(SO_n(\mathbb{C})) = \mathbb{Z}/2$.*

Proof.

Lemma 32.5. *Let X_n be the hypersurface in $\mathbb{C}^n$ given by the equation $z_1^2 + ... + z_n^2 = 1$. Then for any $1 \le k \le n-2$ we have $\pi_k(X_n) = 0$, i.e., every continuous map $S^k \to X_n$ contacts to a point. E.g., X_n is connected for $n \ge 2$, simply connected for $n \ge 3$, doubly connected for $n \ge 4$, etc.*

Proof. The surface X_n is the complexification of the $n-1$-sphere, $X_n^{\mathbb{R}} := X_n \cap \mathbb{R}^n = S^{n-1}$. We will define a continuous family of maps $f_t : X_n \to X_n$ such that $f_1 = \mathrm{Id}$ and f_0 lands in $X_n^{\mathbb{R}}$. This will show that $X_n^{\mathbb{R}}$ is a retract of X_n, so X_n has the required properties since so does $X_n^{\mathbb{R}}$ (indeed, any map $\gamma = f_1 \circ \gamma : S^k \to X_n$ is homotopic to the map $f_0 \circ \gamma$ in $X_n^{\mathbb{R}}$, the homotopy being $f_t \circ \gamma$).

Let $z = x + iy \in \mathbb{C}^n$. Then $z^2 = 1$, so $x^2 - y^2 = 1, xy = 0$.

$$(x + tiy)^2 = x^2 - t^2y^2 = 1 + (1 - t^2)y^2 \geq 1.$$

So we may define

$$f_t(z) := \frac{x + tiy}{\sqrt{x^2 - t^2y^2}}.$$

Then $f_t(z)^2 = 1$, $f_1(z) = z$, and $f_1(z) = \frac{x}{|x|}$ lands in the sphere S^{n-1}, as needed. □

In particular, for $n = 4$, changing coordinates, we see that the surface $ad - bc = 1$ is doubly connected, i.e., $SL_2(\mathbb{C})$ is doubly connected and thus $\pi_1(SO_3(\mathbb{C})) = \mathbb{Z}/2$ (which we already knew).

Now, the group $SO_n(\mathbb{C})$ acts on X_n transitively with stabilizer $SO_{n-1}(\mathbb{C})$, so we have a fibration $SO_n \to X_n$ with fiber SO_{n-1}. Therefore, we have an exact sequence

$$\pi_2(X_n) \to \pi_1(SO_{n-1}(\mathbb{C})) \to \pi_1(SO_n(\mathbb{C})) \to \pi_1(X_n).$$

By Lemma 32.5, the first and the last group in this sequence are trivial for $n \geq 4$ which implies that in this case $\pi_1(SO_{n-1}(\mathbb{C})) \cong \pi_1(SO_n(\mathbb{C}))$, so we conclude by induction that $\pi_1(SO_n(\mathbb{C})) = \mathbb{Z}/2$ for all $n \geq 3$ (using the case $n = 3$ as the base). □

Corollary 32.6. *The group* $\mathrm{Spin}_{2n+1}(\mathbb{C})$ *is simply connected for* $n \geq 1$.

Exercise 32.7. Use a similar argument to show that the groups $SL_{n+1}(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$ are simply connected for $n \geq 1$ (consider their action on nonzero vectors in the vector representation and compute the stabilizer).

Type D_n. We have $\mathfrak{g} = \mathfrak{so}_{2n}$, preserving the quadratic form

$$Q = \sum_{i=1}^{n} x_i x_{i+n}.$$

A Cartan subalgebra consists of matrices $\mathrm{diag}(a_1, ..., a_n, -a_1, ..., -a_n)$. So the representation $\wedge^i V$, $1 \leq i \leq n$, where V is the $2n$-dimensional vector representation, have highest weight $(1, ..., 1, 0, ...0)$ (i ones), which is ω_i if $i \leq n - 2$.

Exercise 32.8. Show that the representation $\wedge^i V$ is irreducible for $0 \leq i \leq n - 1$.

Thus $L_{\omega_i} = \wedge^i V$ for $i \le n-2$. On the other hand, while the representation $L_{(1,...,1,0)}$ is irreducible, it is not fundamental, as $(1,...,1,0) = \omega_{n-1} + \omega_n$, where $\omega_{n-1} = (\frac{1}{2},...,\frac{1}{2},\frac{1}{2})$ and $\omega_n = (\frac{1}{2},...,\frac{1}{2},-\frac{1}{2})$. The fundamental representations $L_{\omega_{n-1}}, L_{\omega_n}$ are called the **spin representations** and denoted S_+, S_-; their elements are called **spinors**. Similarly to the odd dimensional case, they have dimensions 2^{n-1} and characters

$$\chi_{S_\pm} = \left((x_1^{\frac{1}{2}} + x_1^{-\frac{1}{2}})...(x_n^{\frac{1}{2}} + x_n^{-\frac{1}{2}})\right)_+$$

where the subscript $\pm$ means that we take the monomials with odd (for $-$), respectively even (for $+$) number of minuses. This shows that, similarly to the odd dimensional case, S_+, S_- don't occur in $V^{\otimes N}$ and don't lift to $SO_{2n}(\mathbb{C})$ but require the double covering $\mathrm{Spin}_{2n}(\mathbb{C})$, called the **spin group**. Proposition 32.4 implies

Corollary 32.9. *The group* $\mathrm{Spin}_{2n}(\mathbb{C})$ *is simply connected for* $n \ge 2$.

Example 32.10. Consider the spin groups and representations for small dimensions. We have seen that $\mathrm{Spin}_3 = SL_2$, $S = \mathbb{C}^2$. We also have $\mathrm{Spin}_4 = SL_2 \times SL_2$, with S_+, S_- being the 2-dimensional representations of the factors. We have $\mathrm{Spin}_5 = \mathrm{Sp}_4$, with S being the 4-dimensional vector representation. So $SO_5 = \mathrm{Sp}_4/(\pm 1)$. Finally, $\mathrm{Spin}_6 = SL_4$, with S_+, S_- being the 4-dimensional representation V and its dual V^*. Thus $SO_6 = SL_4/(\pm 1)$.

Exercise 32.11. Let V be a finite dimensional vector space with a nondegenerate inner product. Consider the algebra SV of polynomial functions on V^*. Let $x_1,...,x_n$ be an orthonormal basis of V, so that $SV \cong \mathbb{C}[x_1,...,x_n]$, and let $R^2 := \sum_{i=1}^n x_i^2 \in S^2V$ be the "squared radius". Also let $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ be the **Laplace operator.** Note that the Lie algebra $\mathfrak{so}(V)$ acts on SV by automorphisms and R^2 and Δ are $\mathfrak{so}(V)$-invariant. A polynomial $P \in SV$ is called **harmonic** if $\Delta P = 0$.

(i) Show that the operator of multiplication by R^2 and the Laplace operator Δ define an action of $\mathfrak{sl}_2$ on SV which commutes with $\mathfrak{so}(V)$. Namely, they are proportional to f, e respectively. Compute the operator h (it will be a first order differential operator in x_i).

(ii) Let $H_m \subset S^mV$ be the space of harmonic polynomials of degree m (a representation of $\mathfrak{so}(V)$). Show that as an $\mathfrak{so}(V) \oplus \mathfrak{sl}_2$-module, SV decomposes as

$$SV = \oplus_{m=0}^\infty H_m \otimes W_m,$$

where W_m are irreducible (infinite dimensional) representations of $\mathfrak{sl}_2$. Find the dimensions of H_m.

(iii) Show that H_m is irreducible, in fact $H_m = L_{m\omega_1}$. Decompose S^mV into a direct sum of irreducible representations of $\mathfrak{so}(V)$.

(iv) Show that W_m are Verma modules and compute their highest weights.

(v) For $s \in \mathbb{C}$ consider the algebra

$$A_s := \mathbb{C}[x_1, ..., x_n]/(x_1^2 + ... + x_n^2 - s),$$

the algebra of polynomial functions on the hypersurface $x_1^2 + ... + x_n^2 = s$ (here (f) denotes the principal ideal generated by f). This algebra has a natural action of $\mathfrak{so}(V)$. Decompose A into a direct sum of irreducible representations of $\mathfrak{so}(V)$.

33. Lecture 8

33.1. **The Clifford algebra.** It is important to be able to realize the spinor representations explicitly. The reason it is tricky is that these representations don't occur in tensor powers of V (as they have half-integer weights). However, the tensor product of a spinor representation with its dual, $S \otimes S^*$, has integer weights and does express in terms of V. So we need to extract "the square root" from this representation, in the sense that "the space of vectors of size n is the square root of the space of square matrices of size n". This is the idea behind the Clifford algebra construction.

Definition 33.1. Let V be a finite dimensional vector space over an algebraically closed field k of characteristic $\neq 2$ with a nondegenerate symmetric inner product $(,)$. The **Clifford algebra** $\mathrm{Cl}(V)$ is the algebra generated by vectors $v \in V$ with defining relations

$$v^2 = \tfrac{1}{2}(v, v), v \in V.$$

Thus for $a, b \in V$ we have

$$ab + ba = (a + b)^2 - a^2 - b^2 = \tfrac{1}{2}((a + b, a + b) - (a, a) - (b, b)) = (a, b).$$

This is a deformation of the exterior algebra $\wedge V$ which is defined in the same way but $v^2 = 0$. More precisely, $\mathrm{Cl}(V)$ has a filtration (defined by setting $\deg(v) = 1$, $v \in V$) such that the associated graded algebra receives a surjective map $\phi : \wedge V \to \mathrm{grCl}(V)$. We will show that this is a nice ("flat") deformation, in the sense that $\dim \mathrm{Cl}(V) = \dim \wedge V = 2^{\dim V}$, so that ϕ is an isomorphism. This is a kind of Poincaré-Birkhoff-Witt theorem (namely, it is similar to the PBW theorem for Lie algebras, and in fact a special case of one if you pass from Lie algebras to more general Lie superalgebras). Namely, we have the following theorem.

Theorem 33.2. *The algebra* $\mathrm{Cl}(V)$ *is isomorphic to* $\mathrm{Mat}_{2^n}(k)$ *if* $\dim V = 2n$ *and to* $\mathrm{Mat}_{2^n}(k) \oplus \mathrm{Mat}_{2^n}(k)$ *if* $\dim V = 2n + 1$.

Proof. Let us start with the even case. Pick a basis $a_1, ..., a_n, b_1, ..., b_n$ of V so that the inner product is given by

$$(a_i, a_j) = (b_i, b_j) = 0, \ (a_i, b_j) = \delta_{ij}.$$

We have $a_i a_j + a_j a_i = 0$, $b_i b_j + b_j b_i = 0$, $b_i a_j + a_j b_i = 1$. Define the $\mathrm{Cl}(V)$-module $M = \wedge(a_1, ..., a_n)$ with the action of $\mathrm{Cl}(V)$ defined by

$$\rho(a_i)w = a_i w, \ \rho(b_i)w = \tfrac{\partial w}{\partial a_i},$$

where

$$\tfrac{\partial}{\partial a_i} a_{k_1} ... a_{k_r} = (-1)^{j-1} a_{k_1} ... \widehat{a_{k_j}} ... a_{k_r}$$

if $i = k_j$ for some j (where hat means that the term is omitted), and otherwise the result is zero. It is easy to check that this is indeed a representation.

Now for $I = (i_1 < ... < i_k)$, $J = (j_1 < ... < j_m)$ consider the elements $c_{IJ} = a_{i1} ... a_{i_k} b_{j_1} ... b_{j_m} \in \mathrm{Cl}(V)$. It is easy to see that these elements span $\mathrm{Cl}(V)$. Also it is not hard to do the following exercise.

Exercise 33.3. Show that the operators $\rho(c_{IJ})$ are linearly independent.

Thus $\rho : \mathrm{Cl}(V) \to \mathrm{End} M$ is an isomorphism, which proves the proposition in even dimensions.

Now, if $\dim V = 2n + 1$, we pick a basis as above plus an additional element z such that $(z, a_i) = (z, b_i) = 0$, $(z, z) = 2$. So we have

$$z a_i + a_i z = 0, \ z b_i + b_i z = 0, \ z^2 = 1.$$

Now we can define the module $M_\pm$ on which a_i, b_i act as before and $zw = \pm(-1)^{\deg w} w$. It is easy to see as before that the map

$$\rho_+ \oplus \rho_- : \mathrm{Cl}(V) \to \mathrm{End} M_+ \oplus \mathrm{End} M_-.$$

is an isomorphism. This takes care of the odd case. □

We will now construct an inclusion of the Lie algebra $\mathfrak{so}(V)$ into the Clifford algebra. This will allow us to regard representations of the Clifford algebra as representations of $\mathfrak{so}(V)$, which will give us a construction of the spinor representations.

Consider the linear map $\xi : \wedge^2 V = \mathfrak{so}(V) \to \mathrm{Cl}(V)$ given by the formula

$$\xi(a \wedge b) = \tfrac{1}{2}(ab - ba) = ab - \tfrac{1}{2}(a, b).$$

The then

$$[\xi(a \wedge b), \xi(c \wedge d)] = [ab, cd] = abcd - cdab = (b, c)ad - acbd - cdab =$$

$$(b, c)ad - (b, d)ac + acdb - cdab =$$

$$(b, c)ad - (b, d)ac + (a, c)db - cadb - cdab = (b, c)ad - (b, d)ac + (a, c)db - (a, d)cb =$$

$$(b,c)\xi(a\wedge d)-(b,d)\xi(a\wedge c)+(a,c)\xi(d\wedge b)-(a,d)\xi(c\wedge b)=\xi([a\wedge b,c\wedge d]).$$

Thus ξ is a homomorphism of Lie algebras and we can define the representations ξ^*M for even $\dim V$ and $\xi^*M_\pm$ for odd $\dim V$ by $\rho_{\xi^*M}(a) := \rho_M(\xi(a))$.

The representation ξ^*M is reducible, namely

$$\xi^*M = (\xi^*M)_0 \oplus (\xi^*M)_1,$$

where subscripts 0 and 1 indicate the even and odd degree parts.

Exercise 33.4. (i) Show that for even $\dim V$, the representations $(\xi^*M)_0, (\xi^*M)_1$ are isomorphic to S_+, S_- respectively.

(ii) Show that for odd $\dim V$, the representations ξ^*M_+ and ξ^*M_- are both isomorphic to S.

Hint. Find the highest weight vector for each of these representations and compute the weight of this vector. Then compare dimensions.

33.2. **Duals of irreducible reoporesentations.** Now let $\mathfrak{g}$ be any complex semisimple Lie algebra. How to compute the dual of the irreducible representation L_λ? It is clear that the highest weight of L_λ^* equals $-\mu$, where μ is the lowest weight of L_λ, so we should compute the latter. For this purpose, recall that the Weyl group W of $\mathfrak{g}$ contains a unique element w_0 which maps dominant weights to antidominant weights, i.e., maps positive roots to negative roots. This is the maximal element, which is the unique element whose length is $|R_+|$. For example, if $-1 \in W$ then clearly $w_0 = -1$. It is easy to see that the lowest weight of L_λ is $w_0\lambda$.

Thus we get

Proposition 33.5. *We have* $L_\lambda^* = L_{-w_0\lambda}$.

The map $-w_0$ permutes fundamental (co)weights and simple (co)roots, so it is induced by an automorphism of the Dynkin diagram of $\mathfrak{g}$. So if $\mathfrak{g}$ is simple and its Dynkin diagram has no nontrivial automorphisms, we have $w_0 = -1$, so $-w_0 = 1$ and thus $L_\lambda^* = L_\lambda$ for all λ. This happens for A_1, B_n, C_n, G_2, F_4, E_7 and E_8. For A_n, $n \geq 2$, we have $L_{\omega_1}^* = V^* = \wedge^n V = L_{\omega_n}$ (as $\dim V = n+1$), so $-w_0$ is the flip of the chain. For E_6, one can check that $-w_0$ is also the flip.

Exercise 33.6. (i) Show that for D_{2n+1} we have $S_+^* = S_-$ while for D_{2n} we have $S_+^* = S_+$, $S_-^* = S_-$.

(ii) Show that the restriction of the spinor representation S of $\mathfrak{so}_{2n+1}$ to $\mathfrak{so}_{2n}$ is $S_+ \oplus S_-$.

(iii) Show that there exist unique up to scaling nonzero **Clifford multiplication** homomorphisms

$$V\otimes S \to S,\ V\otimes S_+ \to S_-,\ V\otimes S_- \to S_+.$$

(iv) Compute the decomposition of the tensor products

$$S \otimes S^*,\ S_+ \otimes S_+^*,\ S_- \otimes S_-^*,\ S_+ \otimes S_-^*$$

into irreducible representations.

Hint. In the odd dimensional case, use that $\mathrm{Cl}(V) = 2S \otimes S^*$ as an $\mathfrak{so}(V)$-module, that $\mathrm{grCl}(V) = \wedge V$, and that representations of $\mathfrak{so}(V)$ are completely reducible.

The even case is similar:

$$\mathrm{Cl}(V) = S_+ \otimes S_+^* \oplus S_- \otimes S_-^* \oplus S_- \otimes S_+^* \oplus S_+ \otimes S_-^*.$$

If $\dim V = 2n$ and n is even, use that all representations of $\mathfrak{so}(V)$ are self-dual to conclude that the last two summands are isomorphic. (If n is odd, they will not be isomorphic).

Also in this case you need to pay attention to the middle exterior power - it should split into two parts. Namely, if $\dim V = 2n$ then on $\wedge^n V$ we have two invariant bilinear forms: one symmetric coming from the one on V, denoted $B(\xi, \eta)$, and the other given by wedge product $\wedge : \wedge^n V \times \wedge^n V \to \wedge^{2n} V = \mathbb{C}$, which is symmetric for even n and skew-symmetric for odd n. Since the wedge product form is nondegenerate, there is a unique linear operator $* : \wedge^n V \to \wedge^n V$ called the **Hodge *-operator** such that $B(\xi, \eta) = \xi \wedge *\eta$. You should show that $*^2 = 1$ in the even case and $*^2 = -1$ in the odd case (use an orthonormal basis of V). Thus we have an eigenspace decomposition $\wedge^n V = \wedge_+^n V \oplus \wedge_-^n V$, into eigenspaces of $*$ with eigenvalues ± 1 in the even case (called selfdual and anti-selfdual forms respectively) and $\pm i$ in the odd case. You will see that these pieces are irreducible and isomorphic to each other in the odd case but not in the even case, and that one of them (which?) goes into $S_+ \otimes S_+^*$ and the other into $S_- \otimes S_-^*$.

33.3. **Principal $\mathfrak{sl}_2$, exponents.** Let $\mathfrak{g}$ be a simple Lie algebra and let $e = \sum_i e_i$ and $h \in \mathfrak{h}$ be such that $\alpha_i(h) = 2$ for all i (i.e., $h = 2\rho^\vee$). We have $[h, e] = 2e$ and $h = \sum_i (2\rho^\vee, \omega_i) h_i$. So defining $f := \sum_i (2\rho^\vee, \omega_i) f_i$, we have $[h, f] = -2f$, $[e, f] = h$. So e, f, h span an $\mathfrak{sl}_2$ subalgebra of $\mathfrak{g}$ called the **principal $\mathfrak{sl}_2$-subalgebra.**

Example 33.7. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Show that the restriction of the $n+1$-dimensional vector representation V of $\mathfrak{g}$ to the principal $\mathfrak{sl}_2$-subalgebra is the irreducible representation L_n.

Consider now $\mathfrak{g}$ as a module over its principal $\mathfrak{sl}_2$-subalgebra. How does it decompose? To see this, we can look at the weight decomposition of $\mathfrak{g}$ under h. We have $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and these summands correspond to negative, zero and positive weights, respectively. Moreover, all weights are even and for $m > 0$, $\dim \mathfrak{g}[2m] = r_m$ is the number of positive roots of height m, i.e., representable as a sum of m simple roots.

Definition 33.8. m is called an **exponent** of $\mathfrak{g}$ if $r_m > r_{m+1}$. The multiplicity of m is $r_m - r_{m+1}$.

Since r_m is zero for large m while $r_1 = r$, the rank of $\mathfrak{g}$, there are r exponents counting multiplicities. The exponents of $\mathfrak{g}$ are denoted m_i and are arranged in non-decreasing order: $m_1 \le m_2 \le ... \le m_r$ (including multiplicities). Since roots of height 2 are $\alpha_i + \alpha_j$ where i, j are connected by an edge, we have $r_2 = r - 1$, so $m_1 = 1$ and $m_2 > 1$. We also have $m_r = (\rho^\vee, \theta) = h_\mathfrak{g} - 1$, where θ is the maximal root and $h_\mathfrak{g}$ is the **Coxeter number** of $\mathfrak{g}$. Finally, we have $\sum_{i=1}^r m_i = |R_+|$.

Proposition 33.9. *The restriction of* $\mathfrak{g}$ *to the principal* $\mathfrak{sl}_2$*-subalgebra decomposes as* $\oplus_{i=1}^r L_{2m_i+1}$.

Proof. This easily follows from the representation theory of $\mathfrak{sl}_2$ and the definition of m_i. ☐

Example 33.10. The exponents of $\mathfrak{sl}_n$ are $1, 2, ..., n-1$.

33.4. **Complex, real and quaternionic type.**

Definition 33.11. An irreducible finite dimensional $\mathbb{C}$-representation V of a group G or Lie algebra $\mathfrak{g}$ is **complex type** when $V \not\cong V^*$, **real type** if there is a symmetric isomorphism $V \to V^*$ (i.e., an invariant symmetric inner product of V), and **quanternionic type** if there is a skew-symmetric isomorphism $V \to V^*$ (i.e., an invariant skew-symmetric inner product of V).

It is easy to see that any irreducible finite dimensional representation is of one of these three types.

Exercise 33.12. Let V be an irreducible finite dimensional representation of a finite group G.

(i) Show that $\mathrm{End}_{\mathbb{R}G} V$ is $\mathbb{C}$ of complex type, $\mathrm{Mat}_2(\mathbb{R})$ for real type and the quaternion algebra $\mathbb{H}$ for quaternionic type. This explains the terminology.

(ii) Show that V is of real type if and only if in some basis of V the matrices of all elements of G have real entries.

You may find helpful to look at `http://www-math.mit.edu/~etingof/reprbook.pdf`, Problem 5.1.2 (it contains a hint).

Example 33.13. Let L_n be the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight n (i.e., of dimension $n+1$). Then L_n is of real type for even n and quaternionic type for odd n. Indeed, $L_n = S^nV$, where $V = L_1 = \mathbb{C}^2$, so the invariant form on L_n is S^nB, where B is the invariant form on V, which is skew-symmetric.

Now let $\mathfrak{g}$ be any simple Lie algebra and $\lambda \in P_+$ be such that $\lambda = -w_0\lambda$, so that L_λ is self-dual. How to tell if it is of real or quaternionic type?

Proposition 33.14. *L_λ is of real type if $(2\rho^\vee, \lambda)$ is even and of quaternionic type if it is odd.*

Proof. The number $n := (2\rho^\vee, \lambda)$ is the eigenvalue of the element h of the principal $\mathfrak{sl}_2$-subalgebra on the highest weight vector v_λ. All the other eigenvalues are strictly less. Thus the restriction of L_λ to the principal $\mathfrak{sl}_2$-subalgebra is of the form $L_n \oplus \bigoplus_{m<n} k_m L_m$. The invariant form on L_λ restricts to an invariant form on L_n which must be nonzero, so it is skew if n is odd and symmetric if n is even. $\square$

Example 33.15. Consider $\mathfrak{g} = \mathfrak{so}_{2n}$. Then we have

$$\rho^\vee = \rho = \sum_i \omega_i = (n-1, n-2, ..., 1, 0).$$

So $(2\rho^\vee, \omega_{n-1}) = (2\rho^\vee, \omega_n) = \frac{n(n-1)}{2}$. This is odd if $n = 2, 3$ modulo 4 and even if $n = 0, 1$ modulo 4. Thus $S_\pm$ carry a symmetric form when $n = 0$ mod 4 and a skew-symmetric form if $n = 2$ mod 4.

Consider now $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Then $2\rho^\vee = \sum_i \omega_i^\vee = (n, n-1, ..., 1)$. So $(2\rho^\vee, \omega_n) = \frac{n(n-1)}{2}$. So S carries a skew-symmetric form if $n = 1, 2$ mod 4 and a symmetric form if $n = 0, 3$ mod 4.

We obtain the following "Bott periodicity" theorem.

Theorem 33.16. *The behavior of the spin representations of the orthogonal Lie algebra $\mathfrak{so}_m$ is determined by the remainder r of m modulo 8. Namely:*

For $r = 1, 7$, S is of real type.
For $r = 3, 5$, S is of quaternionic type.
For $r = 0$, S_+, S_- are of real type.
For $r = 2, 6$, $S_+^* = S_-$ (complex type).
For $r = 4$, S_+, S_- are of quaternionic type.

34. Lecture 9

34.1. **Reminder on differential forms.** Let M be a smooth n-dimensional real manifold. Recall that a differential k-form on M is a smooth section of the vector bundle $\wedge^i T^*M$, i.e., a skew-symmetric $(n, 0)$-tensor field. Thus for example a 1-form is a section of T^*M. If $x_1, ..., x_n$ are local coordinates on M near some point $p \in M$ then the differentials $dx_1, ..., dx_n$ form a basis in fibers of T^*M near this point, so a general 1-form in these coordinates has the form

$$\omega = \sum_{i=1}^{n} f_i(x_1, ..., x_n) dx_i.$$

If we change the coordinates $x_1, ..., x_n$ to $y_1, ..., y_n$ then x_i are smooth functions of $y_1, ..., y_n$ and in the new coordinates ω looks like

$$\omega = \sum_{i,j=1}^{n} f_i(x_1, ..., x_n) \frac{\partial x_i}{\partial y_j} dy_j.$$

Similarly, a differential k-form in the coordinates x_i looks like

$$\omega = \sum_{1 \le i_1 < ... < i_k \le n} f_{i_1,...,i_k}(x_1, ..., x_n) dx_{i_1} \wedge ... \wedge dx_{i_k}$$

where $f_{i_1,...,i_k}$ are smooth functions, and in the coordinates y_j it looks like

$$\omega = \sum_{1 \le i_1 < ... < i_k \le n} \sum_{1 \le j_1 < ... < j_k \le n} f_{i_1,...,i_k}(x_1, ..., x_n) \det \left(\frac{\partial x_{i_r}}{\partial y_{j_s}} \right) dy_{j_1} \wedge ... \wedge dy_{j_k}.$$

The space of differential k-forms on M is denoted $\Omega^k(M)$. For instance, $\Omega^0(M) = C^\infty(M)$ and $\Omega^k(M) = 0$ for $k > n$. Consider now the extremal case $k = n$. The bundle $\wedge^n T^*M$ is a line bundle, so locally any differential n-form in coordinates x_i has the form

$$\omega = f(x_1, ..., x_n) dx_1 \wedge ... \wedge dx_n,$$

which in coordinates y_j takes the form

$$\omega = f(x_1, ..., x_n) \det \left(\frac{\partial x_i}{\partial y_j} \right) dy_1 \wedge ... \wedge dy_n.$$

We have a canonical differentiation operator $d : \Omega^0(M) \to \Omega^1(M)$ given in local coordinates by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

It is easy to check that this operator does not depend on the choice of coordinates (this becomes obvious if you define it without coordinates, $df(v) = \partial_v f$ for $v \in T_pM$). Also $\Omega^\bullet(M) := \oplus_{k=0}^{n} \Omega^k(M)$ is a graded algebra under wedge product, and d naturally extends to a degree 1 derivation $d : \Omega^\bullet(M) \to \Omega^\bullet M$ defined in coordinates by

$$d(f(x_1, ..., x_n) dx_{i_1} \wedge ... \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}.$$

This is independent on choices and gives rise to a derivation in the "graded" sense:

$$d(a \wedge b) = da \wedge b + (-1)^{\deg a} a \wedge db.$$

A form ω is **closed** if $d\omega = 0$ and **exact** if $\omega = d\eta$ for some η. It is easy to check that $d^2 = 0$, so any exact form is closed. However, not every closed form is exact: on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ the form dx is closed but the function x is defined only up to adding integers, so dx is not exact. The space $\Omega^k_{\text{closed}}(M)/\Omega^k_{\text{exact}}(M)$ is called the k-th **de Rham cohomology** of M, denoted $H^k(M)$.

If $f : M \to N$ is a differentiable mapping then for a differential form $\omega \in \Omega^k(N)$ we can define the pullback $f^*\omega \in \Omega^k(N)$, given by $(f^*\omega)(v_1, , , .v_k) = \omega(f_*v_1, ..., f_*v_k)$ for $v_1, ..., v_k \in T_pM$. This operation commutes with wedge product and the differential, and $(f \circ g)^* = g^* \circ f^*$.

34.2. **Partitions of unity.** Let M be a manifold and $\{U_i, i \in I\}$ be an open cover of M.

Definition 34.1. A smooth **partition of unity** subordinate to $\{U_i, i \in I\}$ is a collection $\{f_s, s \in S\}$ of smooth nonnegative functions on M such that

(i) for all s the support of f_s is contained in U_i for some $i = i(s)$;

(ii) Any $y \in M$ has a neighborhood in which all but finitely many f_s are zero;

(iii) $\sum_s f_s = 1$.

Note that the sum in (iii) makes sense because of condition (ii).

Note also that given any partition of unity $\{f_s\}$ subordinate to $\{U_i\}$, we can define

$$F_i := \sum_{s:i(s)=i} f_s,$$

and this is a new partition of unity subordinate to the same cover now labeled by the set I, with the support of F_i contained in U_i.

Finally, note that in every partition of unity on M, the set of s such that f_s is not identically zero is countable, and moreover finite if M is compact. This follows from the fact that any open cover of a manifold M has a countable subcover, which can be chosen finite for compact M (applied to the neighborhoods from condition (ii)).

Proposition 34.2. *For any open cover there exists a partition of unity subordinate to this cover.*

Proof. Define a function $h : [0, \infty] \to \mathbb{R}$ given by $h(t) = 0$ for $t \geq 1$ and $h(t) = \exp(\frac{1}{t-1})$ for $t < 1$. It is easy to check that h is smooth.

Thus we can define the smooth **hat function** $H(x) := h(|x|^2)$ on $\mathbb{R}^n$, supported on the closed unit ball $\overline{B(0,1)}$.

If $\phi : \overline{B(0,1)} \to M$ is a C^∞-map which is a diffeomorphism onto the image, we will say that the image of ϕ is a **closed ball** in M. Thus given a closed ball $\overline{B}$ on M (equipped with a diffeomorphism $\phi : \overline{B(0,1)} \to \overline{B}$), we have a hat function $H_B(y) := H(\phi^{-1}(y))$ on $\overline{B}$, which we extend by zero to a smooth function on M whose support is $\overline{B}$ and which is strictly positive in its interior $B \subset \overline{B}$.

Now suppose that $\{\overline{B}_s, s \in S\}$ are closed balls in M such that their interiors B_s cover M. Let us say this collection is **locally finite** if every $y \in M$ has a neighborhood which has empty intersection with all but finitely many $\overline{B}_s$. Given such a locally finite collection, and picking diffeomorphisms $\phi_s : \overline{B(0,1)} \to \overline{B}_s$, we can define the smooth function $F(y) := \sum_s H_{B_s}(y)$, which is strictly positive on M since B_s cover M (this makes sense by the local finiteness). Now define the smooth functions $f_s(y) := \frac{H_{B_s}(y)}{F(y)}$. If for an open cover $\{U_i\}$ of M we have $\overline{B}_s \subset U_i$ for some $i = i(s)$, this collection is a partition of unity subordinate to this cover, as desired. So it remains to construct such a locally finite collection.

To this end, for every $y \in M$ fix a closed ball $\overline{B}_y \subset M$ such that $y \in B_y$. Let $\{V_j, j \geq 1\}$ be a countable base of M. For every $y \in M$ there is $j(y)$ such that $y \in V_{j(y)} \subset B_y$. Let J be the range of the function $y \mapsto j(y)$, and for each $k \in J$ pick $y_k \in M$ such that $j(y_k) = k$. Given $y \in M$, let $k = j(y)$, then $y \in V_{j(y)} = V_{j(y_k)} \subset B_{y_k}$. Thus $\{B_{y_k}, k \in J\}$ is an open cover of M. This allows us to represent M as a nested union of compact sets:

$$M = \cup_{r=1}^{\infty} M_r,\ M_r := \cup_{j=k}^{r} \overline{B}_{y_k}.$$

Consider now the compact sets $K_r = \overline{M_r \setminus M_{r-1}}$; we have $M = \cup_{r=1}^{\infty} K_r$. Let $m(r)$ be the largest integer m for which

$$M_m \subset B_1 \cup ... \cup B_{r-1},$$

then $K_r \cap M_{m(r)} = \emptyset$. For each $y \in M_{m(r)}$ pick a closed ball

$$\overline{B}(y) \subset M \setminus K_r$$

such that $y \in B(y)$, and pick finitely many such balls whose interiors still cover $M_{m(r)}$ (which can be done since $M_{m(r)}$ is compact). The union of these interiors is a neighborhood W_r of $M_{m(r)}$ such that $K_r \cap \overline{W}_r = \emptyset$. Thus for each $z \in K_r$ is contained in a closed ball $\overline{B}(r, z) \subset M$ contained in some U_i and having empty intesection with $\overline{W}_r$. Since K_r is compact, we can pick finitely many of them, $\overline{B}(r, z_k)$, $1 \leq k \leq N_r$, whose interiors cover K_r (this can be done since K_r is compact). Since M_m are compact, $m(r) \to \infty$ as $r \to \infty$, so every point $y \in M$ has a neighborhood intersecting only finitely many $\overline{B}(r, z_k)$. Also $B(r, z_k)$ cover M and for each z, k, $\overline{B}(r, z_k) \subset U_i$ for some i. Thus the collection $\{\overline{B}(r, z_k)\}$ has the desired properties and the proposition is proved. $\square$

34.3. **Integration on manifolds.** An important operation with top degree differential forms is **integration**. Namely, if ω is a differential n-form on an open set $U \subset \mathbb{R}^n$ (with the usual orientation), $\omega = f(x_1, ..., x_n)dx_1 \wedge ... \wedge dx_n$, then we can set

$$\int_U \omega := \int_U f(x_1, ..., x_n)dx_1...dx_n.$$

(provided this integral is absolutely convergent). This, however, is not completely canonical: if we change coordinates (so that U maps diffeomorphically to U'), the change of variable formula in a multiple integral tells us that

$$\int_U f(x_1, ..., x_n)dx_1...dx_n = \int_{U'} f(x_1, ..., x_n)|\det\left(\frac{\partial x_i}{\partial y_j}\right)|dy_1 \wedge ... \wedge dy_n,$$

while the transformation law for ω is the same but without the absolute value. This shows that our definition is invariant only under orientation preserving transformations of coordinates, i.e., ones whose Jacobian $\det\left(\frac{\partial x_i}{\partial y_j}\right)$ is positive. Consequently, we will only be able to define integration of top differential forms on **oriented manifolds**, i.e., ones equipped with an atlas of charts in which transition maps have a positive Jacobian; such an atlas defines an **orientation** on M. To fix an orientation, we just need to say which local coordinate systems (or bases of tangent spaces) are right-handed, and do so in a consistent way. But this cannot always be done globally (the classic counterexamples are Möbius strip and Klein bottle).

Now let us proceed to define integration of a continuous top form ω over an oriented manifold M. For this pick an atlas of local charts $\{U_i, i \in I\}$ on M and pick a partition of unity $\{f_s\}$ subordinate to this cover. First assume that ω is nonnegative, i.e., $\omega(v_1, ..., v_n) \geq 0$ for a right-handed basis v_i of any tangent space of M. Then define

$$\int_M \omega := \sum_s \int_{U_{i(s)}} f_{i(s)}\omega \tag{4}$$

where in each U_i we use a right-handed coordinate system to compute the corresponding integral. This makes sense (as a nonnegative real number or $+\infty$), and is also independent of the choice of a partition of unity. Indeed, it is easy to see that for two partitions of unity $\{f_s\}, \{g_t\}$ the answer is the same, by comparing both to the answer for the partition of unity $\{f_s g_t\}$. In fact, this makes sense for any measurable ω (i.e., given by a measurable function in every local chart) if we use Lebesgue integration.

Now, if ω is not necessarily nonnegative, we may define the nonnegative form $|\omega|$ which is ω at points where ω is nonnegative and $-\omega$ otherwise. Then, if

$$\int_M |\omega| < \infty,$$

we can define $\int_M \omega$ by the same formula (4) which will now be a not necessarily positive but absolutely convergent series (a finite sum in the compact case).

Importantly, the same definition works for manifolds M with boundary ∂M (an $n-1$-manifold); the only difference is that at boundary points the manifold looks like $\mathbb{R}^n_+$ (the space of vectors with nonnegative last coordinate) rather than $\mathbb{R}^n$. Note that the boundary of an oriented manifold carries a canonical orientation as well (a basis of $T_p\partial M$ is right-handed if adding to it a vector looking inside M produces a right-handed basis of T_pM).

34.4. **Nonvanishing forms.** Let us say that a top degree continuous differential form ω on M is **non-vanishing** if for any $x \in M$, $\omega_x \in \wedge^n T_x^*M$ is nonzero. In this case, ω defines an orientation on M by declaring a basis $v_1, ..., v_n$ of T_xM right-handed if $\omega(v_1, ..., v_n) > 0$ (in particular, there are no non-vanishing top forms on non-orientable manifolds). Thus we can integrate top differential forms on M, and in particular ω defines a positive measure μ_ω on G, namely

$$\mu_\omega(U) = \int_U \omega$$

for an open set $U \subset M$ (this integral may be $+\infty$, but is finite if U is a small enough neighborhood of any point $x \in M$). Thus we can integrate functions on M with respect to this measure:

$$\int_M f d\mu = \int_M f\omega.$$

This, of course, only makes sense if f is measurable and $\int_M |f| d\mu < \infty$, i.e., if $f \in L^1(M, \mu)$. Note also that if $\lambda \in \mathbb{R}^\times$ then $\mu_{\lambda\omega} = |\lambda|\mu_\omega$.

Example 34.3. If M is an open set in $\mathbb{R}^n$ with the usual orientation and $\omega = dx_1 \wedge ... \wedge dx_n$ then $\int_M \omega = \int_M dx_1 ... dx_n$ is just the volume of M. For this reason top differential forms are often called **volume forms**, especially when they are non-vanishing and thus define an orientation and a measure on M, and in the latter case $\int_M \omega$, if finite, is called the **volume of** M with respect to ω.

Proposition 34.4. *If M is compact and ω is non-vanishing then M has finite volume under the measure $\mu = \mu_\omega$, and every continuous function on M is in $L^1(M, \mu)$.*

Proof. For each $x \in M$ choose a neighborhood U_x of x such that $\mu(U_x) < \infty$. The collection of sets U_x forms an open cover of M, so it has a finite subcover $U_1, ..., U_N$, and $\mu(M) \le \mu(U_1) + ... + \mu(U_N) < \infty$. Then $\int_M |f| d\mu < \max f \cdot \mu(M) < \infty$ for continuous f. □

A central result about integration of differential forms is

Theorem 34.5. *(Stokes formula) If M is an n-dimensional oriented manifold with boundary and ω a differential $n-1$-form on M of class C_1 then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

In particular, if M is closed (has no boundary) then $\int_M d\omega = 0$, and if ω is closed ($d\omega = 0$) then $\int_{\partial M} \omega = 0$.

When M is an interval in $\mathbb{R}$, this reduces to the fundamental theorem of calculus. If M is a region in $\mathbb{R}^2$, this reduces to Green's formula. If M is a surface in $\mathbb{R}^3$, this reduces to the classical Stokes formula from vector calculus. Finally, if M is a region in $\mathbb{R}^3$ then this reduces to the Gauss formula.

The proof of the Stokes formula is not difficult. Namely, by writing ω as $\sum_s f_s \omega$ for some partition of unity, it suffices to prove the formula for M being a box in $\mathbb{R}^n$, which easily follows from the fundamental theorem of calculus.

Remark 34.6. Smooth positive measures (or densities), unlike nonvanishing top forms, exist and can be integrated even on non-orientable manifolds. Namely, we can define the absolute value of the orientation bundle, $| \wedge^n T^*M|$, which is a real line bundle having transition functions $|g_{ij}(x)|$, where $g_{ij}(x)$ are the transition functions of $\wedge^n T^*M$. Using partitions of unity, it is not hard to show that this bundle is trivial (check it!). A positive section of this bundle (i.e., positive in every chart) therefore exists and is nothing but a positive smooth measure on M, and any two such measures differ by multiplication by a positive smooth function. And given such a measure μ and a measurable function f on M such that $\int_M |f| d\mu < \infty$ (i.e., $f \in L^1(M, \mu)$), we can define $\int_M f d\mu$ as usual.

35.1. **Integration on Lie groups.** Now let G be a real Lie group of dimension n. In this case given any $\xi \in \wedge^n \mathfrak{g}^*$, we can extend it to a left-invariant skew-symmetric tensor field (i.e., top differential form) ω_ξ on G. Also, if $\xi \neq 0$ then $\omega = \omega_\xi$ is non-vanishing and thus defines an orientation and a left-invariant positive measure μ_ω on G. Note that ξ is unique up to scaling by a real number $\lambda \in \mathbb{R}^\times$. So, since $\mu_{\lambda\omega} = |\lambda|\mu_\omega$, we see that μ_ω is defined uniquely up to scaling by positive numbers. This measure is called the **left-invariant Haar measure** and we'll denote it just by μ_L (assuming that the normalization has been chosen somehow).

In a similar way we can define the **right invariant Haar measure** μ_R on G. One may ask if these measures coincide (or, rather, are proportional, since they are defined only up to normalization). This question is answered by the following proposition.

Given a 1-dimensional real representation V of a group G, let $|V|$ be the representation of G on the same space with $\rho_{|V|}(g) = |\rho_V(g)|$, where $\rho : G \to \mathrm{Aut}(V) = \mathbb{R}^\times$.

Proposition 35.1. $\mu_L = \mu_R$ *if and only if* $| \wedge^n \mathfrak{g}^*|$ *(or, equivalently,* $| \wedge^n \mathfrak{g}|$*) is a trivial representation of* G*.*

Proof. It is clear that $\mu_L = \mu_R$ if and only if the left-invariant top volume form ω on G is also right invariant up to sign. This is equivalent to saying that ω is conjugation invariant up to sign, i.e., that $\omega_1 \in \wedge^n \mathfrak{g}^*$ is invariant up to sign under the action of G. This implies the statement. □

If $\mu_L = \mu_R$ then G is called **unimodular**. In this case we have a **bi-invariant Haar measure** $\mu = \mu_L = \mu_R$ (under some normalization) on G.

Example 35.2. If G is a discrete countable group then G is unimodular and μ is the counting measure: $\mu(U) = |U|$ (number of elements in U).

In particular, we see that if G has no nontrivial characters $G \to \mathbb{R}^\times$ then it is unimodular.

Exercise 35.3. (i) Let us say that a finite dimensional real Lie algebra $\mathfrak{g}$ of dimension n is unimodular if $\wedge^n \mathfrak{g}$ is a trivial representation of $\mathfrak{g}$. Show that a connected Lie group G is unimodular if and only if so is Lie$G$.

(ii) Show that a perfect Lie algebra (such that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$) is unimodular. In particular, a semisimple Lie algebra is unimodular.

(ii) Show that if $\mathfrak{g}_1, \mathfrak{g}_2$ are unimodular then so is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Deduce that a reductive Lie algebra is unimodular.

(iii) Show that a nilpotent (in particular, abelian) Lie algebra is unimodular.

(iv) Show that the Lie algebra of upper triangular matrices of size n is **not** unimodular for $n > 1$. Give an example of a Lie algebra $\mathfrak{g}$ and ideal I such that I and $\mathfrak{g}/I$ are unimodular but $\mathfrak{g}$ is not.

(v) Give an example of a non-unimodular Lie group G such that its connected component of the identity is unimodular (try groups of the form $\mathbb{Z} \ltimes \mathbb{R}$).

For a unimodular Lie group G, we will sometimes denote the integral of a function f with respect to the Haar measure by

$$\int_G f(g)dg.$$

Proposition 35.4. *A compact Lie group is unimodular.*

Proof. The representation of G on $|\wedge^n \mathfrak{g}^*|$ defines a continuous homomorphism $\rho : G \to \mathbb{R}^+$. Since G is compact, the image $\rho(G)$ of ρ is a compact subgroup of $\mathbb{R}^+$. But the only such subgroup is the trivial group. This implies the statement. □

Thus, on a compact Lie group we have a (bi-invariant) Haar measure μ. Moreover, in this case $\int_G d\mu = \text{Volume}(G) < \infty$, so we have a canonical normalization of μ by the condition that it is a probability measure:

$$\int_G d\mu = 1.$$

E.g., for finite groups this normalization is the averaging measure, which is $|G|^{-1}$ times the counting measure. This is the normalization we will use if G is compact.

35.2. **Representations of compact Lie groups.** Now we can extend to compact groups the result that representations of finite groups are unitary. Namely, let G be a finite dimensional complex representation of a compact Lie group G.

Proposition 35.5. *V admits a G-invariant unitary structure.*

Proof. Fix a positive Hermitian form B on V and define a new Hermitian form on V by

$$B_{\text{av}}(v, w) = \int_G B(\rho_V(g)v, \rho_V(g)w)dg.$$

This form is well defined since G is compact and is G-invariant by construction (since the measure dg is invariant). Also $B_{\mathrm{av}}(v,v) > 0$ for $v \neq 0$ since $B(w,w) > 0$ for any $w \neq 0$. □

Corollary 35.6. *Every finite dimensional representation V of a compact Lie group G is completely reducible.*

Proof. Let $W \subset V$ be a subrepresentation and B be an invariant positive Hermitian form on V. Let $W^\perp \subset V$ be the orthogonal complement of W under B. Then $V = W \oplus W^\perp$, which implies the statement. □

In particular, this applies to the special unitary group $SU(n)$. Recall that $SU(n)/SU(n-1) = S^{2n-1}$, which implies that $SU(n)$ is simply connected. Thus representations of $SU(n)$ is the same thing as representations of the Lie algebra $\mathfrak{su}(n)$ or its complexification $\mathfrak{sl}_n$. Thus we get a new, analytic proof that finite dimensional representations of $\mathfrak{sl}_n$ are completely reducible (this is called "Weyl's unitary trick"). In fact, we will see that complete reducibility of finite dimensional representations of all semisimple Lie algebras can be proved in this way.

35.3. **Matrix coefficients.** Let V be an irreducible representation of a compact Lie group G. As shown above, it has an invariant positive Hermitian inner product, which we'll denote by $(,)$. Moreover, this product is unique up to scaling. Pick an orthonormal basis of V under this inner product: $v_1, ..., v_n$. The **matrix coefficients** of V in this basis are smooth functions

$$\psi_{V,ij}(g) := (\rho_V(g)v_i, v_j).$$

Note that these functions are independent on the normalization of $(,)$.

Suppose now that we also have another such representation W with orthonormal basis w_i.

Theorem 35.7. *(Orthogonality of matrix coefficients) We have*

$$\int_G \psi_{V,ij}(g)\overline{\psi_{W,kl}(g)}dg = \frac{\delta_{VW}\delta_{ik}\delta_{jl}}{\dim V},$$

where $\delta_{VW} = 0$ if V is not isomorphic to W, and $\delta_{VW} = 1$, $v_i = w_i$ if $V = W$.

Proof. We have

$$\int_G \psi_{V,ij}(g)\overline{\psi_{W,kl}(g)}dg = \int_G ((\rho_V(g) \otimes \rho_{\overline{W}}(g))(v_i \otimes w_k), v_j \otimes w_l)dg,$$

so we need to compute

$$P := \int_G (\rho_V(g) \otimes \rho_{\overline{W}}(g))dg = \int_G \rho_{V\otimes\overline{W}}(g)dg.$$

Since W is unitary, $\overline{W} = W^*$, so we have

$$P = \int_G \rho_{V\otimes W^*}(g)dg : V \otimes W^* \to V \otimes W^*.$$

By construction, $\mathrm{Im}(P) \subset (V \otimes W^*)^G$, which is zero if $V \ncong W$. Thus we have proved the proposition in this case.

It remains to consider the case $V = W$. In this case the only invariant in $V \otimes W^* = V \otimes \overline{V}$ up to scaling is $\mathbf{u} := \sum_k v_k \otimes v_k$. Also P is conjugation invariant under G, so by decomposing $V \otimes V^*$ into irreducibles we see that it is the orthogonal projector to $\mathbb{C}\mathbf{u}$:

$$P\mathbf{x} = \frac{(\mathbf{x}, \mathbf{u})}{(\mathbf{u}, \mathbf{u})}\mathbf{u} = \frac{(\mathbf{x}, \mathbf{u})\mathbf{u}}{\dim V}.$$

In particular,

$$(P(v_i \otimes w_k), v_j \otimes w_l) = \frac{\delta_{ik}\delta_{jl}}{\dim V},$$

as claimed. □

Thus we see that the functions $\psi_{V,ij}$ for various V, i, j form an orthonormal system in the Hilbert space $L^2(G) = L^2(G, dg)$ of measurable functions $f : G \to \mathbb{C}$ such that

$$||f||^2 = \int_G |f(g)|^2 dg < \infty.$$

A fundamental result about compact Lie groups is that this system is, in fact, complete:

Theorem 35.8. *(Peter-Weyl theorem) The functions $\psi_{V,ij}$ form an orthonormal basis of $L^2(G)$.*

This theorem may be reformulated as follows. Given a finite dimensional irreducible representation V of G, it is easy to see that we have a natural isomorphism $\mathrm{Hom}_G(V, L^2(G)) \cong V^*$ as G-modules. Indeed, an element $f \in \mathrm{Hom}_G(V, L^2(G))$ can be viewed as a G-invariant L^2-function $f : G \to V^*$, i.e. such that $f(gx) = \rho_{V^*}(g)f(x)$, $g, x \in G$. But then $f(g) = \rho_{V^*}(g)f(1)$, which is smooth in g, hence lies in $L^2(G)$. Thus we have an inclusion

$$\xi : \oplus_{V\in\mathrm{Irrep}(G)} V \otimes \mathrm{Hom}_G(V, L^2(G)) = \oplus_{V\in\mathrm{Irrep}(G)} V \otimes V^* \to L^2(G),$$

which is actually an embedding of $G \times G$-modules, and we will denote the image of ξ by $L^2_{\mathrm{alg}}(G)$ (the "algebraic part" of $L^2(G)$). Note that if $\psi \in L^2(G)$ generates a finite dimensional representation U under the action of G by left translations then ψ belongs to the image of a homomorphism $U \to L^2(G)$, hence to $L^2_{\mathrm{alg}}(G)$. Thus $L^2_{\mathrm{alg}}(G)$ is just the subspace of $\psi \in L^2(G)$ which generate a finite dimensional representation under left translations by G. We also see that it may be equivalently characterized as the subspace of $\psi \in L^2(G)$ which generate a finite dimensional representation under right translations by G.

Theorem 35.9. *(Peter-Weyl theorem, alternative formulation) The space $L^2_{\mathrm{alg}}(G)$ is dense in $L^2(G)$. In other words, the map ξ gives rise an isomorphism*

$$\widehat{\oplus}_{V \in \mathrm{Irrep}(G)} V \otimes V^* \to L^2(G)$$

where the first copy of G acts on V and the second one on V^* and the hat denotes the Hilbert space completion of the direct sum.

Note that this is again an instance of the double centralizer property!

For example, let $G = S^1$. Then the irreducible representations of G are the characters $\psi_n(\theta) = e^{in\theta}$. So the Peter-Weyl theorem in this case says that $\{e^{in\theta}\}$ is an orthonormal basis of $L^2(S^1)$ with norm

$$||f||^2 := \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta,$$

which is the starting point for Fourier analysis. So the Peter-Weyl theorem is similarly a starting point for **nonabelian Fourier (or harmonic) analysis**.

Exercise 35.10. Let G be a compact Lie group and $H \subset G$ a closed subgroup. Then we have a compact homogeneous space G/H and the Haar measure on G descends to a probability measure on G/H. So we can define the infinite dimensional unitary representation $L^2(G/H)$ of G.

(i) Show that have a decomposition

$$L^2(G/H) = \widehat{\oplus}_{V \in \mathrm{Irrep} G} N_H(V) V,$$

where $N_H(V) = \dim V^H$, the dimension of the space of H-invariants of V.

(ii) Let $G = SO(3)$, so the irreducible representations are L_{2m} for $m \geq 0$. Thus

$$L^2(G/H) = \widehat{\oplus}_{m \geq 0} N_H(m) L_{2m}.$$

Compute this decomposition (i.e., the numbers $N_H(m)$) for $H = \mathbb{Z}/n\mathbb{Z}$ acting by rotations around an axis by angles $2\pi k/n$ (rotations of a regular n-gon).

(iii) Do the same for the dihedral group $H = \mathbf{D}_n$ of symmetries of the regular n-gon (where reflections in the plane are realized as rotations around a line in this plane).

(iv) Do the same for the groups $H = SO(2)$ and $H = O(2)$ of rotations and symmetries of the circle.

(v) Do the same for H being the group of symmetries of a platonic solid (tetrahedron, cube, icosahedron).

It may be more convenient to give $N_V(m)$ in the form of the generating function $\sum_m N_V(m)t^m$.

Exercise 35.11. Let $G = GL_n(\mathbb{C})$. A **regular algebraic function** on G is a polynomial of X_{ij} and $\det(X)^{-1}$ for $X \in G$. Denote by $\mathcal{O}(G)$ the algebra of regular algebraic functions on G.

(i) Show that $G \times G$ acts on $\mathcal{O}(G)$ by left and right multiplication.

(ii) (Algebraic Peter-Weyl theorem) Show that as a $G \times G$-module, we have

$$\mathcal{O}(G) = \oplus_{V \in \mathrm{Irrep}(G)} V \otimes V^*.$$

Hint. Compute $\mathrm{Hom}_G(V, \mathcal{O}(G))$ where G acts on $\mathcal{O}(G)$ by right translations. For this, interpret elements of this space as equivariant functions $G \to V^*$ and show that such functions are automatically regular algebraic.

(iii) Generalize (i) and (ii) to orthogonal and symplectic groups.

Corollary 35.12. *Let $\chi_V(g) = \mathrm{Tr}(\rho_V(g))$ be the character of V. Then $\{\chi_V(g), V \in \mathrm{Irrep}G\}$ is an orthonormal basis of $L^2(G)^G$, the space of conjugation-invariant functions in $L^2(G)$ (i.e., such that $f(gxg^{-1}) = f(x)$).*

Proof. We have $\chi_V(g) = \sum_i \psi_{V,ii}(g)$, so by orthogonality of matrix coefficients χ_V are orthonormal in $L^2(G)^G$. So it remains to show that they are complete. For this observe that $L^2_{\mathrm{alg}}(G)^G = \xi(\oplus_V (V \otimes V^*)^G) = \oplus_V \mathbb{C}\chi_V$. Thus our job is to show that $L^2_{\mathrm{alg}}(G)^G$ is dense in $L^2(G)^G$. To this end, for $\psi \in L^2(G)^G$ fix a sequence $\psi_n \in L^2_{\mathrm{alg}}(G)$ such that $\psi_n \to \psi$ as $n \to \infty$. Such a sequence exists by the Peter-Weyl theorem. Let

$$\psi_n^{\mathrm{av}}(x) = \int_G \psi_n(gxg^{-1})dg.$$

It is easy to see that $\psi_n^{\mathrm{av}} \in L^2_{\mathrm{alg}}(G)$. Also $||\psi_n^{\mathrm{av}} - \psi|| \le ||\psi_n - \psi|| \to 0$, $n \to \infty$, as claimed. □

36. Lecture 11

36.1. **Compact operators and the Hilbert-Schmidt theorem.** To prove the Peter-Weyl theorem, we will use the Hilbert-Schmidt theorem – the spectral theorem for compact self-adjoint operators in a Hilbert space.

Recall that a **bounded** operator $A : H \to H$ on a Hilbert space H is a linear operator such that for some $C \geq 0$ we have $||A\mathbf{v}|| \leq C||\mathbf{v}||$, $\mathbf{v} \in H$. The smallest constant C with this property is called the **norm** of A and denoted $||A||$. Recall also that A is **compact** if there is a sequence of finite rank operators $A_n : H \to H$ such that $||A_n - A|| \to 0$ as $n \to \infty$. In other words, the space $K(H)$ of compact operators on H is the closure of the space $K_f(H)$ of finite rank operators under the norm $A \mapsto ||A||$ on the space of bounded operators $B(H)$.

Lemma 36.1. *If A is compact then it maps bounded sets to pre-compact sets (i.e., ones whose closure is compact). In other words, for every bounded sequence $\mathbf{v}_n \in H$, the sequence $A\mathbf{v}_n$ has a convergent subsequence.*[6]

Proof. Let $\mathbf{v}_n \in H$, $||\mathbf{v}_n|| \leq 1$. Pick a sequence of finite rank operators A_n such that $||A_n - A|| < \frac{1}{n}$. Let $\mathbf{v}_n^1$ be a subsequence of $\mathbf{v}_n$ such that $A_1\mathbf{v}_n^1$ is convergent. Let $\mathbf{v}_n^2$ be a subsequence of $\mathbf{v}_n^1$ such that $A_2\mathbf{v}_n^2$ is convergent, and so on. Finally, let $\mathbf{w}_n = \mathbf{v}_n^n$. Note that

$$||A\mathbf{v}_i^k - A\mathbf{v}_j^k|| \leq ||A_k\mathbf{v}_i^k - A_k\mathbf{v}_j^k|| + ||A - A_k|| \cdot ||\mathbf{v}_i^k - \mathbf{v}_j^k||$$

$$\leq ||A_k\mathbf{v}_i^k - A_k\mathbf{v}_j^k|| + \frac{2}{k} - \varepsilon_k.$$

for some $\varepsilon_k > 0$. Since $A_k\mathbf{v}_i^k, i \geq 1$ is convergent, it is a Cauchy sequence, so there is M_k such that for $i, j > M_k$ we have

$$||A\mathbf{v}_i^k - A\mathbf{v}_j^k|| \leq \frac{2}{k}.$$

But $\mathbf{w}_n$ is a subsequence of $\mathbf{v}_n^k$ starting from the k-th term. So there is N_k such that

$$||A\mathbf{w}_i - A\mathbf{w}_j|| \leq \frac{2}{k}.$$

In other words, the sequence $A\mathbf{w}_n$ is Cauchy. Hence it is convergent, as desired. $\square$

Proposition 36.2. *Let M be a compact manifold with positive smooth probability measure $d\mathbf{x}$ and $K(\mathbf{x}, \mathbf{y})$ a continuous function on M. Then the operator*

$$(A\psi)(\mathbf{y}) := \int_M K(\mathbf{x}, \mathbf{y})\psi(\mathbf{x})d\mathbf{x}.$$

on $L^2(M)$ is compact.

[6]The converse statement also holds, but we will not need it.

Proof. By using a partition of unity, the problem can be reduced to the case when M is replaced by the hypercube $[0,1]^n$. Let us split it in m^n pixels of sidelength $\frac{1}{m}$ and approximate $K(\mathbf{x},\mathbf{y})$ by its maximal value on each of the m^{2n} pixels in $[0,1]^{2n}$. Denote the corresponding approximation by $K_m(\mathbf{x},\mathbf{y})$ and the corresponding operator by A_m; it has rank $\le m^n$. Let $\varepsilon_m := \sup|K-K_m|$, then $||A-A_m|| \le \varepsilon_m$. Finally, by Cantor's theorem, K is uniformly continuous, which implies that $\varepsilon_m \to 0$ as $m \to \infty$, hence the statement. $\square$

Recall that a bounded operator A is **self-adjoint** if $(A\mathbf{v},\mathbf{w}) = (\mathbf{v},A\mathbf{w})$ for $\mathbf{v},\mathbf{w} \in H$.

Theorem 36.3. *(Hilbert-Schmidt) Let $A : H \to H$ be a compact self-adjoint operator. Then there is an orthogonal decomposition*

$$H = \mathrm{Ker} A \oplus \widehat{\bigoplus}_\lambda H_\lambda,$$

where λ runs over non-zero eigenvalues of A, and $A|_{H_\lambda} = \lambda \mathrm{Id}$. Moreover, the spaces H_λ are finite dimensional and the eigenvalues λ are real and either form a finite set or a sequence going to 0.

Note that for finite rank operators, this obviously reduces to the standard theorem in linear algebra: a self-adjoint (Hermitian) operator on a finite dimensional space V with a positive Hermitian form has an orthogonal eigenbasis, and its eigenvalues are real.

Proof. We first prove the theorem for the operator A^2. Let $\beta := ||A||^2 = \sup_{||\mathbf{v}||=1}(A^2\mathbf{v},\mathbf{v})$. We may assume without loss of generality that $\beta > 0$. Let A_n be a sequence of self-adjoint finite rank operators converging to A, and let $\beta_n = ||A_n||^2$, which is also the maximal eigenvalue of A_n^2. We have $\beta_n \to \beta$. Let $\mathbf{v}_n$ be a sequence of unit vectors in H such that $A_n^2\mathbf{v}_n = \beta_n\mathbf{v}_n$. By Lemma 36.1, the sequence $A^2\mathbf{v}_n$ has a convergent subsequence, so passing to this subsequence we may assume that $A^2\mathbf{v}_n$ is convergent to some $\mathbf{w} \in H$. Hence $A_n^2\mathbf{v}_n \to \mathbf{w}$, so $\mathbf{v}_n \to \beta^{-1}\mathbf{w}$. Thus $A^2\mathbf{w} = \beta\mathbf{w}$. We can now replace H with the orthogonal complement of $\mathbf{w}$ and iterate this procedure.

As a result we'll get a sequence of numbers $\beta_1 > \beta_2 > .... > 0$, which is either finite or tends to 0 (by compactness of A^2), and the corresponding finite dimensional orthogonal eigenspaces H_{β_k}. Let $\mathbf{v}$ be a vector orthogonal to all H_{β_k}. Then $||A\mathbf{v}|| \le \beta_k||\mathbf{v}||$ for all k, so $A\mathbf{v} = 0$, as desired.

Now, we have $H = \mathrm{Ker}A^2 \oplus \widehat{\bigoplus}_n H_{\beta_n}$, and A preserves this decomposition, acting by 0 on $\mathrm{Ker}A^2$ and with eigenvalues $\pm\sqrt{\beta_n}$ on H_{β_n}. This implies the theorem. $\square$

36.2. **Proof of the Peter-Weyl theorem.** Let M be a manifold with a smooth positive measure dx and $x_0 \in M$. Define a **delta-like sequence** on M around x_0 to be a sequence of continuous "hat" functions h_N on M such that $h_N \geq 0$, for every neighborhood U of x_0 the supports of almost all h_N are contained in U, and $\int_M h_N(x)dx = 1$.

It is clear that delta-like sequences exist. Namely, we can pick a sequence satisfying the first two conditions and then normalize it to satisfy the third one.

Now let G be a compact Lie group and h_N a delta-like sequence around 1 on G. By replacing $h_N(x)$ with $\frac{1}{2}(h_N(x) + h_N(x^{-1}))$, we may assume that h_N is invariant under inversion. Define the **convolution operators** B_N on $L^2(G)$ by

$$(B_N\psi)(y) = \int_G h_N(x)\psi(x^{-1}y)dx = \int_G h_N(yz^{-1})\psi(z)dz.$$

As we have shown, these operators are compact (as the kernel $K(y,z) := h(yz^{-1})$ is continuous), Moreover, they are clearly self-adjoint (as $h_N(x) = h_N(x^{-1})$ and h_N is real) and commute with right translations by G. So by the Hilbert-Schmidt theorem, we have the corresponding spectral decomposition

$$L^2(G) = \mathrm{Ker} B_N \oplus \widehat{\bigoplus_\lambda} H_{N,\lambda}$$

invariant under right translations. Since $H_{N,\lambda}$ are finite dimensional and invariant under right translations, they are contained in $L^2_{\mathrm{alg}}(G)$ (this is the key step of the proof). Thus the closure $\overline{L^2_{\mathrm{alg}}(G)}$ contains the image of B_N. So for any $\psi \in L^2(G)$ we can find $\psi_N \in L^2_{\mathrm{alg}}(G)$ such that $||B_N\psi - \psi_N|| < \frac{1}{N}$.

Now let $\psi \in C(G)$. By Cantor's theorem, ψ is uniformly continuous. It follows that $B_N\psi$ uniformly converges to ψ as $N \to \infty$ (check it!). Thus

$$||\psi - \psi_N|| \leq ||\psi - B_N\psi|| + ||B_N\psi - \psi_N|| < ||\psi - B_N\psi|| + \tfrac{1}{N} \to 0$$

as $N \to \infty$. So $\overline{L^2_{\mathrm{alg}}(G)}$ contains $C(G)$. But $C(G)$ is dense in $L^2(G)$ (namely, by using partition of unity this reduces to the case of a box in $\mathbb{R}^n$, where it is well known). Thus $\overline{L^2_{\mathrm{alg}}(G)} = L^2(G)$. This completes the proof of the Peter-Weyl theorem.

36.3. **Existence of faithful representations.**

Lemma 36.4. *Let G be a compact Lie group and $G = G_0 \supset G_1 \supset ...$ be a nested sequence of closed subgroups without repetitions. Then this sequence is finite.*

Proof. Assume the contrary, i.e. that it is infinite. The dimensions must stabilize, so we may assume that $\dim G_n$ are all the same. Then $K = G_n^\circ$ is independent on n, and we have a nested sequence

$$G_0/K \supset G_1/K \supset ...$$

of finite groups, without repetitions. But such a sequence can't have length bigger than $|G_0/K|$, contradiction. □

Corollary 36.5. *Any compact Lie group has a faithful finite dimensional representation, so it is isomorphic to a closed subgroup of the unitary group $U(n)$.*

Proof. Pick a finite dimensional representation V_1 of G, and let G_1 be the kernel of this representation. Now pick another representation V_2 of G which is nontrivial as a G_1-representation, and let G_2 be the kernel of V_2 in G_1, and so on. By the previous lemma, at some point we will have a subgroup $G_k \subset G$ such that every finite dimensional representation of G is trivial when restricted to G_k. But then by the Peter-Weyl theorem, $G_k = 1$. Thus $V_1 \oplus ... \oplus V_k$ is a faithful G-representation. □

Remark 36.6. Conversely, any closed subgroup of $U(n)$ is a compact Lie group, see Exercise 36.11 below.

Exercise 36.7. Show that any compact Lie group admits a structure of a metric space such that the metric is invariant under left and right translations.

Let us call a finite dimensional representation V of a group G **unimodular** if $\wedge^{\dim V} V \cong \mathbb{C}$ is the trivial representation.

Proposition 36.8. *Let V be a faithful finite dimensional representation of a compact Lie group G, and Y an irreducible finite dimensional representation of G. Then for some n, m, the representation Y is a direct summand in $V^{\otimes n} \otimes V^{*\otimes m}$. Moreover, if V is unimodular then one may take $m = 0$.*

Proof. It suffices to establish the unimodular case since we may replace V with the unimodular representation $V \oplus V^*$. By assumption, G is a closed subgroup of $SU(V) \subset V \otimes V^*$. Consider the algebra $A \subset C(G)$ generated by $V^* \otimes V$ (i.e., by the coordinate functions). By definition, this algebra separates points on G, and it is invariant under complex conjugation since for a unitary matrix with determinant 1 one has $g^\dagger = g^{-1} = \wedge^{\dim V - 1} g$. Therefore by the Stone-Weierstrass theorem, A is dense in $C(G)$, hence in $L^2(G)$. Thus by the Peter-Weyl theorem, the algebra A (on which G acts on right multiplication) contains any irreducible G-module, which implies the statement. □

36.4. **Density in continuous functions.** In fact, we can now prove an even stronger version of the Peter-Weyl theorem:

Theorem 36.9. *The space $L^2_{\rm alg}(G)$ is dense in the space of continuous functions $C(G)$ in the supremum norm*

$$||f|| = \max_{g\in G} |f(g)|.$$

Proof. Consider the closure $\mathcal{A}$ of $L^2_{\rm alg}(G)$ inside $C(G)$ (under the supremum norm). Note that $L^2_{\rm alg}(G)$ is a unital algebra, hence so is $\mathcal{A}$. Also $\mathcal{A}$ is invariant under complex conjugation, and by the existence of faithful representations it separates points on G. Therefore, by the Stone-Weierstrass theorem, $\mathcal{A} = C(G)$. □

Remark 36.10. If $G = S^1$, this is the usual theorem of uniform approximation of continuous functions on the circle by trigonometric polynomials. If we restrict to even functions, this will be just the usual Weierstrass theorem on approximation of continuous functions on an interval by polynomials.

Exercise 36.11. In this exercise you will show that a closed subgroup of a Lie group G is a closed Lie subgroup (Theorem 2.16).

Clearly, it suffices to assume that G is connected. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $H \subset G$ be a closed subgroup.

(i) Let $\mathfrak{h}$ be the set of vectors $a \in \mathfrak{g}$ such that there is a sequence $h_n \in H$, $h_n \to 1$, and nonzero real numbers c_n such that

$$c_n \log h_n \to a,\ n \to \infty.$$

This is clearly a subset of $\mathfrak{g}$ invariant under scalar multiplication (since we can rescale c_n). Show that $\mathfrak{h}$ consists of all $a \in \mathfrak{g}$ for which the 1-parameter subgroup $\exp(ta)$ is contained in H. (Consider the elements $h_n^{[c_n]}$, where $[c]$ is the floor of c).

(ii) Show that $\mathfrak{h}$ is a subspace of $\mathfrak{g}$. (For $a, b \in \mathfrak{h}$ consider the elements $h_N := \exp(\frac{a}{N})\exp(\frac{b}{N})$ to show that $a + b \in \mathfrak{h}$).

(iii) Show that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. (For $a, b \in \mathfrak{h}$ consider the elements

$$h_N := \exp(\tfrac{a}{N})\exp(\tfrac{b}{N})\exp(-\tfrac{a}{N})\exp(-\tfrac{b}{N})$$

to show that $[a, b] \in \mathfrak{h}$).

(iv) Let $H_0 \subset G$ be the connected Lie subgroup with Lie algebra $\mathfrak{h}$. Given a sequence $h_N \in H$, $h_N \to 1$, show that $h_N \in H_0$ for $N \gg 1$. To this end, pick a transverse slice $S \subset G$ to H_0 near 1, and write $h_N = s_N h_{N,0}$, where $h_{N,0} \in H_0$, $s_N \in S$. Look at the asymptotics of $\log s_N$ as $N \to \infty$, and deduce that $s_N = 1$ for large enough N.

(v) Conclude that G/H is a manifold, and S defines a local chart on this manifold near 1. Deduce that H is a closed Lie subgroup of G, and $H_0 = H^\circ$.

37. Lecture 12

37.1. **Compact topological groups.** One can generalize integration theory to arbitrary compact and even to locally compact topological groups. For simplicity we will describe this generalization in the case of compact 2nd countable compact topological groups.

Namely, let X be a compact Hausdorff topological space, and assume that X is 2nd countable (i.e. has a countable base). For compact Hausdorff spaces this is equivalent to being metrizable. Let $C(X,\mathbb{R})$ be the space of continuous real-valued functions on X. This is a real Banach space with norm $||f|| = \max_{x\in X}|f(x)|$. Recall that by the **Riesz-Markov-Kakutani representation theorem**, a finite Borel measure μ on X is the same thing as a positive continuous linear functional $I : C(X,\mathbb{R}) \to \mathbb{R}$ (i.e., such that $I(f) \geq 0$ for $f \geq 0$), namely,

$$I(f) = \int_X f d\mu.$$

Moreover, μ is a probability measure if and only if $I(1) = 1$, and any $\mu \neq 0$ has positive volume and so can be normalized to be a probability measure.

Now let G be a compact 2nd countable topological group. It acts on $C(G,\mathbb{R})$ by left and right translations, so acts on nonnegative probability measures of G.

Theorem 37.1. *(Haar, von Neumann) G admits a unique left-invariant probability measure.*

This measure is also automatically right-invariant (since it is unique) and is called the **Haar measure** on G.

Remark 37.2. A unique up to scaling left-invariant regular Haar measure (albeit of infinite volume and not always right-invariant in the non-compact case) exists more generally for any locally compact group G (not necessarily 2nd countable).[7] We will not prove this here, but we remark that Haar measures on Lie groups are a special case of this.

Proof. Let $g_i, i \geq 1$ be a dense sequence in G (it exists since G has a countable base, hence is separable, as you can pick a point in every open set of this base). Let c_i be a sequence of positive numbers

[7]Note that a finite Borel measure on a compact 2nd countable space is necessarily regular.

such that $\sum_i c_i = 1$. To this data attach the **averaging operator** $A : C(G, \mathbb{R}) \to C(G, \mathbb{R})$ given by

$$(Af)(x) = \sum_i c_i f(xg_i).$$

This operator can be interpreted as follows: we have a Markov chain with states being points of G and the transition probability from x to xg_i equal to c_i, then $(Af)(x)$ is the expected value of f after one transition starting from x. It is clear that that A is a left-invariant bounded operator (of norm 1). Moreover, A acts by the identity on the line $L \subset C(G, \mathbb{R})$ of constant functions.

For $f \in C(G, \mathbb{R})$ denote by $\nu(f)$ the distance from f to L, i.e.,

$$\nu(f) = \tfrac{1}{2}(\max f - \min f).$$

Then $\nu(Af) < \nu(f)$ unless $f \in L$. Indeed, if f is not constant and $x \in G$, pick j such that $f(xg_j) < \max f$ (exists since the sequence xg_i is dense in G), then

$$(Af)(x) = \sum_i c_i f(xg_i) \le (1 - c_j) \max f + c_j f(xg_j) < \max f.$$

So $\max(Af) < \max f$. Similarly, $\min(Af) > \min f$.

Now fix $f \in C(G, \mathbb{R})$ and consider the sequence $f_n := A^n f$, $n \ge 0$. This means that we let our Markov chain run for n steps. We know that for finite Markov chains there is an asymptotic distribution, and we'll show that this is also the case in the situation at hand, giving rise to a construction of the invariant integral.

Obviously, the sequence f_n is uniformly bounded by $\max |f|$. Also it is **equicontinuous**: for any $\varepsilon > 0$ there exists a neighborhood $1 \in U \subset G$ such that for any $x \in G$ and $u \in U$,

$$|f_n(x) - f_n(ux)| < \varepsilon.$$

Indeed, it suffices to show that f is uniformly continuous, i.e., for any ε find U such that for all $x \in G, u \in U$ we have $|f(x) - f(ux)| < \varepsilon$; this U will then work for all f_n. But this is guaranteed by Cantor's theorem; let us recall its proof. Assume the contrary, that there is no such U. Then there are two sequences $x_i, u_i \in G$, $u_i \to 1$, with $|f(x_i) - f(u_i x_i)| \ge \varepsilon$. The sequence x_i has a convergent subsequence, so we may assume without loss of generality that $x_i \to x \in G$. Then taking the limit $i \to \infty$, we get that $\varepsilon \le 0$, a contradiction.

Therefore, by the **Ascoli-Arzela theorem** the sequence f_n has a convergent subsequence. Let us remind the proof of this theorem. We construct subsequences f_n^k of f_n inductively by picking f_n^k from f_n^{k-1} so that $f_n^k(g_k)$ converges (with $f_n^0 = f_n$), which can be done by the boundedness assumption, and then set $h_m := f_m^m = f_{n(m)}$. Then $h_m(g_i)$ converges, hence Cauchy, for all i, which by equicontinuity implies that $h_m(x)$ is a Cauchy sequence in $C(G,\mathbb{R})$, hence converges to some $h \in \mathbb{C}(G,\mathbb{R})$.

We claim that $h \in L$. Indeed, we have

$$\nu(f_{n(m)}) \geq \nu(f_{n(m)+1}) = \nu(Af_{n(m)}) \geq \nu(f_{n(m+1)}),$$

so taking the limit when $m \to \infty$, we get

$$\nu(h) \geq \nu(Ah) \geq \nu(h),$$

i.e., $\nu(Ah) = \nu(h)$. The assignment $f \mapsto h$ is therefore a continuous left-invariant positive linear functional $I : C(G,\mathbb{R}) \to L = \mathbb{R}$, and $I(1) = 1$, as claimed.

Similarly, we may construct a right-invariant integral

$$I_* : C(G,\mathbb{R}) \to L = \mathbb{R}$$

with $I_*(1) = 1$, and by construction for any left invariant integral J we have $J(f) = J(I_*(f))$. Thus for every left invariant integral J with $J(1) = 1$ we have $J(f) = I_*(f)$; in particular $I(f) = I_*(f)$. This shows that I is unique, invariant on both sides and independent on the choice of g_i, c_i, and hence that $A^n f \to I(f)$ as $n \to \infty$. □

Example 37.3. A basic example of a (2nd countable) compact topological group which is, in general, not a Lie group, is a **profinite group**. Namely, let $G_1, G_2, ...$ be finite groups and $\phi_i : G_{i+1} \to G_i$ be surjective homomorphisms. Then the **inverse limit** $G := \varprojlim G_n$ is the group consisting of sequences $g_1 \in G_1, g_2 \in G_2, ...$ where $\phi_i(g_{i+1}) = g_i$. This group G has projections $p_n : G \to G_n$ and a natural topology, for which a base of neighborhood of 1 consists of $\mathrm{Ker}(p_n)$. (This topology can be defined by a bi-invariant mertic: $d(\mathbf{a},\mathbf{b}) = C^{n(\mathbf{a},\mathbf{b})}$, where $n(\mathbf{a},\mathbf{b})$ is the first position at which $\mathbf{a}, \mathbf{b}$ differ, and $0 < C < 1$). A sequence $\mathbf{a}^n$ converges to $\mathbf{a}$ in this topology if for each k, a_k^n eventually stabilizes to a_k. It is easy to show that G is compact.

Profinite groups are ubiquitous in mathematics. For example, the p-**adic integers** $\mathbb{Z}_p$ for a prime p form a profinite group, namely the inverse limit of $\mathbb{Z}/p^n\mathbb{Z}$; in fact, it is a profinite ring. The multiplicative group of this ring $\mathbb{Z}_p^\times$ is also a profinite group. One may also consider non-abelian profinite groups $GL_n(\mathbb{Z}_p)$, $O_n(\mathbb{Z}_p)$, $Sp_{2n}(\mathbb{Z}_p)$, etc. Finally, absolute Galois groups, such as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, are (very complicated) profinite groups.

Note that infinite profinite groups are uncountable and **totally disconnected**, i.e., $G^\circ = 1$.

More generally, the inverse limit makes sense if G_i are compact Lie groups. In this case G is equipped with the product topology, so also compact (by Tychonoff's theorem).

Corollary 37.4. *Finite dimensional (continuous) representations of a compact topological group G are unitary and completely reducible.*

The proof is the same as for Lie groups, once we have the integration theory, which we now do.

Theorem 37.5. *(i) (Peter-Weyl theorem) Let G be a 2nd countable compact topological group. Then the set* $\mathrm{Irrep}G$ *is countable, and*

$$L^2(G) = \widehat{\oplus}_{V\in \mathrm{Irrep}(G)} V\otimes V^*$$

as a $G\times G$-module.

(ii) The subspace $L^2_{\rm alg}(G) = \oplus_{V\in \mathrm{Irrep}(G)} V\otimes V^*$ is dense in $C(G)$ in the supremum norm.

The proof is also analogous to Lie groups, using a delta-like sequence of continuous hat functions. Namely, we may take

$$h_N(x) = \max(\tfrac{1}{N} - d(x,1), 0).$$

where d is some metric defining the topology of G.

Remark 37.6. If G is profinite then finite dimensional representations of G are just representations of G_n for various n:

$$\mathrm{Irrep}G = \cup_{n\ge 1}\mathrm{Irrep}G_n$$

(nested union).

Corollary 37.7. *Any compact 2nd countable topological group is an inverse limit of compact Lie groups.*

Proof. Let $V_1, V_2, ...$ be the irreducible representations of G. Let $K_m = \mathrm{Ker}(\rho_{V_1}\oplus ...\oplus \rho_{V_m}) \subset G$, a closed normal subgroup. Then $G/K_m \subset U(V_1\oplus ...\oplus V_n)$ is a compact Lie group, and $\cap_m K_m = 1$, so G is the inverse limit of G/K_m. □

Exercise 37.8. (i) Let $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$ be the field of p-adic numbers, i.e., the field of fractions of $\mathbb{Z}_p$. Construct the Haar measure $|dx|$ on $\mathbb{Q}_p$ in which the volume of $\mathbb{Z}_p$ is 1 using the Haar measure on $\mathbb{Z}_p$.

(ii) Show that $\mathbb{Q}\subset\mathbb{Q}_p$ and $\mathbb{Q}_p = \mathbb{Q}+\mathbb{Z}_p$, and use this to define an embedding $\mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$. Show that $\mathbb{Q}/\mathbb{Z} = \oplus_{p \text{ prime}}\mathbb{Q}_p/\mathbb{Z}_p$.

(iii) Define the additive character $\psi : \mathbb{Q}_p \to U(1) \subset \mathbb{C}^\times$ by $\psi(x) := \exp(2\pi i\overline{x})$, where $\overline{x}$ is the image of x in $\mathbb{Q}/\mathbb{Z}$. Use ψ to label the characters (=irreducible representations) of $\mathbb{Z}_p$ by $\mathbb{Q}_p/\mathbb{Z}_p$.

(iv) Let $|x|$ be the p-adic norm of $x \in \mathbb{Q}_p$ ($|x| = p^{-n}$ if $x \in p^n\mathbb{Z}_p$ but $x \notin p^{n+1}\mathbb{Z}_p$, and $|0| = 0$). For which $s \in \mathbb{C}$ is the function $|x|^s$ in $L^2(\mathbb{Z}_p)$?

(v) The Peter-Weyl theorem in particular implies that any L^2 function f on a compact 2nd countable abelian group G can be expanded in a Fourier series

$$f(x) = \sum_j c_j \psi_j(x),$$

where ψ_j are the characters of G. Write the Fourier expansion of $|x|^s$ when it is in $L^2(\mathbb{Z}_p)$.

(vi) Show that $\frac{|dx|}{|x|}$ is a Haar measure on the multiplicative group $\mathbb{Q}_p^\times = GL_1(\mathbb{Q}_p)$. More generally, show that $|dX| := \frac{\prod_{1\le i,j\le n}|dx_{ij}|}{|\det(X)|^n}$ is a Haar measure on $GL_n(\mathbb{Q}_p)$ (where $X = (x_{ij})$).

(vii) Classify characters of $\mathbb{Z}_p^\times$.

(viii) Let S be the the space of locally constant functions on $\mathbb{Q}_p$ with compact support (i.e., linear combinations of indicator functions of sets of the form $a + p^n\mathbb{Z}_p$, $a \in \mathbb{Q}_p$). Show that the Fourier transform operator

$$\mathcal{F}(f) = \int_{\mathbb{Q}_p} \psi(xy) f(y)|dy|$$

maps S to itself, and $(\mathcal{F}^2 f)(x) = f(-x)$. Show that $\mathcal{F}$ preserves the integration pairing on S, $(f,g) = \int_{\mathbb{Q}_p} f(x)\overline{g(x)}|dx|$, and therefore extends to a unitary operator $L^2(\mathbb{Q}_p) \to L^2(\mathbb{Q}_p)$.

38. Lecture 13

38.1. **The hydrogen atom.** Let us now apply our knowledge of nonabelian harmonic analysis to solve a basic problem in quantum mechanics – describe the dynamics of the hydrogen atom.

The mechanics of the hydrogen atom is determined by motion of a charged quantum particle (electron) in a rotationally invariant attracting electric field. The potential of such a field is $-\frac{1}{r}$, where $r^2 = x^2 + y^2 + z^2$ (since this theory does not have nontrivial dimensionless quantities, we may choose the units of measurement so that all constants are equal to 1). Thus, the wavefunction $\psi(x, y, z, t)$ for our particle obeys the **Schrödinger equation**

$$i\partial_t \psi = H\psi,$$

where H is the quantum Hamiltonian

$$H := -\frac{1}{2}\Delta - \frac{1}{r},$$

and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator. Recall also that for each t, the function $\psi(-,-,-,t)$ is in $L^2(\mathbb{R}^3)$ and $||\psi|| = 1$. The problem is to solve this equation given the initial value $\psi(x,y,z,0)$.

The Schrödinger equation can be solved by separation of variables as follows. Suppose we have an orthonormal basis ψ_N of $L^2(\mathbb{R}^3)$ such that $H\psi_N = E_N\psi_N$. Then if

$$\psi(x,y,z,0) = \sum_N c_N \psi_N(x,y,z)$$

(i.e., $c_N = (\psi, \psi_N)$) then

$$\psi(x,y,z,t) = \sum_N c_N(t)\psi_N(x,y,z),$$

where $c_N(t) = c_N e^{-iE_N t}$. So our job is to find such basis ψ_N, i.e., diagonalize the self-adjoint operator H.

Note that the operator H is unbounded and defined only on a dense subspace of $L^2(\mathbb{R})$, and although it is symmetric ($(H\psi, \eta) = (\psi, H\eta)$ for compactly supported functions), it is very nontrivial to say what precisely it means that H is self-adjoint. Also, this operator turns out to have both discrete and continuous spectrum, which means that there is actually no basis with the desired propeties – eigenfunctions of H which lie $L^2(\mathbb{R}^3)$ span a proper closed subspace of this Hilbert space. However, this will not be a problem for our calculation.

We first focus on **bound states**, i.e., solutions with $E < 0$ (this is the situation when the electron does not have enough energy to escape from the nucleus). We should thus look for solutions of the **stationary Schrödinger equation**

$$H\psi = E\psi$$

in $L^2(\mathbb{R}^3)$. To do so, let us utilize the rotational symmetry and write this equation in spherical coordinates. For this we just need to write the Laplacian Δ in spherical coordinates. Let us write $\mathbf{r} = r\mathbf{u}$, where $\mathbf{u} \in S^2$ (i.e., $|\mathbf{u}| = 1$). We have

$$\Delta = \Delta_r + \tfrac{1}{r^2}\Delta_{\mathrm{sph}}$$

where

$$\Delta_{\mathrm{sph}} = \tfrac{1}{\sin^2\phi}\partial_\theta^2 + \tfrac{1}{\sin\phi}\partial_\phi \sin\phi\,\partial_\phi$$

is a differential operator on S^2 (the **spherical Laplacian**, or the **Laplace-Beltrami operator**) and

$$\Delta_r = \partial_r^2 + \frac{2}{r}\partial_r$$

is the **radial part** of Δ (check it!). So our equation looks like

$$\partial_r^2\psi + \tfrac{2}{r}\partial_r\psi + \tfrac{2}{r}\psi + \tfrac{1}{r^2}\Delta_{\mathrm{sph}}\psi = -2E\psi.$$

This equation can be solved by again applying separation of variables. Namely, we look for solutions in the form

$$\psi(r,\mathbf{u}) = f(r)\xi(\mathbf{u}),$$

where

$$\Delta\xi + \lambda\xi = 0 \tag{5}$$

Then we obtain the following equation for f:

$$f''(r) + \tfrac{2}{r}f'(r) + (\tfrac{2}{r} - \tfrac{\lambda}{r^2} + 2E)f(r) = 0. \tag{6}$$

So now we have to solve equation (5) and in particular determine which values of λ occur.

To this end, recall that the operator Δ_{sph} is rotationally invariant, so it preserves the decomposition of $L^2(S^2) = \oplus_{\ell\ge 0} L_{2\ell}$ into representations of $SO(3)$. Moreover, on each $L_{2\ell}$ it acts by a certain scalar $-\lambda_\ell$. To compute this scalar, consider the vector Y_ℓ^0 in $L_{2\ell}$ of weight zero. This vector invariant under $SO(2)$ changing θ, so it depends only on ϕ; in fact, it is a polynomial of degree ℓ in $\cos\phi$: $Y_\ell^0 = P_\ell(\cos\phi)$. Also orthogonality of the decomposition implies that

$$\int_{-1}^{1} P_k(z)P_n(z)dz = 0,\ k \ne n.$$

This means that P_n are the **Legendre polynomials**. Also

$$\Delta_{\mathrm{sph}}P_\ell(z) = \partial_z(1-z^2)\partial_z P_\ell(z) = -\lambda_\ell P_\ell(z),$$

which shows (by looking at the leading term) that

$$\lambda_\ell = \ell(\ell+1),\ \ell\in\mathbb{Z}_{\ge 0},$$

and the space of solutions of (5) with $\lambda = \lambda_\ell$ is $2\ell+1$-dimensional and is isomorphic to $L_{2\ell}$ as an $SO(3)$-module.

Consider now the vector $Y_\ell^m \in L_{2\ell}$ of any integer weight $-\ell \le m \le \ell$ (so $Y_\ell = Y_\ell^0$). We will be interested in these vectors up to scaling. We have

$$Y_\ell^m(\phi,\theta) = e^{im\theta}P_\ell^m(\cos\phi),$$

where $P_{\ell,m}$ are certain functions. These functions are called **spherical harmonics**. Moreover, it follows from representation theory of $SO(3)$ that Y_ℓ^m are trigonometric polynomials which are even for even m and odd for odd m (check it!), so P_ℓ^m are polynomials when m is even and are of the form $(1-z^2)^{1/2}$ times a polynomial when m is odd.

Let us calculate the functions P_ℓ^m. Since they are eigenfunctions of the spherical Laplacian, we obtain that P_ℓ^m satisfy the **Legendre differential equation**

$$\partial_z(1-z^2)\partial_z P - \frac{m^2}{1-z^2}P + \ell(\ell+1)P = 0.$$

Exercise 38.1. Show that this equation has a unique up to scaling continuous solution on $[-1,1]$ when $-\ell \le m \le \ell$ and m is an integer, given by the formula

$$P_\ell^m(z) = (1-z^2)^{m/2}\partial_z^{\ell+m}(1-z^2)^\ell.$$

These functions are called **associated Legendre polynomials** (even though they are not quite polynomials when m is odd).

Now we can return to equation (6). It now has the form

$$f''(r) + \tfrac{2}{r}f'(r) + (\tfrac{2}{r} - \tfrac{\ell(\ell+1)}{r^2} + 2E)f(r) = 0. \tag{7}$$

To simplify this equation, write

$$f(r) = r^\ell e^{-\frac{r}{n}} h(\tfrac{2r}{n}).$$

Then for h we get the equation

$$\rho h''(\rho) + (2\ell + 2 - \rho)h'(\rho) + (n - \ell - 1 + \tfrac{1}{4}(1 + 2En^2)\rho)h(\rho) = 0.$$

We see that the equation simplifies when $n = \frac{1}{\sqrt{-2E}}$, i.e., $E = -\frac{1}{2n^2}$. Then we have

$$\rho h''(\rho) + (2\ell + 2 - \rho)h'(\rho) + (n - \ell - 1)h(\rho) = 0,$$

which is the **generalized Laguerre equation**. Moreover, we have $||\psi||^2 < \infty$, which translates to

$$\int_0^\infty \rho^{2\ell+2}e^{-\rho}|h(\rho)|^2 d\rho < \infty \tag{8}$$

(the factor ρ^2 comes from the Jacobian of the spherical coordinates).

How do solutions of the generalized Laguerre equation behave at $\rho = 0$? Let us look for a solution of the form $\rho^s(1 + o(1))$. The characteristic equation for s then has the form

$$s(s + 2\ell + 1) = 0,$$

which gives $s = 0$ or $s = -2\ell - 1$. Thus, for $\ell \ge 1$ the solution $\rho^{-2\ell-1}(1 + o(1))$ does not satisfy (8), so we are left with a unique solution $h_n(\rho)$ which is regular at $\rho = 0$ and $h_n(0) = 1$. On the other hand, if $\ell = 0$, the solution $\rho^{-1}(1 + o(1))$, even though it satisfies (8), gives rise to a rotationally invariant function $\psi \sim \frac{1}{r}$ as $r \to 0$, so we don't get $H\psi = E\psi$, but rather get $H\psi = E\psi + C\delta_0$, where δ_0 is the delta function concentrated at zero. So ψ does not really satisfy the stationary Schrödinger equation as a distribution and has to be discarded, leaving us, as before, with the unique solution $h_n(\rho)$ such that $h_n(0) = 1$.

Using the power series method, we obtain

$$h_n(\rho) = \sum_{k=0}^{\infty} \frac{(1+\ell-n)...(k+\ell-n)}{(2\ell+2)...(2\ell+1+k)} \frac{\rho^k}{k!}.$$

It is easy to see that this series converges for all ρ and

$$\lim_{\rho\to+\infty} \frac{\log h_n(\rho)}{\rho} = 1$$

(to check the latter, show that the Taylor coefficients a_k of h_n are bounded below by $\frac{1}{(k+N)!}$ for some N). So it fails (8) **unless the series terminates**, which happens iff $n - \ell - 1$ is a nonnegative integer. In this case,

$$h_n(\rho) = \sum_{k=0}^{n-\ell-1} \frac{(1+\ell-n)...(k+\ell-n)}{(2\ell+2)...(2\ell+1+k)} \frac{\rho^k}{k!} = L_{n-\ell-1}^{2\ell+1}(\rho),$$

the $n - \ell - 1$-th **generalized Laguerre polynomial** with parameter $\alpha = 2\ell + 1$, a polynomial of degree $n - \ell - 1$. Namely, the generalized Laguerre polynomials L_N^{α} are defined by the formula

$$L_N^{\alpha}(\rho) := \sum_{k=0}^{N} (-1)^N \frac{N...(N-k+1)}{(\alpha+1)...(\alpha+k)} \frac{\rho^k}{k!}.$$

Thus we obtain the following theorem.

Theorem 38.2. *The bound states of the hydrogen atom, up to scaling, are*

$$\psi_{n\ell m}(r,\phi,\theta) = r^{\ell} e^{-\frac{r}{n}} L_{n-\ell-1}^{2\ell+1}(\tfrac{2r}{n}) Y_{\ell}^{m}(\theta,\phi),$$

where $Y_\ell^m(\theta,\phi) = e^{im\theta} P_\ell^m(\phi)$ *are spherical harmonics, where* $n \in \mathbb{Z}_{>0}$, ℓ *an integer between* 0 *and* $n-1$, *and* m *is an integer between* ℓ *and* $-\ell$. *The energy of the state* $\psi_{n\ell m}$ *is* $-\frac{1}{2n^2}$.

The number n is called the **principal quantum number**; it characterizes the energy of the state. The number ℓ is called the **azimuthal quantum number**; it characterizes the eigenvalue of the spherical Laplacian $\Delta_{\rm sph}$, which has the physical interpretation as (minus) the **orbital angular momentum operator** $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$. Note that the operators iL_x, iL_y and iL_z are just the generators of the Lie algebra Lie$(SO(3))$ acting on $\mathbb{R}^3$, i.e., we have

$$[L_x, L_y] = -iL_z,\ [L_y, L_z] = -iL_x,\ [L_z, L_x] = -iL_y.$$

Thus, $\mathbf{L}^2$ is simply a Casimir of Lie$(SO(3))$. Recall that the standard Casimir acts on $L_{2\ell}$ as $2\ell(2\ell+2)/2 = 2\ell(\ell+1)$, so $\mathbf{L}^2 = \frac{1}{2}C$.

Finally, m is called the **magnetic quantum number**, and it is the eigenvalue of $L_z = -i\partial_\theta$ (in spherical coordinates).

Corollary 38.3. *The space W_n of states with principal quantum number n has dimension n^2.*

Proof. By the theorem, this dimension is $\sum_{\ell=0}^{n-1}(2\ell+1) = n^2$. □

In fact, this analysis applies not just to hydrogen but to other chemical elements whose nucleus has charge > 1, if we neglect interaction between electons. Thus it can potentially be used to explain patterns of the periodic table.

We note, however, that $\psi_{n\ell m}$ **do not** form a basis of $L^2(\mathbb{R}^3)$. Instead, they span (topologically) a proper closed subspace of $L^2_0(\mathbb{R}^3)$ of $L^2(\mathbb{R}^3)$ on which the operator H is bounded and negative definite. So if a smooth function φ on $\mathbb{R}^3$ (say, with compact support away from the origin) satisfies $(H\varphi,\varphi) \geq 0$ then $\varphi \notin L^2_0(\mathbb{R}^3)$. It is easy to construct such examples: let φ be a hat function and $\varphi_s(\mathbf{r}) = \varphi(\mathbf{r}+s\mathbf{a})$, where $\mathbf{a}$ is any nonzero vector. We then have

$$(H\varphi_s,\varphi_s) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi(\mathbf{r})|^2 dV - \int_{\mathbb{R}^3} \frac{|\varphi(\mathbf{r})|^2}{|\mathbf{r}-s\mathbf{a}|} dV,$$

and we observe that the first term is positive and the second one goes to zero as $s \to \infty$, so for large s this expression is positive. This happens because besides bound states the hydrogen atom also has **continuous spectrum** $[0,\infty)$ corresponding to free electrons which are not captured by the nucleus. This part of the spectrum can be computed similarly to the discrete (bound state) spectrum, except that the energy will take arbitrary **nonnegative** values. The corresponding wavefunctions are not normalizable (i.e., not in L^2), and are given by similar formulas to bound states but with imaginary n. Their continuous linear combinations satisfying appropriate boundary conditions are called **Coulomb waves**.

Also, the answer n^2 for the number of states in the n-th energy level does not quite agree with the periodic table, which suggests it should rather be $2n^2$: the numbers of electrons at each level are $2, 8, 18, ...$. This is because the Schrödinger model which we computed is not quite right, as it does not take into account an additional degree of freedom called **spin** (a sort of intrinsic angular momentum). Namely, it turns out that the space of states of an electron is not $L^2(\mathbb{R}^3)$ but rather $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, with the same Hamiltonian as before but the Lie algebra

Lie$(SO(3))$ acting diagonally. Thus the space of states of the n-th energy level taking spin into account is

$$V_n = (L_0 \oplus L_2 \oplus ... \oplus L_{2n-2}) \otimes L_1 = 2L_1 \oplus 2L_3 \oplus ... \oplus 2L_{2n-3} \oplus L_{2n-1}$$

and $\dim V_n = 2n^2$. In other words, we have the additional **spin operator**, which is just the operator

$$S = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

acting on the $\mathbb{C}^2$ factor. So the **total spin** (=angular momentum) of a state is $m+s$, where s is the eigenvalue of S, and we have basic states $\psi_{n\ell m+}$ and $\psi_{n\ell m-}$ with spins $m+\frac{1}{2}$ and $m-\frac{1}{2}$ respectively.

Note also that V_n is **not** a representation of $SO(3)$ but only is a representation of its double cover $SU(2)$ where $-\mathrm{Id}$ acts by -1. However, this **anomaly** does not mean a violation of the $SO(3)$ symmetry, since true quantum states are unit vectors in the Hilbert space **up to a phase factor**.

Suppose now that we have k electrons, each at the n-th energy level. If the electrons had been marked, the space of states for them would have been $V_n^{\otimes k}$. But in real life they are indistinguishable, so we need to mod out by permutations. So we might think the space of states is S^kV_n. However, as electrons are **fermions**, this answer turns out to be not correct: the correct answer is $\wedge^kV_n$ rather than S^kV_n. In other words, when two identical electrons are switched, the corresponding vector changes sign. This is another example of a sign which does not violate symmetry since states are well defined only up to a phase factor.

In particular, this implies that if $k > 2n^2$ then the space of states is zero, i.e., there cannot be more than $2n^2$ electrons at the n-th energy level (the **Pauli exclusion principle**). This is exactly the kind of pattern we see in the periodic table.

Exercise 38.4. Let $\mathbf{r} = (x, y, z)$ and $\mathbf{p} = (-i\partial_x, -i\partial_y, -i\partial_z)$ be the position and momentum operators in $\mathbb{R}^3$. Let $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ be the angular momentum operator (these are actually vectors whose components are operators on functions in $\mathbb{R}^3$). Let $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ (the operator of multiplication by this function) and $H = \frac{1}{2}\mathbf{p}^2 + U(r) = -\frac{1}{2}\Delta + U(r)$ be a rotationally symmetric Schrödinger operator on $\mathbb{R}^3$ with potential $U(r)$ (smooth for $r > 0$).

(i) Show that the components of $i\mathbf{L}$ are vector fields that define the action of the Lie algebra Lie$(SO(3))$ on the functions on $\mathbb{R}^3$ induced by rotations. Deduce that $[\mathbf{L}, \mathbf{p}^2] = 0$ (componentwise).

(ii) Let $\mathbf{A}_0 = \frac{1}{2}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p})$. Show that $[\mathbf{A}_0, \mathbf{p}^2] = 0$ (again componentwise).

(iii) Let $\mathbf{A} := \mathbf{A}_0 + \phi(r)\mathbf{r}$. Show that there exists a function ϕ such that $[\mathbf{A}, H] = 0$ if and only if U is the Coulomb potential $\frac{C}{r} + D$, and then ϕ is uniquely determined, and compute ϕ. The corresponding operator $\mathbf{A}$ is called the **quantum Laplace-Runge-Lenz vector**.[8]

(iv) (Hidden symmetry of the hydrogen atom). By virtue of (iii), the components of $\mathbf{A}$ act (by second order differential operators) on functions on $\mathbb{R}^3$ commuting with H. In particular, they act on each W_n (note that in this problem we ignore spin). Use these components to define an action of $\mathfrak{so}_4 = \mathfrak{so}_3 \times \mathfrak{so}_3$ on W_n so that the geometric one (generated by the components of $\mathbf{L}$) is the diagonal copy.

(v) Show that $W_n = L_{n-1} \boxtimes L_{n-1}$ as a representation of $\mathfrak{so}_3 \times \mathfrak{so}_3$.

(vi) Now include spin by tensoring with the representation $\mathbb{C}^2$ of $SU(2)$ and show that $V_n = L_{n-1} \boxtimes L_{n-1} \boxtimes L_1$ as a representation of $\mathfrak{so}_4 \oplus \mathfrak{su}_2$. This representation is irreducible, which explains why the n-th energy level of H is degenerate, with dimension $2n^2$.

Exercise 38.5. Let $H = -\frac{1}{2}\Delta + \frac{1}{2}r^2$ be the Hamiltonian of the quantum harmonic oscillator in $\mathbb{R}^n$, where $r = \sqrt{x_1^2 + ... + x_n^2}$. Compute the eigenspaces of H in $L^2(\mathbb{R}^n)$ as representations of $SO(n)$ and find the eigenvalues of H with multiplicities and an orthogonal eigenbasis.

Hint. Show that the operator $e^{r^2/2} \circ H \circ e^{-r^2/2}$ preserves the space of polynomials $\mathbb{C}[x_1, ..., x_n]$, and find an eigenbasis $P_{i_1 i_2 \dots i_n}$ for this operator in this space (these should express via Hermite polynomials). This will give orthogonal eigenfunctions

$$\psi_{i_1 \dots i_n}(\mathbf{r}) = P_{i_1 \dots i_n}(\mathbf{r}) e^{-r^2/2}$$

in $L^2(\mathbb{R}^n)$. Using properties of Hermite polynomials, conclude that these are complete. Then use Exercise 32.11.

39. Lecture 14

39.1. **Automorophisms of semisimple Lie algebras.** We have shown previously that for a complex semisimple $\mathfrak{g}$, the group $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathfrak{g}$ (Corollary 15.9). We also showed that its connected component of the identity $\mathrm{Aut}(\mathfrak{g})^\circ$ acts transitively on the set of Cartan subalgebras in $\mathfrak{g}$ (Theorem 18.10). This group is called the **adjoint group** attached to $\mathfrak{g}$, and we will denote it by G_{ad}.

[8]In the classical mechanics setting, the existence of this conservation law is the reason why orbits for Coulomb potential are periodic (Kepler's law), while this is not so for other rotationally invariant potentials, except harmonic oscillator. It was discovered many times over the last 300 years. This is one of the most basic examples of "hidden symmetry".

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $H \subset G_{\text{ad}}$ be the corresponding connected Lie subgroup. This subgroup can be viewed as the group of linear operators $\mathfrak{g} \to \mathfrak{g}$ which act by 1 on $\mathfrak{h}$ and by $e^{\alpha(x)}$, $x \in \mathfrak{h}$, on each $\mathfrak{g}_\alpha$. Thus the exponential map $\mathfrak{h} \to H$ defines an isomorphism $\mathfrak{h}/2\pi i P^\vee \cong H$. The group H is called the **maximal torus** of G_{ad} corresponding to $\mathfrak{h}$. Since $\mathfrak{h}$ is a maximal commutative Lie subalgebra of $\mathfrak{g}$, H is a maximal commutative subgroup of G_{ad}, hence the terminology.

Proposition 39.1. *The normalizer $N(H)$ in G_{ad} coincides with the stabilizer of $\mathfrak{h}$ and contains H as a normal subgroup, so that $N(H)/H$ is naturally isomorphic to the Weyl group W.*

Proof. First note that since $SL_2(\mathbb{C})$ is simply connected, for any simple root α_i we have a homomorphism $\eta_i : SL_2(\mathbb{C}) \to G_{\text{ad}}$ which identifies $\text{Lie}SL_2(\mathbb{C})$ with the $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ corresponding to this simple root. Let $S_i := \eta_i\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$. Given $w \in W$, pick a decomposition $w = s_{i_1}...s_{i_n}$, and let $\widetilde{w} := S_{i_1}...S_{i_n} \in G_{\text{ad}}$.[9] Note that $\widetilde{w}$ acts on $\mathfrak{h}$ by w. So if $w = w_1w_2 \in W$ then $\widetilde{w} = \widetilde{w}_1\widetilde{w}_2h$, where h preserves the root decomposition and acts trivially on $\mathfrak{h}$. Thus if $h|_{\mathfrak{g}_{\alpha_j}} = \exp(b_j)$ then $h = \exp(\sum_j b_j\omega_j^\vee) \in H$. So the elements $\widetilde{w}$ and H generate a subgroup $N \subset N(H)$ of G_{ad} such that $N/H \cong W$.

It remains to show that $N(H) = N$. To this end, for $x \in N(H)$, let $\alpha_i' = x(\alpha_i)$. Then α_i' form a system of simple roots, so there exists $w \in W$ such that $w(\alpha_i') = \alpha_{p(i)}$, where p is some permutation. Then $\widetilde{w}x(\alpha_i) = \alpha_{p(i)}$. So $\widetilde{w}x$ defines a Dynkin diagram automorphism of $\mathfrak{g}$. Since this automorphism is defined by an element of G_{ad}, it stabilizes all fundamental representations, so $p = \text{id}$, hence $\widetilde{w}x \in H$, as claimed. $\square$

Remark 39.2. Note that $N(H)$ is **not** isomorphic to $W \ltimes H$, in general; it can be a non-split extension of W by H.

Another obvious subgroup of $\text{Aut}(\mathfrak{g})$ is the finite group $\text{Aut}(D)$ of automorphisms of the Dynkin diagram of $\mathfrak{g}$, which just permutes the generators e_i, f_i, h_i in the Serre presentation. Thus we have a natural homomorphism

$$\xi : \text{Aut}(D) \ltimes G_{\text{ad}} \to \text{Aut}(\mathfrak{g}),$$

which is the identity map on the connected components of 1. This homomorphism is clearly injective, since the center of G_{ad} is trivial and any nontrivial element of $\text{Aut}(D)$ nontrivially permutes fundamental representations of $\mathfrak{g}$.

[9]The element $\widetilde{w}$ in general depends on the decomposition of w as a product of simple reflections. One can show it does not if we take only reduced decompositions, but we will not need this.

Proposition 39.3. *ξ is an isomorphism.*

Proof. Our job is to show that ξ is surjective, i.e. for $a \in \mathrm{Aut}(\mathfrak{g})$ show that $a \in \mathrm{Im}\xi$. By Theorem 18.10, we may assume without loss of generality that a preserves a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (as this can be arranged by multiplying by an element of G_{ad}). Then by multiplying by an element of $\mathrm{Aut}(D) \cdot N(H)$ we can make sure that a acts trivially on $\mathfrak{h}$ and $\mathfrak{g}_{\alpha_i}$. Then $a = 1$, which implies the proposition. □

39.2. **Forms of semisimple Lie algebras.** We have classified semisimple Lie algebras over $\mathbb{C}$, but what about other fields (say of characteristic zero), notably $\mathbb{R}$?

To address this question, note that the Serre presentation of semisimple Lie algebras contains only integers, so it defines a Lie algebra over any such field, by imposing the same generators and relations. Such a Lie algebra is called **split**. So for example, over an algebraically closed field of characteristic zero, any semisimple Lie algebra is automatically split.

Now let $\mathfrak{g}$ be a semisimple Lie algebra over a field K of characteristic zero which splits over a Galois extension L of K, i.e., $\mathfrak{g} \otimes_K L = \mathfrak{g}_L$ is split (i.e., corresponds to a Dynkin diagram). Can we classify such $\mathfrak{g}$? To this end, let $\Gamma = \mathrm{Gal}(L/K)$ and observe that we can recover $\mathfrak{g}$ as $\mathfrak{g}_L^\Gamma$. So $\mathfrak{g}$ is determined by the action of Γ on the split semisimple Lie algebra $\mathfrak{g}_L$. Note that this action is **twisted-linear**, i.e., additive and $g(\lambda x) = g(\lambda)g(x)$ for $x \in \mathfrak{g}_L$, $\lambda \in L$. The simplest example of such an action is the action $\rho_0(g)$ which preserves all the generators e_i, f_i, h_i and just acts on the scalars, which corresponds to the split form of $\mathfrak{g}$. So any twisted-linear action ρ can be written as

$$\rho(g) = \eta(g)\rho_0(g)$$

for some map

$$\eta : \Gamma \to \mathrm{Aut}(\mathfrak{g}_L).$$

In order that ρ be a homomorphism, we need

$$\eta(gh)\rho_0(gh) = \eta(g)\rho_0(g)\eta(h)\rho_0(h),$$

which is equivalent to

$$\eta(gh) = \eta(g) \cdot g(\eta(h)).$$

In other words, η is a 1-**cocycle**. We will denote the Lie algebra attached to such cocycle η by $\mathfrak{g}_\eta$.

It remains to determine when $\mathfrak{g}_{\eta_1}$ is isomorphic to $\mathfrak{g}_{\eta_2}$. This will happen exactly when the corresponding representations ρ_1 and ρ_2 are isomorphic, i.e., there is $a \in \mathrm{Aut}(\mathfrak{g}_L)$ such that $\rho_1(g)a = a\rho_2(g)$, i.e.,

$$\eta_1(g)\rho_0(g)a = a\eta_2(g)\rho_0(g),$$

or

$$\eta_1(g) = a\eta_2(g)g(a)^{-1}.$$

Two 1-cocycles related in this way are called **cohomologous** (obviously, an equivalence relation), and the set of equivalence classes of cohomologous cocycles is called the **first Galois cohomology** of Γ with coefficients in $\mathrm{Aut}(\mathfrak{g}_L)$ and denoted by $H^1(\Gamma, \mathrm{Aut}(\mathfrak{g}_L))$. Note that this is cohomology with coefficients in a nonabelian group, so it is just a set and not a group.

So we obtain

Proposition 39.4. *Semisimple Lie algebras $\mathfrak{g}$ over K with fixed $\mathfrak{g}_L$ are classified by the first Galois cohomology $H^1(\Gamma, \mathrm{Aut}(\mathfrak{g}_L))$.*

Remark 39.5. There is nothing special about Lie algebras here – this works for any kind of linear algebraic structures, such as associative algebras, algebraic varieties, schemes, etc.

39.3. **Real forms of a semisimple Lie algebra.** Let us now make this classification more concrete in the case $K = \mathbb{R}$, $L = \mathbb{C}$, which is relevant to classification of real semisimple Lie groups. In this case, $\Gamma = \mathbb{Z}/2$ generated by complex conjugation and, as we have shown, $\mathrm{Aut}(\mathfrak{g}_L) = \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$, where D is the Dynkin diagram of $\mathfrak{g}$ and G_{ad} is the corresponding connected adjoint complex Lie group. Also since we always have $\eta(1) = 1$, the cocycle η is determined by the element $s = \eta(-1) \in \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$. Moreover, s must satisfy the cocycle condition

$$s\overline{s} = 1$$

and the corresponding real Lie algebra, up to isomorphism, depends only on the cohomology class of s, which is the equivalence class modulo transformations $s \mapsto as\overline{a}^{-1}$. We thus obtain the following theorem.

Theorem 39.6. *Real semisimple Lie algebras whose complexification is $\mathfrak{g}$ (i.e.,* **real forms** *of $\mathfrak{g}$) are classified by $s \in \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$ such that $s\overline{s} = 1$ modulo equivalence $s \mapsto as\overline{a}^{-1}$, $a \in \mathrm{Aut}(\mathfrak{g})$, where complex conjugation acts trivially on $\mathrm{Aut}(D)$.*

We denote the real form of $\mathfrak{g}$ corresponding s by $\mathfrak{g}_s$. Namely, $\mathfrak{g}_s = \{x \in \mathfrak{g} : \overline{x} = s(x)\}$. For example, $\mathfrak{g}_1$ is the split form, consisting of real $x \in \mathfrak{g}$, i.e., such that $\overline{x} = x$.

Alternatively, one may define the **antilinear involution** $\sigma_s(x) = \overline{s(x)}$, and $\mathfrak{g}_s$ is the set of fixed points of σ_s.

In particular, such s defines an element $s_0 \in \mathrm{Aut}(D)$ such that $s_0^2 = 1$. Note that the conjugacy class of s_0 is invariant under equivalences. The element s_0 permutes connected components of D, preserving some and matching others into pairs. Thus every semisimple real Lie algebra is a direct sum of simple ones, and each simple one either has a connected Dynkin diagram D (i.e., the complexified Lie algebra $\mathfrak{g}$ is still simple) or consists of two identical components (i.e., the complexified Lie algebra is $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}$ for some simple complex $\mathfrak{a}$). In the latter case $s = (g, \overline{g}^{-1})s_0$ where s_0 is the transposition and $g \in \mathrm{Aut}(\mathfrak{a})$, so s is cohomologous to s_0 by taking $a = (g, 1)$. Thus in this case $\mathfrak{g}_s = \mathfrak{g}_{s_0} = \mathfrak{a}$, complex simple Lie algebra regarded as a real Lie algebra.

It remains to consider the case when D is connected.

Definition 39.7. (i) A real form $\mathfrak{g}_s$ of a complex semisimple Lie algebra $\mathfrak{g}$ is said to be **inner** to $\mathfrak{g}_{s'}$ if $s' = gs$ up to equivalence, where $g \in \mathrm{G}_{\mathrm{ad}}$ (i.e., s and s' differ by an inner automorphism). The **inner class** of $\mathfrak{g}_s$ is the collection of all real forms inner to $\mathfrak{g}_s$. In particular, an **inner form** is a form inner to the split form.

(ii) $\mathfrak{g}_s$ is called **quasi-split** if $s = s_0 \in \mathrm{Aut}(D)$ (modulo equivalence).

So in particular any real form is inner to a unique quasi-split form, and a real form that is both inner and quasi-split is split.

39.4. **The compact real form.** An important example of a real form of simple complex Lie algebra $\mathfrak{g}$ is the **compact real form**. It is determined by the automorphism ω (called the **Cartan involution**) defined by the formula

$$\omega(h_j) = -h_j,\ \omega(e_j) = -f_j,\ \omega(f_j) = -e_j.$$

Let us denote this real form $\mathfrak{g}_\omega$ by $\mathfrak{g}^c$.

Proposition 39.8. *The Killing form of $\mathfrak{g}^c$ is negative definite.*

Proof. We have an orthogonal decomposition

$$\mathfrak{g}^c = (\mathfrak{h} \cap \mathfrak{g}^c) \oplus \bigoplus_{\alpha \in R_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c.$$

Moreover, the Killing form is clearly negative definite on $\mathfrak{h} \cap \mathfrak{g}^c$, since the inner product on the coroot lattice is positive definite, and $\{i\alpha_j^\vee\}$ is a basis of $\mathfrak{h} \cap \mathfrak{g}^c$. So it suffices to show that the Killing form is negative definite on $(\mathfrak{g}_\alpha \cap \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c$ for any $\alpha \in R_+$.

First consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Then $\mathfrak{g}^c$ is spanned by the Pauli matrices ih, $e - f$, $i(e + f)$, so $\mathfrak{g}^c = \mathfrak{su}(2)$. It follows that the trace form of any finite dimensional representation of $\mathfrak{g}^c$ is negative definite.

Thus for a general $\mathfrak{g}$, the elements S_i preserve $\mathfrak{g}^c$; this follows since the matrix $S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ belongs to $SU(2)$, and $\mathrm{Lie}(SU(2)_i) \subset \mathfrak{g}^c$. It follows that for any $w \in W$ the element $\widetilde{w}$ preserves $\mathfrak{g}^c$. Thus the restriction of the Killing form of $\mathfrak{g}^c$ to $\mathfrak{g}^c \cap (\mathfrak{sl}_2)_\alpha$ is negative definite for any root α (since it is so for simple roots, as follows from the case of $\mathfrak{sl}_2$). This implies the statement. □

Now consider the group $\mathrm{Aut}(\mathfrak{g}^c)$. Since the Killing form on $\mathfrak{g}^c$ is negative definite, it is a closed subgroup in the orthogonal group $O(\mathfrak{g}^c)$, hence is compact. Moreover, it is a Lie group with Lie algebra $\mathfrak{g}^c$. Thus we obtain

Corollary 39.9. *Let $G^c_{\mathrm{ad}} = \mathrm{Aut}(\mathfrak{g}^c)^\circ$. Then G^c_{ad} is a connected compact Lie group with Lie algebra $\mathfrak{g}^c$.*

In particular, this gives a new proof that representations of a finite dimensional semisimple Lie algebra are completely reducible (by using Weyl's unitary trick).

Exercise 39.10. (i) Show that if $\mathfrak{g} = \mathfrak{sl}_n$ then $G^c_{\mathrm{ad}} = PSU(n) = SU(n)/\mu_n$, where μ_n is the group of roots of unity of order n.

(ii) Show that if $\mathfrak{g} = \mathfrak{so}_n$ then $G^c_{\mathrm{ad}} = SO(n)$ for odd n and $SO(n)/\pm 1$ for even n.

(iii) Show that if $\mathfrak{g} = \mathfrak{sp}_{2n}$ then $G^c_{\mathrm{ad}} = U(n, \mathbb{H})/\pm 1$, where $U(n, \mathbb{H})$ is the quaternionic unitary group $Sp_{2n}(\mathbb{C}) \cap U(2n)$ (see Exercise 5.9).

Exercise 39.11. (i) Compute the signature of the Killing form of the split form $\mathfrak{g}_{\mathrm{spl}}$ of a complex simple Lie algebra $\mathfrak{g}$ in terms of its dimension and rank, and show that the compact form is never split.

(ii) Show that the compact form is inner to the quasi-split form defined by the flip of the Dynkin diagram corresponding to taking the dual representation (i.e., induced by $-w_0$), but is never quasi-split itself (show that the quasi-split form contains nonzero nilpotent elements). For which simple Lie algebras is the compact form inner?

40. Lecture 15

40.1. **Examples of real forms.** So let us list real forms of simple Lie algebras that we know so far.

1. Type A_{n-1}. We have the split form $\mathfrak{sl}_n(\mathbb{R})$, the compact form $\mathfrak{su}(n)$, and also for $n > 2$ the quasi-split form associated to the automorphism $s(A) = -JA^TJ^{-1}$, where $J_{ij} = (-1)^i \delta_{i,n+1-j}$ (this automorphism sends e_i, f_i, h_i to $e_{n+1-i}, f_{n+1-i}, h_{n+1-i}$). So the corresponding real Lie algebra is the Lie algebra of traceless matrices preserving the

hermitian or skew-hermitian form defined by the matrix J, which has signature (p,p) if $n = 2p$ and $(p+1,p)$ or $(p,p+1)$ if $n = 2p+1$. Thus in the first case we have $\mathfrak{su}(p,p)$ and in the second case we have $\mathfrak{su}(p+1,p)$. Note that for $n = 2$ we have $\mathfrak{su}(1,1) = \mathfrak{sl}_2(\mathbb{R})$, so in this special case this form is not new. We also observe that for $n \geq 4$ there are other forms, e.g. $\mathfrak{su}(n-p,p)$ with $1 \leq p \leq \frac{n}{2} - 1$.

2. Type B_n. We have the split form $\mathfrak{so}(n+1,n)$, the compact form $\mathfrak{so}(2n+1)$. The Dynkin diagram has no nontrivial automorphisms, so there are no non-split quasi-split forms. It follows that $\mathfrak{so}(3) = \mathfrak{su}(2)$ and $\mathfrak{so}(2,1) = \mathfrak{su}(1,1)$.

3. Type C_n. We have the split form $\mathfrak{sp}_{2n}(\mathbb{R})$ and compact form $\mathfrak{u}(n,\mathbb{H})$. The Dynkin diagram has no nontrivial automorphisms, so there are no non-split quasi-split forms. It follows that $\mathfrak{so}(3,2) = \mathfrak{sp}_4(\mathbb{R})$ and $\mathfrak{so}(5) = \mathfrak{u}(2,\mathbb{H})$.

4. Type D_n. We have the split form $\mathfrak{so}(n,n)$, the compact form $\mathfrak{so}(2n)$. Moreover, in this case we have a unique nontrivial involution of the Dynkin diagram. More precisely, this is true for $n \neq 4$, while for $n = 4$ we have $\mathrm{Aut}(D) = S_3$, but there is still a unique non-trivial involution up to conjugation. So we also have a quasi-split form. To compute it, recall that the split form is defined by the equation $A = -JA^TJ^{-1}$ where $J_{ij} = \delta_{i,2n+1-j}$. The quasisplit form is obtained by replacing J by $J' = gJ$, where g permutes e_n and e_{n+1} (this is the automorphism that switches α_{n-1} and α_n while keeping other simple roots fixed). The signature of the form defined by J' is $(n+1,n-1)$, so we get that the non-split quasi-split form is $\mathfrak{so}(n+1,n-1)$. In particular, for $n = 2$ we get $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\mathfrak{so}(3,1) = \mathfrak{sl}_2(\mathbb{C})$ (the Lie algebra of the Lorentz group of special relativity), $\mathfrak{so}(2,2) = \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$. Also, for $n = 3$ we get $\mathfrak{so}(6) = \mathfrak{su}(4)$, $\mathfrak{so}(3,3) = \mathfrak{sl}_4(\mathbb{R})$, and $\mathfrak{so}(4,2) = \mathfrak{su}(2,2)$.

5. Type G_2. We have the split and compact forms $G_2(\mathbb{R}), G_2^c$.

6. Type F_4. We have the split and compact forms $F_4(\mathbb{R}), F_4^c$.

7. Type E_6. We have the split and compact forms $E_6(\mathbb{R}), E_6^c$ and the quasi-split form E_6^{qs} attached to the non-trivial automorphism.

8. Type E_7. We have the split and compact forms $E_7(\mathbb{R}), E_7^c$.

9. Type E_8. We have the split and compact forms $E_8(\mathbb{R}), E_8^c$.

40.2. **Classification of real forms, continued.** However, we are not done with the classification of real forms yet, as we still need to find all real forms and show there are no others. To this end, consider a complex simple Lie algebra $\mathfrak{g} = \mathfrak{g}^c \otimes_{\mathbb{R}} \mathbb{C}$. We have the compact antilinear involution ω of $\mathfrak{g}$ which acts trivially on $\mathfrak{g}^c$. Another real structure on $\mathfrak{g}$ is then defined by the antilinear involution $\sigma = \omega \circ g$, where $g \in \mathrm{Aut}(\mathfrak{g})$ is such that $\omega(g)g = 1$. But it is easy to see that

$$\omega(g) = (g^\dagger)^{-1},$$

where $x^\dagger$ is the adjoint to $x \in \mathrm{End}(\mathfrak{g})$ under the positive Hermitian form $(X, Y) = \mathrm{tr}(\mathrm{ad}X\mathrm{ad}\omega(Y))$ (the Hermitian extension of the Killing form on $\mathfrak{g}^c$ to $\mathfrak{g}$). It follows that the operator g is self-adjoint. Thus it is diagonalizable with real eigenvalues, and we have a decomposition

$$\mathfrak{g} = \oplus_{\gamma \in \mathbb{R}} \mathfrak{g}(\gamma),$$

where $\mathfrak{g}(\gamma)$ is the γ-eigenspace of g, such that $[\mathfrak{g}(\beta), \mathfrak{g}(\gamma)] = \mathfrak{g}(\beta\gamma)$. Now consider the operator $|g|^t$ for any $t \in \mathbb{R}$. It acts on $\mathfrak{g}(\gamma)$ by $|\gamma|^t$, so $|g|^t = \exp(t \log |g|) \in G_{\mathrm{ad}}$ is a 1-parameter subgroup. Now define $\theta := g|g|^{-1}$. We have $\theta \circ \omega = \omega \circ \theta$ and $\theta^2 = 1$. Also g and θ define the same real structure since $\theta = |g|^{-1/2} g \omega(|g|^{1/2})$. This shows that without loss of generality we may assume that $g = \theta$ with $\theta \circ \omega = \omega \circ \theta$ (i.e., $\theta \in \mathrm{Aut}(\mathfrak{g}^c)$) and $\theta^2 = 1$.

Moreover, another such element ξ defines the same real form if and only if $\xi = x\theta\omega(x)^{-1}$ for some $x \in \mathrm{Aut}(\mathfrak{g})$. Thus we get

$$x\theta\omega(x)^{-1} = \omega(x)\theta x^{-1},$$

so setting $z := \omega(x)^{-1}x$, we get $\omega(z) = z^{-1}$, $\theta z = z^{-1}\theta$. Note that $z = x^\dagger x$ is positive definite. So setting $y = xz^{-1/2}$, we have $\omega(y) = \omega(x)z^{1/2} = xz^{-1/2} = y$, i.e., $y \in \mathrm{Aut}(\mathfrak{g}^c)$, and $\xi = x\theta\omega(x)^{-1} = x\theta z x^{-1} = xz^{-1/2}\theta z^{1/2}x^{-1} = y\theta y^{-1}$. Thus we obtain

Theorem 40.1. *Real forms of $\mathfrak{g}$ are in bijection with conjugacy classes of involutions $\theta \in \mathrm{Aut}(\mathfrak{g}^c)$, via $\theta \mapsto \sigma_\theta = \theta \circ \omega$.*

Thus we have a canonical (up to automorphisms of $\mathfrak{g}^c$) decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, into the eigenspaces of θ with eigenvalues 1 and -1, such that $\mathfrak{k}$ is a Lie subalgebra, $\mathfrak{p}$ is a module over $\mathfrak{k}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. We also have the corresponding decomposition for the underlying real Lie algebra $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$. Moreover, the real form $\mathfrak{g}_\sigma$ is just $\mathfrak{k}^c \oplus i\mathfrak{p}^c$.

Exercise 40.2. Show that $\mathfrak{k}$ is a reductive Lie algebra. Does it have to be semisimple?

Proposition 40.3. *There exists a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ invariant under θ.*

Proof. Take a generic $x \in \mathfrak{k}^c$; as $\mathfrak{k}$ is reductive, it is regular semisimple. Let $\mathfrak{h}_+^c$ be the centralizer of x in $\mathfrak{k}^c$. Then $\mathfrak{h}_+ := \mathfrak{h}_+^c \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{k}$ is a Cartan subalgebra. Let $\mathfrak{h}_-^c$ be a maximal subspace of $\mathfrak{p}^c$ for the property that $\mathfrak{h}^c := \mathfrak{h}_+^c \oplus \mathfrak{h}_-^c$ is a commutative Lie subalgebra of $\mathfrak{g}^c$.

We claim that $\mathfrak{h} := \mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra in $\mathfrak{g}$. Indeed, it obviously consists of semisimple elements (as all elements in $\mathfrak{g}^c$ are semisimple, being anti-hermitian operators on $\mathfrak{g}^c$). Now, if $z \in \mathfrak{g}$ commutes with $\mathfrak{h}$ then $z = z_+ + z_-$, $z_+ \in \mathfrak{k}$ and $z_- \in \mathfrak{p}$, and both z_+, z_- commute with $\mathfrak{h}$. Thus $z_+ \in \mathfrak{h}_+$ and $z_- = x + iy$, where $x, y \in \mathfrak{p}^c$ and both commute with $\mathfrak{h}$. Hence $x, y \in \mathfrak{h}^c$. Thus $z \in \mathfrak{h}$, as claimed. It is clear that $\mathfrak{h}$ is θ-stable, so the proposition is proved. □

Thus we have a decomposition $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$, and θ acts by 1 on $\mathfrak{h}_+$ and by -1 on $\mathfrak{h}_-$.

Lemma 40.4. *The space $\mathfrak{h}_-$ does not contain any coroots of $\mathfrak{g}$.*

Proof. Suppose that $\alpha^\vee \in \mathfrak{h}_-$ is a coroot. Thus $\theta(\alpha^\vee) = -\alpha^\vee$, so $\theta(e_\alpha) = e_{-\alpha}$ and $\theta(e_{-\alpha}) = e_\alpha$ for some nonzero $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$. Let $x = e_\alpha + e_{-\alpha}$. We have $\theta(x) = x$, so $x \in \mathfrak{k}$. On the other hand, $x \notin \mathfrak{h}_+$ (as x is orthogonal to $\mathfrak{h}_+$ and nonzero) and $[\mathfrak{h}_+, x] = 0$ since α vanishes on $\mathfrak{h}_+$. This is a contradiction, since $\mathfrak{h}_+$ is a maximal commutative subalgebra of $\mathfrak{k}$. □

It follows that a generic element $t \in \mathfrak{h}_+$ is regular semisimple. So let us pick one for which $\mathrm{Re}(t, \alpha^\vee)$ is nonzero for any coroot $\alpha^\vee$ of $\mathfrak{g}$, and use it to define a polarization of R: set $R_+ := \{\alpha \in R : \mathrm{Re}(t, \alpha^\vee) > 0\}$. Then $\theta(R_+) = R_+$. So $\theta(\alpha_i) = \alpha_{\theta(i)}$, where $\theta(i)$ is the action of θ on the Dynkin diagram D of $\mathfrak{g}$. Thus if $\theta(i) = i$ then $\theta(e_i) = \pm e_i$, $\theta(h_i) = h_i$, $\theta(f_i) = \pm f_i$ while if $\theta(i) \neq i$, we can normalize $e_i, e_{\theta(i)}, f_i, f_{\theta(i)}$ so that $\theta(e_i) = e_{\theta(i)}$, $\theta(f_i) = f_{\theta(i)}$, $\theta(h_i) = h_{\theta(i)}$. Thus θ can be encoded in a marked Dynkin diagram of $\mathfrak{g}$: we connect vertices i and $\theta(i)$ if $\theta(i) \neq i$ and paint a θ-stable vertex i white if $\theta(e_i) = e_i$ (i.e., $e_i \in \mathfrak{k}$, a **compact root**), and black if $\theta(e_i) = -e_i$ (i.e., $e_i \in \mathfrak{p}$, a **non-compact root**). Such a decorated Dynkin diagram is called a **Vogan diagram**. So we see that every Vogan diagram gives rise to a real form, and every real form is defined by some Vogan diagram. This is great, but we are not finished yet with the classification of real forms since different Vogan diagrams can define the same real form (they could arise from different choices of R_+ coming from different choices of the element t).

Exercise 40.5. (i) Show that the signature of the Killing form of a real form $\mathfrak{g}_\sigma$ of a complex semisimple Lie algebra $\mathfrak{g}$ corresponding to involution θ (i.e., $\sigma = \omega \circ \theta$) equals $(\dim \mathfrak{p}, \dim \mathfrak{k})$. In particular, the Killing form of $\mathfrak{g}_\sigma$ is negative definite if and only if $\mathfrak{g}_\sigma$ is the compact form.

(ii) Deduce that for the split form $\dim \mathfrak{k} = |R_+|$, the number of positive roots of $\mathfrak{g}$.

(iii) Show that for a real form of $\mathfrak{g}$ in the compact inner class, we have $\mathrm{rank}(\mathfrak{k}) = \mathrm{rank}(\mathfrak{g})$.

40.3. **Real forms of classical groups.** We are now ready, however, to classify real forms of classical groups.

1. Type A_{n-1}, compact inner class. In this case θ is an inner automorphism, conjugation by an element of order ≤ 2 in $PSU(n)$. Obviously, such an element can be lifted to $g \in U(n)$ such that $g^2 = 1$, so $\theta(x) = gxg^{-1}$. Thus $g = \mathrm{Id}_p \oplus (-\mathrm{Id}_q)$ where $p + q = n$ and we may assume that $p \ge q$. It is easy to see that this defines the real form $\mathfrak{g}_\sigma = \mathfrak{su}(p, q)$, and $\mathfrak{k} = \mathfrak{gl}_p \oplus \mathfrak{sl}_q$. These are all pairwise non-isomorphic since the corresponding automorphisms θ are not conjugate to each other. So we get $[\frac{n}{2}] + 1$ real forms. Note that for $n = 2$ this exhausts all real forms, so we have only two – $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1) = \mathfrak{sl}_2(\mathbb{R})$ with $\mathfrak{k} = \mathfrak{gl}_1$.

2. Type A_{n-1}, $n > 2$, the split inner class. If n is odd, there is no choice as all the vertices of the Vogan diagram are connected into pairs, so we only get the split form $\mathfrak{g}_\sigma = \mathfrak{sl}_n(\mathbb{R})$. However, if $n = 2k$ is even, there is one unmatched vertex in the middle of the Vogan diagram, which can be either white or black. It is easy to check that in the first case (white vertex) $\mathfrak{k} = \mathfrak{g}^\theta = \mathfrak{sp}_{2k}$ and in the second one (black vertex) $\mathfrak{k} = \mathfrak{g}^\theta = \mathfrak{so}_{2k}$. So the first case is $\mathfrak{g}_\sigma = \mathfrak{sl}(k, \mathbb{H})$, the Lie algebra of quaternionic matrices of size k whose trace has zero real part, while the second case is the split form $\mathfrak{g}_\sigma = \mathfrak{sl}_n(\mathbb{R})$ (See Subsection 5.2).

3. Type B_n. Then θ is an inner automorphism, given by an element of order ≤ 2 in $SO(2n + 1)$. So $\theta = \mathrm{Id}_{2p+1} \oplus (-\mathrm{Id}_{2q})$ where $p + q = n$. Thus all the real forms are $\mathfrak{so}(2p+1, 2q)$ (all distinct), $\mathfrak{k} = \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2q}$.

4. Type C_n. Then θ is an inner automorphism, given by an element of $g \in \mathrm{Sp}_{2n}(\mathbb{C})$ such that $g^2 = 1$ and $g^2 = -1$. In the first case the 1-eigenspace of g has dimension $2p$ and the -1-eigenspace has dimension $2q$ (since they are symplectic), where $q + p = n$, and we may assume $p \ge q$ (replacing g by $-g$ if needed). So the real form we get is $\mathfrak{g}_\sigma = \mathfrak{u}(p, q, \mathbb{H})$, the quaternionic pseudo-unitary Lie algebra for a quaternionic Hermitian form (see Subsection 5.2). In this case $\mathfrak{k} = \mathfrak{sp}_{2p} \times \mathfrak{sp}_{2q}$. On the other hand, if $g^2 = -1$ then $\mathbb{C}^{2n} = V(i) \oplus V(-i)$ (eigenspaces of g, which in this case are Lagrangian subspaces), so $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$. The corresponding real form is the split form $\mathfrak{g}_\sigma = \mathfrak{sp}_{2n}(\mathbb{R})$.

5. Type D_n, compact inner class. We again have an inner automorphism θ given by $g \in SO(2n)$ such that $g^2 = \pm 1$. If $g^2 = 1$ then $\mathbb{C}^{2n} = V(1) \oplus V(-1)$, the direct sum of eigenspaces, and since $\det(g) = 1$, the eigenspaces are even-dimensional, of dimensions $2p$ and $2q$ where $p+q = n$, and, as in the case of type C_n, we may assume $p \ge q$.

So the corresponding real form is $\mathfrak{g}_\sigma = \mathfrak{so}(2p, 2q)$ with $\mathfrak{k} = \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2q}$. On the other hand, if $g^2 = -1$ then we have $\mathbb{C}^{2n} = V(i) \oplus V(-i)$, and these are Lagrangian subspaces of dimension n. So $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$. The corresponding real form is the quaternionic orthogonal Lie algebra (symmetries of a quaternionic skew-Hermitian form), $\mathfrak{g}_\sigma = \mathfrak{so}^*(2n)$ (see Subsection 5.2).

6. Type D_n, the other inner class. In this case θ is given by an element g of $O(2n)$ such that $\det(g) = -1$ and $g^2 = \pm 1$. Note that if $g^2 = -1$ then , as shown above, $\det(g) = 1$, so in the case at hand we always have $g^2 = 1$. Then $\mathbb{C}^{2n} = V(1) \oplus V(-1)$, but now the dimensions of these spaces are odd, $2p+1$ and $2q-1$ where $p+q = n$, and we may assume that $p+1 \geq q$. So the real form is $\mathfrak{g}_\sigma = \mathfrak{so}(2p+1, 2q-1)$, with $\mathfrak{k} = \mathfrak{so}_{2p+1} \times \mathfrak{so}_{2q-1}$. Note that for $n = 3$, $D_3 = A_3$, so we have $\mathfrak{so}(5, 1) = \mathfrak{sl}(2, \mathbb{H})$. Note also that this agrees with what we found before: the split form $\mathfrak{so}(n, n)$ is in the compact inner class for even n and in the other one for odd n, and the quasi-split form $\mathfrak{so}(n+1, n-1)$ the other way around.

Exercise 40.6. Compute the subalgebras $\mathfrak{k}$ for all the real forms of classical groups.

41. Lecture 16

41.1. **Real forms of exceptional Lie algebras.** For exceptional Lie algebras, it is convenient to make systematic use of Vogan diagrams (we could do this also for classical Lie algebras, but there we can also do everything explicitly using linear algebra). Recall that any real form comes from a certain Vogan diagram, but different Vogan diagrams may be equivalent, i.e., define the same real form. So our job is to describe this equivalence relation.

First consider the case of the compact inner class. In this case the Vogan diagram is just the Dynkin diagram with black and white vertices (i.e., no matched vertices). Moreover, the case of all white vertices corresponds to the compact form, while the case when there are black vertices to noncompact forms. So let us focus on the latter case. Thus we have an element $\theta \in H \subset G_{\mathrm{ad}}$ such that $\theta \neq 1$ but $\theta^2 = 1$, but we are allowed to conjugate θ by elements of $N(H)$, i.e., transform it by elements of the Weyl group W. So how do simple reflections s_i act on θ (in terms of its Vogan diagram)?

The Vogan diagram of θ is determined by the numbers $\alpha_j(\theta) = \pm 1$: if this number is 1 then j is white, and if it is -1 then j is black. Now, we have

$$\alpha_j(s_i(\theta)) = (s_i\alpha_j)(\theta) = (\alpha_j - a_{ij}\alpha_i)(\theta).$$

This equals $\alpha_j(\theta)$ unless $\alpha_i(\theta) = -1$ and a_{ij} is odd. Thus we obtain the following lemma.

Lemma 41.1. *Suppose the Vogan diagram of θ contains a black vertex j. Then changing the colors of all neighbors i of j such that a_{ij} is odd gives an equivalet Vogan diagram.*

The same lemma holds, with the same proof, in the case of another inner class (which for exceptional Lie algebras is possible only for E_6), except we should ignore the vertices matched into pairs (so i and j should be θ-stable vertices).

1. Type G_2. We have two color configurations up to equivalence: WW and (BW,WB,BB). The first corresponds to the compact form G_2^c and the second to the split G_2^{spl}. It is easy to check that in the second case $\mathfrak{k} = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ (indeed, it has dimension 6 and rank 2). So we don't have other real forms.

2. Type F_4. Let α_1, α_2 be short roots and α_3, α_4 long roots. Then all nonzero off-diagonal a_{ij} are odd except $a_{23} = -2$. So we may change the colors of the neighbors of any black vertex, except that if the black vertex is 2 then we should not change the color of 3. By such changes, we can bring the colors at $3, 4$ into the form WW or WB, and then bring the colors at $1, 2$ to the form WW or WB. So we are down to four configurations: WWWW, BWWW, WWWB, and BWWB.

Moreover, the fourth case, BWWB, is actually equivalent to the third one, WWWB. This is seen from the chain of equivalences

$$WWWB = WWBB = WBBW =$$

$$BWBW = BBBB = BBWB = BWWB.$$

Thus we are left with three variants, WWWW, BWWW, WWWB.

The first configuration, WWWW, corresponds to the compact form F_4^c.

In the second case, BWWW, $\alpha(\theta) = -1$ exactly when the root α has half-integer coordinates (recall that there are 16 such roots). Thus the Lie algebra $\mathfrak{k}$ is comprised by the root subspaces for roots with integer coordinates and the Cartan subagebra, i.e., $\mathfrak{k} = \mathfrak{so}_9$ (type B_4). Also in this case $\mathfrak{p} = S$, the spinor representation of $\mathfrak{so}_9$. This is not the split form, since for the split form $\dim \mathfrak{k}$ should be 24 and here it is 36. Let us denote this form F_4^1.

Thus, the third case, WWWB, must be the split form, F_4^{spl}. We see that $\mathfrak{k}$ contains the 21-dimensional Lie algebra $\mathfrak{sp}_6 = C_3$ (generated by the simple roots $\alpha_1, \alpha_2, \alpha_3$), so given that $\mathfrak{k}$ has rank 4 and dimension 24, we have $\mathfrak{k} = \mathfrak{sp}_6 \oplus \mathfrak{sl}_2$.

3. Type E6, inner class of split form. In this case in the Vogan diagram two pairs of vertices are connected, so we can only color the two remaining vertices. So we have two equivalence classes of colorings – WW and (BB,BW,WB). Let us show that this corresponds to two different real forms. Consider first the WW case. In this case θ is simply the diagram automorphism, so we have $\mathfrak{k} = F_4$, as the Dynkin diagram of F_4 is obtained by folding the Dynkin diagram of E_6 (check it!). This is not the split form since $\dim \mathfrak{k} = 52$, but for the split form it is 36; denote this form by E_6^1. So the split form E_6^{spl} corresponds to the second equivalence class (BB, BW, WB). One can show that in this case $\mathfrak{k} = \mathfrak{sp}_8$, i.e., type C_4 (check it!).

4. E_6, E_7, E_8**, inner class of compact form.** In this case the Vogan diagram has no arrows and just is the usual Dynkin diagram with vertices colored black and white. One option is that all vertices are white, this corresponds to the compact form E_6^c, E_7^c, E_8^c $(\theta = 1)$. If there is at least one black vertex, then by using equivalence transformations we can make sure that the nodal vertex is black. Then flipping the color of its neighbors if needed, we can make sure that the vertex on the shortest leg is also black. This allows us to change the color of the nodal vertex whenever we want (as long as the vertex on the shortest leg remains black).

We now want to unify the coloring of the long leg. We can bring the long leg to the following normal forms:

E6: WW, BW=BB=WB. But by flipping the colors on the neighbors of the nodal vertex, we see that BW and WW are equivalent, so all patterns are equivalent to BB.

E7: WWW, BWW=BBW=WBB=WWB, BWB=BBB=WBW. But by flipping the colors on the neighbors of the nodal vertex, we see that all patterns are equivalent to BBB.

E8: WWWW, BWWW=BBWW=WBBW=WWBB=WWWB, BWWB=BWBB=WBWW=BBBW=BWBW=BBBB=BBWB= WBBB=WBWB=WWBW. But by flipping the colors on the neighbors of the nodal vertex, we see that all patterns are equivalent to BBBB.

Thus we can always arrange all vertices on the long leg except possibly the neighbor of the node to be black, while the short leg and the node also remain black. In addition, in the cases E6 and E8, these two configurations are equivalent by transformations inside the leg.

Now we can consider the configurations on the remaining leg (of length 2). The equivalence classes are WW and BW=WB=BB.

So in the case of E6 and E8 we get just two cases. It turns out that both for E6 and E8 these give two different real forms, one of which is split in the case of E8.

Consider first the E6 case. One option is to take the Vogan diagram with just one black vertex, at the end of the long leg. Then $\mathfrak{k} = \mathfrak{so}_{10} \oplus \mathfrak{so}_2$ (as the black vertex corresponds to a minuscule weight). We denote this real form by E_6^2. On the other hand, if there is only one black vertex on the short leg, then $\mathfrak{k}$ contains $\mathfrak{sl}_6$, so this real form is different (as $\mathfrak{sl}_6$ is not a Lie subalgebra of $\mathfrak{so}_{10}$). It's not difficult to show that in this case $\mathfrak{k} = \mathfrak{sl}_6 \times \mathfrak{sl}_2$. We denote this real form by E_6^3.

Now consider the E8 case. Again one option is the Vogan diagram with just one black vertex, at the end of the long leg. Then $\mathfrak{k}$ contains E_7, so this is not the split form since $\dim \mathfrak{k} \geq 133$ but for the split form it should be 120. In fact, it is not hard to see that $\mathfrak{k} = E_7 \oplus \mathfrak{sl}_2$. We denote this real form by E_8^1. The second form is the split one, E_8^{spl}. It can, for example, be obtained if we color black only one vertex, at the end of the leg of length 2. In fact, it's not hard to show that the algebra $\mathfrak{k}$ in this case is $\mathfrak{so}_{16}$.

Finally, consider the E_7 case. In this case we have four options, but two of them end up being equivalent. Namely, we have

$$WB[B,B]WBB = BW[W,B]WBB = BW[B,B]WBB = BB[B,W]BBB =$$
$$WB[W,W]BBB = WB[B,W]BWB = WW[B,B]WWB = WW[B,B]WBB.$$

So we are left with three cases, which all turn out different. The first one is just one black vertex at the end of the long leg. In this case $\mathfrak{k}$ contains E_6, so this is not the split form, as $\dim \mathfrak{k} \geq 78$ but for the split form it is 63. It is easy to see that $\mathfrak{k} = E_6 \times \mathfrak{so}_2$ in this case (black vertex corresponds to the minuscule weight). We denote this real form by E_7^1. The second option is a black vertex at the end of the middle leg. Then $\mathfrak{k}$ contains $\mathfrak{so}_{12}$, of dimension 66, so again not the split form. One can show that for this form $\mathfrak{k} = \mathfrak{so}_{12} \times \mathfrak{sl}(2)$. We denote it by E_7^2. Finally, the split form E_7^{spl} is obtained when one colors black just the end of the short leg. Then $\mathfrak{k}$ contains $\mathfrak{sl}_7$ and one can show that $\mathfrak{k} = \mathfrak{sl}_8$.

Exercise 41.2. Work out the details of computation of $\mathfrak{k}$ for real forms of exceptional Lie algebras.

Exercise 41.3. Let $\mathfrak{sl}_3 \subset \mathfrak{g}_2$ be the Lie subalgebra generated by long root elements and $SU(3) \subset G_2^c$ be the corresponding subgroup. Show that $G_2/SU(3) \cong S^6$. Deduce that there are embeddings $G_2^c \subset SO(7)$ and $G_2^c \subset \mathrm{Spin}(7)$.

Hint. Consider the 7-dimensional representation of G_2^c. Show that it is of real type (obtained by complexifying a real representation V)

and then consider the action of G_2^c on the set of unit vectors in V under a positive invariant inner product. Then compute the Lie algebra of the stabilizer and use that the sphere is simply connected.

Exercise 41.4. Show that $\mathrm{Spin}(7)/G_2^c = S^7$ and $SO(7)/G_2^c = \mathbb{RP}^7$.

Hint. Let S be the spin representation of Spin(7). Use that it is of real type (this was shown in a previous exercise) and then consider the action of Spin(7) on vectors of norm 1 in $S_{\mathbb{R}}$. Compute the Lie algebra of the stabilizer and use that the sphere is simply connected.

42. Lecture 17

42.1. **Classification of connected compact Lie groups.** We are now ready to classify connected compact Lie groups.

Exercise 42.1. Show that if K is a compact Lie group then $\mathfrak{k} = \mathrm{Lie}K_{\mathbb{C}}$ is a reductive Lie algebra.

Hint. First use integration over K to show that $\mathfrak{k}$ has a K-invariant positive definite Hermitian form. Then show that if I is an ideal in $\mathfrak{k}$ then its orthogonal complement $I^{\perp}$ is also an ideal.

Now let $\mathfrak{g}$ be a semisimple complex Lie algebra and G the corresponding simply connected complex Lie group (the universal cover of G_{ad}). Let Z be the kernel of the covering map $G \to G_{\mathrm{ad}}$, which is also $\pi_1(G_{\mathrm{ad}})$ and the center of G. The finite dimensional representations of G are the same as those of $\mathfrak{g}$, so the irreducible ones are L_λ, $\lambda \in P_+$. The center Z acts by a certain character $\chi_\lambda : Z \to \mathbb{C}^\times$ on each L_λ. Since $L_{\lambda+\mu}$ is contained in $L_\lambda \otimes L_\mu$, we have $\chi_{\lambda+\mu} = \chi_\lambda\chi_\mu$, so χ uniquely extends to a homomorphism $\chi : P \to \mathrm{Hom}(Z, \mathbb{C}^\times)$. Also, by definition $\chi_\theta = 1$ (since the maximal root θ is the highest weight of the adjoint representation on which Z acts trivially).

Now, by Exercise 31.10, if $\lambda(h_i)$ are sufficiently large then for every root α of $\mathfrak{g}$ we have $L_{\lambda+\alpha} \subset L_\lambda \otimes \mathfrak{g}$. Thus $\chi_{\lambda+\alpha} = \chi_\lambda$, hence $\chi_\alpha = 1$. So χ is trivial on Q, i.e., χ defines a homomorphism $P/Q \to \mathrm{Hom}(Z, \mathbb{C}^\times)$, or, equivalently, $Z \to P^\vee/Q^\vee$.

Note that the same argument works for G_{ad}^c, its universal cover G^c, and its center Z^c instead of G_{ad}, G, Z.

Proposition 42.2. *A representation L_λ of $\mathfrak{g}$ with highest weight $\lambda \in P_+$ lifts to a representation of G_{ad} (or, equivalently, G_{ad}^c) if and only if $\lambda \in P_+ \cap Q$.*

Proof. We have just shown that if $\lambda \in P_+ \cap Q$ then L_λ lifts. The converse follows from Proposition 36.8. $\square$

Now we can proceed with the classification of semisimple compact connected Lie groups. We begin with the following lemma from topology.

Lemma 42.3. *If X is a connected compact manifold then the fundamental group $\pi_1(X)$ is finitely generated.*

Proof. (sketch) Cover X by small balls, pick a finite subcover, connect the centers. We get a finite graph whose fundamental group maps surjectively to $\pi_1(X)$. $\square$

Theorem 42.4. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $G^c_{\rm ad}$ the corresponding adjoint compact group. Then $\pi_1(G^c_{\rm ad}) = P^\vee/Q^\vee$. Thus the universal cover G^c of $G^c_{\rm ad}$ is a compact Lie group.*

Proof. Let K be a finite cover of $G^c_{\rm ad}$, and $Z_K \subset K$ be the kernel of the projection $K \to G^c_{\rm ad}$. Then finite dimensional irreducible representations of K are a subset of finite dimensional irreducible representations of $\mathfrak{g}$, labeled by a subset $P_+(K) \subset P_+$ containing $P_+ \cap Q$ (as by Proposition 42.2 these are highest weights of representations of $G^c_{\rm ad}$). Let $P(K) \subset P$ generated by $P_+(K)$. Let χ_λ be the character by which Z_K acts on the irreducible representation L_λ of K. By Proposition 42.2, χ defines an injective homomorphism $\xi : P(K)/Q \to Z_K^\vee$. Since K is compact, by the Peter-Weyl theorem this homomorphism is surjective, hence is an isomorphism.

It remains to show that $\pi_1(G^c_{\rm ad})$ is finite (then we can take K to be the universal cover of $G^c_{\rm ad}$, in which case $P(K) = P$, so we get $P/Q \cong Z^\vee$, hence $Z = \pi_1(G_{\rm ad}) \cong P^\vee/Q^\vee$). To this end, note that by Lemma 42.3, $\pi_1(G^c_{\rm ad})$ is a finitely generated abelian group. Take a subgroup of finite index N in $\pi_1(G^c_{\rm ad})$ and let K be the corresponding cover. As we have shown, then $N = |Z_K| \le |P(K)/Q| \le |P/Q|$. But for finitely generated abelian groups this implies that the group is finite. $\square$

This immediately implies the following corollary.

Corollary 42.5. *(i) If $\mathfrak{g}$ is a simple complex Lie algebra then the simply connected Lie group G^c corresponding to the Lie algebra $\mathfrak{g}^c$ is compact, and its center is $P^\vee/Q^\vee$, which also equals $\pi_1(G^c_{\rm ad})$.*

(ii) Let $\Gamma \subset P^\vee/Q^\vee$ be a subgroup. Then the irreducible representations of G/Γ are L_λ such that λ defines the trivial character of Γ.

(iii) Let G_i^c be the simply connected compact Lie group corresponding to a simple summand $\mathfrak{g}_i$ of a semisimple Lie algebra $\mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_i$. Then any connected Lie group with Lie algebra $\mathfrak{g}^c$ is compact and has the form $(\prod_{i=1}^n G_i^c)/Z$, where $Z = \pi_1(G^c)$ is a subgroup of $\prod_i Z_i$, and $Z_i =$

$P_i^\vee/Q_i^\vee$ are the centers of G_i^c. Moreover, every semisimple connected compact Lie group has this form.

In particular, it follows that simply connected semisimple compact Lie groups are of the form $\prod_{i=1}^n G_i^c$, where G_i^c are simply connected and simple.

Corollary 42.6. *Any connected compact Lie group is the quotient of $T \times K$ by a finite central subgroup, where $T = (S^1)^m$ is a torus and K is semisimple and simply connected.*

Proof. Let L be such a group, $\mathfrak{l}$ its Lie algebra. It is reductive, so we can uniquely decompose $\mathfrak{l}$ as $\mathfrak{t} \oplus \mathfrak{k}$ where $\mathfrak{t}$ is the center and $\mathfrak{k}$ is semisimple. Let $T, K \subset L$ be the connected Lie subgroups corresponding to $\mathfrak{t}, \mathfrak{k}$. It is clear that $\mathrm{Lie}\overline{T} = \mathfrak{t}$, so T is closed, hence compact, hence a torus. Also K is compact, so also closed, with $\mathrm{Lie}K = \mathfrak{k}$. Thus we have a surjective homomorphism $T \times K \to L$ whose kernel is finite, as desired. □

43. Lecture 18

43.1. **Polar decomposition.** Now let us study the structure of the connected Lie subgroup $G_\sigma \subset G_{\mathrm{ad}}$ corresponding to the real form $\mathfrak{g}_\sigma \subset \mathfrak{g}$ of a semisimple complex Lie algebra $\mathfrak{g}$. It is clear that this subgroup is closed ($\mathrm{Lie}\overline{G_\sigma} = \mathfrak{g}_\sigma$). Let $K_\sigma \subset G_\sigma$ be the connected subgroup corresponding to the Lie subalgebra $\mathfrak{k}^c \subset \mathfrak{g}_\sigma$ (the subgroup of elements acting on $\mathfrak{g}$ by unitary operators). The subgroup K_σ is closed since $\mathrm{Lie}(\overline{K}) = \mathfrak{k}^c$, hence it is compact. Also let $P_\sigma = \exp(i\mathfrak{p}^c) \subset G_\sigma$ (note that P_σ is not a group, in general!). Since $i\mathfrak{p}^c$ acts on $\mathfrak{g}$ by Hermitian operators, the exponential map $\exp : i\mathfrak{p}^c \to P_\sigma$ is a diffeomorphism, so $P_\sigma \subset G_\sigma$ is a closed embedded submanifold (the set of elements acting on $\mathfrak{g}$ by positive Hermitian operators).

Theorem 43.1. *(Polar decomposition for G_σ) The multiplication map $\mu : K_\sigma \times P_\sigma \to G_\sigma$ is a diffeomorphism. Thus $G_\sigma \cong K_\sigma \times \mathbb{R}^{\dim \mathfrak{p}}$ as a manifold (in particular, G_σ is homotopy equivalent to K_σ).*

Proof. Recall that every invertible complex matrix A can be uniquely written as a product $A = U_A R_A$, where $U = U_A$ is a unitary matrix and $R = R_A$ a positive Hermitian matrix, namely $R = (A^\dagger A)^{1/2}$, $U = A(A^\dagger A)^{-1/2}$ (the classical polar decomposition). Let us consider this decomposition for $g \in G_\sigma \subset \mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$. Since $g^\dagger g$ is an automorphism of $\mathfrak{g}$, so is $(g^\dagger g)^{1/2} = R_g$, so $R_g \in P_\sigma$ (a positive self-adjoint element in G_σ). Also since U_g is unitary, it belongs to K_σ. Thus the regular map $g \mapsto (U_g, R_g)$ is the inverse to μ (using the uniqueness of the polar decomposition). □

Corollary 43.2. *The multiplication map defines a diffeomorphism*

$$G_{\rm ad}^c \times \mathbf{P} \cong G_{\rm ad},$$

where $\mathbf{P}$ *is the set of elements acting on* $\mathfrak{g}$ *by positive Hermitian operators. In particular,* $\pi_1(G_{\rm ad}) = \pi_1(G_{\rm ad}^c) = P^\vee/Q^\vee$.

Corollary 43.3. *If* G *is a semisimple complex Lie group then the center* Z *of* G *is contained in* G^c*, i.e., coincides with the center* Z^c *of* G^c*. Thus the restriction of finite dimensional representations from* G *to* G^c *is an equivalence of categories.*

This also implies that by taking coverings the polar decomposition applies verbatim to the real form $G_\sigma \subset G$ of any complex semisimple Lie group G instead of $G_{\rm ad}$. We note, however, that if G is simply connected, then G_σ need not be. In fact, its fundamental group could be infinite. The simplest example is $G = SL_2(\mathbb{C})$, then for the split form $G_\sigma = SL_2(\mathbb{R})$, which as we showed is homotopy equivalent to $SO(2) = S^1$, i.e. its fundamental group is $\mathbb{Z}$.

Example 43.4. 1. For $G_\sigma = SL_n(\mathbb{C})$ we have $K_\sigma = SU(n)$ and P_σ is the set of positive Hermitian matrices of determinant 1, so the polar decomposition in this case is the usual polar decomposition of complex matrices.

2. For $G_\sigma = SL_n(\mathbb{R})$ we have $K_\sigma = SO(n)$ and P_σ is the set of positive symmetric matrices of determinant 1, so the polar decomposition in this case is the usual polar decomposition of real matrices.

43.2. **Connected complex reductive groups.**

Definition 43.5. A connected complex Lie group G is **reductive** if it is of the form $((\mathbb{C}^\times)^r \times G_0)/Z$ where G_0 is semisimple and Z is a finite central subgroup. A complex Lie group G is reductive if G° is reductive and G/G° is finite.

Example 43.6. $GL_n(\mathbb{C}) = (\mathbb{C}^\times \times SL_n(\mathbb{C}))/\mu_n$ is reductive.

It is clear that the Lie algebra Lie$G$ of any complex reductive Lie group G is reductive, and any complex reductive Lie algebra is the Lie algebra of a connected complex reductive Lie group. However, a simply connected complex Lie group with a reductive Lie algebra need not be reductive (e.g. $G = \mathbb{C}$).

If $G = ((\mathbb{C}^\times)^r \times G_0)/Z$ is a connected complex reductive Lie group then as we have shown, $Z \subset (S^1)^r \times G_0^c \subset (\mathbb{C}^\times)^r \times G_0$, so we can define the compact subgroup $K \subset G$ by $K := ((S^1)^r \times G_0^c)/Z$. Then it is easy to see that restriction of finite dimensional representations from G to K is an equivalence, so representations of G are completely reducible. The irreducible representations are parametrized by collections $(n_1, ..., n_r, \lambda)$, $\lambda \in P_+(G_0)$, $n_i \in \mathbb{Z}$, which define the trivial character of Z.

43.3. **Linear groups.** A connected Lie group G (real or complex) is called **linear** if it can be realized as a Lie subgroup of $GL_n(\mathbb{R})$, respectively $GL_n(\mathbb{C})$. We have seen that any complex semisimple group is linear. However, for real semisimple groups this is not so (e.g. the universal cover of $SL_2(\mathbb{R})$ is not linear). In fact, we see that we can characterize real semisimple groups as follows.

Proposition 43.7. *Suppose $\mathfrak{g}_\sigma$ is a real form of a semisimple complex Lie algebra $\mathfrak{g}$, G a complex Lie group with Lie algebra $\mathfrak{g}$, and G_σ its Lie subgroup corresponding to the real Lie subalgebra $\mathfrak{g}_\sigma \subset \mathfrak{g}$. Then G_σ is a linear group. Moreover, every connected real semisimple linear Lie group has this form.*

Exercise 43.8. Classify simply connected real semisimple **linear** Lie groups.

43.4. **Maximal tori in connected compact Lie groups.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{g}_c$ its compact form, G a Lie group with Lie algebra $\mathfrak{g}$, $G^c \subset G$ its compact part, as above.

A **Cartan subalgebra** $\mathfrak{h}^c \subset \mathfrak{g}^c$ is a maximal commutative Lie subalgebra (note that it automatically consists of semisimple elements since all elements of $\mathfrak{g}^c$ are semisimple). In other words, it is a subspace such that $\mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$.

Recall that all Cartan subalgebras of $\mathfrak{g}$ are conjugate, even if equipped with a system of simple roots. Namely, given two such subalgebras $(\mathfrak{h}, \Pi)$ and $(\mathfrak{h}', \Pi')$, there is $g \in G$ such that $g(\mathfrak{h}, \Pi)g^{-1} = (\mathfrak{h}', \Pi')$. The same result holds for $\mathfrak{g}^c$.

Lemma 43.9. *Any two Cartan subalgebras in $\mathfrak{g}^c$ equipped with systems of simple roots are conjugate under G^c.*

Proof. Given $(\mathfrak{h}^c, \Pi)$ and $(\mathfrak{h}^{c'}, \Pi')$, there is $g \in G$ such that $g(\mathfrak{h}^c, \Pi)g^{-1} = (\mathfrak{h}^{c'}, \Pi')$. Then we also have $\overline{g}(\mathfrak{h}^c, \Pi)\overline{g}^{-1} = (\mathfrak{h}^{c'}, \Pi')$, where $\overline{g} := \omega(g)$. So $\overline{g}^{-1}g$ commutes with $\mathfrak{h}^c$, i.e., $\overline{g}h = g$, where $h \in H$. Writing $g = kp$, where $k \in G^c$, $p \in \mathbf{P}$, we have $kp^{-1}h = kp$, so $h = p^2$. Since p is positive, $p = h^{1/2}$, so it commutes with $\mathfrak{h}^c$, thus $k(\mathfrak{h}^c, \Pi)k^{-1} = (\mathfrak{h}^{c'}, \Pi')$, as claimed. □

Note that for every Cartan subalgebra $\mathfrak{h}^c \subset \mathfrak{g}^c$, $H^c = \exp(\mathfrak{h}^c) \subset G^c$ is a torus, which is clearly maximal (as the Lie algebra of a larger torus would be a larger commutative subalgebra than $\mathfrak{h}$). Conversely, if $H^c \subset G^c$ is a maximal torus then $\mathrm{Lie}(H_c)$ can be included in a Cartan subalgebra, hence it is itself a Cartan subalgebra. So we have a bijection between Cartan subalgebras in $\mathfrak{g}^c$ and maximal tori in G^c. Similarly, there is a bijection between Cartan subalgebras in $\mathfrak{g}$ and maximal tori in G.

This implies

Corollary 43.10. *Any two maximal tori in G or G^c equipped with systems of simple roots are conjugate.*

Theorem 43.11. *Every element of a connected compact Lie group K is contained in a maximal torus, and all maximal tori in K are conjugate (even when equipped with a system of simple roots).*

Proof. We may assume without loss of generality that K is semisimple, i.e., $K = G^c$ for a semisimple complex Lie group G, which implies the second statement. To prove the first statement, let $K' \subset K$ be the set of elements contained in a maximal torus. Fix a maximal torus $T \subset K$ and consider the map $f : K \times T \to K$ given by $f(k,t) = ktk^{-1}$, whose image is K'. This implies that K' is compact, hence closed. On the other hand, say that $x \in K$ is **regular** if its centralizer $\mathfrak{z}_x$ in $\mathfrak{g}^c$ has dimension $\le r$. This set is open and nonempty since there are plenty of regular elements in $\mathfrak{g}^c$. But every regular element is contained in the maximal torus $\exp(\mathfrak{z}_x)$, so the elements of $K \setminus K'$ are all non-regular. But the set of non-regular elements is defined by polynomial equations (minors of $\mathrm{Ad}(g)$ of codimension $\le r$ all vanish), so $K \setminus K'$ must be empty. ☐

Corollary 43.12. *The exponential map* $\exp : \mathfrak{g}^c \to G^c$ *is surjective.*

Exercise 43.13. Is the exponential map surjective for the group $SL_2(\mathbb{C})$?

43.5. **Semisimple and unipotent elements.** Let G be a connected reductive complex Lie group. An element $g \in G$ is called **semisimple** if it acts in every finite dimensional representation of G by a semisimple (=diagonalizable) operator, and **unipotent** if it acts in every finite dimensional representation of G by a unipotent operator (all eigenvalues are 1).

Exercise 43.14. Let Y be a faithful finite dimensional representation of G (we know it exists). Show that $g \in G$ is semisimple if and only if it acts semisimply on Y, and unipotent if and only if it acts unipotently on Y.

Hint: Use Proposition 36.8.

Exercise 43.15. Show that if G is semisimple then the exponential map defines a homeomorphism between the set of nilpotent elements in $\mathfrak{g} = \mathrm{Lie}G$ and the set of unipotent elements in G.

Exercise 43.16. Let Z be the center of a connected complex reductive group G.

(i) Show that the homomorphism $\pi : G \to G/Z$ defines a bijection between unipotent elements of G and G/Z.

(ii) Show that the set of semisimple elements of G is the preimage under π of the set of semisimple elements of G/Z.

Proposition 43.17. *(Jordan decomposition in G). Every element $g \in G$ has a unique factorization $g = g_s g_u$, where $g_s \in G$ is semisimple, $g_u \in G$ is unipotent and $g_s g_u = g_u g_s$.*

Exercise 43.18. Prove Proposition 43.17.

Hint. Use Exercise 43.16 to reduce to the case when $G = G_{\mathrm{ad}}$ is a semisimple adjoint group. In this case, write $\mathrm{Ad}(g)$ as su, where s is a semisimple and u a unipotent operator with $su = us$ (Jordan decomposition for matrices). Show that $s = \mathrm{Ad}(g_s)$ and $u = \mathrm{Ad}(g_u)$ for some commuting $g_s, g_u \in G_{\mathrm{ad}}$. Then establish uniqueness using the uniqueness of Jordan decomposition of matrices.

43.6. **The Cartan decomposition.** Let G be a complex semisimple group, $G_\sigma \subset G$ a real form, $\mathfrak{g}_\sigma \subset \mathfrak{g}$ their Lie algebras. We have the polar decomposition $G_\sigma = K_\sigma P_\sigma$ and the additive version $\mathfrak{g}_\sigma = \mathfrak{k}^c \oplus i\mathfrak{p}^c$. Recall that we have fixed a Cartan subalgebra $\mathfrak{h}^c = \mathfrak{h}^c_+ \oplus \mathfrak{h}^c_-$, $\mathfrak{h}_c^+ \subset \mathfrak{k}^c$, $\mathfrak{h}_c^- \subset \mathfrak{p}^c$. Let $A = \exp(i\mathfrak{h}^c_-) \subset P_\sigma \subset G_\sigma$. This is a subgroup isomorphic to $\mathbb{R}^n$, where $n = \dim \mathfrak{h}^c_-$.

Lemma 43.19. *$\mathfrak{h}^c_-$ is a maximal abelian subalgebra of $\mathfrak{p}^c$, and all maximal abelian subalgebras of $\mathfrak{p}^c$ are conjugate to $\mathfrak{h}^c_-$.*

The proof of this lemma will be a homework problem.

Theorem 43.20. *(The Cartan decomposition) We have $G_\sigma = K_\sigma A K_\sigma$. In other words, every element $g \in G_\sigma$ has a factorization $g = k_1 a k_2$, $k_1, k_2 \in K_\sigma$, $a \in A$.*[10]

Proof. It suffices to show that every K_σ-orbit on P_σ intersects A. To do so, take $x \in P_\sigma$ and let $p = -i\log(x) \in \mathfrak{p}^c$. Let $\mathfrak{h}^c_{p-}$ be a maximal abelian subalgebra of $\mathfrak{p}$ containing p. By Lemma 43.19, there is $g \in K_\sigma$ such that $\mathrm{Ad}(g)\mathfrak{h}^c_{p-} = \mathfrak{h}^c_-$. Then $\mathrm{Ad}(g)x \in A$, as claimed. ☐

Remark 43.21. This theorem has a straightforward generalization to reductuve groups

[10] This factorization is not unique.

Example 43.22. 1. For $G_\sigma = GL_n(\mathbb{C})$, this theorem says that any invertible complex matrix can be written as U_1DU_2, where U_1, U_2 are unitary and D is diagonal with positive entries.

2. For $G_\sigma = GL_n^+(\mathbb{R})$ (matrices with positive determinant), this theorem says that any invertible real matrix can be written as O_1DO_2, where O_1, O_2 are orthogonal with determinant 1 and D is diagonal with positive entries.

43.7. **Integral form of the Weyl character formula.**

Proposition 43.23. *Let f be a conjugation invariant function on a compact connected Lie group K with a maximal torus $T \subset K$ and Haar probability measure dk. Then*

$$\int_K f(k)dk = \frac{1}{|W|}\int_T f(x)|\Delta(t)|^2 dt,$$

where $\Delta(t)$ is the Weyl denominator,

$$\Delta(t) = \prod_{\alpha \in R^+}(1-\alpha(t)).$$

Proof. Since characters of irreducible representations are dense invariant functions, it suffices to check this for $f = \chi_\lambda$, the character of the irreducible representation L_λ. Then the left hand side is $\delta_{0\lambda}$ by orthogonality of characters. On the other hand, the Weyl character formula implies that the right hand side also equals $\delta_{0\lambda}$. □

Example 43.24. Let f be a function on $U(n)$. Then

$$\int_{U(n)} f(k)dk = \frac{1}{(2\pi)^n n!}\int_{|z_1|=...=|z_n|=1} f(\mathrm{diag}(z_1, ..., z_n))\prod_{m<j}|z_m - z_j|^2 d\theta_1....d\theta_n$$

where $z_j = e^{i\theta_j}$.

Thus we see that the orthogonality of characters can be written as

$$\frac{1}{|W|}\int_T \chi_\lambda(t)\overline{\chi}_\mu(t)|\Delta(t)|^2 dt = \delta_{\lambda,\mu}.$$

44. Lecture 19

44.1. **Topology of Lie groups.** We would now like to study topology of connected Lie groups. As we know, any real semisimple Lie group G_σ is diffeomorphic to the product of its compact subgroup K_σ and a Euclidean space. This combined with Levi decomposition implies that topology of connected Lie groups essentially reduces to topology of compact ones.

So let us study cohomology of compact connected Lie groups.

We first recall some generalities on cohomology of manifolds. As we mentioned before, the cohomology of an n-dimensional manifold M can be computed by the **de Rham complex**

$$0 \to \Omega^0(M) \to \Omega^1(M) \to ... \to \Omega^n(M) \to 0,$$

where $\Omega^i(M)$ is the space of smooth (complex-valued) differential i-forms on M. The maps in this complex are given by the differential $d : \Omega^i(M) \to \Omega^{i+1}(M)$, which satisfies the equation $d^2 = 0$. Namely, we define the i-th **de Rham cohomology** of M as the quotient

$$H^i(M, \mathbb{C}) := \Omega^i_{\text{closed}}(M)/\Omega^i_{\text{exact}}(M)$$

where $\Omega^i_{\text{closed}}(M) \subset \Omega^i(M)$ is the space of **closed forms** (such that $d\omega = 0$) and $\Omega^i_{\text{exact}}(M) \subset \Omega^i(M)$ is the space of **exact forms** (such that $\omega = d\eta$ for some $\eta \in \Omega^{i-1}(M)$).

If M is compact then the spaces $H^i(M, \mathbb{C})$ are known to be finite dimensional, so we can define the **Betti numbers** of M, $b_i(M) := \dim H^i(M, \mathbb{C})$. Note that $b_0(M)$ is the number of connected components of M, so if M is connected then $b_0(M) = 1$.

The wedge product of differential forms descends to the cohomology, which makes $H^\bullet(M, \mathbb{C}) := \oplus_{i=0}^n H^i(M, \mathbb{C})$ into a graded algebra. This algebra is associative and graded-commutative: $ab = (-1)^{\deg(a)\deg(b)}ba$ (since the wedge product of differential forms has these properties). Moreover, if $f : M \to N$ is a differentiable map of manifolds then we have the pullback map $f^* : \Omega^i(N) \to \Omega^i(M)$ which commutes with d and hence descends to the cohomology. Also f^* preserves the wedge product, hence defines a graded algebra homomorphism $f^* : H^\bullet(N, \mathbb{C}) \to H^\bullet(M, \mathbb{C})$.

Exercise 44.1. Let $f : [0, 1] \times M \to N$ be a differentiable map and $f_t : M \to N$ be given by $f_t(x) = f(t, x)$. Then $f_0^* = f_1^*$ on $H^\bullet(N, \mathbb{C})$. In other words, f^* is invariant under (smooth) homotopies of f.

Lemma 44.2. *(Cartan magic formula) Let v be a vector field on M, $L_v : \Omega^i(M) \to \Omega^i(M)$ the Lie derivative and $\iota_v : \Omega^i(M) \to \Omega^{i-1}(M)$ the contraction operator. Then*

$$L_v = \iota_v d + d\iota_v.$$

Proof. It suffices to check this identity on local charts. It is easy to see that both sides are derivations, so it suffices to check the equation on functions (0-forms) and on 1-forms of the form df where f is a function. For functions we have $L_v f = \iota_v df$, which is essentially the definition of L_v, while for $\omega = df$ we have

$$L_v(df) = d(L_v f) = d\iota_v(df) = (\iota_v d + d\iota_v)(df),$$

since $d^2 = 0$. □

Corollary 44.3. *L_v maps closed forms to exact forms, hence acts trivially in cohomology.*

Corollary 44.4. *If a connected Lie group G acts on a manifold M then G acts trivially on $H^\bullet(M, \mathbb{C})$.*

Theorem 44.5. *Suppose a compact connected Lie group G acts on a manifold M. Then the cohomology $H^\bullet(M, \mathbb{C})$ is computed by the complex of invariant differential forms $\Omega^\bullet(M)^G$.*

Proof. We have the averaging operator $P : \Omega^\bullet(M) \to \Omega^\bullet(M)$ over G which commutes with d, so we have a decomposition

$$\Omega^\bullet(M) = \Omega^\bullet(M)^G \oplus \Omega^\bullet(M)_0$$

where the first summand is the image of P and the second one is the kernel of P. Now, if $\omega \in \Omega^i(M)_0$ is closed than the cohomology class $[\omega]$ of ω coincides with the cohomology class of $[g\omega]$ for all $g \in G$. Thus

$$[\omega] = \int_G [g\omega] dg = \Big[\int_G g\omega dg \Big] = 0.$$

Thus $\omega = d\eta$ for some $\eta \in \Omega^i(M)$. Then $\omega = (1-P)\omega = d(1-P)\eta$, and $(1-P)\eta \in \Omega^i(M)_0$. So the complex $\Omega^\bullet(M)_0$ is exact, which implies the statement. □

Corollary 44.6. *If G is a compact Lie group then $H^\bullet(G, \mathbb{C})$ is computed by the complex $\Omega^\bullet(G)^G$ of left-invariant differential forms on G.*

The complex $\Omega^\bullet(G)^G$ is called the **Chevalley-Eilenberg complex** of G, and it can be described purely algebraically. To this end, we will need another lemma from basic differential geometry.

Lemma 44.7. *(Cartan differentiation formula) Let $\omega \in \Omega^m(M)$ and $v_0, ..., v_m$ be vector fields on M. Then*

$$d\omega(v_0, ..., v_m) = \sum_i (-1)^i L_{v_i}(\omega(v_0, ..., \widehat{v}_i, ..., v_m)) +$$

$$\sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_0, ..., \widehat{v}_i, ..., \widehat{v}_j, ..., v_m).$$

Proof. It is easy to show that the right hand side is linear over functions on M with respect to each v_i (the first derivatives of the function cancel out). Therefore, it suffices to assume that $v_i = \frac{\partial}{\partial x_{k_i}}$ (in local coordinates), and $\omega = f dx_{j_1} \wedge ... \wedge x_{j_m}$. Then the second summand on the RHS vanishes and the verification is straightforward. □

Corollary 44.8. *Let G be a Lie group and $\omega \in \Omega^m(G)^G$ be a left-invariant differential form. Then for any left-invariant vector fields $v_0, ..., v_m$ we have*

$$d\omega(v_0, ..., v_m) = \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_0, ..., \widehat{v_i}, ..., \widehat{v_j}, ..., v_m). \tag{9}$$

Proof. This follows since the functions $\omega(v_0, ..., \widehat{v_i}, ..., v_m)$ are constant. □

Now observe that $\Omega^m(G)^G = \wedge^m \mathfrak{g}^*$. Thus we get

Corollary 44.9. *For any Lie group G the complex $\Omega^\bullet(G)^G$ coincides with the complex*

$$0 \to \mathbb{C} \to \mathfrak{g}^* \to (\wedge^2 \mathfrak{g})^* \to ...(\wedge^m \mathfrak{g})^* \to ...$$

with differential defined by (9)*, where $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$.*

This purely algebraic complex can be defined for a Lie algebra $\mathfrak{g}$ over any field (the equality $d^2 = 0$ follows from the Jacobi identity).[11] It is called the **standard complex** or the **Chevalley-Eilenberg complex** of $\mathfrak{g}$, denoted $CE^\bullet(\mathfrak{g})$, and its cohomology is called the **Lie algebra cohomology** of $\mathfrak{g}$, denoted $H^\bullet(\mathfrak{g})$.

Also note that the complex $CE^\bullet(\mathfrak{g})$ has wedge product multiplication, which descends to the cohomology. Thus $H^\bullet(\mathfrak{g})$ is a graded algebra. Also, if $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ for a compact connected Lie group G then $H^\bullet(\mathfrak{g}) \cong H^\bullet(G, \mathbb{C})$ as a graded algebra. However, this may fail even at the level of vector spaces (i.e., Betti numbers) if G is not compact.

Example 44.10. Let $\mathfrak{g}$ be abelian, $\dim \mathfrak{g} < \infty$. Then $CE^\bullet(\mathfrak{g}) = \wedge^\bullet \mathfrak{g}^*$, with zero differential, so $H^\bullet(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g})^*$. So if $G = (S^1)^n$ is a torus then we get $H^\bullet(G, \mathbb{C}) = \wedge^\bullet(\xi_1, ..., \xi_n)$ where ξ_i have degree 1. It particular, $H^\bullet(S_1) = \wedge^\bullet(\xi)$. However, for the universal cover $\mathbb{R}$ of S^1 this is clearly false.

Remark 44.11. Corollary 44.9 implies that for compact Lie groups G, K the map $\Omega^\bullet(G) \otimes \Omega^\bullet(K) \to \Omega^\bullet(G \times K)$ (i.e., in components, $\Omega^i(G) \otimes \Omega^j(K) \to \Omega^{i+j}(G \times K)$) defines an isomorphism of cohomology rings $H^\bullet(G, \mathbb{C}) \otimes H^\bullet(K, \mathbb{C}) \to H^\bullet(G \times K)$. This is a special case of the **Künneth theorem**, which actually holds for any manifolds (and more generally for sufficiently nice topological spaces), which need not have any group structure. We warn the reader, however, that the **tensor product of algebras here is in the graded sense**, i.e.

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')}(aa' \otimes bb').$$

[11] Note that if $\mathfrak{g}$ is finite dimensional then $\wedge^i \mathfrak{g}^* = (\wedge^i \mathfrak{g})^*$.

Theorem 44.12. *If G is a connected compact Lie group with $\mathrm{Lie}(G)_{\mathbb{C}} = \mathfrak{g}$ then $H^\bullet(G, \mathbb{C}) \cong (\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}}$ as a ring.*

Proof. We have an action of $G \times G$ on G, so the cohomology of G is computed by the complex of invariants $\Omega^\bullet(G)^{G\times G} = (\wedge^\bullet \mathfrak{g}^*)^G$. So our job is to show that the differential in this complex is actually zero. But this follows immediately from the definition of the differential in $\wedge^\bullet \mathfrak{g}^*$. □

Corollary 44.13. *If G is a compact connected Lie group and $\Gamma \subset G$ a finite subgroup then π^* defines an isomorphism $H^\bullet(G/\Gamma, \mathbb{C}) \to H^\bullet(G, \mathbb{C})$.*

Thus it suffices to determine the cohomology of simple, simply connected compact Lie groups.

To understand the algebra $R := H^\bullet(G) = H^\bullet(G, \mathbb{C})$ better, note that the multiplication map $G \times G \to G$ induces the graded algebra homomorphism $\Delta : H^\bullet(G) \to H^\bullet(G \times G) = H^\bullet(G) \otimes H^\bullet(G)$, which is coassociative: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. Such a map is called a **coproduct** since it defines an algebra structure on R^*. We also have the augmentation map $\varepsilon : R \to \mathbb{C}$ such that

$$(\varepsilon \otimes 1)(\Delta(x)) = (1 \otimes \varepsilon)(\Delta(x)) = x$$

for all $x \in R$. Such a structure is called a **graded bialgebra**. [12]

Exercise 44.14. (Hopf theorem) Use the graded bialgebra structure on R to show that R is a **free** graded commutative algebra on some homogeneous generators. Deduce that all generators must be of odd degree, so $R = \wedge^\bullet(\xi_1, ..., \xi_r)$ with $\deg \xi_i = 2m_i + 1$ for some nonnegative integers m_i. Thus $\dim R = 2^r$.

Hint. An element $x \in R$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$. Show that any homogeneous primitive x has odd degree and $x^2 = 0$, and that R^* is generated by homogeneous primitive elements. Then show that primitive elements commute and there can be no nontrivial relations between them (take a relation of lowest degree, compute its coproduct and find a relation of even lower degree, getting a contradiction).

For more hints see `http://preprints.ihes.fr/2006/M/M-06-40.pdf`, subsection 2.4.

Let us now determine the number r. We have $2^r = \dim(\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}}$. But this dimension can be computed using the Weyl character formula.

[12] Moreover, we have an algebra homomorphism $S : R \to R$ induced by the inversion map $G \to G$ called the **antipode**. This makes R into what is called a **graded Hopf algebra**.

Namely, the character of $\wedge^\bullet \mathfrak{g}^*$ is

$$\chi_{\wedge^\bullet \mathfrak{g}^*}(t) = 2^{\mathrm{rank}\mathfrak{g}} \prod_{\alpha>0} (1+\alpha(t))(1+\alpha(t)^{-1}),$$

where $T \subset G$ is a maximal torus and $t \in T$. So

$$\dim(\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}} = \frac{2^{\mathrm{rank}\mathfrak{g}}}{|W|} \int_T \prod_{\alpha>0} (1-\alpha(t^2))(1-\alpha(t^2)^{-1}) dt = 2^{\mathrm{rank}\mathfrak{g}}.$$

So $r = \mathrm{rank}(\mathfrak{g})$.

Thus we have

$$H^\bullet(G) = H^\bullet(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}} = \wedge^\bullet(\xi_1, ..., \xi_r),$$

where $r = \mathrm{rank}(\mathfrak{g})$. and $\deg(\xi_i) = 2m_i + 1$. Moreover, it suffices to consider the case when $\mathfrak{g}$ is simple. What are the numbers m_i in this case?

Let us order m_i as follows: $m_1 \le m_2 \le ... \le m_r$. We know that $r + 2\sum m_i = \dim \mathfrak{g}$, so $\sum_i m_i = |R_+|$. Also we know that $m_1 = 1$, $m_2 > 1$:

Exercise 44.15. Show that for a simple Lie algebra $\mathfrak{g}$ we have $(\wedge^3 \mathfrak{g}^*)^{\mathfrak{g}} = \mathbb{C}$, spanned by the triple product $([xy], z)$.

Hint. Let $\omega \in (\wedge^3 \mathfrak{g}^*)^{\mathfrak{g}}$.

1. Show that

$$\omega(e_i, [f_i, h_i], h) + \omega(e_i, h_i, [f_i, h]) = 0$$

for $h \in \mathfrak{h}$ and deduce that

$$\omega(e_i, f_i, h) = \frac{1}{2}\alpha_i(h)\omega(e_i, f_i, h_i).$$

2. Take $y, z \in \mathfrak{h}$ and show that

$$\omega(h_i, y, z) + \omega(f_i, [e_i, y], z) + \omega(f_i, y, [e_i, z]) = 0.$$

Deduce that $\omega(x, y, z) = 0$ for $x, y, z \in \mathfrak{h}$. Conclude that ω is completely determined by $\omega(e_\alpha, e_{-\alpha}, h)$ for all roots α and $h \in \mathfrak{h}$. Use the Weyl group to reduce to $\omega(e_i, f_i, h)$ and then to $\omega(e_i, f_i, h_i)$.

3. Finally, use that

$$\omega([e_i, e_j], f_i, f_j) = \omega(e_j, f_j, h_i) = \omega(e_i, f_i, h_j)$$

to show that all possible ω are proportional.

Example 44.16. We get $m_2 = 2$ for A_2, $m_2 = 3$ for $B_2 = C_2$, $m_2 = 5$ for G_2.

In fact, we have the following theorem, which we will not prove here:

Theorem 44.17. *The numbers m_i are the the exponents of $\mathfrak{g}$ that we defined earlier. In other words, the degrees $2m_i + 1$ of generators of the cohomology ring are the dimensions of simple modules occurring in the decomposition of $\mathfrak{g}$ over its principal $\mathfrak{sl}_2$-subalgebra.*

Remark 44.18. The Poincaré polynomial $P(q)$ of $(\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}}$ is given by the formula

$$P(z) = \frac{(1+q)^r}{|W|} \int_T \prod_{\alpha \in R} (1 + q\alpha(t))(1 - \alpha(t)).$$

So Theorem 44.17 is equivalent to the statement that this integral equals $\prod_i (1 + q^{2m_i+1})$.

We will prove Theorem 44.17 in the case of type A.

Corollary 44.19. *For $\mathfrak{g} = \mathfrak{sl}_n$ we have $m_i = i$. Equivalently, the same is true for $\mathfrak{g} = \mathfrak{gl}_n$ if we add $m_0 = 0$.*

Proof. Let $\mathfrak{g} = \mathfrak{gl}_n$, $V = \mathbb{C}^n$. We need to compute the Poincare polynomial of $\wedge^\bullet (V \otimes V^*)^{\mathfrak{g}}$. To this end, we will employ **skew-Howe duality**:

Exercise 44.20. Show that $\wedge^\bullet (V \otimes V^*) = \oplus S^\lambda V \otimes S^{\lambda^t} V^*$ where λ^t is the conjugate partition to λ.

Hint. Mimick the proof of the usual Howe duality. Use that for a partition λ of n, if π_λ is the corresponding irreducible representation of S_n then $\pi_\lambda \otimes \text{sign} = \pi_{\lambda^t}$ (you don't have to prove this fact).

Now we get that the Poincaré polynomial in question is

$$P(q) = \sum_{\lambda = \lambda^t} q^{|\lambda|},$$

where the summation is over λ with $\le n$ parts. But there are exactly 2^n such symmetric partitions λ: they consist of a sequence of hooks $(k, 1^{k-1})$ with decreasing values of k. The degree of such a hook is $2k - 1$, which implies that

$$P(z) = (1+q)(1+q^3)...(1+q^{2n-1}).$$

□

Thus we get that the cohomology $H^\bullet(U(n))$ is $\wedge^\bullet(\xi_1, \xi_3, ..., \xi_{2n-1})$ (where subscripts are degrees) and $H^\bullet(SU(n)) = \wedge^\bullet(\xi_3, ..., \xi_{2n-1})$.

45. Lecture 20

45.1. **Cohomology of homogeneous spaces.** Let G be a compact Lie group, $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ and $K \subset G$ a closed subgroup, $\mathfrak{k} = \mathrm{Lie}(K)_{\mathbb{C}}$, and consider the homogeneous space G/K. How to compute the cohomology $H^\bullet(G/K)$?

Since the group G acts on G/K, this cohomology is computed by the complex $\Omega^\bullet(G/K)^G = (\wedge^\bullet(\mathfrak{g}/\mathfrak{k})^*)^K$. Let us denote this complex by $CE^\bullet(\mathfrak{g}, K)$. It is called the **relative Chevalley-Eilenberg complex**.

For example, if $K = \Gamma$ is finite, this is just the Γ-invariant part of the usual Chevalley-Eilenberg complex. But Γ acts trivially on the cohomology, so we get $H^\bullet(G/\Gamma) = H^\bullet(G)$ (as already noted above).

But what happens if $\dim K > 0$? Can we describe the differential in this complex algebraically as we did for $K = 1$?

This is answered by the following proposition. Let $\mathfrak{k} \subset \mathfrak{g}$ be a pair of Lie algebras (over any field, not necessarily finite dimensional). Denote by $CE^i(\mathfrak{g}, \mathfrak{k})$ the spaces $((\wedge^\bullet(\mathfrak{g}/\mathfrak{k}))^*)^{\mathfrak{k}}$.

Proposition 45.1. *$CE^\bullet(\mathfrak{g}, \mathfrak{k})$ is a subcomplex of $CE^\bullet(\mathfrak{g})$.*

Exercise 45.2. Prove Proposition 45.1.

Definition 45.3. The complex $CE^\bullet(\mathfrak{g}, \mathfrak{k})$ is called the **relative Chevalley-Eilenberg complex**, and its cohomology is called the **relative Lie algebra cohomology**, denoted by $H^\bullet(\mathfrak{g}, \mathfrak{k})$.

Now note that, going back to the setting of compact Lie groups, we have $CE^\bullet(\mathfrak{g}, K) = CE^\bullet(\mathfrak{g}, \mathfrak{k})^{K/K^\circ}$, so we have

Corollary 45.4. *$H^\bullet(G/K) \cong H^\bullet(\mathfrak{g}, \mathfrak{k})^{K/K^\circ}$ as algebras.*

Thus, the computation of the cohomology of G/K reduces to the computation of the relative Lie algebra cohomology, which is a again purely algebraic problem.

Corollary 45.5. *Suppose $z \in K$ is an element that acts by -1 on $\mathfrak{g}/\mathfrak{k}$. Then $(\wedge^i(\mathfrak{g}/\mathfrak{k})^*)^K = 0$ for odd i. Hence the differential in $CE^\bullet(\mathfrak{g}, K)$ is 0 and thus $H^\bullet(G/K) \cong (\wedge^\bullet(\mathfrak{g}/\mathfrak{k})^*)^K$, with cohomology present only in even degrees.*

46. Lecture 21

46.1. **Grassmannians.** Let $G = U(m+n), K = U(n) \times U(m)$, so that G/K is the **Grassmannian** $\mathrm{G}_{m+n,n}(\mathbb{C}) \cong \mathrm{G}_{m+n,m}(\mathbb{C})$ (the manifold of m-dimensional or n-dimensional subspaces of $\mathbb{C}^{m+n}$). The element $z = I_n \oplus (-I_m)$ acts by -1 on $\mathfrak{g}/\mathfrak{k} = V \otimes W^* \oplus W \otimes V^*$, where V, W

are the tautological representations of $U(n)$ and $U(m)$. So we get that the Grassmannian has cohomology only in even degrees, and

$$H^{2i}(\mathrm{G}_{m+n,m}(\mathbb{C})) = \wedge^{2i}(V\otimes W^*\oplus W\otimes V^*)^{U(n)\times U(m)}.$$

We can therefore use the skew Howe duality to see that

$$\dim H^{2i}(\mathrm{G}_{m+n,m}(\mathbb{C})) = N_i(n,m),$$

where $N_i(n,m)$ is the number of partitions λ whose Young diagrams has i boxes and fits into the rectangle $m\times n$.

To compute $N_i(m,n)$, consider the generating function

$$f_{n,m}(q)=\sum_i N_i(n,m)q^i.$$

Then, denoting by p_i the jumps of our partition, we have

$$\sum_{n\ge 0} f_{n,m}(q)z^n =$$

$$\sum_{p_0,p_1,\dots,p_m\ge 0} z^{p_0+p_1+\dots+p_m}q^{p_1+2p_2+\dots+mp_m}=\prod_{j=0}^m\frac{1}{1-q^jz}.$$

So the Betti numbers of Grassmannians are the coefficients of this series. For example, if $m=1$ we get

$$\sum_{n\ge 0} f_{n,m}(q)z^n=\frac{1}{(1-z)(1-qz)}=\sum_n(1+q+\dots+q^n)z^n.$$

So we recover the Poincare polynomial $1+q+\dots+q^n$ of the complex projective space $\mathbb{CP}^n$.

The polynomials $f_{n,m}(q)$ are called the **Gaussian binomial coefficients** and they can be computed explicitly. Namely, we have

$$f_{m,n}(q)=\binom{m+n}{n}_q=\binom{m+n}{m}_q=\frac{[m+n]_q!}{[m]_q![n]_q!},$$

where $[m]_q:=\frac{q^m-1}{q-1}$ and $[m]_q!:=[1]_q\dots[m]_q$. In other words, we have the **q-binomial theorem**

$$\sum_{n\ge 0}\binom{m+n}{n}_q z^n=\prod_{j=0}^m\frac{1}{1-q^jz}. \tag{10}$$

Note that setting $q=1$, we get the familiar formula from calculus:

$$\sum_{n\ge 0}\binom{m+n}{m}z^n=\frac{1}{(1-z)^{m+1}}.$$

Exercise 46.1. Compute the Betti numbers of $\mathrm{G}_{N,2}(\mathbb{C})$.

Exercise 46.2. Prove (10).

Hint. Let $F(z)$ be the RHS of this identity. Write a q-difference equation expressing $F(qz)$ in terms of $F(z)$. Show that this equation has a unique solution such that $F(0) = 1$. Then prove that the LHS satisfies the same equation.

46.2. **Schubert cells.** There is actually a more geometric way to obtain the same result. This way is based on decomposing the Grassmannians into **Schubert cells**. Namely, let $F_i \subset \mathbb{C}^{m+n}$ be spanned by the first i basis vectors $e_1, ..., e_i$; thus

$$0 = F_0 \subset F_1 \subset ... \subset F_{m+n} = \mathbb{C}^{m+n}.$$

Given an m-dimensional subspace $V \subset \mathbb{C}^{m+n}$, let ℓ_j be the smallest integer for which $\dim(F_{\ell_j} \cap V) = j$. Then

$$1 \le \ell_1 < \ell_2 < ... < \ell_m \le m + n,$$

which defines a partition with parts

$$\lambda_1 = \ell_m - m, \lambda_2 = \ell_{m-1} - m + 1, ..., \lambda_m = \ell_1 - 1$$

fitting in the $m \times n$ box. Let $S_\lambda \subset \mathrm{G}_{m+n,m}(\mathbb{C})$ be the set of V giving such numbers λ_i.

Exercise 46.3. Show that S_λ is an embedded (non-closed) complex submanifold of the Grassmannian isomorphic to the affine space $\mathbb{C}^{|\lambda|}$ of dimension $|\lambda| = \sum_i \lambda_i$.

Hint. Show that for $V \in S_\lambda$, the elements $f_k := e^*_{\ell_k}|_V$ form a basis of V^*. For $\ell_j + 1 \le i \le \ell_{j+1}$ (with $\ell_{m+1} := m + n$), show that $e^*_i|_V$ is a linear combination of f_k, $j+1 \le k \le m$, and denote the corresponding coefficients by $a_{ik}(V)$. Show that the assignment $V \mapsto (a_{ik}(V))$ is an isomorphism $S_\lambda \cong \mathbb{C}^{|\lambda|}$.

Definition 46.4. The subset S_λ of the Grassmannian is called the **Schubert cell** corresponding to λ.

So we see that $\mathrm{G}_{m+n,m}(\mathbb{C})$ has a **cell decomposition** into a disjoint union of Schubert cells.

Now we can rederive the same formula for the Poincaré polynomial of the Grassmannian from the following fact from algebraic topology:

Proposition 46.5. *If X is a connected cell complex which only has even-dimensional cells, then the cohomology of X vanishes in odd degrees, and the groups $H^{2i}(X, \mathbb{Z})$ are free abelian groups of ranks $b_{2i}(X)$, where the Betti number $b_{2i}(X)$ is just the number of cells in X of dimension i. Moreover, X is simply connected.*

So we obtain an even stronger statement than before:

Corollary 46.6. *$H^{2i}(\mathrm{G}_{m+n,n}(\mathbb{C}),\mathbb{Z})$ are free abelian groups of ranks given by coefficients of $\binom{m+n}{m}_q$, and the odd cohomology groups are zero. Moreover, Grassmannians are simply connected.*

In particular, this gives Betti numbers over any field (including positive characteristric), not just $\mathbb{C}$.

46.3. **Flag manifolds.** The **flag manifold** $\mathcal{F}_n(\mathbb{C})$ is the space of all **complete flags** $0 = V_0 \subset V_1 \subset ... \subset V_n = \mathbb{C}^n$, where $\dim V_i = i$. Note that the flag manifold is a homogeneous space: $\mathcal{F}_n = G/T$, where $G = U(n)$ and $T = U(1)^n$ is a maximal torus in G.

We have fibrations $\pi : \mathcal{F}_n(\mathbb{C}) \to \mathbb{CP}^{n-1}$ sending $(V_1, ..., V_{n-1})$ to V_{n-1}, whose fiber is the space of flags in V_{n-1}, i.e., $\mathcal{F}_{n-1}(\mathbb{C})$. This shows, by induction, that flag manifolds can be decomposed into even-dimensional cells isomorphic to $\mathbb{C}^r$ (also called Schubert cells). Thus the Betti numbers of $\mathcal{F}_n$ vanish in odd degrees, and in even degrees are given by the generating function

$$\sum b_{2i}(\mathcal{F}_n)q^n = [n]_q! = (1+q)(1+q+q^2)...(1+q+...+q^{n-1}).$$

Remark 46.7. We have a map $\pi_m : \mathcal{F}_{m+n}(\mathbb{C}) \to \mathrm{G}_{m+n,m}(\mathbb{C})$ sending $(V_1, ..., V_{m+n-1})$ to V_m. This is a fibration with fiber $\mathcal{F}_m(\mathbb{C}) \times \mathcal{F}_n(\mathbb{C})$. Thus we get another proof of the formula for Betti numbers of the Grassmannian.

We can also define the **partial flag manifold** $\mathcal{F}_S(\mathbb{C})$, where $S \subset [1, n-1]$ is a subset, namely the space of **partial flags** $(V_s, s \in S)$, $V_s \subset \mathbb{C}^n$, $\dim V_s = s$, $V_s \subset V_t$ if $s < t$.

Exercise 46.8. Let $S = \{n_1, n_1+n_2, ..., n_1+...+n_{k-1}\}$, and $n_k = n - n_1 - ... - n_{k-1}$. Show that the even Betti numbers of the partial flag manifold are the coefficients of the polynomial

$$P_S(q) := \frac{[n]_q!}{[n_1]_q!...[n_k]_q!}$$

(and the odd Betti numbers vanish). Show that the partial flag manifold is simply connected.

47. Lecture 22

47.1. **Cohomology of Lie algebras with coefficients.** The definition of cohomology of Lie algebras may be generalized to define the cohomology with coefficients in a module, so that the cohomology considered above is the one for the trivial module.

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. The **Chevalley-Eilenberg (or standard) complex of $\mathfrak{g}$ with coefficients in V** is the defined by

$$CE^\bullet(\mathfrak{g}, V) := \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, V)$$

with differential defined by the full Cartan formula (without dropping the first term):

$$d\omega(a_0, ..., a_m) = \sum_i (-1)^i a_i \omega(a_0, ..., \widehat{a}_i, ..., a_m) +$$

$$\sum_{i<j} (-1)^{i+j} \omega([a_i, a_j], a_0, ..., \widehat{a}_i, ..., \widehat{a}_j, ..., a_m).$$

The cohomology of this complex is called the **cohomology of $\mathfrak{g}$ with coefficients in V** and denoted $H^\bullet(\mathfrak{g}, V)$. Note that the previously defined cohomology $H^\bullet(\mathfrak{g})$ is $H^\bullet(\mathfrak{g}, \mathbb{C})$.

If $\mathfrak{g}$ is the Lie algebra of a Lie group G (or its complexification) and V is finite dimensional, then we simply have $CE^\bullet(\mathfrak{g}, V) := (\Omega^\bullet(G) \otimes V)^{\mathfrak{g}}$ (and the differential is just the de Rham differential). So in particular we have

Proposition 47.1. *(i) If G is compact and V is a nontrivial irreducible representation then*

$$H^i(\mathfrak{g}, V) = 0, \ i > 0.$$

In particular, this is so for any non-trivial irreducible finite dimensional reporesentation V of a semisimple Lie algebra $\mathfrak{g}$.

(ii) (Whitehead's lemma) For semisimple $\mathfrak{g}$ and any finite dimensional V we have $H^1(\mathfrak{g}, V) = H^2(\mathfrak{g}, V) = 0$.

However, this cohomology is non-trivial in general if $\mathfrak{g}$ is not semisimple or V is infinite dimensional.

Let us explore the meaning of $H^i(\mathfrak{g}, V)$ for small i.

1. We have $H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$, the $\mathfrak{g}$-invariants in V.

2. $H^1(\mathfrak{g}, V)$ is the quotient of the space $Z^1(\mathfrak{g}, V)$ of 1-cocycles $\omega : \mathfrak{g} \to V$, i.e., linear maps satisfying

$$\omega([x, y]) = x\omega(y) - y\omega(x)$$

by the space of 1-coboundaries $B^1(\mathfrak{g}, V)$, of the form $\omega(x) = xv$ for some $v \in V$.

Proposition 47.2. *(i) If V, W are representations of $\mathfrak{g}$ then* $\mathrm{Ext}^1(V, W) = H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(V, W))$.

(ii) Consider the action of the additive group of V on the Lie algebra $\mathfrak{g} \ltimes V$ (with trivial commutator on V) by

$$v \circ (x, w) = (x, w + xv).$$

Then $H^1(\mathfrak{g}, V)$ classifies Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{g} \ltimes V$ of the form $x \mapsto (x, \omega(x))$ modulo this action.

Proof. (i) Suppose the space $W \oplus V$ is equipped with the action of $\mathfrak{g}$ so that W is a submodule and V the quotient. Thus the action of $\mathfrak{g}$ on $W \oplus V$ is given by

$$\rho(x) = \begin{pmatrix} \rho_W(x) & \omega(x) \\ 0 & \rho_V(x) \end{pmatrix},$$

where $\omega : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{k}}(V, W)$. So the identity $\rho([x, y]) = [\rho(x), \rho(y)]$ translates into

$$\omega([x, y]) = \rho_W(x)\omega(y) - \omega(x)\rho_V(y),$$

i.e., $\rho \in Z^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(V, W))$. Also it is easy to check that for two such representations ρ_1, ρ_2 there is an isomorphism $\rho_1 \to \rho_2$ acting trivially on W and V/W if and only if the corresponding maps ω_1, ω_2 differ by a coboundary: $\omega_1 - \omega_2 \in B^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(V, W))$. This implies the statement.

(ii) We leave this to the reader as an exercise. □

3. $Z^1(\mathfrak{g}, \mathfrak{g})$ is the Lie algebra of derivations of $\mathfrak{g}$, and $B^1(\mathfrak{g}, \mathfrak{g})$ is the ideal of inner derivation. So $H^1(\mathfrak{g}, \mathfrak{g})$ is the Lie algebra of **outer derivations**, the quotient of all derivations by inner derivations.

4. Suppose we want to define an **abelian extension** $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ by V, i.e., a Lie algebra which can be included in the short exact sequence

$$0 \to V \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

where V is an abelian ideal. To classify such extension, pick a vector space splitting $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus V$, then the commutator looks like

$$[(x, v), (y, w)] = ([x, y], xw - yv + \omega(x, y)),$$

where $\omega : \wedge^{\mathfrak{g}} \to V$ is a linear map. The Jacobi identity is then equivalent to ω being in the space $Z^2(\mathfrak{g}, V)$ of 2-cocycles. Moreover, it is easy to check that for two such extensions $\widetilde{\mathfrak{g}}_1, \widetilde{\mathfrak{g}}_2$ there is an isomorphism $\phi : \widetilde{\mathfrak{g}}_1 \to \widetilde{\mathfrak{g}}_2$ which acts trivially on V and $\mathfrak{g}$ if and only if the corresponding cocycles ω_1, ω_2 differ by a coboundary: $\omega_1 - \omega_2 \in B^2(\mathfrak{g}, V)$. Thus, we get

Proposition 47.3. *Abelian extensions of $\mathfrak{g}$ by V modulo isomorphisms which act trivially on $\mathfrak{g}$ and V are classified by $H^2(\mathfrak{g}, V)$. For example, the space $H^2(\mathfrak{g}, \mathbb{C})$ classifies* 1**-dimensional central extensions** *of $\mathfrak{g}$:*

$$0 \to \mathbb{C} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

Example 47.4. Let $\mathfrak{g} = \mathbb{C}^2$ be the 2-dimensional abelian Lie algebra. Then we have seen that the Poincaré polynomial of the cohomology of $\mathfrak{g}$ is $1 + 2q + q^2$ (cohomology of the 2-torus). So $H^2(\mathfrak{g}, \mathbb{C}) = \mathbb{C}$. The only cocycle up to scaling is given by $\omega(x, y) = 1$, where x, y is a basis of $\mathfrak{g}$, and all coboundaries are zero. So we have a central extension of $\mathfrak{g}$ defined by this cocycle with basis x, y, c and $[x, y] = c$, $[x, c] = [y, c] = 0$. This is the **Heisenberg Lie algebra**, which is isomorphic to the Lie algebra of strictly upper-triangular 3 by 3 matrices.

5. Let us now study deformations of Lie algebras. Suppose $\mathfrak{g}$ is a Lie algebra and we want to deform the bracket, with deformation parameter t. So the new bracket will be

$$[x, y]_t = [x, y] + tc_1(x, y) + t^2c_2(x, y) + ...,$$

where $c_i : \wedge^2\mathfrak{g} \to \mathfrak{g}$ are linear maps. This bracket should satisfy the Jacobi identity, i.e., define a new Lie algebra structure on $\mathfrak{g}[[t]]$. Such deformations are distinguished up to linear isomorphisms

$$a = 1 + ta_1 + t^2a_2 + ...$$

where $a_i \in \mathrm{End}_{\mathbf{k}}(\mathfrak{g})$.

In particular, in first order, i.e., modulo t^2, we get a new Lie algebra structure on $\mathfrak{g}[t]/t^2\mathfrak{g}[t] = \mathfrak{g} \oplus t\mathfrak{g}$ such that this Lie algebra can be included in the short exact sequence

$$0 \to t\mathfrak{g} \to \mathfrak{g} \oplus t\mathfrak{g} \to \mathfrak{g} \to 0$$

where $t\mathfrak{g}$ is an abelian ideal. So this is an abelian extension of $\mathfrak{g}$ by $t\mathfrak{g}$, and we know that such extensions are classified by $H^2(\mathfrak{g}, \mathfrak{g})$. So we conclude that we have

Proposition 47.5. *First-order deformations of $\mathfrak{g}$ as a Lie algebra are classified by $H^2(\mathfrak{g}, \mathfrak{g})$.*

Thus if $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, every deformation is isomorphic to the trivial one, with $c_1 = c_2 = ... = 0$. Indeed, applying automorphisms $a = 1 + ta_1 + t^2a_2 + ...$, we can kill successively c_1, then c_2, then c_3, and so on. Thus we have

Corollary 47.6. *If $\mathfrak{g}$ is semisimple then it is rigid, i.e., has no non-trivial Lie algebra deformations.*

Example 47.7. Let $\mathfrak{g}$ be the 2-dimensional abelian Lie algebra. Then $H^2(\mathfrak{g}, \mathfrak{g}) = \mathbb{C}^2$, and we get a 2-parameter family of deformations with bracket $[x, y] = tx + sy$. These, however, turn out to be all equivalent (for $(t, s) \neq (0, 0)$) under the action of $GL_2(\mathbb{C})$: they are all isomorphic to the Lie algebra with basis x, y and commutator $[x, y] = y$.

However, not all first order deformations of a Lie algebra lift to second order, i.e., modulo t^3. Namely, The Jacobi identity in the second order tells us that $dc_2 = [c_1, c_1]$, where $[c_1, c_1]$ is the **Schouten bracket** of c_1 with itself, a certain quadratic expression in c_1. This expression is automatically a cocycle, but we need it to be a coboundary. So the cohomology class of $[c_1, c_1]$ in $H^3(\mathfrak{g}, \mathfrak{g})$ is an obstruction to lifting the deformation modulo t^3. Thus the space $H^3(\mathfrak{g}, \mathfrak{g})$ is the home for **obstructions to deformations**.

6. In a similar way we can study deformations $V[[t]]$ of a module V over $\mathfrak{g}$:

$$\rho_t(x) = \rho(x) + t\rho_1(x) + t^2\rho_2(x) + ...$$

Modulo t^2 we get a $\mathfrak{g}$-module structure on $V[t]/t^2V[t] = V \oplus tV$ such that we have a short exact sequence

$$0 \to tV \to V \oplus tV \to V \to 0.$$

Thus first order deformations of V are classified by $\mathrm{Ext}^1_{\mathfrak{g}}(V, V) = H^1(\mathfrak{g}, \mathrm{End}_{\mathbf{k}} V)$. Again, lifting of this deformation modulo t^3 is not automatic, and we get an obstruction in $\mathrm{Ext}^2_{\mathfrak{g}}(V, V) = H^2(\mathfrak{g}, \mathrm{End}_{\mathbf{k}}(V))$.

47.2. **Levi decomposition.**

Theorem 47.8. *(Levi decomposition) Over real or complex numbers we have* $\mathfrak{g} \cong \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}$*, where* $\mathfrak{g}_{\mathrm{ss}} \subset \mathfrak{g}$ *is a semisimple subalgebra (but not necessarily an ideal); i.e.,* $\mathfrak{g}$ *is isomorphic to the semidirect product* $\mathfrak{g}_{\mathrm{ss}} \ltimes \mathrm{rad}(\mathfrak{g})$*. In other words, the projection* $p : \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}}$ *admits an (in general, non-unique) splitting* $q : \mathfrak{g}_{\mathrm{ss}} \to \mathfrak{g}$*, i.e., a Lie algebra map such that* $p \circ q = \mathrm{Id}$.

Proof. We can write $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathrm{rad}(\mathfrak{g})$ as a vector space. Then the commutator looks like

$$[(a, x), (b, y)] = ([x, b]-[y, a]+[a, b]+\omega(x, y), [x, y]) \ x, y \in \mathfrak{g}_{ss}, a, b \in \mathrm{rad}(\mathfrak{g}).$$

Let $\mathrm{rad}(\mathfrak{g}) = D^0 \supset D^1 \supset ...$ be the upper central series of $\mathrm{rad}(\mathfrak{g})$, i.e., $D^{i+1} = [D^i, D^i]$. Suppose $D^n \neq 0$ but $D^{n+1} = 0$ (so D^n is an abelian ideal). Using induction in dimension of $\mathfrak{g}$ and replacing $\mathfrak{g}$ by $\mathfrak{g}/D^n$, we may assume that $\omega(x, y) \in D^n$. But then $\omega \in Z^2(\mathfrak{g}_{ss}, D^n)$, which equals $B^2(\mathfrak{g}_{ss}, D^n)$ by Whitehead's lemma, i.e., $\omega = d\eta$. Using η, we can modify the splitting $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathrm{rad}(\mathfrak{g})$ to make sure that $\omega = 0$. This implies the statement. ∎

47.3. **The nilradical.** Consider now a solvable Lie algebra $\mathfrak{a}$ and its adjoint representation. By Lie's theorem, in some basis $\mathfrak{a}$ acts in this representation by upper triangular matrices. Let $\mathfrak{n} \subset \mathfrak{a}$ be the subset of nilpotent elements (the **nilradical** of $\mathfrak{a}$). Thus $\mathfrak{n}$ is the set of $x \in \mathfrak{a}$ that act in this basis by strictly upper triangular matrices. In particular, $\mathfrak{n} \supset [\mathfrak{a}, \mathfrak{a}]$, so $\mathfrak{a}/\mathfrak{n}$ is abelian. Moreover each diagonal entry of these upper triangular matrices gives rise to a character $\lambda \in (\mathfrak{a}/\mathfrak{n})^*$. By definition of $\mathfrak{n}$, these characters form a spanning set in $(\mathfrak{a}/\mathfrak{n})^*$.

Proposition 47.9. *If $d : \mathfrak{a} \to \mathfrak{a}$ is a derivation then $d(\mathfrak{a}) \subset \mathfrak{n}$.*

Proof. Since there are finitely many characters λ, for each of them we have $e^{td}\lambda = \lambda$. It follows that d acts on $\mathfrak{a}/\mathfrak{n}$ trivially. □

Thus if $\mathfrak{a} = \mathrm{rad}(\mathfrak{g})$ is the radical of $\mathfrak{g}$ then $\mathfrak{g}$ acts trivially on $\mathfrak{a}/\mathfrak{n}$.

47.4. **Exponentiating nilpotent and solvable Lie algebras and the third Lie theorem.** Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Theorem 47.10. *There is a simply connected Lie group G over $\mathbb{K}$ with $\mathrm{Lie}(G) = \mathfrak{g}$, such that the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Moreover, if $\mathfrak{g}$ is nilpotent and if we identify G with $\mathfrak{g}$ under this diffeomorphism then the multiplication map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is polynomial.*

Example 47.11. Let $\mathfrak{g}$ be the Heisenberg Lie algebra, i.e. the Lie algebra of strictly upper triangular matrices. Then the multiplication map has the form

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + \tfrac{1}{2}(xy' - x'y)).$$

Proof. The proof is by induction in $\dim \mathfrak{g}$. Namely, fix a nonzero homomorphism $\chi : \mathfrak{g} \to \mathbb{K}$ (which exists since $\mathfrak{g}$ is solvable), and let $\mathfrak{g}_0 = \mathrm{Ker}\chi$. Then we have $\mathfrak{g} = \mathbb{K}d \ltimes \mathfrak{g}_0$, the semidirect product, where $d \in \mathfrak{g}$ acts as a derivation on $\mathfrak{g}_0$. Let G_0 be the simply connected Lie group corresponding to $\mathfrak{g}_0$, which is defined by the induction assumption, and $\exp : \mathfrak{g}_0 \to G_0$ is a diffeomorphism. So we have a regular multiplication map $P : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$, and we have a 1-parameter group of automorphisms $e^{td} : \mathfrak{g}_0 \to \mathfrak{g}_0$. Thus we can define a group structure on $\mathfrak{g}$ by the formula

$$(x, t) * (y, s) = (P(x, e^{td}(y)), t + s), \ x, y \in \mathfrak{g}_0, \ t, s \in \mathbb{K}.$$

It is not difficult to check that the exponential map for this group is a diffeomorphism, and the multiplication written using this identification is given by a polynomial if $\mathfrak{g}$ is nilpotent. Namely,

$$\exp(td + x) = (td, x_t),$$

where

$$x_t = \sum_{n \geq 1} \frac{t^{n-1} d^{n-1}(x)}{n!} = \frac{e^u - 1}{u}|_{u=td}(x).$$

□

Definition 47.12. The simply connected Lie group whose Lie algebra is nilpotent is called **unipotent**.[13]

Corollary 47.13. *(Third Lie theorem) For any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$ there is a simply connected Lie group G with* $\mathrm{Lie}(G) = \mathfrak{g}$.

Proof. By the previous theorem, we have such a group A for $\mathfrak{a} = \mathrm{rad}(\mathfrak{g})$. Moreover, by the Levi decomposition, the simply connected semisimple group G_{ss} corresponding to $\mathfrak{g}_{ss}$ acts on $\mathrm{rad}(\mathfrak{g})$. Hence by the second Lie theorem, G_{ss} acts on A, and the simply connected Lie group $G_{ss} \ltimes A$ has the Lie algebra $\mathfrak{g}_{ss} \ltimes \mathrm{rad}(\mathfrak{g}) = \mathfrak{g}$. □

Corollary 47.14. *A simply connected complex Lie group G is of the form $G_{ss} \ltimes A$, where A is solvable simply connected, hence diffeomorphic to $\mathbb{C}^n$. Thus G has the homotopy type of G^c_{ss}.*

48. Lecture 23

48.1. **Algebraic Lie algebras.** Let us say that a finite dimensional complex Lie algebra $\mathfrak{g}$ is **algebraic** if $\mathfrak{g}$ is the Lie algebra of a group $G = K \ltimes N$, where K is a reductive group and N a unipotent group.

Example 48.1. Let $\mathfrak{g}_1$ be a 3-dimensional Lie algebra with basis d, x, y and $[x, y] = 0$, $[d, x] = x$, $[d, y] = \sqrt{2}y$. Similarly, let $\mathfrak{g}_2$ have basis d, x, y with $[x, y] = 0$, $[d, x] = x$, $[d, y] = y + x$. Then $\mathfrak{g}_1, \mathfrak{g}_2$ are not algebraic (check it!).

Proposition 48.2. *Any finite dimensional complex Lie algebra is a Lie subalgebra of an algebraic one.*

Proof. Let us say that $\mathfrak{g}$ is n**-algebraic** if it is the Lie algebra of a group $G := K \ltimes A$, where K is reductive and $\mathfrak{a} = \mathrm{Lie}(A)$ is solvable with $\dim(\mathfrak{a}/\mathfrak{n}) \le n$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{a}$. Thus 0-algebraic is the same as algebraic. Note that for any $\mathfrak{g}$ we have the Levi decomposition $\mathfrak{g} = \mathfrak{g}_{ss} \ltimes \mathfrak{a}$, where $\mathfrak{a} = \mathrm{rad}(\mathfrak{g})$, which shows that any $\mathfrak{g}$ is n-algebraic for some n. So it suffices to show that any n-algebraic Lie algebra for $n > 0$ embeds into an $n-1$-algebraic one.

To this end, let $\mathfrak{g} = \mathrm{Lie}(G)$ be n-algebraic, with $G = K \ltimes A$ and A simply connected. Let $\mathfrak{a} = \mathrm{Lie}(A)$, so $\dim(\mathfrak{a}/\mathfrak{n}) = n$. Pick $d \in \mathfrak{a}$, $d \notin \mathfrak{n}$ such that d is K-invariant. This can be done since K acts trivially on $\mathfrak{a}/\mathfrak{n}$ and its representations are completely reducible. We have a

[13]The reason for this terminology is that these groups act by unipotent operators on the adjoint representation.

decomposition $\mathfrak{a} = \oplus_{i=1}^r \mathfrak{a}[\beta_i]$ of $\mathfrak{a}$ into generalized eigenspaces of d. It is clear that K preserves each $\mathfrak{a}[\beta_i]$. Pick a character $\chi : \mathfrak{a} \to \mathbb{C}$ such that $\chi(d) = 1$.

Consider the subgroup Γ of $\mathbb{C}$ generated by β_i and let $\alpha_1, ..., \alpha_m$ be a basis of Γ, so that $\beta_i = \sum_j b_{ij}\alpha_j$ for $b_{ij} \in \mathbb{Z}$. Let $T = (\mathbb{C}^\times)^m$ and make T act on G so that it commutes with K and acts on $\mathfrak{a}[\beta_i]$ by $(z_1, ..., z_m) \mapsto \prod_j z_j^{\beta_{ij}}$. Now consider the group $\widetilde{G} := (K \times T) \ltimes A$. Let $\mathfrak{a}' \subset \mathrm{Lie}(T) \ltimes \mathfrak{a} \subset \mathrm{Lie}(\widetilde{G})$ be spanned by $\mathrm{Ker}\chi$ and $d - \alpha$ where $\alpha = (\alpha_1, ..., \alpha_m) \in \mathrm{Lie}(T)$. Then the nilradical $\mathfrak{n}'$ of $\mathfrak{a}'$ is spanned by $\mathfrak{n}$ and $d - \alpha$ (as the latter is nilpotent). Moreover, if A' is the simply connected group corresponding to $\mathfrak{a}'$, then $(K \times T) \ltimes A \cong (K \ltimes T) \ltimes A'$ Thus, the Lie algebra $\widetilde{\mathfrak{g}} := \mathrm{Lie}(\widetilde{G})$ is $n-1$-algebraic (as $\dim(\mathfrak{a}'/\mathfrak{n}') = n - 1$), and it contains $\mathfrak{g}$, as claimed. ☐

Example 48.3. The Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ in the previous example are 1-algebraic.

To embed $\mathfrak{g}_1$ into an algebraic Lie algebra, add element δ with $[\delta, x] = 0$, $[\delta, y] = y$, $[\delta, d] = 0$. Then the Lie algebra $\mathfrak{g}_1'$ spanned by δ, d, x, y is $\mathfrak{b} \oplus \mathfrak{b}$, where $\mathfrak{b}$ is the non-abelian 2-dimensional Lie algebra (so it is algebraic). Namely, the first copy of $\mathfrak{b}$ is spanned by δ, y and the second by $d - \sqrt{2}\delta, x$.

To embed $\mathfrak{g}_2$ into an algebraic Lie algebra, add element δ with $[\delta, x] = 0$, $[\delta, y] = x$, $[\delta, d] = 0$. Then the Lie algebra $\mathfrak{g}_2'$ spanned by δ, d, x, y is $\mathbb{C} \ltimes \mathcal{H}$, where $\mathcal{H}$ is the 3-dimensional Heisenberg Lie algebra with basis δ, x, y, and $\mathbb{C}$ is spanned by $d - \delta$ (so it is algebraic, as $d - \delta$ acts diagonalizably with integer eigenvalues).

48.2. **Faithful representations of nilpotent Lie algebras.** Let $\mathfrak{n}$ be a finite dimensional nilpotent Lie algebra over $\mathbb{C}$. In this subsection we will show that $\mathfrak{n}$ has a finite dimensional faithful representation.

To this end, recall that $\mathfrak{n} = \mathrm{Lie}(N)$ where N is a simply connected Lie group, and the exponential map $\exp : \mathfrak{n} \to N$ is bijective. Moreover, the multiplication law of N, when rewritten on $\mathfrak{n}$ using the exponential map, is given by polynomials.

Proposition 48.4. *Let $\mathcal{O}(N)$ be the space of polynomial functions on $N \cong \mathfrak{n}$ (identified using the exponential map). Then $\mathcal{O}(N)$ is invariant under the action of $\mathfrak{n}$ by left-invariant vector fields. Moreover, we have a canonical filtration $\mathcal{O}(N) = \cup_{n\geq 1} V_n$, where $V_n \subset \mathcal{O}(N)$ are finite dimensional subspaces such that $V_1 \subset V_2 \subset ...$ and $\mathfrak{n}V_n \subset V_{n-1}$.*

Proof. Let $P : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$ be the polynomial multiplication law. Let $x \in \mathfrak{n}$ and L_x the corresponding left-invariant vector field. Let $f \in$

$\mathcal{O}(N) = S\mathfrak{n}^*$. Then for $y \in \mathfrak{n}$ we have

$$(L_x f)(y) = \frac{d}{dt}|_{t=0} f(P(y, tx)).$$

Since f and P are polynomials, this is clearly a polynomial in y. Thus $L_x : \mathcal{O}(N) \to \mathcal{O}(N)$.

We have a lower central series filtration on $\mathfrak{n}$:

$$\mathfrak{n} = D_0(\mathfrak{n}) \supset [\mathfrak{n}, \mathfrak{n}] = D_1(\mathfrak{n}) \supset ... \supset D_m(\mathfrak{n}) = 0.$$

This gives an ascending filtration

$$0 = D_0(\mathfrak{n})^\perp \subset .... \subset D_m(\mathfrak{n})^\perp = \mathfrak{n}^*.$$

We assign to $D_j(\mathfrak{n})^\perp$ filtration degree d^j, where d is a sufficiently large positive integer. This gives rise to an ascending filtration $F^\bullet$ on $S\mathfrak{n}^* = \mathcal{O}(N)$. Note that

$$P(x, y) = x + y + \sum_{i \geq 1} Q_i(x, y),$$

where $Q_i : \mathfrak{n} \times \mathfrak{n} \to [\mathfrak{n}, \mathfrak{n}]$ has degree i in x. Thus

$$(L_x f)(y) = (\partial_x f)(y) + (\partial_{Q_1(x,y)} f)(y).$$

The first term clearly lowers the degree, and so does the second one if d is large enough. So we may take $V_n = F_n(S\mathfrak{n}^*)$ to be the space of polynomials of degree $\leq n$, then $L_x V_n \subset V_{n-1}$, as claimed. □

Example 48.5. We illustrate this proof on the example of the Heisenberg algebra $\mathcal{H} = \langle x, y, c \rangle$ with $[x, y] = c$ and $[x, c] = [y, c] = 0$. In this case

$$e^{tx} e^{sy} = e^{tx+sy+\frac{1}{2}tsc},$$

so writing $u = px + qy + rc \in \mathcal{H}$, we get

$$P(p_1, q_1, r_1, p_2, q_2, r_2) = (p_1 + p_2, q_1 + q_2, r_1 + r_2 + \tfrac{1}{2}(p_1 q_2 - p_2 q_1)).$$

Thus

$$L_c = \partial_r,\ L_x = \partial_p - \tfrac{1}{2} q \partial_r,\ L_y = \partial_q + \tfrac{1}{2} p \partial_r.$$

We have $D_1(\mathcal{H}) = \mathbb{C}c$, so $D_1(\mathcal{H})^\perp$ is spanned by p, q. Thus we have $\deg(p) = \deg(q) = d$, $\deg(r) = d^2$. So for any $d > 1$, L_c, L_x, L_y lower the degree. So setting $V_n = F_n(S\mathcal{H}^*)$ to be the (finite dimensional) space of polynomials of degree $\leq n$, we see that L_c, L_x, L_y map V_n to V_{n-1}.

Corollary 48.6. *Every finite dimensional nilpotent Lie algebra $\mathfrak{n}$ over $\mathbb{C}$ has a faithful finite dimensional representation where all its elements act by nilpotent operators. Thus $\mathfrak{n}$ is isomorphic to a subalgebra of the Lie algebra of strictly upper triangular matrices of some size.*

Proof. By definition, $\mathcal{O}(N)$ is a faithful $\mathfrak{n}$-module. Hence so is V_n for some n. □

48.3. **Faithful representations of general finite dimensional Lie algebras.**

Theorem 48.7. *(Ado's theorem) Every finite dimensional Lie algebra over $\mathbb{C}$ has a finite dimensional faithful representation.*

Proof. Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. Since $\mathfrak{g}$ can be embedded into an algebraic Lie algebra, we may assume without loss of generality that $\mathfrak{g}$ is algebraic. Thus $\mathfrak{g} = \mathrm{Lie}(G)$ where $G = K \ltimes N$ for reductive K and unipotent N. Also we may assume that $\mathfrak{g} \neq \mathfrak{g}' \oplus \mathfrak{g}''$ for $\mathfrak{g}', \mathfrak{g}'' \neq 0$, otherwise the problem reduces to a smaller algebraic Lie algebra (indeed if V', V'' are faithful representations of $\mathfrak{g}', \mathfrak{g}''$ then $V' \oplus V''$ is a faithful representation of $\mathfrak{g}' \oplus \mathfrak{g}''$). Then $\mathfrak{k} = \mathrm{Lie}(K)$ acts faithfully on $\mathfrak{n} = \mathrm{Lie}(N)$. Now, $\mathfrak{g}$ acts on $\mathcal{O}(N)$ preserving the subspaces V_n ($\mathfrak{n} = \mathrm{Lie}(N)$ acts by left invariant vector fields and $\mathfrak{k}$ by adjoint action).

As we have shown, $\mathfrak{n}$ acts faithfully on V_n for some n. We claim that this V_n is, in fact, a faithful representation of the whole $\mathfrak{g}$, which implies the theorem. Indeed, let $\mathfrak{a} \subset \mathfrak{g}$ be the ideal of elements acting by zero on V_n, and let $\overline{\mathfrak{a}}$ be the projection of $\mathfrak{a}$ to $\mathfrak{k}$ (an ideal in $\mathfrak{k}$). Since $\mathfrak{n}$ acts faithfully on V_n, we have $\mathfrak{a} \cap \mathfrak{n} = 0$. Given $a \in \mathfrak{a}$, we have $a = \overline{a} + b$ where $\overline{a} \in \overline{\mathfrak{a}}$ is the projection of a and $b \in \mathfrak{n}$. For $x \in \mathfrak{n}$ we have $[a, x] \in \mathfrak{a} \cap \mathfrak{n} = 0$. Thus $[\overline{a}, x] = -[b, x]$. Hence the operator $x \mapsto [\overline{a}, x]$ on $\mathfrak{n}$ is nilpotent. So $\overline{\mathfrak{a}}$ acts on $\mathfrak{n}$ by nilpotent operators. Since K is reductive and $\overline{\mathfrak{a}} \subset \mathfrak{k}$ is an ideal, this means that $\overline{\mathfrak{a}}$ acts on $\mathfrak{n}$ by zero. Thus $\overline{\mathfrak{a}} = 0$ and $\mathfrak{a} \subset \mathfrak{n}$. Hence $\mathfrak{a} = 0$. □

49. Lecture 24

49.1. **Borel subgroups and subalgebras.** Let G be a connected complex reductive Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$. Fix a Cartan subalegbra $\mathfrak{h} \subset \mathfrak{g}$ with a system of positive roots Π, and consider the corresponding triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+$ is spanned by positive root elements and $\mathfrak{n}_-$ by negative root elements. Let H be the maximal torus in G corresponding to $\mathfrak{h}$, N_+ the unipotent subgroup of G corresponding to $\mathfrak{n}_+$, and $B_+ = HN_+$ the solvable subgroup with $\mathrm{Lie}(B_+) = \mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$; these are all closed Lie subgroups.

Definition 49.1. A **Borel subalgebra** of $\mathfrak{g}$ is a Lie subalgebra conjugate to $\mathfrak{b}_+$. A **Borel subgroup** of G is a Lie subgroup conjugate to B_+. A **parabolic subalgebra (subgroup)** is a Lie subalgebra of $\mathfrak{g}$ (subgroup of G) containing a Borel subalgebra (subgroup).

Since all pairs $(\mathfrak{h}, \Pi)$ are conjugate, this definition does not depend on the choice of $(\mathfrak{h}, \Pi)$.

Lemma 49.2. *B_+ is its own normalizer in G.*

Proof. Let $\gamma \in G$ be such that $\mathrm{Ad}\gamma(B_+) = B_+$. Let $H' = \mathrm{Ad}\gamma(H) \subset B_+$. It is easy to show that we can conjugate H' back into H inside B_+, so we may assume without loss of generality that $H' = H$. Then $\gamma \in N(H)$, and it preserves positive roots. Hence the image of γ in W is 1, so $\gamma \in H$, as claimed. $\square$

49.2. **Flag manifold.** Thus the set of all Borel subalgebras (or subgroups) in G is the homogeneous space G/B_+, a complex manifold. It is called the **flag manifold** of G. Note that it only depends on the semisimple part $\mathfrak{g}_{ss} \subset \mathfrak{g}$.

Let $G^c \subset G$ be the compact form of G, with Lie algebra $\mathfrak{g}_c \subset \mathfrak{g}$. It is easy to see that $\mathfrak{g}_c + \mathfrak{b}_+ = \mathfrak{g}$. Thus the G^c-orbit $G^c \cdot 1$ of $1 \in G/B_+$ contains a neighborhood of 1 in G/B_+. Hence the same holds for any point of this orbit, i.e., $G^c \cdot 1 \subset G/B_+$ is a submanifold of the same dimension, i.e., an open subset. But it is also compact, since G^c is compact, hence closed. As G/B_+ is connected, we get that $G^c \cdot 1 = G/B_+$, i.e., G^c acts transitively on G/B_+.

Also the Cartan involution maps positive roots to negative ones, so $G^c \cap B_+ \subset w_0(B_+) \cap B_+ = H$. Thus $G^c \cap B_+ = H^c$, a maximal torus in G_c. So we get

Proposition 49.3. *We have $G/B_+ = G^c/H^c$. In particular, G/B_+ is a compact complex manifold of dimension $|R_+| = \frac{1}{2}(\dim \mathfrak{g} - \mathrm{rank}\mathfrak{g})$.*

Let $A = \exp(i\mathfrak{h}^c) \subset H$, $K = G^c$, $N = N_+$.

Corollary 49.4. *(The* **Iwasawa decomposition** *of G) The multiplication map $K \times A \times N \to G$ is a diffeomorphism. In particular, we have $G = KAN$.*

Example 49.5. 1. For $G = SL_2$ we have $G/B_+ = SU(2)/U(1) = S^2$, the Riemann sphere.

2. For $G = GL_n$ we have $G/B_+ = F_n$, the set of flags in $\mathbb{C}^n$ that we considered before.

Another realization of the flag manifold is one as the G-orbit of the line spanned by the highest weight vector in an irreducible representation with a regular highest weight. Namely, let $\lambda \in P_+$ be a dominant integral weight with $\lambda(h_i) \geq 1$ for all i (so the smallest such λ is $\lambda = \rho$). Let L_λ be the corresponding irreducible representation. We have $\mathfrak{b}_+ \cdot \mathbb{C}v_\lambda = \mathbb{C}v_\lambda$, but $e_\alpha v_\lambda \neq 0$ for any $\alpha \in R_-$, so $\mathfrak{b}_+$ is the stabilizer of $\mathbb{C}v_\lambda$ in $\mathfrak{g}$. Thus any $g \in G$ which preserves $\mathbb{C}v_\lambda$ belongs to the normalizer of $\mathfrak{b}_+$ (or, equivalently, B_+), i.e., $g \in B_+$. Thus $\mathcal{O} := G \cdot \mathbb{C}v_\lambda \subset \mathbb{P}L_\lambda$ is identified with G/B_+. This shows that $\mathcal{O}$ is compact, hence closed, i.e., $\mathcal{O} = G/B_+$ is a complex projective variety.

49.3. **Borel fixed point theorem.**

Theorem 49.6. *Let $\mathfrak{a}$ be a solvable Lie algebra over $\mathbb{C}$, V a finite dimensional $\mathfrak{a}$-module. Let $X \subset \mathbb{P}V$ be a closed $\mathfrak{a}$-invariant subset. Then there exists $x \in X$ fixed by $\mathfrak{a}$.*

Proof. The proof is by induction in $n = \dim \mathfrak{a}$. The base $n = 0$ is trivial, so we only need to justify the induction step. Since $\mathfrak{a}$ is solvable, it has an ideal $\mathfrak{a}'$ of codimension 1. By the induction assumption, $Y := X^{\mathfrak{a}'}$ is a nonempty closed subset of X, so it suffices to show that the 1-dimensional Lie algebra $\mathfrak{a}/\mathfrak{a}'$ has a fixed point on Y. Thus it suffices to prove the theorem for $n = 1$.

So let $\mathfrak{a}$ be 1-dimensional, spanned by $a \in \mathfrak{a}$. We can choose the normalization of a so that all eigenvalues of a on V have different real parts. Fix $x_0 \in X$ and consider the curve $e^{ta}x_0$ for $t \in \mathbb{R}$. It is easy to see that there exists $x := \lim_{t\to\infty} e^{ta}x_0$. Then x is fixed by $\mathfrak{a}$, as desired. $\square$

49.4. **Maximal solvable subalgebras.** Note that $\mathfrak{b}_+$ is a maximal solvable subalgebra of $\mathfrak{g}$; any bigger (parabolic) subalgebra will contain some negative root vector, hence the corresponding $\mathfrak{sl}_2$-subalgebra, so it will not be solvable. Moreover, B_+ is a maximal solvable subgroup of G: if $P \supset B_+$ then some element $g \in P$ does not normalize $\mathfrak{b}_+$, so $\mathrm{Lie}(P)$ has to be larger than $\mathfrak{b}_+$, hence not solvable. Thus any Borel subalgebra (subgroup) is a maximal solvable one. It turns out that the converse also holds.

Proposition 49.7. *Any solvable subalgebra of $\mathfrak{g}$ (connected solvable subgroup of G) is contained in a Borel subalgebra (subgroup).*

Proof. Let $\mathfrak{a} \subset \mathfrak{g}$ be a solvable subalgebra. By the Borel fixed point theorem, $\mathfrak{a}$ has a fixed point $\mathfrak{b} \in G/B_+$. Thus $\mathfrak{a}$ normalizes $\mathfrak{b}$. Hence $\mathfrak{a} \subset \mathfrak{b}$, as claimed. $\square$

Corollary 49.8. *Any element of $\mathfrak{g}$ is contained in a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$.*

Let us say that a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is **a nilpotent subalgebra** if it consists of nilpotent elements. Note that this is a stronger condition than just being nilpotent as a Lie algebra.

Corollary 49.9. *Any nilpotent subalgebra of* $\mathfrak{g}$ *is conjugate to a Lie subalgebra of* $\mathfrak{n}_+$*. Thus* $\mathfrak{n}_+$ *is a maximal nilpotent subalgebra of* $\mathfrak{g}$*, and any maximal nilpotent subalgebra of* $\mathfrak{g}$ *is conjugate to* $\mathfrak{n}_+$*.*

Proof. There is $g \in G$ such that $\mathrm{Ad}(g)\mathfrak{a} \subset \mathfrak{b}_+$, but since $\mathfrak{a}$ is nilpotent we actually have $\mathrm{Ad}(g)\mathfrak{a} \subset \mathfrak{n}_+$. □

A similar result holds for groups, with the same proof:

Corollary 49.10. *Any unipotent subgroup of* G *is conjugate to a (closed) Lie subgroup of* N_+*. Thus* N_+ *is a maximal unipotent subgroup of* G*, and any maximal unipotent subgroup of* G *is conjugate to* N_+*.*

We also have

Proposition 49.11. *The normalizer of* $\mathfrak{n}_+$ *and* N_+ *in* G *is* B_+*. Thus every maximal nilpotent subalgebra (unipotent subgroup) is contained in a unique Borel. Hence such subalgebras (subgroups) are parametrized by the flag manifold* G/B_+*.*

Proof. Clearly B_+ is contained in the normalizer of N_+, so this normalizer is a parabolic subgroup. We have seen that such subgroup, if larger than B_+, must have a larger Lie algebra that $\mathfrak{b}_+$, so it contains some root $\mathfrak{sl}_2$-subalgebra. But the group corresponding to such subalgebra does not normalize $\mathfrak{n}_+$, a contradiction. □