MODULAR DATA

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The goal of this note (which contains no new results) is to give an introduction to the basic theory of modular data. We loosely follow [G] (see also [L]).[1]

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1. The definition of a modular datum

1.1. **Fusion rings and categories.** Recall that a $\mathbb{Z}_+$-ring is a ring R, finitely generated and free as a $\mathbb{Z}$-module, which is equipped with a distinguished basis B containing 1 in which the structure constants are nonnegative. Given a $\mathbb{Z}_+$-ring R and an element $a \in R$, let $\eta(a)$ be the coefficient of 1 in the decomposition of a with respect to the distinguished basis. Recall that R is called a based ring (or a fusion ring) if there is an involution $* : B \to B$ (called dualization) such that $\eta(ab^*) = \delta_{ab}$ for any two elements $a, b \in B$. Clearly, if such an involution exists, it is unique.

The Grothendieck ring $R = K_0(\mathcal{C})$ of a fusion category $\mathcal{C}$ is a fusion ring. In this situation, $\mathcal{C}$ is said to be a categorification of R.

1.2. **Modular data.** Let k be an algebraically closed field of characteristic zero.

Definition 1.1. A modular datum M over k is a finite set I with a distinguished element $0 \in I$ and functions $S : I \times I \to k$ (called the S-matrix) and $T : I \to k$ (called the twist function), satisfying the following conditions:

(1) S is symmetric, $d_i := S_{0i} \neq 0\ \forall i \in I$, and $s_{00} = 1$;

(2) The functions $X_i : I \to k$ defined by $X_i(j) := \frac{S_{ij}}{S_{0j}}$ span inside $\mathrm{Fun}(I, k)$ a based ring R_M (in which they are the distinguished basis);

(3) $d_i = d_{i^*}$;

(4) If we regard S and T as linear operators on $\mathrm{Fun}(I, k)$ (by setting $(Sf)(i) = \sum S_{ij} f(j)$, $(Tf)(i) = T(i)f(i)$), then they define a projective representation of the modular group $SL_2(\mathbb{Z})$. That is, there exists $\tau \in k^\times$ such that $(ST)^3 = \tau S^2$, and this operator commutes with T. Also, $T(0) = 1$.

The numbers d_i are called the quantum dimensions of M.

Proposition 1.2. *If $\mathcal{C}$ is a modular tensor category, then its set of simple objects I, the S-matrix S and the twist function T form a modular datum.*

Definition 1.3. A categorification of a modular datum M is a categorification $\mathcal{C}$ of the based ring R_M which is a modular category with S-matrix S and twists T_i, $i \in I$.

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[1] We note that our definition of a modular datum is slightly less restrictive than the definition of a modular datum in [G] (where positivity of dimensions is assumed), and the definition of a fusion datun in [L] (where there is an additional involution on the set of labels).

Theorem 1.4. *(i) Let $D=\sum_i d_i^2$. Then $D\neq 0$, and one has $S^2=DC$, where $C_{ij}=\delta_{ij^*}$*

(ii) The numbers S_{ij}/d_j (in particular, d_i) and D/d_i^2 are algebraic integers.

(iii) d_i are totally real and $D-1$ is totally positive. The matrix $S/\sqrt{D}$ (well defined up to sign) is totally unitary.

Proof. (i) By (2), S is invertible. So we have $\delta_j = s_{0j}\sum_i (S^{-1})_{ji}X_i$. Thus for $j\neq m$ we have

$$\sum_{i,r}(S^{-1})_{ji}(S^{-1})_{mr}X_iX_r=0.$$

In particular, applying η to this equality and using the symmetry of S, we get

$$\sum_i (S^{-1})_{ji}(S^{-1}_{i^*m})=0.$$

Thus, $S^{-1}CS^{-1}$ is a diagonal matrix, hence so is SCS.

Now, let N_{ij}^m be the structure constants of R_M. Then by definition

$$S_{ir}S_{jr}=\sum N_{ij}^m S_{mr}S_{0r}.$$

So since $S_{rj}=S_{jr}$ and $S_{0r}=S_{r0}=S_{r^*0}$, we get

$$S_{ir}S_{rj}=\sum N_{ij}^m S_{mr}S_{r^*0}.$$

Summing this over r, and using that SCS is diagonal, we get

$$(S^2)_{ij}=\delta_{ij^*}D.$$

(ii) By definition, S_{ij}/d_j is an eigenvalue of the integer matrix N_i of multiplication by X_i in R_M, thus it is an algebraic integer. Now,

$$\frac{D}{d_i^2}=\sum_j \frac{s_{ij}}{d_i}\frac{s_{ji^*}}{d_i^*},$$

so it is also an algebraic integer.

(iii) $2d_i=d_i+d_{i^*}$ is an eigenvalue of the matrix $N_i+N_{i^*}$, which is a symmetric integer matrix. So d_i is totally real. Next, $(S_{ij}+S_{i^*j})/d_j$ is also an eigenvalue of $N_i+N_{i^*}$, so it is also totally real and hence S_{i^*j} is totally complex conjugate to S_{ij}. This together with (i) implies that under any complex embedding of $\mathbb{Q}(S_{ij})$, $SS^\dagger=D$, where $\dagger$ is the Hermitian adjoint to S, as desired. □

2. The Anderson-Moore-Vafa property

Let $\mathcal{C}$ be a modular tensor category with simple objects X_i. Consider the space $E_{ijm}^r=\mathrm{Hom}(X_r,X_i\otimes X_j\otimes X_m)$. Using the braiding this space is identified with E_{jim}^r. The hexagon relation for the braiding implies that

$$\det(\beta^2_{ij,m}|_{E^r_{ijm}})=\det(\beta^2_{jm}|_{E^r_{ijm}})\det(\beta^2_{im}|_{E^r_{jim}}).$$

Recall that β^2 acts on $\mathrm{Hom}(X_s,X_p\otimes X_q)$ by the scalar T_pT_q/T_s. Therefore, the last identity can be rewritten in the form

$$\prod_p (T_pT_m/T_r)^{N_{ij}^pN_{pm}^r}=\prod_p (T_iT_m/T_p)^{N_{im}^pN_{jp}^r}\prod_p (T_jT_m/T_p)^{N_{jm}^pN_{ip}^r}.$$

This implies the following result.

Theorem 2.1. *(Anderson-Moore-Vafa) For a categorifiable modular datum, one has*

$$(T_iT_jT_mT_r)^{N_{ijm}^r} = \prod_p T_p^{N_{ij}^p N_{pm}^r + N_{im}^p N_{jp}^r + N_{jm}^p N_{ip}^r},$$

where $N_{ijm}^r := \sum_p N_{ij}^p N_{pm}^r$.

Corollary 2.2. *([E]) For a categorifiable modular datum,* $T^N = 1$ *for some* N *dividing* $D^{5/2}$*. Therefore,* $\tau/\sqrt{D}$ *is a root of unity of order dividing* $4D^{5/2}|I|$.

Proof. The Anderson-Moore-Vafa equation can be written as the following equation in $\mathbb{Z} \otimes_{\mathbb{Z}} k^\times$:

$$N_{ijm}^r \otimes T_iT_jT_mT_r = \sum_p (N_{ij}^p N_{pm}^r + N_{im}^p N_{jp}^r + N_{jm}^p N_{ip}^r) \otimes T_p.$$

Let R be the ring of integers of the number field generated by d_i. Multiplying the last equation by d_r, and taking the sum over r (remembering that addition in the second component is really multiplication), we get the following equation in $R \otimes_{\mathbb{Z}} k^\times$:

$$d_id_jd_m \otimes T_iT_jT_m + \sum d_r N_{ijm}^r \otimes T_r = \sum_p (N_{ij}^p d_m d_p + N_{im}^p d_j d_p + N_{jm}^p d_i d_p) \otimes T_p.$$

Let us now multiply this equation by d_i and sum over i. Then we get

$$\sum_i d_i^2 d_j d_m \otimes T_i + D d_j d_m \otimes T_j T_m + \sum_r d_j d_m d_r^2 \otimes T_r = \sum_p (2 d_j d_m d_p^2 + D N_{jm}^p d_p) \otimes T_p.$$

After cancelations we get

$$D d_j d_m \otimes T_j T_m = D \sum_p N_{jm}^p d_p \otimes T_p.$$

Now multiply this by d_j and sum over j. We get

$$D^2 d_m \otimes T_m + \sum_j D d_m d_j^2 \otimes T_j = D d_m \sum_p d_p^2 \otimes T_p$$

After cancellation we get

$$D^2 d_m \otimes T_m = 0,$$

and using that d_m^2 divides D, we find that T_m is a root of unity of order dividing $D^{5/2}$. The second statement is deduced from the first one by taking the determinant of the equation $(ST)^3 = \tau S^2$ and using that $\det(S)^4 = D^{2|I|}$. □

2.1. **The Ising category.** As an example consider the Ising category, with simple objects $X_0 = \mathbf{1}$, $X_1 = \chi$ and $X_2 = X$, where $\chi \otimes \chi = \mathbf{1}$, $\chi \otimes X = X \otimes \chi = X$, and $X \otimes X = \mathbf{1} \oplus \chi$. In this case, we have $s_{01} = s_{11} = 1$, $s_{02} = \sqrt{2}$, $s_{12} = -\sqrt{2}$, and $s_{22} = 0$. In this case, it is easy to check that we must put $T_1 = -1$, but the axioms of a modular datum do not impose any conditions on $\theta = T_2$. On the other hand, the Anderson-Moore-Vafa identity for $i = j = m = r = 2$ gives $\theta^8 = -1$, i.e. θ must be a primitive 16-th root of unity. Thus the Anderson-Moore-Vafa identity does not follow from the axioms of a modular datum. On the other hand, if $\theta^8 = -1$, the corresponding modular datum is categorifiable. (There are eight nondegenerate braided categories with such a fusion ring, and sixteen modular ones, parametrized by choices of $\sqrt{2}$ and by θ; namely, each braided category has two modular structures, differing by sign of T_2).

References

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