Mathematical Ideas and Notions of Quantum Field Theory

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To Pierre Deligne on his 80th birthday with admiration

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Preface

Physics has always been a major source of both motivation and applications for several central fields of mathematics, such as analysis, differential equations, and probability. However, the development of quantum field theory and string theory in the last half-century has taken interactions between these two disciplines to an unprecedented level, incorporating into this area of physics (often called "high-energy physics") such traditionally "pure" areas of mathematics as algebraic topology, category theory, differential and algebraic geometry, representation theory, combinatorics, and even number theory. This interaction has been highly fruitful in both directions, and has led to a necessity for high-energy physicists to know the basics of modern mathematics and for mathematicians to know the basics of modern physics. The physicists have been quick to learn, and nowadays high-energy physicists often understand relevant areas of mathematics as deeply as professional mathematicians. On the other hand, many mathematicians have been dragging their feet, deterred by lack of rigor in physical texts, and, more importantly, by a different manner of presentation. In particular, even the basic setting of quantum field theory, necessary for understanding its more advanced (and more mathematically exciting) parts, is already largely unfamiliar to mathematicians. Nevertheless, many of the basic ideas of quantum field theory can in fact be presented rigorously, and even in cases when a rigorous exposition is impractical, difficult, or impossible, one can still attempt to explain the material in a mathematically natural way. Doing so is the main goal of the present book.

Namely, this book is an expanded version of the lecture notes for a graduate course on basic mathematical structures of quantum field theory that I gave at the MIT Mathematics Department in 2002 and then again in 2023. The reader

should not hope to learn quantum field theory from this text—this is impossible, for instance, because I know less about this subject than a beginning graduate student in high-energy physics. Rather, as mentioned above, its aim is to present the basic setup of quantum field theory in a mathematically motivated manner, highlighting its connections with various fields of mathematics. As such, it could serve to prepare the reader for more advanced texts in this genre (such as [**QFS**]) or for reading a regular QFT textbook or lecture notes (such as [**Co**, **W**, **IZ**, **PS**, **T1**, **T2**, **T3**, **T4**]) from a mathematician's viewpoint. Note that a lot of important material is contained in the exercises, which I strongly recommend that the reader solve while reading the text.

We begin with a general discussion of classical and quantum mechanics and field theory (Chapter 1). Then we proceed to prove the steepest descent and stationary phase formulas in classical asymptotic analysis, which serve as a finite-dimensional model for perturbative computations with path integrals (Chapter 2). Then, in Chapter 3, we develop Feynman calculus, the main combinatorial tool in perturbative quantum field theory. To illustrate Feynman calculus, we give a number of its applications to enumerative combinatorics (the matrix-tree theorem and its specializations).

In Chapter 4, we extend Feynman calculus to matrix integrals, and show that for such integrals Feynman graphs are replaced by fat graphs (surfaces), so that the coefficients of the asymptotic expansion in $1/N$ (where N is the matrix size) are sums over fat graphs of a given genus. This allows us to obtain nontrivial counts of planar graphs.

In Chapter 5, we use matrix integral techniques to prove the Harer-Zagier theorem on the Euler characteristic of the moduli space of curves.

All this material is, however, about quantum field theory in 0 spacetime dimensions, or, as one may jokingly say, in -1-dimensional space. To connect to real physics, we must go up at least one dimension, i.e., consider quantum field theory in $0+1$ spacetime (or 0 space) dimensions, which is quantum mechanics (Chapter 6). We begin with a review of Lagrangian formalism of classical mechanics (Lagrangians, least action principle), and then proceed to quantize this formalism, developing the path integral approach to quantum mechanics. Namely, we describe perturbative expansion of quantum-mechanical path integrals using Feynman diagrams and give several examples. We also explain that quantum mechanical path integrals are related to (rigorously defined) Wiener integrals in the theory of stochastic processes by the Wick rotation of the time, $t \mapsto it$.

In Chapter 7, after reviewing Hamiltonian formalism in classical mechanics, we describe its quantization using von Neumann's theory of unbounded self-adjoint operators, which gives a rigorous basis for nonperturbative quantum mechanics. We also prove the Feynman-Kac formula which relates the

correlation functions obtained in the Lagrangian and Hamiltonian approaches. This chapter includes some standard topics in the mathematical background of quantum mechanics: properties of the ground state, behavior of eigenfunctions of a Schrödinger operator, eigenfunctions for piecewise constant potential, WKB approximation, and the Weyl law.

In Chapters 8 and 9, we discuss the supergeneralization of the material of the previous chapters, i.e., describe classical and quantum mechanics for fermions. We begin with a review of supergeometry, Lie supergroups and superalgebras, and Berezin's integration theory on supermanifolds, and then proceed to extend Feynman calculus to the supercase. Then we discuss classical and quantum mechanics for fermions.

In Chapter 10 we finally get to the actual quantum field theory, in d spacetime dimensions with $d \geq 2$. We start with reviewing Lagrangian classical field theory. In particular, we review the classical theory of spinors (including mod 8 periodicity) and use it to describe the possible kinetic terms and mass terms in fermionic Lagrangians. We then pass to quantization of free classical field theories, in particular describing the theories of free bosons and fermions by computing their two-point functions. Then we describe Hamiltonian formalism of classical field theory. We also discuss classical pure gauge theory and its quantization in the abelian case. Finally, we pass to the Hamiltonian approach to quantum field theory and discuss Wightman axioms, representation-theoretic description of quantum particles, their mass, spin, and helicity. We conclude with describing the quantum theory of a free scalar boson, free fermion, and pure $U(1)$ gauge theory from this point of view, and then discuss normal ordering, composite operators, and operator product expansion in scalar field theory.

Chapter 11 is dedicated to symmetries and geometric aspects of field theory. We begin by discussing field theories on manifolds. Then we discuss symmetries in field theory and Noether's theorem, and we describe the types of symmetry that occur in relativistic field theories. We then briefly discuss spontaneous symmetry breaking, the Higgs mechanism, and dimensional reduction. Then we present Witten's solution of 2-dimensional nonabelian pure quantum gauge theory and discuss its application to geometry of the moduli space of nonabelian flat connections on a closed oriented surface. Finally, we discuss the cobordism picture of QFT, axiomatics of topological QFT, classification of 2-dimensional topological QFT, and conclude with a brief discussion of Chern-Simons theory and quantum invariants of 3-manifolds.

In Chapter 12 we give a brief introduction to 2-dimensional conformal field theory. After a review of classical field theory of a massless scalar in 1+1 dimensions, we quantize it and construct its Hilbert space from the Fock representation of the infinite-dimensional Heisenberg Lie algebra. We show that

the partition function of this theory (normalized using zeta function regularization) is modular invariant, reflecting its conformal symmetry. Then we show that the Hilbert space of the theory carries two commuting projective actions of the Lie algebra W of polynomial vector fields on $\mathbb{C}^\times$, which expresses infinitesimal conformal symmetry. We explain that this action is truly projective, i.e., both copies of W are replaced by its nontrivial central extension—the Virasoro algebra (conformal anomaly). Then we discuss a circle-valued version of this theory, vertex operators, and T-duality. We also briefly discuss the quantum theory of a free fermion in 1+1 dimensions and the Wess-Zumino-Witten model. Finally, we discuss Segal's cobordism picture of CFT (in particular, the semigroup of annuli) and conclude with a discussion of rational CFT.

In Chapter 13, we describe the basics of the perturbative renormalization theory. In particular, we discuss ultraviolet divergences of Feynman amplitudes and regularization of such divergences by introducing counterterms in the Lagrangian depending on the cutoff Λ in the momentum space. We define super-renormalizable, renormalizable and nonrenormalizable theories, critical dimensions for various theories and terms in the Lagrangian, and discuss the key examples. Then we discuss dimensional regularization, renormalization groups, asymptotic freedom, quantum anomaly of scale-invariance, and low energy effective theories, and we conclude by discussing lattice approximations of quantum field theories.

Finally, Chapter 14 is dedicated to supersymmetry. In this chapter we first systematically discuss supersymmetric quantum mechanics and its geometric applications (Atiyah-Singer index theorem, Hodge theory of Riemannian and Kähler manifolds). Then we pass to supersymmetry in higher dimensions (mostly at the classical level) and discuss the main types of supersymmetric theories for various dimensions and amounts of supersymmetry and describe the structure of the corresponding supermultiplets.

The approximate dependencies between chapters are represented in Figure 0.1

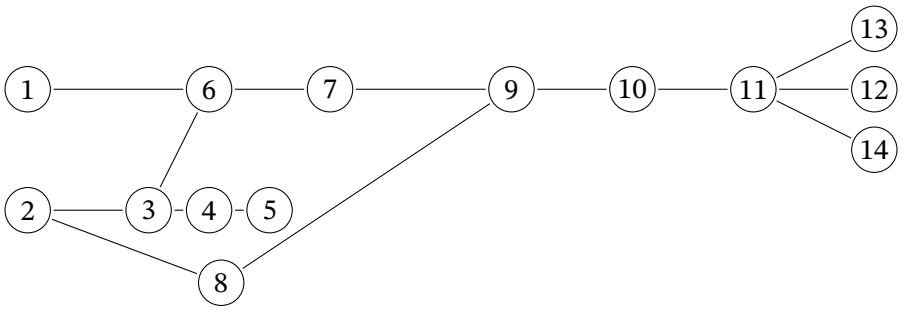

Figure 0.1

The mathematical background needed for reading this book varies from chapter to chapter, but it is mostly covered by advanced undergraduate and sometimes introductory graduate classes. The background helpful in various chapters is as follows:

1. Generalities on analysis, differential equations, calculus of variations, and probability.

2. Real analysis and a bit of 1-dimensional complex analysis.

3. Elementary real and complex analysis, multilinear algebra, and combinatorics.

4. Analysis, elementary multilinear algebra, combinatorics, and 2-dimensional topology.

5. Same as item 4 plus a bit more topology and complex geometry in higher dimensions.

6. Elementary analysis and ordinary differential equations (ODE).

7. Real analysis, differential equations, and a bit of functional analysis (operators in Hilbert space).

8. Elementary analysis, multilinear and abstract algebra.

9. Same as items 7 and 8.

10. Algebra, Lie theory, a bit of functional analysis (operators in Hilbert space, distributions).

11. Algebra, Lie theory, and differential geometry.

12. Representation theory, 1-dimensional complex analysis, and basics on modular forms.

13. Analysis, algebra, Lie theory, and differential geometry.

14. Algebra, Lie theory, differential geometry, and differential topology.

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Finally, I would like to acknowledge the use of several AI platforms (ChatGPT and sometimes Deepseek and QwQ-32B) while preparing the final version of this book. Specifically, I used them to find references, proofread parts of text, propose edits, and occasionally help me choose better wording.

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Chapter 1

Generalities on mechanics and field theory

1.1. Classical mechanics

In the most basic setup of classical mechanics, we study the motion of a particle of mass m in a Euclidean space V. This motion is described by a function of one variable, $q = q(t) \in V$, representing the position of the particle at a time t. This function must satisfy *Newton's equation of motion*,

$$m\ddot{q} = -\nabla U(q), \tag{1.1}$$

where $U : V \to \mathbb{R}$ is the *potential energy*. Thus the particle motion is determined by its initial position $q(0)$ and initial velocity $\dot{q}(0)$.

A crucial property of Newton's equation is the *energy conservation law*. Namely, let $p := m\dot{q}$ be the *momentum*, and define the *Hamiltonian* (*energy function*)

$$H := \frac{m\dot{q}^2}{2} + U(q) = \frac{p^2}{2m} + U(q),$$

the sum of the potential energy $U(q)$ and the *kinetic energy* $\frac{m\dot{q}^2}{2}$. Then the energy conservation law states that *H remains constant in time*. Indeed, (1.1) is equivalent to the system

$$\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q}, \tag{1.2}$$

called *Hamilton's equations*, so the chain rule gives

$$\frac{dH}{dt} = \frac{\partial H}{\partial q} \cdot \dot{q} + \frac{\partial H}{\partial p} \cdot \dot{p} = \frac{\partial H}{\partial q} \cdot \frac{\partial H}{\partial p} - \frac{\partial H}{\partial p} \cdot \frac{\partial H}{\partial q} = 0.$$

In the 1-dimensional case $V = \mathbb{R}$, this allows us to solve Newton's equation. Indeed, every solution satisfies $H = E$ for some constant E, so we have

$$\dot{q} = \pm\sqrt{\tfrac{2}{m}(E - U(q))},$$

where the sign indicates the direction of the particle motion. Hence

$$\pm\frac{dq}{\sqrt{\frac{2}{m}(E - U(q))}} = dt,$$

or

$$t = \pm\int \frac{dq}{\sqrt{\frac{2}{m}(E - U(q))}} + t_0, \tag{1.3}$$

so $q = q(t)$ can be recovered by inverting (1.3).

Another way to express Newton's law of motion is to say that $q(t)$ must be a solution of a certain variational problem. Namely, let us introduce the *Lagrangian*

$$\mathcal{L}(q) := \frac{m\dot{q}^2}{2} - U(q)$$

(the difference of kinetic and potential energy), and the *action* functional

$$S(q) := \int_a^b \mathcal{L}(q)dt$$

(for some fixed $a < b$). The expression $\mathcal{L}(q)dt$ is called the *Lagrangian density*. Newton's law of motion can now be expressed as the *least action principle*: $q(t)$ must be a critical point of S on the space of all functions with given $q(a)$ and $q(b)$; i.e., Newton's equation is the *Euler-Lagrange equation*

$$\frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\dot{q}} = \frac{\delta\mathcal{L}}{\delta q}$$

for a solution of the variational problem defined by S. Indeed, using integration by parts, for $\varepsilon \in C^1[a,b] \otimes V$ with $\varepsilon(a) = \varepsilon(b) = 0$, we have

$$\frac{d}{ds}|_{s=0}\int_a^b \mathcal{L}(q + s\varepsilon)dt = \int_a^b (\frac{\delta\mathcal{L}}{\delta q} \cdot \varepsilon + \frac{\delta\mathcal{L}}{\delta\dot{q}} \cdot \dot{\varepsilon})dt$$

$$= \int_a^b \left(\frac{\delta\mathcal{L}}{\delta q} - \frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\dot{q}}\right) \cdot \varepsilon dt = -\int_a^b (\nabla U(q) + m\ddot{q}) \cdot \varepsilon dt,$$

and this vanishes for all ε iff q satisfies Newton's equation $m\ddot{q} = -\nabla U(q)$.

Remark 1.1. The term "least action principle" comes from the fact that in some cases (for example when the Hessian matrix of U is nonpositive definite: $\mathrm{Hess}(U)(q) \leq 0$), the action is not only extremized but also minimized at the solution $q(t)$. In general, however, this is not the case, and the trajectory of the particle may be not a (local) minimum, but only a critical point of the action. Therefore, Newton's law of motion is better formulated as the "extremal (or stationary) action principle".

Exercise 1.2.

(i) Consider the motion of a particle of mass 1 in a Euclidean space V. Show that if the potential is concave ($\mathrm{Hess}(U)(q) \leq 0$), then for any $\mathbf{a}, \mathbf{b} \in V$ and $a < b \in \mathbb{R}$ there exists at most one solution of the Newton equation with $q(a) = \mathbf{a}$ and $q(b) = \mathbf{b}$, and it is the strict global minimum for the action with these boundary conditions (if it exists).

(ii) Show that the conclusion of (i) holds if $\mathrm{Hess}(U)(q) - \frac{\pi^2}{(b-a)^2}\mathbf{1} < 0$.

Hint. Prove and use Wirtinger's inequality: if $\varepsilon \in C^1[a,b]$, $\varepsilon(a) = \varepsilon(b) = 0$, then

$$\int_a^b \varepsilon'(t)^2 dt \geq \frac{\pi^2}{(b-a)^2} \int_a^b \varepsilon(t)^2 dt.$$

(iii) Compute the unique solution in (i) if $U(q) = -\frac{1}{2}B(q,q)$, where B is a nonnegative definite symmetric bilinear form on V.

(iv) Show that the statements of (i) fail for $\dim V = 1$, $U(q) = \frac{1}{2}q^2$, and $b - a \geq \pi$.

(v) Let $\dim V = 1$ and U be a smooth potential on $\mathbb{R}$. Suppose that $\limsup_{|x|\to\infty} \frac{U(x)}{x^2} \leq 0$. Show that a solution in (i) (possibly nonunique) exists for any $a, b, \mathbf{a}, \mathbf{b}$. Give an example of a smooth potential U for which a solution in (i) does not always exist.[1]

Remark 1.3. Physicists often consider solutions of Newton's equation on the whole line rather than on a fixed interval $[a,b]$. In this case, the naive definition of an extremal does not make sense, since the action integral $S(q) = \int_{\mathbb{R}} \mathcal{L}(q)dt$ is improper and, in general, diverges. Instead, one makes the following "corrected" definition: a function $q(t)$ on $\mathbb{R}$ is an extremal of S if the expression

$$\frac{d}{ds}|_{s=0} \int_{\mathbb{R}} \mathcal{L}(q + s\varepsilon)dt := \int_{\mathbb{R}} (\frac{\delta \mathcal{L}}{\delta q} \cdot \varepsilon + \frac{\delta \mathcal{L}}{\delta \dot{q}} \cdot \dot{\varepsilon})dt$$

(where $\varepsilon(t)$ is any compactly supported perturbation), is identically zero. With this definition, the extremals are exactly the solutions of Newton's equation (which, as before, is easily seen by integration by parts).

[1] One can show using the calculus of variations that for any $\dim V$, if $U(q) \leq 0$ for all q, then the solution always exists.

Remark 1.4. Note that this formalism also describes the motion of a system of n particles if we combine the vectors representing their positions in a Euclidean space V into a single vector in V^n. More generally, we may consider a particle moving on a Riemannian manifold M. In this case $q(t)$ is a path on M, and the motion is still described by Newton's equation, where $\dot{q}(t) \in T_{q(t)}M$ is the velocity vector of the particle at a time t (a tangent vector to M at $q(t)$), and $\ddot{q}$ means the covariant derivative $\nabla_{\dot{q}}\dot{q}$ of $\dot{q}$ with respect to the *Levi-Civita connection*.[2] For example, if $U = 0$, this is the *geodesic flow*, whose trajectories are the geodesics on M. The same applies to a system of n particles on M, in which case $q(t)$ is a path on the configuration space M^n.

Finally, a similar analysis applies to more general Lagrangians which are arbitrary smooth functions of (finitely many) derivatives of q, possibly also depending explicitly on time. However, in physical applications one usually encounters Lagrangians depending only on q and $\dot{q}$.

Remark 1.5. Hamilton's equations (1.2) imply that for any smooth function $F(q, p)$, along the particle trajectory we have

$$\frac{dF}{dt} = \{F, H\}, \tag{1.4}$$

where

$$\{F, G\} := \frac{\partial F}{\partial q} \cdot \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \cdot \frac{\partial G}{\partial q}.$$

It is easy to check that $(F, G) \mapsto \{F, G\}$ is a Lie bracket on the space of functions of q, p, which is called the *Poisson bracket*. The Poisson bracket form (1.4) of Hamilton's equations is important for connecting classical mechanics with symplectic geometry and for quantization.

1.2. Classical field theory

In classical field theory, the situation is similar to classical mechanics, but with infinitely many particles. Namely, in this case we should think not of a single particle or a finite system of particles, but rather of a "continuum of particles" (e.g., a string, a membrane, a jet of fluid, etc.). In other words, there is a particle at every point of some region of space. So in d-dimensional classical field theory the motion is described by a *classical field*—a (vector-valued) function $\phi(t, \mathrm{x})$ depending on both time and space coordinates ($t \in \mathbb{R}$, $\mathrm{x} \in \mathbb{R}^{d-1}$). Consequently, for $d \geq 2$ the equations of motion are *partial differential equations* (PDEs). Also, classical mechanics is then just the special case $d = 1$.

[2] More precisely, to define the covariant derivative $\nabla_{\dot{q}}\dot{q}$ at a point $q_0 = q(t_0) \in M$, it is necessary first to extend $\dot{q}$ from the curve $q = q(t)$ to a vector field on a neighborhood of q_0 in M. The resulting value is independent of the choice of this extension.

For example, for a string or a membrane the equation of motion is the *wave equation*

$$\Box\phi = 0, \tag{1.5}$$

with $\Box$ denoting the *D'Alembertian* $\partial_t^2 - c^2\Delta$, where Δ is the Laplacian with respect to the space coordinates and c is the velocity of wave propagation; e.g., for the string c is proportional to the square root of the string tension, while for electromagnetic waves c is the speed of light.

As in classical mechanics, in classical field theory there is a Lagrangian $\mathcal{L}(\phi)$ (a smooth function of finitely many partial derivatives of ϕ, possibly also explicitly depending on the space and time coordinates, but usually just on ϕ and $\nabla\phi$), whose integral

$$S(\phi) = \int_D \mathcal{L}(\phi) dx dt$$

over a compact region D in the *spacetime* $\mathbb{R}^d$ is called the *action*. As in mechanics, the expression $\mathcal{L}(\phi)dxdt$ is called the *Lagrangian density*. The law of motion can be expressed as the condition that the action must be extremized over any such region D with fixed boundary conditions; so the equations of motion (also called the *field equations*) are the Euler-Lagrange equations for this variational problem. For example, in the case of string or membrane the Lagrangian is

$$\mathcal{L}(\phi) = \tfrac{1}{2}(\phi_t^2 - c^2(\nabla\phi)^2), \tag{1.6}$$

and the Euler-Lagrange equation is the wave equation (1.5).

Remark 1.6. As in mechanics, in field theory solutions of the equations of motion on the whole spacetime (rather than a compact region D) are extremals of the action in the sense that

$$\tfrac{d}{ds}|_{s=0} \int_{\mathbb{R}^d} \mathcal{L}(\phi + s\varepsilon) dx dt = 0,$$

where ε is a compactly supported perturbation.

Remark 1.7. A classical field theory is called *free* if its Lagrangian (hence action) is quadratic in ϕ; in this case, the equations of motion are linear. For example, the theory defined by Lagrangian (1.6) is free. Since Lagrangians in actual physical theories are typically invariant under translations of spacetime, the Euler-Lagrange equations are as well. Thus for free theories these equations have constant coefficients, as in the wave equation (1.5). As a result, free classical field theories are *exactly solvable*—all solutions can be found explicitly by standard methods of the theory of linear PDE with constant coefficients.

The interesting classical field theories are therefore nonfree—they contain cubic and higher terms in the Lagrangian (called *interacting terms*), which renders the equations of motion nonlinear and makes the dynamics much more complex. However, as usual in the theory of PDE, we can learn quite a bit about the behavior of a nonfree (i.e., nonlinear) theory by viewing it as a deformation of a free (linear) one, i.e., approximating the action by its quadratic part near its critical point. Such an approach is called *perturbation theory*.

1.3. Brownian motion

One of the main differences between classical and quantum mechanics is, roughly speaking, that quantum particles do not have to obey the classical equations of motion but can randomly deviate from their classical trajectories. Therefore, given the position and velocity of the particle at a given time, we cannot determine its position at a later time but can only determine the density of probability that at this later time the particle will be found at a given point. In this sense quantum particles are similar to random (Brownian) particles. Brownian particles are a bit easier to understand conceptually, so let us begin with them.

The motion of a Brownian particle in $\mathbb{R}^k$ in a potential field

$$U : \mathbb{R}^k \to \mathbb{R}$$

is described by a stochastic process $q = q(t)$, $q = (q_1, \dots, q_k) \in \mathbb{R}^k$. That is, for each real t we have a random variable $q(t) \in \mathbb{R}^k$ (the position of the particle at a time t), such that the dependence of t is regular in some sense. Namely, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$ the random dynamics of the particle conditioned to have $q(a) = \mathbf{a}$, $q(b) = \mathbf{b}$ is "defined" as follows:[3] if $y : [a, b] \to \mathbb{R}^k$ is a continuously differentiable function with $y(a) = \mathbf{a}$, $y(b) = \mathbf{b}$, then the density of probability that $q(t) = y(t)$ for $t \in [a, b]$ is proportional to $e^{-S(y)/\kappa}$, where

$$S(y) := \int_a^b (\tfrac{1}{2}\dot{y}^2 + U(y))dt$$

is the action and $\kappa > 0$ is the diffusion coefficient. Thus, the likeliest $q(t)$ is the one that minimizes S (in particular, solves the classical equations of motion $\ddot{q} = \nabla U(q)$), while the likelihood of the other paths decays exponentially with the deviation of the action of these paths from the minimal possible.

Remark 1.8.

1. This discussion thus assumes that the extremum of S at q is actually a minimum, which we know is not always the case, but it is so when U is convex, i.e., $\mathrm{Hess}(U)(q) \geq 0$ for all q, by Exercise 1.2 applied to $-U$.

[3] We put the word "defined" in quotation marks because this definition is obviously heuristic and not rigorous; see below for more explanation.

2. The reader must have noticed that compared to the discussion of classical mechanics (with $m = 1$), the sign in front of the potential U has been changed to the opposite one. This is not a misprint! It has to do with the fundamental fact that statistical mechanics is related to usual (quantum) mechanics by the *Wick rotation* $t \mapsto it$, where $i = \sqrt{-1}$. In particular, this means that Brownian motion is well defined in the physically important case of convex potential, such as the multidimensional *harmonic oscillator* potential $\frac{1}{2}B(q,q)$, where B is a positive definite bilinear form.

All the information we can hope to get about the stochastic process $q(t)$ is contained in the *correlation functions*

$$\langle q_{j_1}(t_1)\dots q_{j_n}(t_n)\rangle,$$

which by definition are the expectation values of the products of random variables $q_{j_1}(t_1),\dots,q_{j_n}(t_n)$, (more specifically, by Kolmogorov's theorem the stochastic process $q(t)$ is completely determined by these functions). So such functions should be regarded as the output, or answer, of the theory of the Brownian particle.

Thus the main question is how to compute the correlation functions. Physicists write down the following "answer" motivated by the above definition: given points $t_1,\dots,t_n \in [a,b]$,

$$\langle q_{j_1}(t_1)\dots q_{j_n}(t_n)\rangle = \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1)\dots q_{j_n}(t_n)e^{-\frac{S(q)}{\kappa}}Dq, \tag{1.7}$$

where integration is carried out over the space $P_{\mathbf{a},\mathbf{b}}$ of paths

$$q : [a,b] \to \mathbb{R}^k,\ q(a) = \mathbf{a},\ q(b) = \mathbf{b},$$

and Dq is a Lebesgue measure on this space such that

$$\int_{P_{\mathbf{a},\mathbf{b}}} e^{-\frac{S(q)}{\kappa}}Dq = 1.$$

Alternatively, when they do not want to normalize the Lebesgue measure, they write

$$\langle q_{j_1}(t_1)\dots q_{j_n}(t_n)\rangle = \frac{1}{Z}\int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1)\dots q_{j_n}(t_n)e^{-\frac{S(q)}{\kappa}}Dq, \tag{1.8}$$

where

$$Z := \int_{P_{\mathbf{a},\mathbf{b}}} e^{-\frac{S(q)}{\kappa}}Dq$$

is the *partition function*. Such an integral is called a *path integral*, since it is an integral over the space of paths.

It is clear, however, that such definition and answer are a priori not satisfactory from the mathematical viewpoint, since the infinite-dimensional integration requires justification. In the case of Brownian motion, such a justification is actually possible within the framework of Lebesgue measure theory, and the corresponding integration theory is called the theory of *Wiener integral*. (To be more precise, one cannot define the measure Dq, but one can define the measure $e^{-\frac{S(q)}{\kappa}}Dq$ for sufficiently nice potentials $U(q)$.)

Remark 1.9. As $\kappa \to 0$, the nonoptimal trajectories become increasingly less likely relative to the optimal one, so in the limit we recover the deterministic system

$$\langle q_{j_1}(t_1)\dots q_{j_n}(t_n)\rangle \to \mathbb{q}_{j_1}(t_1)\dots \mathbb{q}_{j_n}(t_n),$$

where $\mathbb{q}(t)$ is the classical trajectory with $\mathbb{q}(a) = \mathbf{a}, \mathbb{q}(b) = \mathbf{b}$ (note that if $U \geq 0$ and convex, then this trajectory is unique by Exercise 1.2).

1.4. Quantum mechanics

Now let us turn to a quantum particle. Quantum mechanics is notoriously difficult to visualize, and the randomness of the behavior of a quantum particle is less intuitive and more subtle than that of a Brownian particle; nevertheless, it was pointed out by Feynman that the behavior of a quantum particle in a potential field $U(q)$ is correctly described by the same model, with the real positive parameter κ replaced by the imaginary number $-i\hbar$, where $\hbar > 0$ is the *Planck constant*, and the time t is replaced by it. In other words, the dynamics of a quantum particle can be expressed (we will discuss later how) via the *correlation functions*

$$\langle q_{j_1}(t_1)\dots q_{j_n}(t_n)\rangle = \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1)\dots q_{j_n}(t_n)e^{\frac{iS(q)}{\hbar}}Dq, \tag{1.9}$$

where Dq is normalized so that

$$\int_{P_{\mathbf{a},\mathbf{b}}} e^{\frac{iS(q)}{\hbar}}Dq = 1, \tag{1.10}$$

and $S(q)$ is now given by the same formula as in classical mechanics (and differing by sign from Brownian motion):

$$S(q) = \int_a^b (\frac{\dot{q}^2}{2} - U(q))dt.$$

As before, we have to make sense of this path integral, and now the theory of Wiener integrals does not work any more: for instance, the absolute value of the integrand in (1.10) does not decay as the path $q(t)$ deviates from the classical trajectory (in fact, it identically equals to 1!). So we will be able to make sense of (1.9) only perturbatively, i.e., with $\hbar$ being a formal parameter.

In fact, an effective mathematically rigorous approach to quantum mechanics is based on a different framework—*Hamiltonian formalism*, a quantization of the Hamiltonian formalism of classical mechanics discussed in more detail in Chapter 7. In this approach the main object of study is the Hilbert space $\mathcal{H}$ of quantum states equipped with a vacuum (ground) state $\Omega \in \mathcal{H}$ and various (in general, unbounded) self-adjoint operators on $\mathcal{H}$ corresponding to physical observables, such as the Hamiltonian H (energy), coordinate operators q_j, momentum operators p_j, etc. The evolution of these operators in time is governed by the *Schrödinger equation*

$$\frac{dA}{dt} = -\frac{i}{\hbar}[A, H],$$

which quantizes Hamilton's equations in their Poisson bracket form (1.4), replacing Poisson bracket by commutator. Thus we have

$$A(t) = e^{\frac{itH}{\hbar}} A(0) e^{-\frac{itH}{\hbar}}.$$

The correlation functions $\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle$ in the Hamiltonian formulation are given by the matrix coefficients $(\Omega, q_{j_1}(t_1) \dots q_{j_n}(t_n)\Omega)$. The equality

$$\int q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{\frac{iS(q)}{\hbar}} Dq = (\Omega, q_{j_1}(t_1) \dots q_{j_n}(t_n)\Omega),$$

which connects the path integral with operator formalism, is known as the *Feynman–Kac formula*.

Still, formula (1.9) is extremely helpful for motivational purposes and with appropriate care can be used for computation.

Remark 1.10. Similarly to Brownian motion (cf. Remark 1.9), in the limit $\hbar \to 0$ we are supposed to recover the classical system

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle \to \mathbb{q}_{j_1}(t_1) \dots \mathbb{q}_{j_n}(t_n).$$

However, now this is achieved not because individual nonoptimal trajectories become less likely, but rather due to cancellation in the oscillatory integral (1.9) which corresponds to the physical phenomenon of *quantum interference*. We will observe how this cancellation occurs in finite-dimensional oscillatory integrals when we discuss the stationary phase formula below.

1.5. Quantum field theory

In quantum field theory, the situation is similar to quantum mechanics, but, as in the classical case, with infinitely many particles. Namely, a quantization of a classical field theory with action $S(\phi)$ is obtained by considering correlation functions

$$\langle \phi_{j_1}(x_1) \dots \phi_{j_n}(x_n) \rangle = \int \phi_{j_1}(x_1) \dots \phi_{j_n}(x_n) e^{\frac{iS(\phi)}{\hbar}} D\phi, \tag{1.11}$$

where $x_k = (t_k, \mathrm{x}_k) \in \mathbb{R}^d$ and the integration measure $D\phi$ is normalized so that $\int e^{\frac{iS(\phi)}{\hbar}} D\phi = 1$.

Of course, from the mathematical viewpoint, this setting is a priori even less satisfactory than the one of quantum mechanics. Indeed, it involves integration with respect to the complex-valued measure $e^{\frac{iS(\phi)}{\hbar}} D\phi$ on the space of functions of ≥ 2 variables, which is ill defined even after the Wick rotation. Yet physicists imagine that certain integrals of this type exist, and come to sensible and interesting conclusions (both physical and mathematical) by working with them. Therefore, making sense of such integrals is an interesting problem for mathematicians which will be in the center of our attention.

Specifically, we will make sense of path integral (1.11) as a power series in $\hbar$. To this end, write the action $S(\phi)$ as the sum of homogeneous parts

$$S(\phi) = \sum_{j \geq 2} S_j(\phi),$$

where $S_j(\phi)$ is homogeneous of degree j (assuming that S contains no constant or linear terms in ϕ, i.e., $\phi = 0$ is a critical point of S with $S(0) = 0$). Then replacing ϕ by $\hbar^{\frac{1}{2}}\phi$ brings (1.11) to the form

(1.12)

$$\hbar^{-\frac{n}{2}} \langle \phi_{j_1}(x_1) \dots \phi_{j_n}(x_n) \rangle = \int \phi_{j_1}(x_1) \dots \phi_{j_n}(x_n) e^{i(S_2(\phi) + \sum_{j \geq 3} \hbar^{\frac{j}{2-1}} S_j(\phi))} D\phi.$$

For $\hbar = 0$ this is the Gaussian path integral

$$I_n(x_1, \dots, x_n) = \int \phi_{j_1}(x_1) \dots \phi_{j_n}(x_n) e^{iS_2(\phi)} D\phi$$

with quadratic action $S_2(\phi)$, which defines a *free quantum field theory*. Such an integral can be easily defined using Wick's formula (Section 3.1), which gives an explicit expression for it as a sum over matchings (fixed-point-free involutions) on the set $[1, n]$ (in particular, it vanishes if n is odd):

$$I_n(x_1, \dots, x_n) = \sum_{\sigma \in \mathrm{Matchings}([1,n])} \prod_{j \in [1,n]: j < \sigma(j)} G(x_j - x_{\sigma(j)}).$$

Here G is the *Green's function* (also called the *propagator*)—the 2-point correlation function, which is just the fundamental solution of the field equations evaluated at $x_1 - x_2$. Thus, as in the classical case, a free quantum field theory is exactly solvable. Moreover, we can now formally expand (1.12) in powers of $\hbar$ as a perturbation of this Gaussian integral using the power series for the exponential and again Wick's formula. This produces, in every order in $\hbar$, a sum of finite-dimensional integrals (*Feynman amplitudes*) attached to certain graphs (*Feynman diagrams*); see Chapter 3 and Section 6.3.

For $d \geq 2$, these finite-dimensional integrals are often divergent, however, and one needs to regularize them in a coherent manner to obtain meaningful

perturbation expansions of correlation functions. This is accomplished by *perturbative renormalization theory* (Chapter 13). Namely, this theory identifies a subclass of Lagrangians, called *renormalizable*, for which such a systematic regularization is possible, and which luckily contains the Lagrangians of physical interest in our $(3+1)$-dimensional world.

The next (much more serious) obstacle, which arises already for $d = 1$ (quantum mechanics) and even $d = 0$ (finite-dimensional integrals) is that such perturbation expansions never converge for $\hbar \neq 0$. Indeed, one can see this on the model example

$$I(\hbar) = \int_{\mathbb{R}} e^{-\frac{x^2+x^4}{2\hbar}} dx \sim \sqrt{2\pi\hbar} \sum_{n\geq 0} \frac{(-1)^n (4n-1)!!}{2^n n!} \hbar^n;$$

this expansion is only asymptotic and diverges for all nonzero $\hbar$ since the coefficients grow faster than a geometric progression. However, it can be summed using *Borel summation* (a method providing, for some asymptotic series, a natural choice of a function with such asymptotics, see Section 2.5). Physicists believe that the same happens in cases of interest in quantum field theory: the appropriate Borel summation of the perturbation expansion evaluated at numerical $\hbar$ should provide actual answers to physically interesting questions. Statements pertaining to such numerical values are called *nonperturbative*.

As in quantum mechanics, a more mathematically robust approach to quantum field theory is through Hamiltonian formalism, in which the main object of study is the Hilbert space $\mathcal{H}$ of quantum states equipped with a vacuum state $\Omega \in \mathcal{H}$ and quantum fields $\phi_j(x)$ valued in operators on $\mathcal{H}$ so that the correlation functions $\langle \phi_{j_1}(x_1) \dots \phi_{j_n}(x_n) \rangle$ are the matrix coefficients $(\Omega, \phi_{j_1}(x_1) \dots \phi_{j_n}(x_n)\Omega)$. Namely, in the Hamiltonian approach it is possible to give a rigorous definition of a quantum field theory (the *Wightman axioms*, Section 10.9) and even to construct some interesting examples of quantum field theories satisfying these axioms. This is the subject of *constructive field theory* ([**GJ**]). Admittedly, many important quantum field theories are not known to or known not to satisfy Wightman axioms. But these axioms are still useful for motivational purposes as a rigorous model framework. Moreover, there are quantum field theories which cannot be defined by quantizing a Lagrangian (e.g., minimal models of conformal field theory). So the Hamiltonian approach is not only mathematically more robust than the Lagrangian (path integral) one but is also more general (at least for theories on $\mathbb{R}^d$). However, a great advantage of the path integral approach is the ability to consider arbitrary spacetime manifolds, which are not equipped with (or even do not admit) a splitting into space and time.

Remark 1.11. A central role in understanding both classical and quantum field theories is played by their symmetries. For instance, note that Lagrangian

(1.6) has a large symmetry group—the semidirect product $\mathrm{SO}_+(1,d-1) \ltimes \mathbb{R}^d$, where the Lorentz group $\mathrm{SO}_+(1,d-1)$ acts by matrices preserving the Minkowskian metric $c^2t^2 - \sum_{i=1}^{d-1} x_i^2$ and positive direction of time, and $\mathbb{R}^d$ acts by translations. This is the *Poincaré symmetry*, which is, in fact, the symmetry of special relativity. In other words, the corresponding field theory is *relativistic*. This will be so for all classical and quantum field theories on $\mathbb{R}^d$ that we will consider in this book. There are also several other important types of symmetry—global symmetry, gauge symmetry, conformal symmetry, supersymmetry, R-symmetry. They are discussed in Section 11.5.

Remark 1.12. As a running example for classical and quantum field theory we have considered the theory of a *scalar field*, where fields are functions $\phi(x)$ on the spacetime. One may also consider fields which are more general geometric objects (spinors, differential forms, connections, etc.). This means that they transform in a more complicated way under Lorentz symmetries. For instance, matter particles (e.g., electrons and positrons) are modeled in quantum field theory by *spinor fields* $\psi(x)$, which are sections of the spin bundle on spacetime. They transform in the spin representation of the Lorentz group, which is in fact only a *projective representation*—a representation of its double cover $\mathrm{Spin}_+(1,d-1)$. Moreover, such fields are *odd*, or *fermionic*, i.e., the values $\psi(x)$ and $\psi(y)$ at $x, y \in \mathbb{R}^d$ anticommute (in the classical field theory setting): $\psi_i(x)\psi_j(y) = -\psi_j(y)\psi_i(x)$. Thus in the corresponding quantum theory we need to integrate over odd vector spaces, where some coordinates are Grassmann (i.e., anticommuting) variables. Such spaces are called *superspaces*, and the theory of integration over them is developed in Chapter 8.

Another example is *pure electrodynamics* (*Maxwell theory*) discussed in Section 10.7. In this theory, the basic field is a $U(1)$-*gauge field* A, representing a *photon*. A $U(1)$-gauge field is a connection on a $U(1)$-bundle over $\mathbb{R}^4$ (concretely, a 1-form), and the field equations are the *Maxwell equations*, but fields are considered up to *gauge symmetries* $A \mapsto A + df$, where f is a function on $\mathbb{R}^4$. This theory is free as Maxwell equations are linear (photons do not interact), but one may add a spinor field $\psi(x)$ transforming under the tautological representation of $U(1)$, with a Lagrangian containing the cubic interacting term $(\overline{\psi}, A\psi)$; this allows us to model interactions involving electrons, positrons, and photons. One may also define a similar theory in which $U(1)$ is replaced by a general (not necessarily abelian) compact Lie group G (*pure Yang-Mills theory*) which is no longer free, as well as its extensions containing matter fields (scalars and spinors) modeling various particles. In Yang-Mills theory, the basic field is a gauge field A which is a connection on a principal G-bundle on $\mathbb{R}^4$ (concretely, a 1-form valued in $\mathfrak{g} = \mathrm{Lie}\, G$) considered up to gauge transformations $g : R^4 \to G$ acting by $A \circ g = g^{-1}dg + g^{-1}Ag$. Field theories, in which fields are considered up to such symmetries, are called *gauge theories*, and they

play a central role in physics, as they describe the main types of interactions of elementary particles—electromagnetic, strong, and weak interactions. The ultimate such theory is the *standard model* which describes all known particles and interactions except gravity.

Remark 1.13. Physicists like to discuss quantum field theories in terms of their behavior at various energy scales. Namely, recall that in classical mechanics we have three basic units of measurement—distance D, time T, and mass M—and all quantities express through these: for example, velocity is D/T, momentum is MD/T, energy is MD^2/T^2, etc. In relativistic mechanics we have a natural unit of velocity—the speed of light c which provides an identification $D \cong T$. Thus momentum and energy now have the same units of measurement as mass (M). Further, in quantum mechanics we have Einstein's formula $E = \hbar\nu$ with Planck constant $\hbar$ connecting energy and frequency. So in relativistic quantum field theory we get an identification $M \cong 1/T$. Thus we are left with just one dimensionful scale (either distance or time) or the reciprocal (mass, momentum, or energy). In particular, *small* distances, times correspond to *large* masses, momenta, energies (the *ultraviolet* (*UV*) *limit*), and *large* distances, times correspond to *small* masses, momenta, energies (the *infrared* (*IR*) *limit*).

Now, renormalizable quantum field theories are defined by the condition that all the coefficients (couplings) g_i in the Lagrangian are measured in units M^{d_i}, where $d_i \geq 0$.[4] This means that under the rescaling $x \mapsto sx$ in the path integral they rescale as $g_i \mapsto g_i s^{d_i}$, which preserves a neighborhood of 0 when $s \leq 1$. This creates a possibility that in the UV limit $s \to 0$, the couplings become small (weak) and the perturbation expansion with respect to g_i (closely related to the $\hbar$-expansion discussed above) becomes asymptotically valid. Namely, if $d_i > 0$ for all i (the super-renormalizable case), then this is automatic, while if there are dimensionless couplings g_i with $d_i = 0$, then one needs an additional property of *asymptotic freedom* (Section 13.9). If so, then it is believed that the theory is *fundamentally defined in the UV limit* as a deformation of a free quantum field theory, meaning that its expansion in g_i can be Borel-summed to get nonperturbative correlation functions, which should converge to those of a free theory at large energies. Then one may study how these functions behave at smaller energy scales that we observe in the real world. This is called "infrared behavior", and its understanding is the main challenge in studying the dynamics of quantum field theories. The passage from larger to smaller energy scales is greatly elucidated by the theory of *renormalization group* (Section 13.8) and the notion of the *low energy effective theory* (Section 13.11).

[4] For instance, if $\mathcal{L}(\phi) = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{6}g\phi^3$, then $(\nabla\phi)^2 dx$ should be dimensionless, so the units for ϕ are $M^{\frac{d-2}{2}}$. Since $g\phi^3 dx$ should also be dimensionless, the units for g are $M^{d-3\frac{d-2}{2}} = M^r$, where $r = 3 - \frac{d}{2}$. This theory is renormalizable for $d \leq 6$ (Section 13.2), which is exactly when $r \geq 0$.

1.6. A note on rigor

A large part of the material in this book can be developed completely rigorously, even though we sometimes choose not to do so, to avoid overly formal presentation and keep the text at reasonable size. Specifically, all the material on steepest descent and stationary phase formulas, Feynman calculus, matrix integrals, classical mechanics, and Hamiltonian quantum mechanics including fermions (Chapters 2–9) is fully rigorous mathematics; for instance, a lot of the mathematical background on Hamiltonian quantum mechanics can be found in [**RS**]. Lagrangian quantum mechanics (path integrals) can be developed rigorously only upon Wick rotation (with integrand $e^{-S/\hbar}$). As explained in Section 1.3, this is based on the theory of stochastic processes. The Feynman-Kac formula connecting the Lagrangian and Hamiltonian picture in this setting (Section 7.6) is also fully rigorous.

In arbitrary spacetime dimension, classical field theory (including fermions, fields defined and taking values on manifolds, and supersymmetry) is also entirely rigorous. The same applies to perturbative quantum field theory, including the theory of renormalization groups (Chapter 13). Also all free quantum field theories (Chapter 10) rigorously exist beyond perturbation theory, as do some super-renormalizable quantum field theories defined in the framework of constructive field theory ([**GJ**]). Also many interesting quantum conformal field theories (minimal models, Wess-Zumino-Witten models, see Chapter 12) are mathematically well defined. However, among (asymptotically free) renormalizable quantum field theories containing dimensionless couplings, only a few are known to exist rigorously (in two spacetime dimensions), mostly thanks to recent developments in probability theory. Notably, nonperturbative 4-dimensional quantum Yang-Mills theory (an enhanced version of which, the *Standard Model*, is believed to describe our universe) still remains very much out of reach for a mathematical study. More precisely, one can define this theory on a lattice (see Section 13.12), but mathematicians can say nothing about what happens when the lattice spacing goes to zero and the lattice degenerates into continuous space, even though some impressive numerical results and predictions can be obtained using the lattice approximation. In fact, this is the subject of one of the seven Millennium Problems ([**JW**]).

Chapter 2

The steepest descent and stationary phase formulas

Now, let us forget for a moment that the integrals (1.7), (1.9), and (1.11) are infinite dimensional and hence problematic to define, and ask ourselves the following question: why should we expect to recover the usual classical mechanics or field theory when the parameter $\varkappa$ or $\hbar$ goes to zero? The answer is that this expectation is based on the *steepest descent* (respectively, *stationary phase*) principle from classical analysis: if $f(x)$ is a function on $\mathbb{R}^d$, then the integrals $\int g(x)e^{-\frac{f(x)}{\varkappa}}dx$, $\int g(x)e^{\frac{if(x)}{\hbar}}dx$ "localize" to minima (respectively, critical points) of the function f as $\varkappa \to 0$ (respectively, $\hbar \to 0$). As this classical fact is of central importance to the whole subject, let us now discuss it in some detail.

2.1. Gaussian integrals

We start with auxiliary facts from linear algebra and analysis. Let V be a real vector space of dimension d. Let $\mathbf{M}(V)$ be the set of nondegenerate complex-valued symmetric bilinear forms on V with nonnegative definite real part. We have an open dense subset $\mathbf{M}^\circ(V) \subset \mathbf{M}(V)$ of forms with positive definite real part. If $B = P + iQ \in \mathbf{M}^\circ(V)$, where P, Q are the real and imaginary parts of B, then $P^{-1}Q : V \to V$ is a self-adjoint operator with respect to P, which therefore has real eigenvalues and diagonalizes in an orthonormal basis. In this basis $B(x,y) = \sum_{j=1}^d a_j x_j y_j$, where $\mathrm{Re}(a_j) = 1$. Thus $B^{-1} \in \mathbf{M}^\circ(V^*)$. It

follows that the map $B \mapsto B^{-1}$ is a homeomorphism $\mathbf{M}(V) \cong \mathbf{M}(V^*)$ which restricts it to a homeomorphism $\mathbf{M}^\circ(V) \cong \mathbf{M}^\circ(V^*)$.

Now fix a translation-invariant volume form dx on V. Then for every complex-valued symmetric bilinear form B on V we can define its determinant $\det B$. Thus we can define a continuous function $(\det B)^{-\frac{1}{2}}$ on $\mathbf{M}(V)$ using the branch of the square root which is positive on positive definite forms (it exists and is unique because $\mathbf{M}(V)$ is star-like with respect to any point of $\mathbf{M}^\circ(V)$, hence simply connected). Note that if $B = iQ$, where Q is a real nondegenerate form, then $(\det(iQ))^{-\frac{1}{2}} = e^{\frac{-\pi i\sigma(Q)}{4}} |\det Q|^{-\frac{1}{2}}$, where σ is the signature of Q. Indeed, it suffices to check the statement for diagonal forms, hence for $d = 1$, in which case it is straightforward.

Let $\mathcal{S}(V)$ be the *Schwartz space* of V, i.e., the space of complex-valued smooth functions on V all of whose derivatives are rapidly decaying at ∞ (faster than any power of $|x|$). In other words, $\mathcal{S}(V)$ is the space of smooth functions f on V such that $D(V)f \subset L^2(V)$, where $D(V)$ is the algebra of differential operators on V with polynomial coefficients. The Schwartz space has a natural Fréchet topology defined by the seminorms $\|Df\|_{L^2}$, $D \in D(V)$. The topological dual space $\mathcal{S}'(V)$ is the space of *tempered distributions* on V. Note that we have natural inclusions

$$\mathcal{S}(V) \subset L^2(V) \subset \mathcal{S}'(V).$$

Recall that the *Fourier transform* is the operator

$$\mathcal{F} : \mathcal{S}(V) \to \mathcal{S}(V^*)$$

given by

$$\mathcal{F}(g)(p) := (2\pi)^{-\frac{d}{2}} \int_V g(x) e^{-i(p,x)} dx,$$

which defines an isometry $L^2(V) \to L^2(V^*)$ such that $(\mathcal{F}^2 g)(x) = g(-x)$. By duality, it defines an operator

$$\mathcal{F} : \mathcal{S}'(V) \to \mathcal{S}'(V^*)$$

which extends $\mathcal{F}$. For any complex symmetric bilinear form B with $\operatorname{Re} B \geq 0$ the function $e^{-\frac{1}{2}B(x,x)}$ belongs to $\mathcal{S}'(V)$, and moreover to $\mathcal{S}(V)$ if and only if $B \in \mathbf{M}^\circ(V)$. Furthermore, it depends continuously on B as an element of these spaces. We will call it the *complex Gaussian distribution*.

Lemma 2.1 (Gaussian integral)**.** *For any $B \in \mathbf{M}(V)$ we have*

$$\mathcal{F}(e^{-\frac{1}{2}B(x,x)}) = (\det B)^{-\frac{1}{2}} e^{-\frac{1}{2}B^{-1}(p,p)}.$$

Proof. By continuity, it suffices to prove this when $\operatorname{Re} B > 0$. In this case B is diagonalizable, so the statement reduces to the case $d = 1$. In this case we have

to show that for every $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx - \frac{1}{2}ax^2} dx = \frac{1}{\sqrt{a}} e^{-\frac{1}{2a}p^2}.$$

Since both sides are holomorphic in a in the region $\operatorname{Re} a > 0$, it is enough to check the statement when a is real. The integral in question can be written as

$$\frac{e^{-\frac{1}{2}a^{-1}p^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(x+ia^{-1}p)^2} dx.$$

But using Cauchy's theorem,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(x+ia^{-1}p)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+ia^{-1}p} e^{-\frac{1}{2}ax^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx.$$

Thus the result follows from the Poisson integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

by rescaling x. □

In the sense of Lemma 2.1 we can say, setting $p = 0$, that

$$(2\pi)^{-\frac{d}{2}} \int_V e^{-\frac{1}{2}B(x,x)} dx = (\det B)^{-\frac{1}{2}}. \tag{2.1}$$

Note that this equality is also true in the sense of absolute convergence if $B \in \mathbf{M}^\circ(V)$ and of conditional convergence otherwise (check it!).

2.2. Gaussian integrals with insertions

Now let $g \in \mathcal{S}(V)$. Consider the integral

$$I_g(\hbar) := \int_V g(\hbar^{\frac{1}{2}}x) e^{-\frac{1}{2}B(x,x)} dx,\ \hbar \geq 0,$$

where for $\hbar = 0$ we use (2.1), so

$$I_g(0) = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} g(0). \tag{2.2}$$

Let $\Delta_B : \mathcal{S}(V) \to \mathcal{S}(V)$ be the Laplace operator corresponding to B: $\Delta_B = \sum_{j=1}^{d} \partial_{B^{-1}e_j^*} \partial_{e_j}$ for a basis $\{e_i\}$ of V, where for $v \in V$, ∂_v is the operator of derivative along v.

Theorem 2.2. *We have*

$$I_g'(\hbar) = I_{\frac{1}{2}\Delta_B g}(\hbar),\ \hbar \geq 0.$$

Thus $I_g \in C^\infty[0,\infty)$. In particular, if g vanishes at the origin to order $2n+1$, then $I_g(0) = \cdots = I_g^{(n)}(0) = 0$.

The rest of the section is occupied by the proof of Theorem 2.2.

Lemma 2.3. *I_g is a continuous function.*

Proof. Only continuity at $\hbar = 0$ requires proof. By Plancherel's theorem and Lemma 2.1,

$$\begin{aligned} I_g(\hbar) &= (g(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) \\ &= \hbar^{-\frac{d}{2}}(\det B)^{-\frac{1}{2}}(\hat{g}(\hbar^{-\frac{1}{2}}p), e^{-\frac{1}{2}B^{-1}(p,p)}) = (\det B)^{-\frac{1}{2}}(\hat{g}(p), e^{-\frac{\hbar}{2}B^{-1}(p,p)}), \end{aligned}$$

where $\hat{g}$ is the Fourier transform of g. But $e^{-\frac{\hbar}{2}B^{-1}(p,p)} \to 1$ in $\mathcal{S}'(V^*)$ as $\hbar \to 0$ (as the complex Gaussian distribution depends continuously on the bilinear form). Thus

$$\lim_{\hbar\to 0} I_g(\hbar) = (\det B)^{-\frac{1}{2}}(\hat{g}(p), 1) = (2\pi)^{\frac{d}{2}}(\det B)^{-\frac{1}{2}}g(0) = I_g(0),$$

as desired. □

Lemma 2.4. *If $\ell \in V^*$ and $f \in \mathcal{S}(V)$, then*

$$I_{\ell f}(\hbar) = \hbar I_{\partial_{B^{-1}\ell} f}(\hbar).$$

Proof. We have

$$\begin{aligned} I_{\ell f}(\hbar) &= \hbar^{\frac{1}{2}}(\ell(x)f(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) = \hbar^{\frac{1}{2}}(f(\hbar^{\frac{1}{2}}x), \ell(x)e^{-\frac{1}{2}B(x,x)}) \\ &= -\hbar^{\frac{1}{2}}(f(\hbar^{\frac{1}{2}}x), \partial_{B^{-1}\ell}e^{-\frac{1}{2}B(x,x)}) = \hbar^{\frac{1}{2}}(\partial_{B^{-1}\ell}f(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) \\ &= \hbar((\partial_{B^{-1}\ell}f)(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) = \hbar I_{\partial_{B^{-1}\ell}f}(\hbar). \end{aligned}$$

This proves the lemma. □

Now we prove Theorem 2.2. If $\hbar > 0$, then by direct differentiation we get

$$I_g'(\hbar) = \tfrac{1}{2}\hbar^{-1}I_{Eg}(\hbar),$$

where $E := \sum_{j=1}^d e_j^*\partial_{e_j}$ is the Euler vector field on V. Thus by Lemma 2.4 we have

$$I_g'(\hbar) = I_{\frac{1}{2}\Delta_B g}(\hbar),\ \hbar > 0. \tag{2.3}$$

So, using Lemma 2.3, it suffices to show that $I_g \in C^1[0,\infty)$ (then smoothness will follow by repeated application of (2.3)). To this end, note that if C is a positive definite form on V, then

$$I_{e^{-\frac{1}{2}C(x,x)}}(\hbar) = \int_V e^{-\frac{1}{2}(B+\hbar C)(x,x)}dx = (2\pi)^{\frac{d}{2}}\det(B+\hbar C)^{-\frac{1}{2}},$$

which is analytic, hence continuously differentiable on $[0,\infty)$. So subtracting from g a multiple of such function, it suffices to prove that $I_g \in C^1[0,\infty)$ when $g(0) = 0$. In this case g is well known to be a linear combination of functions

of the form ℓf, where $f \in \mathcal{S}(V)$ and $\ell \in V^*$. So it suffices to check that $I_g \in C^1[0,\infty)$ for $g = \ell f$. But then by Lemma 2.4 $I_g'(0) = I_{\partial_{B^{-1}\ell} f}(0) = I_{\frac{1}{2}\Delta_B g}(0)$, as

$$\tfrac{1}{2}\Delta_B g(0) = \tfrac{1}{2}\Delta_B(\ell f)(0) = \sum_j \ell(e_j)\partial_{B^{-1}e_j^*} f(0) = \partial_{B^{-1}\ell} f(0).$$

This completes the proof.

Exercise 2.5. Let $\mathcal{S}_m(V) \subset C^m(V)$ be the subspace of functions whose derivatives of order $\le m$ are rapidly decaying. Prove that the differentiation formula of Theorem 2.2 holds for $g \in \mathcal{S}_2(V)$. Deduce that if $g \in \mathcal{S}_{2n}(V)$, then $I \in C^n[0,\infty)$, and that if moreover g vanishes at 0 to order $2n+1$, then $I_g(0) = \cdots = I_g^{(n)}(0) = 0$.

2.3. The steepest descent formula

Let $a < b$ be real numbers and let $f, g : [a,b] \to \mathbb{R}$ be continuous functions which are smooth on (a,b).

Theorem 2.6 (Steepest descent formula)**.** *Assume that f attains a global minimum at a unique point $c \in [a,b]$, such that $a < c < b$ and $f''(c) > 0$. Then one has*

$$\int_a^b g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{-\frac{f(c)}{\hbar}} I(\hbar), \tag{2.4}$$

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ such that

$$I(0) = \sqrt{2\pi}\frac{g(c)}{\sqrt{f''(c)}}.$$

Proof. Without loss of generality we may put $c = 0$, $f(c) = 0$. Denote $f''(c)$ by M. Making a change of variable, we may reduce this to a situation where $f(x) = \frac{M}{2}x^2$ when x is in some neighborhood U of 0. Let h be a "bump" function, a smooth function supported in U which equals 1 in a smaller neighborhood $0 \in U' \subset U$. Write $g = g_1 + g_2$, where $g_1 = hg$ and $g_2 = (1-h)g$. Let I be defined by equation (2.4), and let I_1, I_2 be defined by the same equation for g replaced by g_1, g_2, so $I = I_1 + I_2$. Since f has a unique global minimum, we see by direct differentiation that for all n, $I_2^{(n)}(\hbar)$ is rapidly decaying as $\hbar \to 0$. Thus for $g = g_2$ (i.e., if $g = 0$ near c) the result is obvious, as $\inf_{x: g(x)\neq 0} f(x) > 0$. Thus we may assume without loss of generality that $g = g_1$ and $g_2 = 0$. We extend g by zero to the whole real line.

Let us make a change of variable $y := \hbar^{-\frac{1}{2}}x$. Then we get

$$I(\hbar) = \int_{-\infty}^{\infty} g(\hbar^{\frac{1}{2}}y) e^{-\frac{M}{2}y^2} dy. \tag{2.5}$$

Thus the result follows from (2.2) and Theorem 2.2. □

Remark 2.7. Theorem 2.6, in fact, provides an explicit formula for the Taylor coefficients of $I(\hbar)$. Namely, as in the proof of Theorem 2.6, assume that $c = 0$ and $f(x) = \frac{1}{2}p(x)^2$ near 0, where

$$p'(0) = \sqrt{f''(0)} > 0.$$

Ignoring limits of integration (which, as we have seen, are irrelevant for the asymptotic expansion of $I(\hbar)$), we have[5]

$$I(\hbar) = \hbar^{-\frac{1}{2}} \int g(x) e^{-\frac{p(x)^2}{2\hbar}} dx \sim \int_{-\infty}^{\infty} \tilde{g}(\hbar^{\frac{1}{2}} y) e^{-\frac{y^2}{2}} dy$$

where

$$\tilde{g}(z) := g(p^{-1}(z))(p^{-1})'(z) = \frac{g(p^{-1}(z))}{p'(p^{-1}(z))}.$$

By Theorem 2.2, the first $N+1$ terms of the Taylor expansion of this integral are given by the integral

$$I_N(\hbar) := \int_{-\infty}^{\infty} \tilde{g}_N(\hbar^{\frac{1}{2}} y) e^{-\frac{y^2}{2}} dy,$$

where $\tilde{g}_N$ is the $2N$th Taylor polynomial of $\tilde{g}$ at 0. Thus if $\tilde{g}(z) \sim \sum_{n=0}^{\infty} b_n z^n$, then

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} \hbar^n \int_{-\infty}^{\infty} y^{2n} e^{-\frac{y^2}{2}} dy.$$

But, setting $u = \frac{y^2}{2}$, we have

(2.6)
$$\int_{-\infty}^{\infty} y^{2n} e^{-\frac{y^2}{2}} dy = 2^{n+\frac{1}{2}} \int_0^{\infty} u^{n-\frac{1}{2}} e^{-u} du = 2^{n+\frac{1}{2}} \Gamma(n+\tfrac{1}{2}) = (2\pi)^{\frac{1}{2}} (2n-1)!!,$$

where $(2n-1)!! := \prod_{1 \le j \le n} (2j-1)$. Hence

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} 2^{n+\frac{1}{2}} \Gamma(n+\tfrac{1}{2}) \hbar^n.$$

2.4. Stationary phase formula

Theorem 2.6 has the following imaginary analogue, called the *stationary phase formula*.

[5] Recall that for $I \in C^\infty[0, \varepsilon)$, we write $I(\hbar) \sim \sum_{n=0}^{\infty} a_n \hbar^n$ if for every $N \ge 0$, we have $I(\hbar) = \sum_{n=0}^{N-1} a_n \hbar^n + O(\hbar^N)$ as $\hbar \to 0$.

Theorem 2.8 (Stationary phase formula)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be smooth functions. Assume that f has a unique critical point $c \in [a, b]$, such that $a < c < b$ and $f''(c) \neq 0$, and g has vanishing derivatives of all orders at a and b. Then*

$$\int_a^b g(x) e^{\frac{if(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{\frac{if(c)}{\hbar}} I(\hbar),$$

where $I(\hbar)$ extends to a smooth function on $[0, \infty)$ such that

$$I(0) = \sqrt{2\pi} e^{\pm\frac{\pi i}{4}} \frac{g(c)}{\sqrt{|f''(c)|}},$$

where $\pm$ is the sign of $f''(c)$.[6]

Remark 2.9. It is important to assume that g has vanishing derivatives of all orders at a and b. Otherwise, we will get additional boundary contributions.

Proof. The proof is analogous to the proof of the steepest descent formula, but slightly more subtle, as we have to keep track of cancellations. First we need the following very simple but important lemma which allows us to do so.

Lemma 2.10 (Riemann's lemma)**.**

(i) *Let $f : [a, b] \to \mathbb{R}$ be a smooth function such that $f'(x) > 0$ for all $x \in [a, b]$, and let $g : [a, b] \to \mathbb{R}$ be a C^n-function such that*

$$g(a) = \cdots = g^{(n-1)}(a) = g(b) = \cdots = g^{(n-1)}(b) = 0.$$

Let

$$I(\hbar) := \int_a^b g(x) e^{\frac{if(x)}{\hbar}} dx.$$

Then $I(\hbar) = O(\hbar^n)$, $\hbar \to 0$.

(ii) *Suppose g is smooth on $[a, b]$ and all derivatives of g at a and b are zero. Then I extends (by setting $I(0) := 0$) to a smooth function on $[0, \infty)$ all of whose derivatives are rapidly decaying as $\hbar \to 0$.*

Proof. (i) By making a change of variable, we may assume without loss of generality that $f(x) = x$. Then the proof is by induction in n. The base case $n = 0$ is obvious. For $n > 0$ note that

$$\int_a^b g(x) e^{\frac{ix}{\hbar}} dx = i\hbar \int_a^b g'(x) e^{\frac{ix}{\hbar}} dx$$

(integration by parts), which justifies the induction step.

(ii) follows from (i) by repeated differentiation.

□

[6] This is called the stationary phase formula because the main contribution comes from the point where the phase $\frac{f(x)}{\hbar}$ is stationary.

Now we proceed to prove the theorem. As in the proof of the steepest descent formula, we may assume that $c = 0$ and $f = \frac{M}{2}x^2$ near 0 for some $M \neq 0$, and write I as the sum $I_1 + I_2$. Moreover, by Lemma 2.10(ii),

$$I_2(\hbar) = \int_a^b g_2(x) e^{\frac{if(x)}{\hbar}} dx$$

is rapidly decaying with all derivatives, as

$$I_2(\hbar) = \int_a^{c_1} g_2(x) e^{\frac{if(x)}{\hbar}} dx + \int_{c_2}^b g_2(x) e^{\frac{if(x)}{\hbar}} dx$$

for some $c_1 < c < c_2$ and by our assumption the function f is monotonic with $f'(x) \neq 0$ on both $[a, c_1]$ and $[c_2, b]$. So it suffices to prove the theorem for $g = g_1$.

Again following the proof of the steepest descent formula, we have

$$I(\hbar) = \int_{-\infty}^{\infty} g(\hbar^{\frac{1}{2}} y) e^{\frac{iM}{2}y^2} dy, \tag{2.7}$$

so as before the result follows from (2.2) and Theorem 2.2. □

Remark 2.11. Since computation of the asymptotic expansion of $I(\hbar)$ is a purely algebraic procedure, the explicit formula for this expansion in the imaginary case is the same as in the real case (Remark 2.7) but with $\hbar$ replaced by $i\hbar$:

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} 2^{n+\frac{1}{2}} \Gamma(n + \tfrac{1}{2}) (i\hbar)^n.$$

2.5. Nonanalyticity of $I(\hbar)$ and Borel summation

Even though $I(\hbar)$ is smooth at $\hbar = 0$, its Taylor series is usually only an asymptotic expansion which diverges for any $\hbar \neq 0$, so that this function is not analytic at 0. To illustrate this, consider the integral

$$\int_{-\infty}^{\infty} e^{-\frac{x^2+x^4}{2\hbar}} dx = \hbar^{\frac{1}{2}} I(\hbar),$$

where

$$I(\hbar) = \int_{-\infty}^{\infty} e^{-\frac{y^2+\hbar y^4}{2}} dy. \tag{2.8}$$

Since this integral is divergent for any $\hbar < 0$, we cannot conclude its analyticity at $\hbar = 0$, and it indeed fails to be so. Namely, as in Remark 2.7, the asymptotic

expansion of integral (2.8) is obtained by expanding the exponential $e^{-\frac{1}{2}\hbar y^4}$ into a Taylor series and integrating termwise using (2.6):

$$I(\hbar) \sim \sum_{n=0}^{\infty} a_n \hbar^n,$$

where

$$a_n = (-1)^n \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{y^{4n}}{2^n n!} dy$$
$$= (-1)^n \frac{2^{n+\frac{1}{2}}\Gamma(2n+\frac{1}{2})}{n!} = (-1)^n \sqrt{2\pi} \frac{(4n-1)!!}{2^n n!}.$$

It is clear that this sequence has super-exponential growth, so the radius of convergence of the series is zero.

Let us now discuss the question: to what extent does the asymptotic expansion of the function $I(\hbar)$ (which we can find using Feynman diagrams as explained below) actually determine this function?

Suppose that

$$\tilde{I}(\hbar) = \sum_{n\geq 0} a_n \hbar^n$$

is a series with a zero radius of convergence. In general, we cannot uniquely determine a function I on $[0, \varepsilon)$ whose expansion is given by such a series: it always exists (a nontrivial but enjoyable exercise!) but in general there is no canonical choice (which is part of the reason why the exercise is nontrivial!). However, assume that the exponential generating function of a_n,

$$g(\hbar) = \sum_{n\geq 0} a_n \frac{\hbar^n}{n!},$$

is convergent in some neighborhood of 0, continues analytically to $[0, \infty)$, and has at most exponential growth as $\hbar \to +\infty$. In this case there is a "canonical" way to construct a smooth function I on $[0, \varepsilon)$ with (asymptotic) Taylor expansion $\tilde{I}$, called the *Borel summation* of $\tilde{I}$. Namely, the function I is defined by the formula

$$I(\hbar) = \int_0^{\infty} g(\hbar u) e^{-u} du = \hbar^{-1} \int_0^{\infty} g(u) e^{-\frac{u}{\hbar}} du,$$

i.e., $I(\hbar) = \hbar^{-1}(\mathcal{L}g)(\hbar^{-1})$, where $\mathcal{L}$ is the Laplace transform (note that since g grows at most exponentially at infinity, this is well defined for small enough $\hbar > 0$). Note that

$$I(\hbar) = \int_{-\infty}^{\infty} |v| g(\hbar v^2) e^{-v^2} dv = \hbar^{-\frac{1}{2}} \int_{-\infty}^{\infty} g_*(\hbar^{\frac{1}{2}} v) e^{-v^2} dv,$$

where $g_*(v) = |v| g(v^2)$. Thus Exercise 2.5 implies that to compute the asymptotic expansion of I, we may replace g by its Taylor polynomials at 0. Hence the identity $\int_0^{\infty} x^n e^{-x} dx = n!$ implies that I has the Taylor expansion given by $\tilde{I}$.

For example, consider the divergent series

$$\tilde{I} := \sum_{n\geq 0} (-1)^n n!\, \hbar^n.$$

Then

$$g(\hbar) = \sum_{n\geq 0} (-1)^n \hbar^n = \frac{1}{1+\hbar}.$$

Hence, the Borel summation yields

$$I(\hbar) = \int_0^\infty \frac{e^{-u}}{1+\hbar u} du = \hbar^{-1} e^{\hbar^{-1}} E_1(\hbar^{-1}),$$

where $E_1(x) := \int_x^\infty \frac{e^{-u}}{u} du$ is the exponential integral.

Physicists expect that in physically interesting situations perturbation expansions in quantum field theory are Borel summable, and the actual answers are obtained from these expansions by Borel summation. The Borel summability of perturbation series has actually been established in a few nontrivial examples of QFT, for example in quantum mechanics (see [**Ni**] and references therein).

Exercise 2.12. Show that the function given by (2.8) equals the Borel sum of its asymptotic expansion.

Hint. The function $g(z)$ in this example is a special case of the hypergeometric function ${}_2F_1$ which does not express in elementary functions, but it satisfies a hypergeometric differential equation. Write down this equation and show that the Laplace transform turns it into another second-order linear differential equation and that the function $I(\hbar)$ given by (2.8) satisfies this equation.

2.6. Application of steepest descent

Let us provide an application of Theorem 2.6. Consider the integral

$$\Gamma(s+1) = \int_0^\infty t^s e^{-t} dt,\ s > 0.$$

By doing a change of variable $t = sx$, we get

$$\frac{\Gamma(s+1)}{s^{s+1}} = \int_0^\infty x^s e^{-sx} dx = \int_0^\infty e^{-s(x-\log x)} dx.$$

Thus, we can apply Theorem 2.6 for $\hbar = \frac{1}{s}$, $f(x) = x - \log x$, $g(x) = 1$. Of course, the interval $[a, b]$ is now infinite, and the function f blows up on the boundary, but one can easily see that the theorem is still applicable, with the

same proof. The function $f(x) = x - \log x$ has a unique critical point on $[0, \infty)$, which is $c = 1$, and we have $f''(c) = 1$. Then we get

$$\Gamma(s+1) \sim s^s e^{-s}\sqrt{2\pi s}(1 + \tfrac{a_1}{s} + \tfrac{a_2}{s^2} + \cdots),\ s \to \infty. \tag{2.9}$$

This is the celebrated *Stirling formula*.

Moreover, we can compute the coefficients $a_1, a_2, \ldots$ using Remark 2.7. Namely,

$$p(x) = \sqrt{2(x - \log(1+x))} = x\sqrt{1 - \tfrac{2x}{3} + \tfrac{x^2}{2} - \ldots} = x - \tfrac{x^2}{3} + \tfrac{7x^3}{36} + \ldots.$$

Thus

$$p^{-1}(z) = z + \tfrac{z^2}{3} + \tfrac{z^3}{36} + \ldots,$$

hence

$$(p^{-1})'(z) = 1 + \tfrac{2z}{3} + \tfrac{z^2}{12} + \ldots,$$

So for instance by Remark 2.7 $a_1 = b_2 = \frac{1}{12}$.

Remark 2.13. Another way to compute this asymptotic expansion is to use the Euler product formula for the Gamma function. Differentiating the logarithm of this formula twice, we obtain (for $z > 0$)

$$(\log\Gamma)''(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} = \sum_{n=0}^{\infty} \int_0^{\infty} te^{-(z+n)t}dt = \int_0^{\infty} \frac{te^{-zt}}{1-e^{-t}}dt.$$

Recall that the *Bernoulli numbers* are defined by the generating function

$$\sum_{n\geq 0} \frac{B_n t^n}{n!} = \frac{t}{1-e^{-t}},$$

e.g., $B_0 = 1, B_1 = \frac{1}{2}, B_{2n+1} = 0$ for $n \geq 1$. Thus we get for $z \to \infty$

$$(\log\Gamma)''(z) \sim \sum_{n\geq 0} B_n z^{-n-1}.$$

Integrating, we get

$$(\log\Gamma)'(z) \sim \log z + C_1 - \sum_{n\geq 1} \frac{B_n}{n} z^{-n},$$

so integrating again and adding $\log z$, we get

$$\log\Gamma(z+1) \sim z\log z - z + C_1 z + \frac{1}{2}\log z + C_2 + \sum_{n\geq 2} \frac{B_n}{n(n-1)} z^{-n+1}.$$

From Stirling's formula we have $C_1 = 0, C_2 = \frac{1}{2}\log(2\pi)$, so in the end we get

$$(\log\Gamma)'(z) \sim \log z - \sum_{n\geq 1} \frac{B_n}{n} z^{-n}, \tag{2.10}$$

$$\log\Gamma(z+1) \sim z\log z - z + \frac{1}{2}\log z + \frac{1}{2}\log(2\pi) + \sum_{n\geq 2} \frac{B_n}{n(n-1)} z^{-n+1}. \tag{2.11}$$

So

$$1 + \tfrac{a_1}{s} + \tfrac{a_2}{s^2} + \cdots = \exp(\sum_{n\geq 2} \tfrac{B_n}{n(n-1)} s^{-n+1}).$$

In particular, since $B_2 = \frac{1}{6}$, we get $a_1 = \frac{1}{12}$.

Exercise 2.14. Calculate $\int_0^\pi \sin^n x dx$ for nonnegative integers n using integration by parts. Then apply steepest descent to this integral and discover a formula for π (the so-called *Wallis formula*).

Exercise 2.15. The Bessel function $I_0(a)$ is defined by the formula

$$I_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{a\cos\theta} d\theta.$$

It is an even entire function with Taylor expansion

$$I_0(a) = \sum_{n=0}^{\infty} \frac{a^{2n}}{2^{2n} n!^2}.$$

Use the steepest descent/stationary phase formulas to find the asymptotic expansion of $I_0(a)$ as $a \to +\infty$ and $a \to i\infty$. Compute the first two terms of the expansion (cf. Remark 2.22).

2.7. Multidimensional versions of steepest descent and stationary phase

Theorems 2.6 and 2.8 have multidimensional analogues. To formulate them, let V be a real vector space of dimension d with a fixed volume element dx, and let $D \subset V$ be a compact region with smooth boundary.[7]

Theorem 2.16 (Multidimensional steepest descent formula)**.** *Let f, g be continuous functions $D \to \mathbb{R}$ which are smooth in the interior of D. Assume that f achieves global minimum on D at a unique point c, such that c is an interior point and* $\mathrm{Hess}(f)(c) > 0$. *Then*

$$\int_D g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{-\frac{f(c)}{\hbar}} I(\hbar), \tag{2.12}$$

[7]The condition of smooth boundary is introduced for simplicity of exposition only and is not essential. The same results and proofs apply with trivial modifications to more general regions, e.g., those whose boundary is only piecewise smooth in an appropriate sense.

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ such that

$$I(0) = (2\pi)^{\frac{d}{2}} \frac{g(c)}{\sqrt{\det \mathrm{Hess}(f)(c)}}.$$

Theorem 2.17 (Multidimensional stationary phase formula)**.** *Let $f, g : D \to \mathbb{R}$ be smooth functions. Assume that f has a unique critical point c in D, such that c is an interior point with $\det \mathrm{Hess}(f)(c) \neq 0$, and g has vanishing derivatives of all orders on ∂D. Then*

$$\int_D g(x) e^{\frac{\mathrm{i} f(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{\frac{\mathrm{i} f(c)}{\hbar}} I(\hbar), \tag{2.13}$$

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ such that

$$I(0) = (2\pi)^{\frac{d}{2}} e^{\frac{\pi \mathrm{i} \sigma}{4}} \frac{g(c)}{\sqrt{|\det \mathrm{Hess}(f)(c)|}},$$

where σ is the signature of the symmetric bilinear form $\mathrm{Hess}(f)(c)$.

2.8. Morse lemma

For the proof of these theorems it is convenient to use a fundamental result in multivariable calculus called the *Morse lemma*. This lemma easily follows by induction in dimension from the following theorem.

Theorem 2.18 (Separation of variables)**.** *Let f be a smooth function on an open ball $0 \in B \subset \mathbb{R}^d$ which has a nondegenerate critical point at 0, and suppose $f(0) = 0$. Then there is a local coordinate system near 0 (possibly defined in a smaller ball) in which*

$$f(x_1, \dots, x_d) = f(x_1, \dots, x_{d-1}) \pm x_d^2.$$

Proof. By making a linear change of variables, we can assume that the quadratic part of f has the form $Q(y) \pm u^2$, where $y := (x_1, \dots, x_{d-1})$, $u := x_d$. Consider the hypersurface S defined by the equation

$$\partial_u f(y,u) = 0.$$

The linear part of $\partial_u f(y,u)$ is $\pm 2u$, so by the implicit function theorem there is a change of coordinates F near 0 (with $dF(0) = 1$) in which u is replaced by $v := \pm\frac{1}{2}\partial_u f(y,u)$ and y is kept unchanged; so $u = g(y,v)$ for some function g with $(\partial_v g)(0,0) \neq 0$. Let

$$f_*(y,v) := f(y,u) = f(y, g(y,v)).$$

Then by the chain rule

$$\partial_v f_*(y,v) = \partial_u f_*(y,v) \frac{\partial u}{\partial v} = \partial_u f(y,u) \frac{\partial u}{\partial v} = \pm 2v \partial_v g(y,v).$$

Thus the hypersurface S in the new coordinates is defined by the equation $v = 0$. So we may assume without loss of generality that S is given by the equation $u = 0$ to start with. Then $(\partial_u f)(y, 0) = 0$, so

$$f(y, u) - f(y, 0) = h(y, u)u^2,$$

where h is a smooth function in B with $h(0, 0) = \pm 1$. By replacing u with $\tilde{u} := \sqrt{|h(y, u)|}u$ and keeping y unchanged, we may assume that $h = \pm 1$. Then

$$f(u, y) = f(0, y) \pm u^2,$$

as claimed. □

Corollary 2.19 (Morse lemma)**.** *Let f be a smooth function on an open ball $0 \in B \subset \mathbb{R}^d$ which has a nondegenerate critical point at 0, and suppose $f(0) = 0$. Then there is a local coordinate system $(x_1, \dots, x_d)$ near 0 (possibly defined in a smaller ball) in which*

$$f = x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_d^2.$$

In other words, near a nondegenerate critical point a smooth function is equivalent by a change of coordinates to its quadratic part.

Proof. As mentioned above, this follows easily from Theorem 2.18 by induction in dimension. □

Exercise 2.20. Let f be a smooth function on $\mathbb{R}^2$ which is a cubic polynomial in x:

$$f(x, y) = a(y) + b(y)x + c(y)x^2 + d(y)x^3.$$

Assume that $a(0) = a'(0) = 0$, $b(0) = b'(0) = 0$, $a''(0) = c(0) = 2$. Find explicitly local coordinates $u = u(x, y), v = v(x, y)$ near 0 in which $f(x, y) = u^2 + v^2$.

2.9. Proof of the multidimensional steepest descent and stationary phase formulas

The proofs of the multidimensional steepest descent and stationary phase formulas are parallel to the proofs of their 1-dimensional versions, using the Morse lemma. Namely, the Morse lemma allows us to assume without loss of generality that f is quadratic near the critical point. After this, the proof of the steepest descent formula is identical to the 1-variable case. The same applies to the stationary phase formula, using the following multivariable analogue of the Riemann lemma.

Lemma 2.21. *Let $f, g : D \to \mathbb{R}$ be smooth functions such that all derivatives of g vanish on ∂D and df does not vanish anywhere on the support of g. Then the*

function

$$I(\hbar) := \int_D g(x) e^{\frac{\mathrm{i} f(x)}{\hbar}} dx$$

extends to a smooth function on $[0, \infty)$ *and has rapidly decaying derivatives of all orders as* $\hbar \to 0$.

Proof. Since df does not vanish on $\mathrm{supp}(g)$, we can cover $\mathrm{supp}(g)$ by local charts U_i in which $f(x)$ is the last coordinate x_d. By compactness this cover can be chosen finite. By using a partition of unity $\{h_i\}$ on $\mathrm{supp}\, g$ subordinate to this cover and replacing g with $h_i g$, we may assume without loss of generality that g is supported on a single chart. Then changing variables, we may also assume that $f(x) = x_d$. Then integrating out the variables $x_1, \ldots, x_{d-1}$, we reduce to the 1-dimensional case covered by Lemma 2.10. □

Remark 2.22. It is clear from the proof of the stationary phase formula that it extends to the case when f may have several critical points but all of them are interior and nondegenerate. In this case the asymptotic expansions coming from different critical points are simply added together. The same applies to the steepest descent formula if the global minimum is attained at several points, all of which are interior and nondegenerate.

Chapter 3

Feynman calculus

3.1. Wick's theorem

Let V be a real vector space of dimension d with volume element dx. Let $S(x)$ be a smooth function on a compact region $D \subset V$ with smooth boundary which attains its minimum at a unique point $c \in D$ in the interior of D, and let g be any smooth function on D. In Chapter 2 we proved the steepest descent formula, which implies that the function

$$I(\hbar) = \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D g(x) e^{-\frac{S(x)}{\hbar}} dx$$

admits an asymptotic power series expansion in $\hbar$:

$$I(\hbar) \sim a_0 + a_1\hbar + \cdots + a_m\hbar^m + \cdots. \tag{3.1}$$

Our main question now will be: how do we compute the coefficients a_i?

Our proof of the steepest descent formula shows that although the problem of computing $I(\hbar)$ is transcendental, the problem of computing the coefficients a_i is, in fact, purely algebraic, and it involves only differentiation of the functions S and g at the point c. Indeed, recalling the proof of equation (3.1), we see that the calculation of a_i is reduced to calculation of integrals of the form

$$\int_V P(x) e^{-\frac{B(x,x)}{2}} dx,$$

where P is a polynomial and B is a positive definite bilinear form (in fact, $B(v,u) = (\partial_v \partial_u S)(c)$). But such integrals can be exactly evaluated. Namely, it is sufficient to consider the case when P is a product of linear functions, in which case the answer is given by the following elementary formula, known to physicists as *Wick's theorem.*

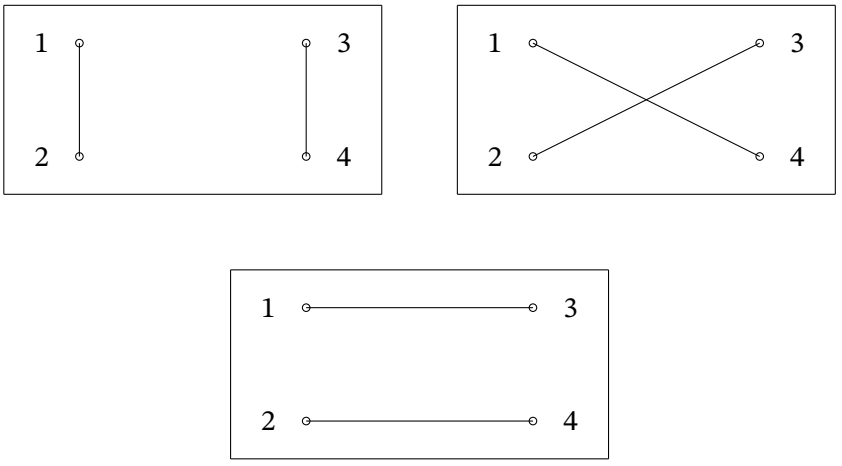


Figure 3.1. Matchings of the set $\{1, 2, 3, 4\}$

For a positive integer k, consider the set $\{1, \dots, 2k\}$. By a *matching* σ on this set, we will mean its partition into k disjoint two-element subsets (pairs). A matching can be visualized by drawing $2k$ points and connecting two points with an edge if they belong to the same pair; see Figure 3.1. This will give k edges which are not connected to each other.

Let us denote the set of matchings on a set T by $\Pi(T)$ and on the set $\Pi(\{1, \dots, 2k\})$ by Π_k. It is clear that $|\Pi_k| = \frac{(2k)!}{2^k \cdot k!} = (2k-1)!!$. For any $\sigma \in \Pi_k$, we can think of σ as a permutation of $\{1, \dots, 2k\}$, such that $\sigma^2 = 1$ and σ has no fixed points. Namely, σ maps any element i to the second element $\sigma(i)$ of the pair containing i.

Theorem 3.1 (Wick's theorem). *Let B^{-1} denote the inverse form to B on V^*, and let $\ell_1, \dots, \ell_N \in V^*$. Then, if N is even, we have*

$$\int_V \ell_1(x) \dots \ell_N(x) e^{-\frac{B(x,x)}{2}} dx = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\sigma \in \Pi_{N/2}} \prod_{i \in \{1,\dots,N\}/\sigma} B^{-1}(\ell_i, \ell_{\sigma(i)}).$$

If N is odd, the integral is zero.

Proof. If N is odd, the statement is obvious, because the integrand is an odd function. So consider the even case $N = 2k$. Since both sides of the equation are symmetric polylinear forms in $\ell_1, \dots, \ell_N$, it suffices to prove the result when $\ell_1 = \dots = \ell_N = \ell$. Further, it is clear that the formula in question is stable under linear changes of variables, so we can choose a coordinate system in such a way that $B(x, x) = x_1^2 + \dots + x_d^2$, and $\ell(x) = x_1$. Therefore, it is sufficient to

assume that $d = 1$ and $\ell(x) = x$. In this case, the theorem says that

$$\int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2}} dx = (2\pi)^{\frac{1}{2}} (2k-1)!!,$$

which is formula (2.6). □

Example 3.2. We have

$$\int_V \ell_1(x)\ell_2(x) e^{-\frac{B(x,x)}{2}} dx = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} B^{-1}(\ell_1, \ell_2),$$

$$\int_V \ell_1(x)\ell_2(x)\ell_3(x)\ell_4(x) e^{-\frac{B(x,x)}{2}} dx$$

$$= \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} (B^{-1}(\ell_1, \ell_2)B^{-1}(\ell_3, \ell_4) + B^{-1}(\ell_1, \ell_3)B^{-1}(\ell_2, \ell_4) + B^{-1}(\ell_1, \ell_4)B^{-1}(\ell_2, \ell_3)).$$

Wick's theorem shows that the problem of computing a_i is of a combinatorial nature. In fact, the central role in this computation is played by certain finite graphs, which are called *Feynman diagrams*. They are the main subject of the remainder of this chapter.

3.2. Feynman diagrams and Feynman's theorem

We come back to the problem of computing the coefficients a_i. Since each particular a_i depends only on a finite number of derivatives of g at c, it suffices to assume that g is a polynomial or, more specifically, a product of linear functions: $g = \ell_1 \dots \ell_N$, $\ell_i \in V^*$. Thus, it suffices to be able to compute the series expansion of the integral

$$\langle \ell_1 \dots \ell_N \rangle := \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D \ell_1(x) \dots \ell_N(x) e^{-\frac{S(x)}{\hbar}} dx. \tag{3.2}$$

Without loss of generality we may assume that $c = 0$ and $S(c) = 0$ (so $dS(c) = 0$). Then the (asymptotic) Taylor expansion of S at c is

$$S(x) = \frac{B(x,x)}{2} - \sum_{i \geq 3} \frac{B_i(x, \dots, x)}{i!},$$

where $B_i := -d^i S(0)$. Therefore, regarding the left-hand side of (3.2) as a power series in $\hbar$ and making a change of variable $x \mapsto \hbar^{\frac{1}{2}} x$ (as in Section 3.1) we get

$$\langle \ell_1 \dots \ell_N \rangle = \hbar^{\frac{N}{2}} \int_V \ell_1(x) \dots \ell_N(x) e^{-\frac{B(x,x)}{2} + \sum_{i \geq 3} \hbar^{\frac{i}{2}-1} \frac{B_i(x,\dots,x)}{i!}} dx.$$

Note that this is only an identity of asymptotic expansions in $\hbar$, as we ignored the rapidly decaying error which comes from replacing the region D by the

whole space. But it implies in particular that $\langle \ell_1 \dots \ell_N \rangle = O(\hbar^{\lceil \frac{N}{2} \rceil})$ as $\hbar \to 0$ (as the expansion contains only integer powers of $\hbar$).

Theorem 3.5, due to Feynman, gives the value of this integral in terms of Feynman diagrams. This theorem is easy to prove but is central in quantum field theory, and it will be one of our main theorems. Before formulating Feynman's theorem, let us introduce some notation.

Let $G_{\geq 3}(N)$ be the set of isomorphism classes of graphs with N 1-valent "external" vertices, labeled by $1, \dots, N$, and a finite number of unlabeled "internal" vertices, of any valency ≥ 3. Note that here and below graphs are allowed to have multiple edges between two vertices and loops from a vertex to itself; see Figure 3.2.

For each graph $\Gamma \in G_{\geq 3}(N)$, we define the *Feynman amplitude* of Γ using the following *Feynman rules*.

1. Put the covector ℓ_j at the jth external vertex.

2. Put the tensor B_i at each i-valent internal vertex.

3. Take the contraction of the tensors along the edges of Γ, using the bilinear form B^{-1}. This will produce a number, called the (*Feynman*) *amplitude* of Γ that is denoted $F_\Gamma(\ell_1, \dots, \ell_N)$.

Remark 3.3. If Γ is not connected, then F_Γ is defined to be the product of the numbers obtained from the connected components. Also, the amplitude of the empty diagram is defined to be 1.

Example 3.4. Let

$$B_3 := \sum_i b_i^{13} \otimes b_i^{23} \otimes b_i^{33}, \; B_4 := \sum_j b_j^{14} \otimes b_j^{24} \otimes b_j^{34} \otimes b_j^{44},$$

where $b_i^{jk} \in V^*$. Then for the graph Γ_3 in Figure 3.2 the amplitude equals

$$\begin{aligned} F_{\Gamma_3}(\ell_1, \ell_2) &= \sum_i B^{-1}(\ell_1, b_i^{13}) B^{-1}(b_i^{23}, b_i^{33}) \\ &\quad \cdot \sum_{i,j} B^{-1}(b_i^{13}, b_j^{14}) B^{-1}(b_i^{23}, b_j^{24}) B^{-1}(b_i^{33}, b_j^{34}) B^{-1}(b_j^{44}, \ell_2). \end{aligned}$$

Theorem 3.5 (Feynman)**.** *One has*

$$\langle \ell_1 \dots \ell_N \rangle = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\Gamma \in G_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N), \tag{3.3}$$

where $b(\Gamma)$ is the number of edges minus the number of internal vertices of the graph Γ.

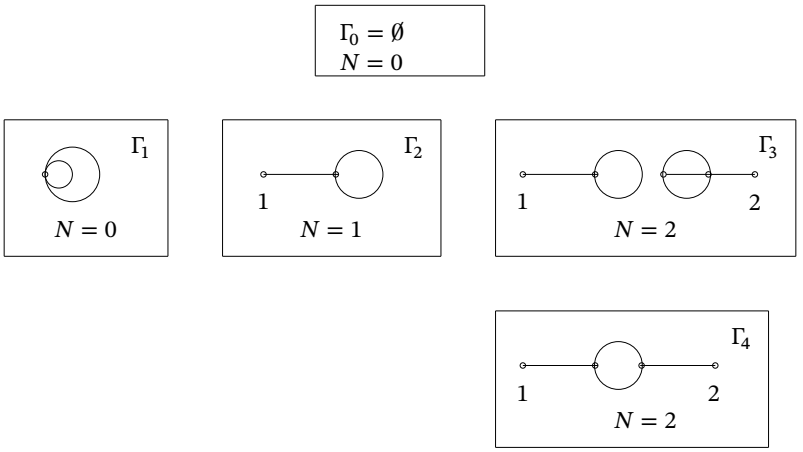


Figure 3.2. Examples of elements of $G_{\geq 3}(N)$.

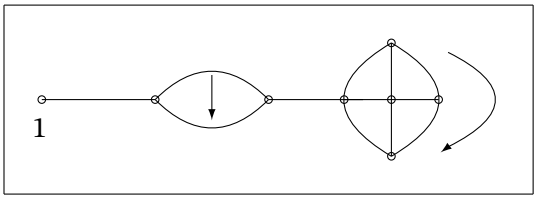


Figure 3.3. An automorphism of a graph

Here $\mathrm{Aut}(\Gamma)$ denotes the group of automorphisms of Γ, and by an automorphism of Γ we mean a permutation of vertices *and* edges (possibly flipping the self-loops) which fixes each external vertex and preserves the graph structure; see Figure 3.3. Thus there can exist nontrivial automorphisms which act trivially on the set of vertices and even ones also acting trivially on the set of edges. For example, there is an automorphism of Γ_4 that flips the upper and lower arc, and an automorphism of Γ_2 that flips the self-loop.

Remark 3.6.

1. Note that this sum is infinite but $\hbar$-adically convergent.

2. Theorem 3.5 is a generalization of Wick's theorem: the latter is obtained if $S(x) = \frac{B(x,x)}{2}$. Indeed, in this case graphs which give nonzero amplitudes do not have internal vertices, and thus reduce to graphs corresponding to matching σ.

Let us now make some comments about the terminology. In quantum field theory, the function $\langle \ell_1 \dots \ell_N \rangle$ is called the *N-point correlation function* and graphs Γ are called *Feynman diagrams*. The form B^{-1} which is put on the edges is called the *propagator*. The cubic and higher terms $\frac{B_i}{i!}$ in the expansion

of the function S are called *interaction terms*, since such terms (in the action functional) describe interaction between particles. The situation in which S is quadratic (i.e., there is no interaction) is called a *free theory*; i.e., for the free theory the correlation functions are determined by Wick's formula.

Remark 3.7. Sometimes it is convenient to consider normalized correlation functions

$$\langle \ell_1 \dots \ell_N \rangle_{\text{norm}} := \frac{\langle \ell_1 \dots \ell_N \rangle}{\langle \emptyset \rangle},$$

where $\langle \emptyset \rangle$ denotes the integral without insertions. Feynman's theorem implies that they are given by the formula

$$\langle \ell_1 \dots \ell_N \rangle_{\text{norm}} = \sum_{\Gamma \in G^*_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N),$$

where $G^*_{\geq 3}(N)$ is the subset of all graphs in $G_{\geq 3}(N)$ which have no components without external vertices.

3.3. A weighted version of Feynman's theorem

Before proving Theorem 3.5, we would like to slightly modify and generalize it. Namely, in quantum field theory it is often useful to consider an interacting theory as a deformation of a free theory. This means that $S(x) = \frac{B(x,x)}{2} + \tilde{S}(x)$, where $\tilde{S}(x)$ is a perturbation

$$\tilde{S}(x) := -\sum_{i \geq 0} g_i \frac{B_i(x, \dots, x)}{i!}$$

in which $g_r, r \geq 0$, are (formal) parameters. One benefit of these parameters is that they will allow us to group the amplitudes of Feynman diagrams in the sum (3.3) by the numbers of vertices of each valency. Namely, consider the *partition function*

$$Z = \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx$$

as a series in g_i. Let $\mathbf{n} = (n_0, n_1, n_2, \dots)$ be a sequence of nonnegative integers, almost all zero. Let $G(\mathbf{n})$ denote the set of isomorphism classes of graphs with n_0 0-valent vertices, n_1 1-valent vertices, n_2 2-valent vertices, etc.; thus, now we are considering graphs without external vertices. For $\Gamma \in G(\mathbf{n})$, let F_Γ be the amplitude of Γ defined as before. Thus

$$F_\Gamma = \prod_i g_i^{n_i} \cdot \mathbb{F}_\Gamma,$$

where $\mathbb{F}_\Gamma$ is the Feynman amplitude computed without the factors g_j.

Theorem 3.8. *One has*

$$Z = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\mathbf{n}} \sum_{\Gamma \in G(\mathbf{n})} \frac{\hbar^{b(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} F_\Gamma$$
$$= \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\operatorname{Aut}(\Gamma)|},$$

where $b(\Gamma) = \sum_i n_i(\frac{i}{2} - 1)$ *is the number of edges minus the number of vertices of* Γ.

Note that we may view Z as an element of the algebra

$$\mathbb{C}[g_0\hbar^{-1}, g_1\hbar^{-\frac{1}{2}}, g_j, j \geq 2][[\hbar^{\frac{1}{2}}]],$$

i.e., it can be specialized to numerical values of

$$g_0\hbar^{-1}, g_1\hbar^{-\frac{1}{2}}, g_2, g_3, \dots,$$

giving an element of $\mathbb{C}[[\hbar^{\frac{1}{2}}]]$. Also Z can be specialized to $\hbar = 1$, giving an element of $\mathbb{C}[[g_j, j \geq 0]]$, and the theorem is, in fact, equivalent to this specialization. Still we choose to keep $\hbar$ to be able to take the classical limit $\hbar \to 0$.

We will prove Theorem 3.8 in Section 3.4. Meanwhile, let us show that Theorem 3.5 is in fact a special case of Theorem 3.8. Indeed, because of symmetry of the correlation functions with respect to $\ell_1, \dots, \ell_N$, it is sufficient to consider the case $\ell_1 = \dots = \ell_N = \ell$. In this case, denote the correlation function $\langle \ell^N \rangle$ (the *expectation value* of ℓ^N). Clearly, to compute $\langle \ell^N \rangle$ for all N, it is sufficient to compute the generating function

$$\langle e^\ell \rangle = \hbar^{-\frac{d}{2}} \int_V e^{\ell(x) - \frac{S(x)}{\hbar}} dx := \sum_{N=0}^{\infty} \frac{\langle \ell^N \rangle}{N!},$$

which up to scaling and multiplication of ℓ by i is the Fourier transform of the Feynman density $e^{-\frac{S(x)}{\hbar}} dx$. But this expectation value is exactly the one given by Theorem 3.8 for $g_j = 1$, $j \geq 3$, $g_0 = g_2 = 0$, $g_1 = \hbar$, $B_1 = \ell$, $B_0 = 0$, $B_2 = 0$. Thus, Theorem 3.8 implies Theorem 3.5 (the factor $N!$ in the denominator is accounted for by the fact that in Theorem 3.8 we consider unlabeled, rather than labeled, 1-valent vertices).

3.4. Proof of Feynman's theorem

Now we will prove Theorem 3.8. Let us make a change of variable $y = \hbar^{-\frac{1}{2}} x$. Expanding the exponential in a Taylor series, we obtain

$$Z = \sum_{\mathbf{n}} Z_{\mathbf{n}},$$

where

$$Z_{\mathbf{n}} = \int_V e^{-\frac{B(y,y)}{2}} \prod_i \frac{g_i^{n_i}}{i!^{n_i}\, n_i!} (\hbar^{\frac{i}{2}-1} B_i(y,\dots,y))^{n_i} dy.$$

Writing B_i as a sum of products of linear functions, and using Wick's theorem, we find that the value of the integral for each $\mathbf{n}$ can be expressed combinatorially as follows.

1. Attach to each factor B_i a "flower"—a vertex with i outgoing edges; see Figure 3.4.

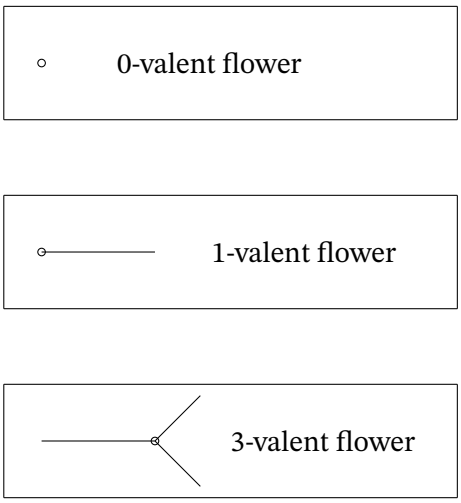


Figure 3.4

2. Consider the set $T_{\mathbf{n}}$ of ends of these outgoing edges (see Figure 3.5), and for any matching σ of this set, consider the corresponding contraction of the tensors B_i using the form B^{-1}. This will produce a scalar $\mathbb{F}(\sigma)$.

3. The integral $Z_{\mathbf{n}}$ is given by

$$Z_{\mathbf{n}} = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \prod_i \frac{g_i^{n_i}}{i!^{n_i}\, n_i!} \hbar^{n_i(\frac{i}{2}-1)} \sum_{\sigma \in \Pi(T_{\mathbf{n}})} \mathbb{F}(\sigma). \tag{3.4}$$

Now, recall that matchings on a set can be visualized by drawing its elements as points and connecting them with edges. If we do this with the set $T_{\mathbf{n}}$, all ends of outgoing edges will become connected with each other in some way, i.e., we will obtain a certain (unoriented) graph $\Gamma = \Gamma_\sigma$; see Figure 3.6. Moreover, it is easy to see that the scalar $\mathbb{F}(\sigma)$ is nothing but the amplitude $\mathbb{F}_\Gamma$.

It is clear that any graph Γ with n_i i-valent vertices for each i can be obtained in this way. However, the same graph can be obtained in many different ways, so if we want to collect identical terms in the sum over σ, and turn it into a sum over Γ, we must find the number of σ which yield a given Γ.

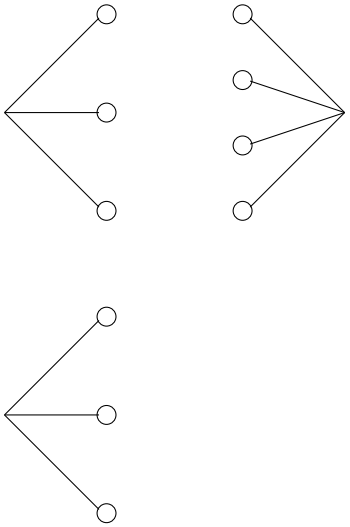


Figure 3.5. The set $T_{\mathbf{n}}$ for $\mathbf{n} = (0, 0, 0, 2, 1, 0, 0, \ldots)$ (the set of white circles)

For this purpose, we will consider the group $\mathbb{G}_{\mathbf{n}}$ of permutations of $T_{\mathbf{n}}$, which preserves "flowers" (i.e., endpoints of any two edges outgoing from the same flower end up again in the same flower). This group involves

1) permutations of "flowers" with a given valency;

2) permutations of the i edges inside each i-valent "flower".

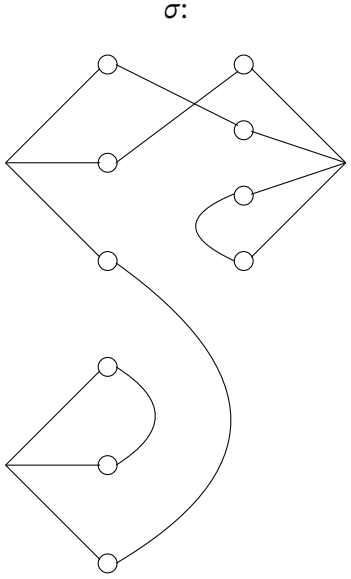


Figure 3.6. A matching σ of $T_{\mathbf{n}}$ and the corresponding graph Γ

More precisely, the group $\mathbb{G}_{\mathbf{n}}$ is the semidirect product of symmetric groups

$$\mathbb{G}_{\mathbf{n}} = \prod_i (S_{n_i} \ltimes S_i^{n_i}).$$

Note that $|\mathbb{G}_{\mathbf{n}}| = \prod_i i!^{n_i} n_i!$, which is the product of the numbers in the denominator of formula (3.4).

The group $\mathbb{G}_{\mathbf{n}}$ acts on the set $\Pi(T_{\mathbf{n}})$ of all matchings σ of $T_{\mathbf{n}}$. Moreover, it acts transitively on the set $\Pi_\Gamma(T_{\mathbf{n}})$ of matchings of $T_{\mathbf{n}}$ which yield a given graph Γ. Furthermore, it is easy to see that the stabilizer of a given matching is $\mathrm{Aut}(\Gamma)$. Thus, the number of matchings giving Γ is

$$N_\Gamma = \frac{\prod_i i!^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|}.$$

Hence,

$$\sum_{\sigma \in \Pi(T_{\mathbf{n}})} \mathbb{F}(\sigma) = \sum_\Gamma \frac{\prod_i i!^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|} \mathbb{F}_\Gamma.$$

Finally, note that the exponent of $\hbar$ in equation (3.4) is $\sum_i n_i(\frac{i}{2} - 1)$, which is the number of edges of Γ minus the number of vertices, i.e., $b(\Gamma)$. Substituting this into (3.4), we get the result.

Example 3.9. Let $d = 1$, $V = \mathbb{R}$, $g_i = a$, $B_i = z^i$ for all $i \geq 0$ (where z is a formal variable), $\hbar = 1$. Then we find the asymptotic expansion

$$\frac{1}{\sqrt{2\pi}} \int e^{-x^2/2 + ae^{zx}}\, dx = \sum_{n,k \geq 0} a^n \sum_{\Gamma \in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}\,\Gamma|},$$

where $G(n,k)$ is the set of isomorphism classes of graphs with n vertices and k edges.[8] Expanding the left-hand side, we get

$$\sum_k \sum_{\Gamma \in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|} = \frac{e^{\frac{z^2 n^2}{2}}}{n!},$$

and hence

$$\sum_{\Gamma \in G(n,k)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{2k}}{2^k k!\, n!}.$$

Exercise 3.10. Check this by direct combinatorics.

[8] This integral converges for $a < 0$, $z \in \mathbb{R}$, but this is not important for us here, since we consider the integral formally.

3.5. Sum over connected diagrams

Now we will show that the logarithm of the partition function Z is also given by summation over diagrams, but with only connected diagrams taken into account. This significantly simplifies the analysis of Z in the first few orders of perturbation theory, since the number of connected diagrams with a given number of vertices and edges is significantly smaller than the number of all diagrams.

Theorem 3.11. *Let* $Z_0 = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}}$. *Then one has*

$$\log \frac{Z}{Z_0} = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G_c(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\operatorname{Aut}(\Gamma)|},$$

where $G_c(\mathbf{n})$ *is the set of connected graphs in* $G(\mathbf{n})$.[9]

Proof. For any graphs Γ_1, Γ_2, let $\Gamma_1\Gamma_2$ stand for the disjoint union of Γ_1 and Γ_2, and for any graph Γ let Γ^n denote the disjoint union of n copies of Γ. Then every graph can be uniquely written as $\Gamma_1^{k_1} \dots \Gamma_l^{k_l}$, where Γ_j are connected nonisomorphic graphs. Moreover, it is clear that $\mathbb{F}_{\Gamma_1\Gamma_2} = \mathbb{F}_{\Gamma_1}\mathbb{F}_{\Gamma_2}$, $b(\Gamma_1\Gamma_2) = b(\Gamma_1) + b(\Gamma_2)$, and

$$|\operatorname{Aut}(\Gamma_1^{k_1} \dots \Gamma_l^{k_l})| = \prod_j |\operatorname{Aut}(\Gamma_j)|^{k_j} k_j!\,.$$

Thus, exponentiating the connected-diagram expansion and using the above facts together with the Taylor series for e^x, we recover the formula of Theorem 3.8. This shows that Theorems 3.8 and 3.11 are equivalent; since Theorem 3.8 is already proved, Theorem 3.11 follows. □

3.6. The loop expansion

Note that since summation in Theorem 3.11 is over connected Feynman diagrams, the number $b(\Gamma)$ is the number of loops in Γ minus 1. In particular, the lowest coefficient in $\hbar$ is that of $\hbar^{-1}$, and it is the sum over all trees; the next coefficient is to $\hbar^0$, and it is the sum over all diagrams with one loop (cycle); the next coefficient to $\hbar$ is the sum over 2-loop diagrams, and so on. Therefore, physicists refer to the expansion of Theorem 3.11 as the *loop expansion*.

Let us study the two most singular terms in this expansion (with respect to $\hbar$), i.e., the terms given by the sum over trees and 1-loop graphs.

Let x_0 be the critical point of the function S. It exists and is unique, since g_i are assumed to be formal parameters, so S is "infinitesimally close" to a quadratic function. Let $G^{(j)}(\mathbf{n})$ denote the set of classes of graphs in $G_c(\mathbf{n})$ with j

[9] We define a connected graph as a graph with exactly one connected component. So the empty graph, which has zero connected components, is not considered connected.

loops. Let

$$\left(\log \frac{Z}{Z_0}\right)_j := \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\Gamma \in G^{(j)}(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\operatorname{Aut}(\Gamma)|},$$

so that

$$\log \frac{Z}{Z_0} = \sum_{j=0}^{\infty} \left(\log \frac{Z}{Z_0}\right)_j \hbar^{j-1}.$$

Theorem 3.12.

$$\left(\log \frac{Z}{Z_0}\right)_0 = -S(x_0), \tag{3.5}$$

and

$$\left(\log \frac{Z}{Z_0}\right)_1 = \frac{1}{2} \log \frac{\det B}{\det \operatorname{Hess}(S)(x_0)}. \tag{3.6}$$

Proof. First note that the statement is purely combinatorial. This means, in particular, that it is sufficient to check that the statement yields the correct asymptotic expansion of the right-hand sides of equations (3.5) and (3.6) in the case when S is a polynomial with real coefficients of the form

$$S(x) = \frac{B(x,x)}{2} - \sum_{i=0}^{N} g_i \frac{B_i(x,\dots,x)}{i!}$$

and $\hbar > 0$. To do so, let $Z := \hbar^{-\frac{d}{2}} \int_{\mathbf{B}} e^{-\frac{S(x)}{\hbar}} dx$, where $\mathbf{B}$ is a ball centered at 0. For sufficiently small g_i, the function S has a unique global minimum point x_0 in $\mathbf{B}$, which is nondegenerate. Thus, by the steepest descent formula, we have

$$\frac{Z}{Z_0} = e^{-\frac{S(x_0)}{\hbar}} I(\hbar),$$

where $I(\hbar) \sim \sqrt{\frac{\det B}{\det \operatorname{Hess} S(x_0)}}(1 + a_1\hbar + a_2\hbar^2 + \cdots)$ (asymptotically). Thus,

$$\log \frac{Z}{Z_0} = -S(x_0)\hbar^{-1} + \frac{1}{2} \log \frac{\det B}{\det \operatorname{Hess}(S)(x_0)} + O(\hbar),\ \hbar \to 0.$$

This implies the result. □

Physicists call the expression $(\log \frac{Z}{Z_0})_0$ the *classical* (*or tree*) *approximation* to the quantum mechanical quantity $\hbar \log \frac{Z}{Z_0}$, and they call the sum $(\log \frac{Z}{Z_0})_0 + \hbar(\log \frac{Z}{Z_0})_1$ the 1-*loop approximation*. Similarly one defines higher loop approximations. Note that the classical approximation is obtained by finding the critical point and value of the classical action $S(x)$, which in the classical mechanics and field theory situation corresponds to solving classical equations of motion.

3.7. Nonlinear equations and trees

As we have noted, Theorem 3.12 does not involve integrals and is purely combinatorial. Therefore, there should exist a purely combinatorial proof of this theorem. Such a proof indeed exists. Here we will give a combinatorial proof of the first statement of Theorem 3.12 (formula (3.5)).

Consider the equation $dS(x) = 0$, defining the critical point x_0. This equation can be written as $x = \beta(x)$, where

$$\beta(x) := \sum_{i\geq 1} g_i \frac{B^{-1}B_i(x,\dots,x,-)}{(i-1)!},$$

where $B^{-1} : V^* \to V$ is the operator corresponding to the form B^{-1}, and $B_i(x,\dots,x,-) \in V^*$.

In the sense of the power series norm, β is a contracting mapping. Thus, $x_0 = \lim_{N\to\infty} \beta^N(x)$, for any initial vector x, for example $x = 0$. In other words, we will obtain x_0 if we keep substituting the series $\beta(x)$ into itself. This leads to summation over trees (explain why!). More precisely, we get the following expression for x_0:

$$x_0 = \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\Gamma\in G^{(0)}(\mathbf{n},1)} \frac{\mathbb{F}_\Gamma}{|\operatorname{Aut}(\Gamma)|},$$

where $G^{(0)}(\mathbf{n}, 1)$ is the set of trees with one external vertex and n_i internal vertices of degree i. Now, since $S(x) = \frac{B(x,x)}{2} - \sum_i g_i \frac{B_i(x,\dots,x)}{i!}$, the expression $-S(x_0)$ equals the sum of expressions $\prod_i g_i^{n_i} \frac{\mathbb{F}_\Gamma}{|\operatorname{Aut}(\Gamma)|}$ over all trees (without external vertices). Indeed, the term $\frac{B(x_0,x_0)}{2}$ corresponds to gluing two trees with external vertices (identifying the two external vertices, so that they disappear); so it corresponds to summing over trees with a marked edge, i.e., counting each tree as many times as it has edges. On the other hand, the term $g_i \frac{B_i(x_0,\dots,x_0)}{i!}$ corresponds to gluing i trees with external vertices together at these vertices (making a tree with a marked vertex). So $\sum_i g_i \frac{B_i(x_0,\dots,x_0)}{i!}$ corresponds to summing over trees with a marked vertex, i.e. counting each tree as many times as it has vertices. But the number of vertices of a tree exceeds the number of edges by 1. Thus, the difference $-S(x_0)$ of the above two contributions corresponds to summing over trees, counting each exactly once. This implies formula (3.5).

3.8. The case $d = 1$

In the case $d = 1$ we can compute the tree sum $-S(x_0)$ even more explicitly. Namely, let

$$S(x) := \frac{x^2}{2} - ah(x),$$

where $h(x) = \sum_{n\geq 0} c_n x^n$ with $c_1 \neq 0$. Then x_0 is the solution of the equation $x = ah'(x)$, i.e., $x_0 = f(a)$, where $x = f(y)$ is the inverse function to $y = \frac{x}{h'(x)}$ (since $h'(0) = c_1 \neq 0$, this function maps 0 to 0 and is invertible near 0). So the tree approximation takes the form $-S(x_0) = F(a)$, where

$$F(a) = -\frac{f(a)^2}{2} + ah(f(a)).$$

Thus

$$F'(a) = -f(a)f'(a) + h(f(a)) + ah'(f(a))f'(a).$$

But $h'(f(a)) = \frac{f(a)}{a}$, so the first and third summands cancel, and we get

$$F'(a) = h(f(a)),$$

hence

$$-S(x_0) = \int_0^a h(f(x))dx. \tag{3.7}$$

3.9. Counting trees and Cayley's theorem

In this section we will apply Theorem 3.12 to tree counting problems, in particular will prove a classical theorem due to Cayley that the number of labeled trees with n vertices is n^{n-2}.

We consider essentially the same situation as we considered in Example 3.9: $d = 1$, $B_i = 1$, $g_i = a$. Thus, we have $S(x) = \frac{x^2}{2} - ae^x$. By Theorem 3.12, we have

$$\sum_{n\geq 0} a^n \sum_{\Gamma\in T(n)} \frac{1}{|\operatorname{Aut}(\Gamma)|} = -S(x_0),$$

where $T(n)$ is the set of isomorphism classes of trees with n vertices, and x_0 is the root of the equation $S'(x) = 0$, i.e. $x = ae^x$.

In other words, let $x = f(y)$ be the function inverse to the function $y = xe^{-x}$ near $x = 0$, then $x_0 = f(a)$. The function $f(y)$ is related to (the principal branch of) the *Lambert function* $W(y)$ by the formula

$$f(y) = -W(-y).$$

By (3.7)

$$-S(x_0) = \int_0^a e^{f(x)}dx = \int_0^a \frac{f(x)}{x}dx.$$

Thus it remains to find the Taylor expansion of f. This expansion is given by the following classical result.

Proposition 3.13. *One has*

$$f(z) = \sum_{n\geq 1} \frac{n^{n-2}}{(n-1)!} z^n.$$

Proof. Let $f(z) = \sum_{n\geq 1} a_n z^n$. Then

$$a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{z^{n+1}}dz = \frac{1}{2\pi i}\oint \frac{w}{(we^{-w})^{n+1}}d(we^{-w})$$
$$= \frac{1}{2\pi i}\oint e^{nw}\frac{1-w}{w^n}dw = \frac{n^{n-1}}{(n-1)!} - \frac{n^{n-2}}{(n-2)!} = \frac{n^{n-2}}{(n-1)!}.$$

Here the contour of integration in all integrals is a small loop going counterclockwise around 0. □

So we get

$$-S(x_0) = \int_0^a \frac{f(x)}{x}dx = \sum_{n\geq 1}\frac{n^{n-2}}{n!}a^n.$$

This shows that

$$\sum_{\Gamma\in T(n)}\frac{1}{|\operatorname{Aut}(\Gamma)|} = \frac{n^{n-2}}{n!}.$$

But each isomorphism class of unlabeled trees with n vertices has $\frac{n!}{|\operatorname{Aut}(\Gamma)|}$ non-isomorphic labelings. Thus we obtain

Corollary 3.14 (Cayley's theorem)**.** *The number of labeled trees with n vertices equals* n^{n-2}.

3.10. Counting trees with conditions

In a similar way we can count labeled trees with conditions on vertices. For example, let us compute the number of labeled trivalent trees with m vertices (i.e., trees that have only 1-valent and 3-valent vertices). Clearly, $m = 2k$, otherwise there is no such tree. The relevant action functional is

$$S(x) = \tfrac{x^2}{2} - a(x + \tfrac{x^3}{6}).$$

Then the critical point x_0 is obtained from the equation $S'(x) = 0$, i.e.

$$x = a(1 + \tfrac{x^2}{2}).$$

This yields

$$x_0 = \frac{1-\sqrt{1-2a^2}}{a}.$$

Thus, by (3.7) the tree sum equals

$$-S(x_0) = \int_0^a \left(\tfrac{1-\sqrt{1-2x^2}}{x} + \tfrac{(1-\sqrt{1-2x^2})^3}{6x^3}\right)dx$$
$$= \frac{2}{3}\int_0^a \tfrac{1-(1+x^2)\sqrt{1-2x^2}}{x^3}dx = \frac{(1-2x^2)^{\frac{3}{2}} - (1-3x^2)}{3x^2}.$$

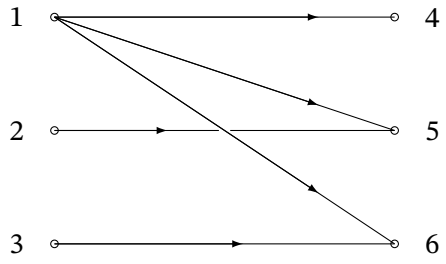


Figure 3.7. A labeled oriented tree with three sources and three sinks.

Expanding this in a Taylor series, we find

$$-S(x_0) = \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{(n+1)!} a^{2n}.$$

Hence, we get

Corollary 3.15. *The number N_k of trivalent labeled trees with $2k$ vertices is* $(2k-3)!\,!\,\frac{(2k)!}{(k+1)!}$.

For example, $N_1 = 1$ (a single edge), $N_2 = 4$ (a single tree with 4! labelings modulo a group of order 6), $N_3 = 90$ (a single tree with 6! labelings modulo a group of order 8), etc.

3.11. Counting oriented trees

Feynman calculus can be used to count not only nonoriented but also oriented graphs. For example, suppose we want to count labeled oriented trees, whose vertices are either sources or sinks; see Figure 3.7. In this case, it is easy to see (check it!) that the relevant integration problem is in two dimensions, with the action

$$S(x, y) = xy - be^x - ae^y$$

(the quadratic form xy is not positive definite, but this is immaterial since our computations are purely formal). So the critical point is found from the equations

$$xe^{-y} = a,\ ye^{-x} = b.$$

As before, look for a solution $(x, y) = (x_0, y_0)$ in the form

$$x = a + \sum_{p\geq 1, q\geq 1} c_{pq} a^p b^q,\ y = b + \sum_{p\geq 1, q\geq 1} d_{pq} a^p b^q.$$

A calculation with residues similar to the one we did for unoriented trees yields

$$c_{pq} = \frac{1}{(2\pi i)^2}\oint\oint \frac{x}{a^{p+1}b^{q+1}}da\wedge db$$
$$= \frac{1}{(2\pi i)^2}\oint\oint \frac{e^{qx+py}}{x^p y^{q+1}}(1-xy)dx\wedge dy = \frac{q^{p-1}p^{q-1}}{(p-1)!\,q!}.$$

Similarly, $d_{pq} = \frac{q^{p-1}p^{q-1}}{p!(q-1)!}$. Now, similarly to the unoriented case, we find that $-a\partial_a S(x,y) = x$, $-b\partial_b S(x,y) = y$, so

$$-S(x,y) = b + \int_0^a \frac{x}{u}du = a + b + \sum_{p,q\geq 1}\frac{p^{q-1}q^{p-1}}{p!\,q!}a^p b^q.$$

This implies that the number of labeled trees with p sources and q sinks is $p^{q-1}q^{p-1}\frac{(p+q)!}{p!q!}$. In particular, if we specify which vertices are sources and which are sinks, the number of labeled trees is $p^{q-1}q^{p-1}$.

Exercise 3.16. Do this calculation in detail.

3.12. The matrix-tree theorem

These calculations can be generalized to compute the number of *colored* labeled trees. Define the *Kirchhoff polynomial* $K_m(\mathbf{u})$ of a symmetric m-by-m matrix $\mathbf{u} \coloneqq (u_{ik})$ by

$$K_m(\mathbf{u}) \coloneqq \det{}_j M(\mathbf{u})$$

for any $1 \leq j \leq m$, where $M(\mathbf{u})$ is the (degenerate) m-by-m matrix with entries

$$M(\mathbf{u})_{i\ell} \coloneqq -u_{i\ell} + \delta_{i\ell}\sum_k u_{k\ell}$$

and $\det_j$ means that the determinant is computed after the jth row and column of the matrix are removed (so M and hence K_m does not depend on the diagonal entries u_{ii}). It is easy to see that this definition does not depend on j. (Check it!) Thus

$$K_m(\mathbf{u}) = \frac{1}{m}\operatorname{Tr}\wedge^{m-1}M(\mathbf{u}) = \frac{1}{m}\lambda_1\ldots\lambda_{m-1},$$

where $\lambda_1,\ldots,\lambda_{m-1}$ are the eigenvalues of $M(\mathbf{u})$ excluding one zero eigenvalue. For instance,

$$K_1 = 1,\ K_2 = u_{12},\ K_3 = u_{12}u_{13} + u_{13}u_{23} + u_{12}u_{23},$$

etc.

Now let $\mathbf{p} = (p_1,\ldots,p_m)$ be an m-tuple of positive integers and $\mathbf{r} = (r_{ij}, 1 \leq i \leq j \leq m)$ be a collection of nonnegative integers with $|\mathbf{r}| = |\mathbf{p}| - 1$, where $|\mathbf{r}| \coloneqq \sum_{i\leq j} r_{ij}$, $|\mathbf{p}| \coloneqq \sum_k p_k$. We want to compute the number $N(\mathbf{p},\mathbf{r})$ of labeled trees on $|\mathbf{p}|$ vertices (thus $|\mathbf{r}|$ edges) with the first p_1 vertices colored with color

1, the next p_2 with color 2,..., the last p_m with color m, and r_{ij} edges going between vertices of colors i and j.

It suffices to compute the polynomial

$$Q_{\mathbf{p}}(\mathbf{z}) := \sum_{\mathbf{r}:|\mathbf{r}|=|\mathbf{p}|-1} N(\mathbf{p},\mathbf{r}) \prod_{i\le j} z_{ij}^{r_{ij}},$$

of an m-by-m symmetric matrix $\mathbf{z} = (z_{ij})$, which is the generating function of these numbers.

Let $\widehat{\mathbf{p}}$ be the diagonal m-by-m matrix with diagonal entries p_i.

Theorem 3.17. *We have*

$$Q_{\mathbf{p}}(\mathbf{z}) = (p_1 \dots p_m)^{-1} \prod_{\ell} (\sum_k p_k z_{k\ell})^{p_\ell - 1} K_m(\widehat{\mathbf{p}}\mathbf{z}\widehat{\mathbf{p}}).$$

Note that for $m = 1$ and $\mathbf{z} = 1$ this recovers Cayley's theorem (Corollary 3.14), while for $m = 2$ and $\mathbf{z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ it recovers our count of oriented trees of Section 3.11.

Proof. We attach to each color i a variable x_i. Then the corresponding action is

$$S(x) = \frac{1}{2} x^T B x - \sum_{i=1}^{m} a_i e^{x_i},$$

where $B = (b_{ij})$ is inverse to the matrix $\mathbf{z}$. Then by Theorem 3.12, $Q_{\mathbf{p}}(\mathbf{z})$ is the coefficient to $\prod_k a_k^{p_k}$ in $-S(x)$ multiplied by $\prod_\ell p_\ell!$, where x is the critical point of S (note that $-S(x) \in \mathbb{C}[[a_1, \dots, a_m]]$). The equations for this critical point are

$$\sum_k b_{ik} x_k = a_i e^{x_i}. \tag{3.8}$$

Let $X_i := \sum_k b_{ik} x_k = a_i e^{x_i}$, then $x_i = \sum_k z_{ik} X_k$. We have

$$-\frac{\partial S(x)}{\partial a_j} = -\sum_{i,k} \frac{\partial x_i}{\partial a_j} b_{ik} x_k + \sum_i \frac{\partial x_i}{\partial a_j} a_i e^{x_i} + e^{x_j}.$$

By (3.8) the first two summands cancel, so we obtain

$$-a_j \frac{\partial S(x)}{\partial a_j} = a_j e^{x_j} = X_j.$$

Let $S_j(x)$ be the sum of all the monomials in $S(x)$ which contain the variable a_j. Then we get

$$-S_j(x) = \int X_j \frac{da_j}{a_j},$$

where

$$\int \prod_k a_k^{p_k} \frac{da_j}{a_j} = \frac{\prod_k a_k^{p_k}}{p_j}$$

if $p_j > 0$.

Thus the coefficient to $\prod_k a_k^{p_k}$ in $-S(x)$ equals the coefficient to the same monomial in X_j divided by p_j, for any $1 \le j \le m$. So, denoting by $D_T(\mathbf{z})$ the principal minor of $\mathbf{z}$ corresponding to a subset $T \subset \{1, \dots, m\}$, we get

$$\begin{aligned}
\frac{Q_{\mathbf{p}}(\mathbf{z})}{\prod_\ell p_\ell!} &= \tfrac{p_j^{-1}}{(2\pi i)^m} \oint X_j \Big(\prod_k a_k^{-p_k-1}\Big) d\mathbf{a} \\
&= \tfrac{p_j^{-1}}{(2\pi i)^m} \oint X_j \Big(\prod_k (X_k e^{-x_k})^{-p_k-1}\Big) d(X_1 e^{-x_1}) \wedge \cdots \wedge d(X_m e^{-x_m}) \\
&= \tfrac{p_j^{-1}}{(2\pi i)^m} \oint \sum_{T \subset \{1,\dots,m\}} (-1)^{|T|} D_T(\mathbf{z}) X_j \Big(\prod_{\ell \notin T} X_\ell^{-1}\Big) \Big(\prod_\ell X_\ell^{-p_\ell}\Big) e^{\sum_{k,\ell} p_k z_{k\ell} X_\ell} dX_1 \wedge \cdots \wedge dX_m \\
&= p_j^{-1} \sum_{T \subset \{1,\dots,m\}} (-1)^{|T|} \frac{p_j - 1 + \delta_{jT^c}}{\sum_k p_k z_{kj}} D_T(\mathbf{z}) \prod_\ell \frac{(\sum_k p_k z_{k\ell})^{p_\ell - \delta_{\ell T}}}{(p_\ell - \delta_{\ell T})!} \\
&= p_j^{-1} \left(\frac{p_j - 1}{\sum_k p_k z_{kj}} \det(\delta_{i\ell} \sum_k p_k z_{k\ell} - z_{i\ell} p_\ell) + \det{}_j(\delta_{i\ell} \sum_k p_k z_{k\ell} - z_{i\ell} p_\ell) \right) \prod_\ell \frac{(\sum_k p_k z_{k\ell})^{p_\ell - 1}}{p_\ell!},
\end{aligned}$$

where $\delta_{\ell T} = 1$ if $\ell \in T$ and 0 otherwise. The first determinant is zero, so we get

$$Q_{\mathbf{p}}(\mathbf{z}) = (p_1 \dots p_m)^{-1} \prod_\ell (\sum_k p_k z_{k\ell})^{p_\ell - 1} \det{}_j(\delta_{i\ell} \sum_k p_k z_{k\ell} p_\ell - p_i z_{i\ell} p_\ell),$$

as claimed. □

Theorem 3.17 is a weighted version of *Kirchhoff's matrix-tree theorem*, which is a generalization of Cayley's theorem. More precisely, take $\mathbf{z} = A_\Gamma$ to be the adjacency matrix of a connected graph Γ, m the number of vertices of Γ, and $p_i = 1$ for all i. Then $Q_{\mathbf{p}}(\mathbf{z}) = N_\Gamma$ is the *number of spanning trees* of Γ, and Theorem 3.17 implies that

$$N_\Gamma = \det{}_j(\Delta_\Gamma),\ \Delta_\Gamma := D_\Gamma - A_\Gamma,$$

where D_Γ is the diagonal matrix with diagonal entries the vertex degrees of Γ. The matrix Δ_Γ is called the *Laplace operator* of Γ. It is easy to check that this matrix is positive semidefinite, with zero eigenvalue occurring with multiplicity 1. Thus we get

Corollary 3.18 (Matrix-tree theorem). *The number of spanning trees of* Γ *equals*

$$N_\Gamma = \frac{1}{m} \lambda_1 \dots \lambda_{m-1},$$

where λ_i *are the nonzero* (*in fact, positive*) *eigenvalues of* Δ_Γ.

Cayley's theorem is obtained from this result when Γ is a complete graph, in which case $\lambda_i = m$ for all i, so we get $N_\Gamma = m^{m-2}$.

3.13. 1-particle irreducible diagrams and the effective action

Let $Z = Z_S$ be the partition function corresponding to the action S. In previous sections we have seen that the "classical" (or "tree") part $(\log \frac{Z_S}{Z_0})_0$ of the quantity $\hbar \log \frac{Z_S}{Z_0}$ is quite elementary to compute—it is just minus the critical value of the action $S(x)$. Thus, if we could find a new "effective" action S_{eff} (a "deformation" of S) such that

$$\hbar^{-1}(\log \tfrac{Z_{S_{\text{eff}}}}{Z_0})_0 = \log \tfrac{Z_S}{Z_0}$$

(i.e., the classical answer for the effective action is the quantum answer for the original one), then we can consider the quantum theory for the action S to be solved. In other words, the problem of solving the quantum theory attached to S (i.e., finding the corresponding integrals) essentially reduces to the problem of computing the effective action S_{eff}.

We will now give a recipe of computing the effective action in terms of amplitudes of Feynman diagrams, and see that it is computationally easier than computing the sum over connected diagrams.

Definition 3.19. An edge e of a connected graph Γ is said to be a *bridge* if the graph $\Gamma \setminus e$ is disconnected. A connected graph without bridges is called 1-*particle irreducible* (1PI).[10]

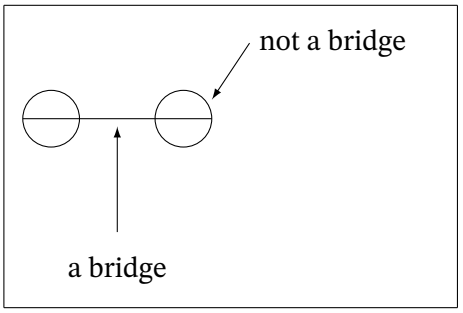
To compute the effective action, we will need to consider graphs with external edges (but having at least one internal vertex). Such a graph Γ (with N external edges) will be called 1-particle irreducible if the corresponding "amputated" graph is also; i.e., the graph obtained from Γ by removal of the external edges. In particular, a graph with one internal vertex is always 1-particle irreducible, while a single edge graph without internal vertices is defined **not** to be 1-particle irreducible. The notions of a bridge and a 1-particle irreducible graph are illustrated in Figure 3.8.

Denote by $G_{1\text{PI}}(\mathbf{n}, N)$ the set of isomorphism classes of 1-particle irreducible graphs with N external edges and n_i i-valent internal vertices for each i (where isomorphisms are not allowed to move external edges).

Theorem 3.20. *The effective action S_{eff} is given by the formula*

$$S_{\text{eff}}(x) = \frac{B(x,x)}{2} - \sum_{i \geq 0} \frac{\mathcal{B}_i(x, \dots, x)}{i!},$$

[10] This is the physical terminology. The mathematical term is "2-edge-connected".



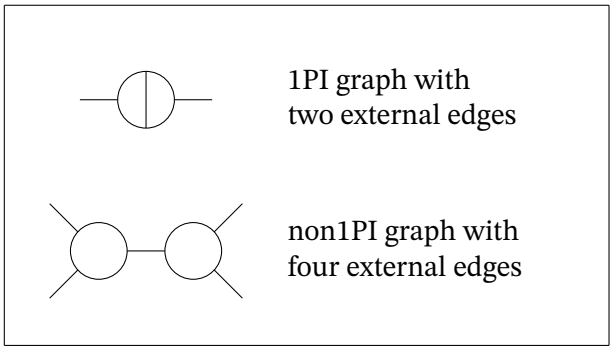


Figure 3.8

where

$$\mathcal{B}_N(x,\dots,x) = \hbar^{1-N/2} \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G_{1\mathrm{PI}}(\mathbf{n},N)} \frac{\mathbb{F}_\Gamma(Bx,\dots,Bx)}{|\operatorname{Aut}(\Gamma)|}.$$

Thus, $S_{\text{eff}} = S + \hbar S_1 + \hbar^2 S_2 + \dots$. The expressions $\hbar^j S_j$ are called the *j-loop corrections* to the effective action.

This theorem allows physicists to worry only about 1-particle irreducible diagrams, and is the reason why you will rarely see other diagrams in a QFT textbook. As before, it is very useful in doing low-order computations, since the number of 1-particle irreducible diagrams with a given number of loops is much smaller than the number of connected diagrams with the same number of loops.

Proof. The proof is based on Lemma 3.21 from graph theory.

Lemma 3.21. *Any connected graph* Γ *can be uniquely represented as a tree whose vertices are* 1*-particle irreducible subgraphs* (*with external edges*)*, and whose edges are the bridges of* Γ.

The lemma is obvious. Namely, let us remove all bridges from Γ. Then Γ will turn into a disjoint union of 1-particle irreducible graphs which should be taken to be the vertices of the said tree.

The tree corresponding to the graph Γ is called the *skeleton* of Γ; see Figure 3.9.

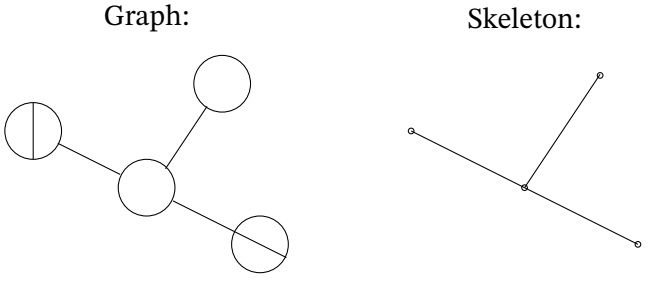


Figure 3.9. The skeleton of a graph.

It is easy to see that Lemma 3.21 implies Theorem 3.20. Indeed, it implies that the sum over all connected graphs occurring in the expression of $\log \frac{Z_S}{Z_0}$ can be written as a sum over skeleton trees, so that the contribution from each tree is (proportional to) the contraction of the tensors $\mathcal{B}_i$ placed at its vertices, and $\mathcal{B}_i$ is the (weighted) sum of amplitudes of all 1-particle irreducible graphs with i external edges. □

3.14. 1-particle irreducible diagrams and the Legendre transform

Recall the notion of the *Legendre transform*. Let f be a smooth function on a vector space Y, such that the map $Y \to Y^*$ given by $x \to df(x)$ is a diffeomorphism. Then one can define the Legendre transform of f as follows. For $p \in Y^*$, let $x_0 = x_0(p)$ be the critical point of the function $(p, x) - f(x)$; i.e., the unique solution of the equation $df(x) = p$. Then the Legendre transform of f is the function on Y^* defined by

$$L(f)(p) = (p, x_0) - f(x_0).$$

It is easy to see that the differential of $L(f)$ is also a diffeomorphism $Y^* \to Y$ (in fact, inverse to $df(x)$), and that $L^2(f) = f$.

Example 3.22. Let $f(x) = \frac{ax^2}{2}$, $a \neq 0$. Then $px - f(x) = px - \frac{ax^2}{2}$ has a critical point at $x = \frac{p}{a}$, and the critical value is $\frac{p^2}{2a}$. Thus $L(\frac{ax^2}{2}) = \frac{p^2}{2a}$. More generally, if $f(x) = \frac{B(x,x)}{2}$, where B is a nondegenerate symmetric form on Y, then $L(f)(p) = \frac{B^{-1}(p,p)}{2}$; e.g., the Legendre transform of a Lagrangian $\frac{mv^2}{2} - U(x)$ of

a particle of mass m with respect to velocity $v = \dot{x}$ is its Hamiltonian (energy) $\frac{p^2}{2m} + U(x)$, and vice versa. This is, in fact, the case in complete generality, which is why Legendre transform plays an important role in classical mechanics and field theory.

Note that the stationary phase formula implies that the Legendre transform is the classical analogue of the Fourier transform. Indeed, the leading term of the asymptotics as $\hbar \to 0$ of the logarithm of the (suitably normalized) Fourier transform $\hbar^{-\frac{d}{2}} \int_V e^{\frac{i(-(p,x)+S(x))}{\hbar}} dx$ of the Feynman density $e^{\frac{iS(x)}{\hbar}} dx$ (where the integral is understood in the sense of distributions) is $-\frac{iL(S)(p)}{\hbar}$.

Now let us consider Theorem 3.20 in the situation of Theorem 3.5. Thus, $S(x) = \frac{B(x,x)}{2} + O(x^3)$, and we look at

$$Z(p) = \hbar^{-\frac{d}{2}} \int_V e^{\frac{(p,x)-S(x)}{\hbar}} dx.$$

By Theorem 3.20, one has

$$\log \frac{Z(p)}{Z_0} = -\hbar^{-1} S_{\text{eff}}(x_0, p),$$

where the effective action $S_{\text{eff}}(x, p)$ is the sum over 1-particle irreducible graphs and $x_0 = x_0(p)$ is its critical point.

Now, we must have $S_{\text{eff}}(x, p) = -p \cdot x + S_{\text{eff}}(x)$, since the only 1PI graph which contains 1-valent internal vertices (corresponding to p) is the graph with one edge, connecting an internal vertex with an external one (so it yields the term $-p \cdot x$, and other graphs contain no p-vertices). This shows that $\hbar \log \frac{Z(p)}{Z_0}$ is the critical value of $p \cdot x - S_{\text{eff}}(x)$. Thus we have proved the following.

Proposition 3.23. *We have*

$$S_{\text{eff}}(x) = L(\hbar \log \tfrac{Z(p)}{Z_0}),\ \hbar \log \tfrac{Z(p)}{Z_0} = L(S_{\text{eff}}(x)).$$

Physicists formulate this result as follows: the effective action is the Legendre transform of $\hbar$ times the logarithm of the generating function for quantum correlators (and vice versa).

Exercise 3.24. Compute the 1-loop contribution to $\log \frac{Z}{Z_0}$ for

$$S(x) = \frac{x^2}{2} - a(x + \frac{x^3}{6}).$$

Using this, compute the number of labeled n-vertex 1-loop graphs with 1-valent and 3-valent vertices only (be careful with double edges and self-loops!). Check your answer by directly enumerating such graphs with small number of vertices.

Exercise 3.25. Find the exponential generating function $\sum_n a_n \frac{z^n}{n!}$ for the numbers a_n of labeled n-vertex trees with 1-valent and 4-valent vertices. You may express the answer via inverse functions to polynomials.

Exercise 3.26. Find the 1-loop contribution to the effective action for $S(x) = \frac{x^2}{2} - \frac{gx^3}{6}$. That is, one has $S_{\text{eff}} = S + \hbar S_1 + O(\hbar^2)$, and you need to find S_1. Which Feynman diagrams need to be considered?

Exercise 3.27. Consider the heat equation $u_t = \frac{1}{2}\Delta_B u$, where Δ_B is the Laplace operator attached to B defined in Section 2.2. It is solved by the heat flow $u(x,t) = e^{\frac{t\Delta_B}{2}} u(x,0)$. Show that the effective action S_{eff} for the action $S(x) = \frac{B(x,x)}{2} - \tilde{S}(x)$ can be computed as the sum of contributions of 1PI Feynman diagrams without self-loops for the action

$$S^\circ(x) := \frac{B(x,x)}{2} - \tilde{S}^\circ(x),$$

where $\tilde{S}^\circ(x) := e^{\frac{\hbar\Delta_B}{2}} \tilde{S}(x)$ is obtained by transforming $\tilde{S}$ by the heat flow for time $\hbar$.

Chapter 4

Matrix integrals

Let $\mathfrak{h}_N$ be the space of Hermitian matrices of size N. The inner product on $\mathfrak{h}_N$ is given by $B(A_1, A_2) = \mathrm{Tr}(A_1 A_2)$. In this chapter we will consider integrals of the form

$$Z_N := \hbar^{-\frac{N^2}{2}} \int_{\mathfrak{h}_N} e^{-\frac{S(A)}{\hbar}} dA,$$

where the Lebesgue measure dA is normalized by the condition

$$\int_{\mathfrak{h}_N} e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA = 1$$

(so we do not have to drag around the $\sqrt{2\pi}$-factors) and

$$S(A) := \frac{\mathrm{Tr}(A^2)}{2} - \sum_{m \geq 1} g_m \frac{\mathrm{Tr}(A^m)}{m}$$

is the action functional.[11] We will be interested in the behavior of the coefficients of the expansion of Z_N in g_i for large N. The study of this behavior will lead us to considering not simply Feynman graphs, but actually fat (or ribbon) graphs, which are in fact 2-dimensional surfaces. Thus, before we proceed further, we need to do some 2-dimensional combinatorial topology.

4.1. Fat graphs

Recall from the proof of Feynman's theorem that given a finite collection of flowers and a matching σ on the set T of endpoints of their edges, we can obtain a graph Γ_σ by connecting (or gluing) the points which fall into the same pair.

[11] Note that we divide by m and not by $m!$. We will see below why such normalization will be more convenient.

Now, given an i-flower, let us inscribe it in a closed disk D (so that the ends of the edges are on the boundary). Then take its small tubular neighborhood in D. This produces a region with piecewise smooth boundary. We will equip this region and its boundary with the standard orientation and call it a *fat i-valent flower*. The boundary of a fat i-valent flower has the form $P_1Q_1P_2Q_2\dots P_iQ_iP_1$, where P_i, Q_i are the angle points, the intervals P_jQ_j are arcs on ∂D, and Q_jP_{j+1} are (smooth) arcs lying inside D; see Figure 4.1.

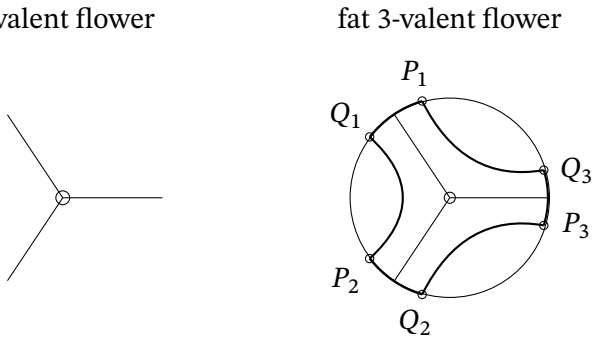


Figure 4.1

Now, given a collection of usual flowers and a matching σ as above, we can consider the corresponding fat flowers, and glue them, respecting the orientation, along intervals P_jQ_j according to σ. This will produce a compact oriented surface with boundary (the boundary is glued from intervals Q_jP_{j+1}). We will denote this surface by $\tilde{\Gamma}_\sigma$, and call it the *fattening* of Γ with respect to σ. A fattening of a graph will be called a *fat* (*or ribbon*) *graph*.

Thus, a fat graph is not just an oriented surface with boundary, but such a surface together with a partition of this surface into fat flowers.

Note that the same graph Γ can have many fattenings which are nonhomeomorphic (albeit homotopy equivalent) surfaces, and in particular the genus g of the fattening is **not** determined by Γ; see Figure 4.2.

4.2. Matrix integrals in large N limit, planar graphs, and the genus expansion

Let us now return to the study of the integral Z_N. We have

$$B_m(A,\dots,A) = (m-1)!\,\mathrm{Tr}(A^m).$$

Thus by Feynman's theorem,

$$\log Z_N = \sum_{\mathbf{n}} \prod_i \frac{(g_i \hbar^{\frac{i}{2}-1})^{n_i}}{i!^{n_i}\, n_i!} \sum_{\sigma\in\Pi_c(T_{\mathbf{n}})} \mathbb{F}(\sigma),$$

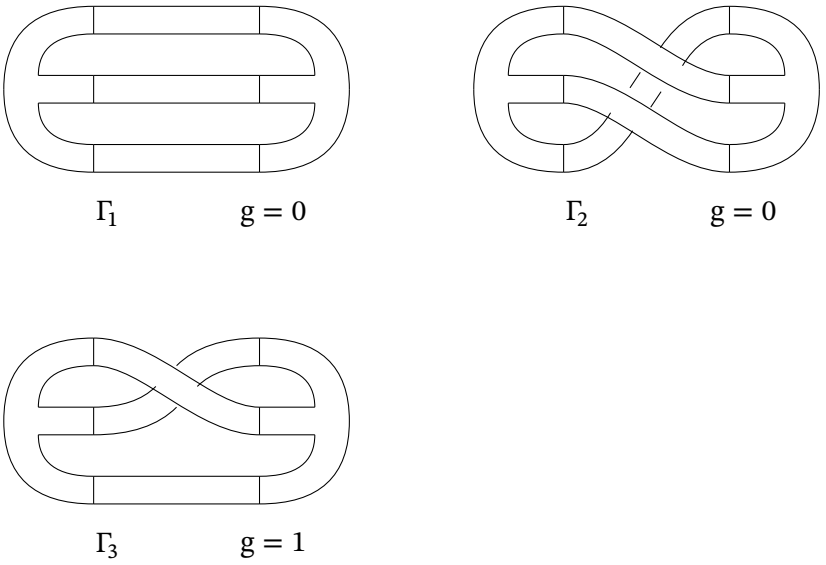


Figure 4.2. Gluing a fat graph from fat flowers

where the summation is taken over the set $\Pi_c(T_{\mathbf{n}})$ of all matchings of $T = T_{\mathbf{n}}$ that produce a connected graph Γ_σ, and $\mathbb{F}(\sigma)$ denotes the contraction of the tensors $(m-1)!\,\mathrm{Tr}(A^m)$ using σ. So let us compute $\mathbb{F}(\sigma)$.

Let $\{e_i\}$ be the standard basis of $\mathbb{C}^N$, and let $\{e_i^*\}$ be the dual basis of the dual space. Then the tensor $\mathrm{Tr}(A^m)$ can be written as

$$\mathrm{Tr}(A^m) = \sum_{i_1,\dots,i_m=1}^{N} (e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*, A^{\otimes m}).$$

Thus

$$B_m = \sum_{s\in S_{m-1}} \sum_{i_1,\dots,i_m=1}^{N} s(e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*),$$

where s permutes the elementary matrices $e_{i_k} \otimes e_{i_{k+1}^*}$ and the sum is over all possible cyclic orderings of edges of an m-valent flower, which bijectively correspond to elements of S_{m-1} (elements s of S_m mapping m to m). Hence

$$\mathbb{F}(\sigma) = \sum_{s\in\prod_i S_{i-1}^{n_i}} \widetilde{\mathbb{F}}(s\sigma),$$

where $\widetilde{\mathbb{F}}(\sigma)$ is obtained by contracting the tensors

$$\sum_{i_1,\dots,i_m=1}^{N} e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*, \tag{4.1}$$

according to the fat graph $\tilde{\Gamma}_\sigma$. It follows that

$$\log Z_N = \sum_{\mathbf{n}} \prod_i \frac{g_i^{n_i} \hbar^{n_i(\frac{i}{2}-1)}}{i!^{n_i}\, n_i!} \sum_{\sigma \in \Pi(T_{\mathbf{n}})} \sum_{s \in \prod_i S_{i-1}^{n_i}} \widetilde{\mathbb{F}}(s\sigma)$$

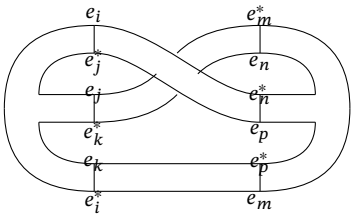
$$= \sum_{\mathbf{n}} \prod_i \frac{g_i^{n_i} \hbar^{n_i(\frac{i}{2}-1)}}{i^{n_i} n_i!} \sum_{\sigma} \widetilde{\mathbb{F}}(\sigma)$$

(the product $\prod_i i!^{n_i}$ in the denominator got replaced by $\prod_i i^{n_i}$ since in the sum $\sum_{s,\sigma} \widetilde{\mathbb{F}}(s\sigma)$ every term $\widetilde{\mathbb{F}}(\sigma)$ occurs $|\prod_i S_{i-1}^{n_i}| = \prod_i (i-1)!^{n_i}$ times).

For a surface Σ with boundary, let $\nu(\Sigma)$ denote the number of connected components of the boundary.

Proposition 4.1. $\widetilde{\mathbb{F}}(\sigma) = N^{\nu(\tilde{\Gamma}_\sigma)}$.

Proof. One can visualize each summand in the sum (4.1) as a labeling of the angle points $P_1, Q_1, \ldots, P_m, Q_m$ on the boundary of a fat m-valent flower by i_1, i_2, $i_2, i_3, \ldots, i_m, i_1$. Now, the contraction using σ of some set of such monomials is nonzero iff the subscript is constant along each boundary component of $\tilde{\Gamma}_\sigma$; see Figure 4.3. This implies the result. □



Contraction nonzero iff
$i = n$, $j = p$, $j = m$,
$k = n$, $k = p$, $i = m$,
that is
$i = n = k = p = j = m$.

Figure 4.3. Contraction defined by a fat graph.

Let $\widetilde{G}_c(\mathbf{n})$ be the set of isomorphism classes of connected fat graphs with n_i i-valent vertices for $i \geq 1$. For $\tilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})$, let $b(\tilde{\Gamma})$ be the number of edges minus the number of vertices of the underlying usual graph Γ.

Corollary 4.2.

$$\log Z_N = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\tilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{\nu(\tilde{\Gamma})}}{|\operatorname{Aut}(\tilde{\Gamma})|}$$

$$= \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\tilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{\nu(\tilde{\Gamma})} \hbar^{b(\tilde{\Gamma})}}{|\operatorname{Aut}(\tilde{\Gamma})|}.$$

Proof. Let $\mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}} := \prod_i (S_{n_i} \ltimes (\mathbb{Z}/i\mathbb{Z})^{n_i})$. This group acts on $T_{\mathbf{n}}$, so that $\tilde{\Gamma}_\sigma = \tilde{\Gamma}_{g\sigma}$, for any $g \in \mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}}$. Moreover, the group acts transitively on the set of σ giving a

fixed fat graph $\tilde{\Gamma}_\sigma$, and the stabilizer of any σ is $\mathrm{Aut}(\tilde{\Gamma}_\sigma)$. This implies the result, as $|\mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}}| = \prod_i i^{n_i} n_i!$ which cancels the denominators. □

Now for any compact connected surface Σ with boundary, let $\mathrm{g}(\Sigma)$ be the genus of Σ. Then for a connected fat graph $\tilde{\Gamma}$,

$$b(\tilde{\Gamma}) = 2\mathrm{g}(\tilde{\Gamma}) - 2 + \nu(\tilde{\Gamma})$$

(minus the Euler characteristic). Thus, defining

$$\widehat{Z}_N(\hbar) := Z_N(\tfrac{\hbar}{N}),$$

we obtain

Theorem 4.3.

$$\log \widehat{Z}_N = \sum_{\mathbf{n}} \prod_i (\mathrm{g}_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\tilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{2-2\mathrm{g}(\tilde{\Gamma})}}{|\mathrm{Aut}(\tilde{\Gamma})|}.$$

This implies the following important result, due to 't Hooft.

Theorem 4.4.

(1) *There exists a limit* $W_\infty := \lim_{N\to\infty} \frac{\log \widehat{Z}_N}{N^2}$. *This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}} \prod_i (\mathrm{g}_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\tilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})[0]} \frac{1}{|\mathrm{Aut}(\tilde{\Gamma})|},$$

where $\widetilde{G}_c(\mathbf{n})[0]$ *denotes the set of planar connected fat graphs, i.e., those which have genus zero.*

(2) *Moreover, there exists an expansion*

$$\frac{\log \widehat{Z}_N}{N^2} = \sum_{\mathrm{g} \in \mathbb{Z}_{\geq 0}} a_{\mathrm{g}} N^{-2\mathrm{g}},$$

where

$$a_{\mathrm{g}} = \sum_{\mathbf{n}} \prod_i (\mathrm{g}_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\tilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})[\mathrm{g}]} \frac{1}{|\mathrm{Aut}(\tilde{\Gamma})|},$$

and $\widetilde{G}_c(\mathbf{n})[\mathrm{g}]$ *denotes the set of connected fat graphs of genus* g.

Remark 4.5. Genus zero fat graphs are said to be planar because the underlying usual graphs can be put on the 2-sphere (and hence on the plane) without self-intersections.

Remark 4.6. 't Hooft's theorem may be interpreted in terms of the usual Feynman diagram expansion. Namely, it implies that for large N, the leading contribution to $\log Z_N(\frac{\hbar}{N})$ comes from the terms in the Feynman diagram expansion corresponding to planar graphs (i.e., those that admit an embedding into the 2-sphere).

4.3. Integration over real symmetric matrices

One may also consider the matrix integral over the space $\mathfrak{s}_N$ of real symmetric matrices of size N. Namely, one puts

$$Z_N = \hbar^{-\frac{N(N+1)}{4}} \int_{\mathfrak{s}_N} e^{-\frac{S(A)}{\hbar}} dA,$$

where S and dA are as above. Let us generalize Theorem 4.4 to this case.

As before, consideration of the large N limit leads to consideration of fat flowers and gluing of them. However, the exact nature of gluing is now somewhat different. Namely, in the Hermitian case we had

$$(e_i \otimes e_j^*, e_k \otimes e_l^*) = \delta_{il}\delta_{jk},$$

which forced us to glue fat flowers preserving orientation. On the other hand, in the real symmetric case $e_i^* = e_i$, and the inner product of the functionals $e_i \otimes e_j$ on the space of symmetric matrices (defined by $(e_k \otimes e_l)(A) := a_{kl}$) is given by

$$(e_i \otimes e_j, e_k \otimes e_l) = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

This means that besides the usual (orientation preserving) gluing of fat flowers, we now must allow gluing with a twist of the ribbon by 180°. Fat graphs thus obtained will be called *twisted fat graphs*. That means, a twisted fat graph is a surface with boundary (possibly not orientable), together with a partition into fat flowers, and orientations on each of them (which may or may not match at the cuts; see Figure 4.4).

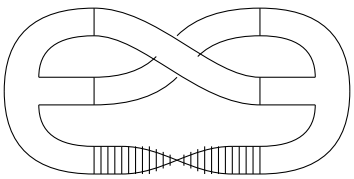


Figure 4.4. Twisted fat graph

Now one can show analogously to the Hermitian case that the $\frac{1}{N}$ expansion of $\log \widehat{Z}_N$ (where $\widehat{Z}_N := Z_N(\frac{2\hbar}{N})$) is given by the same formula as before, but with summation over the set $\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})$ of twisted fat graphs:

Theorem 4.7.

$$\log \widehat{Z}_N = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})} \frac{N^{2-2\mathrm{g}(\widetilde{\Gamma})}}{|\operatorname{Aut}(\widetilde{\Gamma})|}.$$

Here the genus g of a (possibly nonorientable) surface is defined for closed surfaces by $\mathrm{g} := 1 - \frac{\chi}{2}$, where χ is the Euler characteristic. Thus the genus of $\mathbb{RP}^2$ is $\frac{1}{2}$, the genus of the Klein bottle is 1, and so on.

In particular, we have the following version of 't Hooft's theorem.

Theorem 4.8.

(1) *There exists a limit* $W_\infty := \lim_{N\to\infty} \frac{\log \widehat{Z}_N}{N^2}$. *This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\tilde{\Gamma}\in \widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[0]} \frac{1}{|\operatorname{Aut}(\tilde{\Gamma})|},$$

where $\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[0]$ *denotes the set of planar connected twisted fat graphs, i.e., those which have genus zero.*

(2) *Moreover, there exists an expansion*

$$\frac{\log \widehat{Z}_N}{N^2} = \sum_{\mathrm{g}\in \frac{1}{2}\mathbb{Z}_{\geq 0}} a_{\mathrm{g}} N^{-2\mathrm{g}},$$

where

$$a_{\mathrm{g}} = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\tilde{\Gamma}\in \widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[\mathrm{g}]} \frac{1}{|\operatorname{Aut}(\tilde{\Gamma})|},$$

and $\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[\mathrm{g}]$ *denotes the set of connected twisted fat graphs which have genus* g.

Exercise 4.9. Consider the matrix integral over the space $\mathfrak{q}_N$ of quaternionic Hermitian matrices of size N. Show that in this case the results are the same as in the real case, except that each twisted fat graph counts with a sign equal to $(-1)^\nu$, where ν is the number of boundary components. In other words, $\log Z_N^{\mathrm{quat}}(\hbar)$ equals $\log Z_{2N}^{\mathrm{real}}(\hbar)$ with N replaced by $-N$.

Hint. Use that the quaternionic unitary group $U(N, \mathbb{H})$ is a real form of $\mathrm{Sp}(2N)$, and $\mathfrak{q}_N$ is a real form of the representation of $\Lambda^2 V$, where V is the standard (vector) representation of $\mathrm{Sp}(2N)$. Compare to the case of real symmetric matrices, where the relevant representation is $S^2 V$ for $O(N)$, and the case of complex Hermitian matrices, where it is $V \otimes V^*$ for $\mathrm{GL}(N)$.

4.4. The number of ways to glue a surface from a polygon and the Wigner semicircle law

Matrix integrals are so rich that even the simplest possible example reduces to a nontrivial counting problem. Namely, consider the matrix integral Z_N over complex Hermitian matrices with $\hbar = 1$ in the case $S(A) = \frac{\mathrm{Tr}(A^2)}{2} - s\frac{\mathrm{Tr}(A^{2m})}{2m}$,

where $s^2 = 0$ (i.e., we work over the ring $\mathbb{C}[s]/(s^2)$). Then from Theorem 4.4 we get

$$\int_{\mathfrak{h}_N} \mathrm{Tr}(A^{2m}) e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA = P_m(N),$$

where $P_m(N)$ is a polynomial given by the formula

$$P_m(N) = \sum_{g \geq 0} \varepsilon_g(m) N^{m+1-2g},$$

and $\varepsilon_g(m)$ is the number of ways to glue a surface of genus g from a $2m$-gon with labeled sides, i.e., to match the sides and then glue the matching ones to each other in an orientation-preserving manner. Indeed, in this case we have only one fat flower of valency $2m$, which has to be glued with itself; so a direct application of our Feynman rules leads to counting ways to glue a surface of a given genus from a polygon.

The value of this integral is given by the following nontrivial theorem.

Theorem 4.10 (Harer-Zagier [**HZ**])**.**

$$P_m(N) = \frac{(2m)!}{2^m m!} \sum_{p=0}^{m} \binom{m}{p} 2^p \frac{N(N-1)\dots(N-p)}{(p+1)!}.$$

The theorem is proved in the next sections.

Looking at the leading coefficient of P_m, we get

Corollary 4.11. *The number of ways to glue a sphere from a $2m$-gon is the Catalan number $C_m = \frac{(2m)!}{m!(m+1)!} = \frac{1}{m+1}\binom{2m}{m}$.*

Corollary 4.11 actually has another (elementary combinatorial) proof, which is as follows. For each matching σ on the set of sides of the $2m$-gon, let us connect the midpoints of the matched sides by straight lines; see Figure 4.5. It is geometrically evident that if these lines do not intersect, then the gluing will give a sphere. We claim that the converse is true as well. Indeed, assume the contrary, i.e., that for cyclically ordered edges a, b, c, d, the edge a connects to c and the edge b connects to d. Then it is easy to see that gluing these two pairs of edges gives a torus with a hole (or without if $m = 2$). But

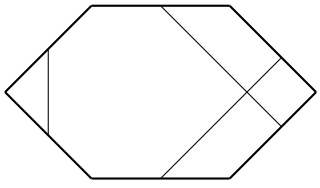


Figure 4.5. Matching of sides of a 6-gon.

an (open) torus with a hole can't be embedded into a sphere (e.g., unlike the sphere, it contains a non-self-intersecting copy of K_5), a contradiction.

Now it remains to count the number of ways to connect midpoints of sides with lines without intersections. Suppose we draw one such line, such that the number of sides on the left is $2k$ and on the right is $2l$ (so that $k + l = m - 1$). Then we face the problem of connecting the two sets of $2k$ and $2l$ sides without intersections. This shows that the number of gluings D_m satisfies the recursion

$$D_m = \sum_{k+l=m-1} D_k D_l,\ D_0 = 1.$$

In other words, the generating function

$$h(x) := \sum_m D_m x^m = 1 + x + \cdots$$

satisfies the equation $h(x) - 1 = xh(x)^2$. This implies that

$$h(x) = \frac{1 - \sqrt{1 - 4x}}{2x},$$

which yields that $D_m = C_m$. We are done.

Corollary 4.11 can be used to derive the following fundamental result from the theory of random matrices, discovered by Wigner in 1955.

Theorem 4.12 (Wigner's semicircle law)**.** *Let f be a continuous function on $\mathbb{R}$ of at most polynomial growth at infinity. Then*

$$\lim_{N\to\infty} \frac{1}{N} \int_{\mathfrak{h}_N} \operatorname{Tr} f(\tfrac{A}{\sqrt{N}}) e^{-\frac{\operatorname{Tr}(A^2)}{2}} = \frac{1}{2\pi} \int_{-2}^{2} f(x)\sqrt{4 - x^2}\,dx.$$

This theorem is called the semicircle law because it says that the graph of the density of eigenvalues of a large random Hermitian matrix distributed according to the "Gaussian unitary ensemble" (i.e., with density $e^{-\frac{\operatorname{Tr}(A^2)}{2}} dA$) is a semicircle. In particular, we see that for large N almost all eigenvalues of A belong to the interval $[-2\sqrt{N}, 2\sqrt{N}]$, so the limit does not depend on the values of f outside $[-2, 2]$.

Proof. By Weierstrass' theorem on uniform approximation of a continuous function on an interval by polynomials, we may assume that f is a polynomial. (Exercise: Justify this step.) Thus, it suffices to check the result if $f(x) = x^{2m}$. In this case, by Corollary 4.11, the left-hand side is C_m. On the other hand, an elementary computation yields

$$\frac{1}{2\pi} \int_{-2}^{2} x^{2m}\sqrt{4 - x^2}\,dx = C_m,$$

which implies the theorem. □

4.5. Hermite polynomials

The proof[12] of Theorem 4.10 given below uses Hermite polynomials. So let us recall their properties.

Hermite polynomials are defined by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

So the leading term of $H_n(x)$ is $(2x)^n$.

We collect the standard properties of $H_n(x)$ in the following theorem.

Theorem 4.13.

(i) *The exponential generating function of $H_n(x)$ is*

$$f(x,t) = \sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}.$$

(ii) *$H_n(x)$ satisfy the differential equation $f'' - 2xf' + 2nf = 0$. In other words, $H_n(x)e^{-x^2/2}$ are eigenfunctions of the operator $L = -\frac{1}{2}\partial^2 + \frac{1}{2}x^2$ (the Hamiltonian of the quantum harmonic oscillator) with eigenvalues $n + \frac{1}{2}$.*

(iii) *$H_n(x)$ are orthogonal:*

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n!\, \delta_{mn}.$$

Moreover, the functions $H_n(x)e^{-\frac{x^2}{2}}$ form an orthogonal basis of $L^2(\mathbb{R})$.

(iv) *One has*

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^{2m} H_{2k}(x) dx = \frac{(2m)!}{(m-k)!} 2^{2(k-m)}$$

(if $k > m$, the answer is zero).

(v) *One has*

$$\frac{H_r^2(x)}{2^r r!} = \sum_{k=0}^{r} \frac{r!}{2^k k!^2\, (r-k)!} H_{2k}(x).$$

Sketch of the proof. (i) This follows immediately from the fact that the operator $\sum_{n\geq 0}(-1)^n \frac{t^n}{n!}\frac{d^n}{dx^n}$ maps a function $g(x)$ to $g(x-t)$.

(ii) This follows from (i) and the fact that the function $f(x,t)$ satisfies the PDE

$$f_{xx} - 2xf_x + 2tf_t = 0.$$

[12] I adopted this proof from D. Jackson's notes.

(iii) The orthogonality follows from (i) by direct integration:

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} f(x,t)f(x,u)e^{-x^2}dx = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{2ut-(x-u-t)^2}dx = e^{2ut}.$$

Thus the functions $H_n(x)e^{-\frac{x^2}{2}}$ form an orthogonal system in $L^2(\mathbb{R})$.

To show that these functions are complete, denote by $E \subset L^2(\mathbb{R})$ the closure of their span $\mathbb{C}[x]e^{-\frac{x^2}{2}}$. By approximating the function e^{ipx} by its Taylor polynomials, it is easy to see that $e^{ipx-\frac{x^2}{2}} \in E$ for any $p \in \mathbb{R}$. Thus for any compactly supported smooth $\phi \in C_0^\infty(\mathbb{R})$, we have

$$\phi(x)e^{-\frac{x^2}{2}} = \int_{\mathbb{R}} \widehat{\phi}(p)e^{ipx-\frac{x^2}{2}}dp \in E,$$

where $\widehat{\phi}$ is the (suitably normalized) Fourier transform of ϕ. In other words, $C_0^\infty(\mathbb{R})$ is dense in E. But $C_0^\infty(\mathbb{R})$ is clearly dense in $L^2(\mathbb{R})$, so $E = L^2(\mathbb{R})$, as claimed.

(iv) By (i), one should calculate $\int_{\mathbb{R}} x^{2m}e^{2xt-t^2}e^{-x^2}dx$. This integral equals

$$\int_{\mathbb{R}} x^{2m}e^{-(x-t)^2}dx = \int_{\mathbb{R}} (y+t)^{2m}e^{-y^2}dy = \sqrt{\pi}\sum_p \binom{2m}{2p}\frac{(2m-2p)!}{2^{2(m-p)}(m-p)!}t^{2p}.$$

The result is now obtained by extracting individual coefficients.

(v) By (iii), it suffices to show that

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} H_r^2(x)H_{2k}(x)e^{-x^2}dx = \frac{2^{r+k}r!^2\,(2k)!}{k!^2\,(r-k)!}.$$

To prove this identity, let us integrate the product of three generating functions. By (i), we have

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} f(x,t)f(x,u)f(x,v)e^{-x^2}dx$$
$$= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{2(ut+uv+tv)-(x-u-t-v)^2}dx = e^{2(ut+tv+uv)}.$$

Extracting the coefficient of $t^r u^r v^{2k}$, we get the result.

□

4.6. Proof of Theorem 4.10

We need to compute the integral

$$\int_{\mathfrak{h}_N} \mathrm{Tr}(A^{2m})e^{-\frac{\mathrm{Tr}(A^2)}{2}}dA.$$

To do this, we note that the integrand is invariant with respect to conjugation by unitary matrices. Therefore, the integral can be reduced to an integral over the eigenvalues $\lambda_1, \dots, \lambda_N$ of A.

More precisely, consider the spectrum map $\sigma : \mathfrak{h}_N \to \mathbb{R}^N/S_N$. It is well known (due to H. Weyl) that the direct image $\sigma_* dA$ is given by the formula $\sigma_* dA = C \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda$, where $C > 0$ is a normalization constant that will not be relevant to us. Thus, we have

$$P_m(N) = \frac{NJ_m}{J_0},\ J_m := \int_{\mathbb{R}^N} (\tfrac{1}{N} \sum_i \lambda_i^{2m}) e^{-\sum_i \frac{\lambda_i^2}{2}} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda.$$

To calculate J_m, we will use Hermite polynomials. Observe that since $H_n(x)$ are polynomials of degree n with highest coefficient 2^n, we have

$$\prod_{i<j} (\lambda_i - \lambda_j) = 2^{-\frac{N(N-1)}{2}} \det(H_k(\lambda_\ell)),$$

where k runs through the set $0, 1, \dots, N-1$ and ℓ runs through $1, \dots, N$. Thus, we find

(4.2)
$$\begin{aligned} J_m &= 2^{m+\frac{N^2}{2}} \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda \\ &= 2^{m-\frac{N(N-2)}{2}} \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \det(H_k(\lambda_j))^2 d\lambda \\ &= 2^{m-\frac{N(N-2)}{2}} \sum_{\sigma,\tau \in S_N} (-1)^\sigma (-1)^\tau \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \prod_i H_{\sigma(i)}(\lambda_i) H_{\tau(i)}(\lambda_i) d\lambda. \end{aligned}$$

(Here $(-1)^\sigma$ denotes the sign of σ.)

Since Hermite polynomials are orthogonal, the only terms in this sum that are nonzero are the terms with $\sigma(i) = \tau(i)$ for $i = 2, \dots, N$. That is, the nonzero terms have $\sigma = \tau$. Thus, we have

(4.3)
$$\begin{aligned} J_m &= 2^{m-\frac{N(N-2)}{2}} \sum_{\sigma \in S_N} \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \prod_i H_{\sigma(i)}(\lambda_i)^2 d\lambda \\ &= 2^{m-\frac{N(N-2)}{2}} (N-1)!\, \gamma_0 \dots \gamma_{N-1} \sum_{j=0}^{N-1} \frac{1}{\gamma_j} \int_{-\infty}^{\infty} x^{2m} H_j(x)^2 e^{-x^2} dx, \end{aligned}$$

where $\gamma_i := \int_{-\infty}^{\infty} H_i(x)^2 e^{-x^2} dx$ are the squared norms of the Hermite polynomials. Applying this for $m = 0$ and dividing NJ_m by J_0, we find

$$P_m(N) = 2^m \sum_{j=0}^{N-1} \frac{1}{\gamma_j} \int_{-\infty}^{\infty} x^{2m} H_j(x)^2 e^{-x^2} dx.$$

Using Theorem 4.13(iii) and (v), we find that $\gamma_i = 2^i i! \sqrt{\pi}$, and hence

$$P_m(N) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \sum_{j=0}^{N-1} \sum_{k=0}^{j} \frac{2^m x^{2m} H_{2k}(x)}{2^k k!^2 (j-k)!} e^{-x^2} dx.$$

Now, using Theorem 4.13(iv), we get

$$\begin{aligned} P_m(N) &= \frac{(2m)!}{2^m} \sum_{j=0}^{N-1} \sum_{k=0}^{j} \frac{2^k j!}{(m-k)!\, k!^2 (j-k)!} \\ &= \frac{(2m)!}{2^m m!} \sum_{j=0}^{N-1} \sum_{k=0}^{j} 2^k \binom{m}{k} \binom{j}{k}. \end{aligned}$$

The sum over k can be represented as the constant term of a polynomial:

$$\sum_{k=0}^{j} 2^k \binom{m}{k} \binom{j}{k} = C.T.((1+z)^m (1+2z^{-1})^j).$$

Therefore, summation over j (using the formula for the sum of the geometric progression) yields

$$\begin{aligned} P_m(N) &= \frac{(2m)!}{2^m m!} C.T. \left((1+z)^m \frac{(1+2z^{-1})^N - 1}{2z^{-1}} \right) \\ &= \frac{(2m)!}{2^m m!} \sum_{p=0}^{m} 2^p \binom{m}{p} \binom{N}{p+1}. \end{aligned}$$

We are done.

Exercise 4.14. Find the number of ways to glue an orientable surface of genus $g \geq 1$ from a 4g-gon (the gluing must preserve orientation), and justify your answer.

Answer: $\frac{(4g-1)!!}{2g+1}$.

Exercise 4.15. Consider a random Hermitian matrix $A \in \mathfrak{h}_N$, distributed with Gaussian density $e^{-\operatorname{Tr}(A^2)} dA$. Show that the most likely eigenvalues of A are the roots of the Nth Hermite polynomial H_N.

Hint.

(a) Write down the system of algebraic equations for the maximum of the density on eigenvalues, by equating its logarithmic partial derivatives to zero.

(b) Introduce the polynomial $P(z) = \prod_i (z - \lambda_i)$, where λ_i are the most likely eigenvalues. Let $f = P'/P$. Compute $f' + f^2$ (look at the poles).

(c) Reduce the obtained Riccati equation for f to a second-order linear differential equation for P. Show that this equation is Hermite's equation, and deduce that $P = \frac{H_N}{2^N}$.

4.7. Matrix integrals and counting planar diagrams

Now let us apply the matrix integral techniques to counting planar graphs. Let

$$U(x) := \frac{x^2}{2} - \sum_{j\geq 1} g_j \frac{x^j}{j}$$

(with g_j being formal parameters), and consider the matrix integral

$$Z_N(\hbar) = \hbar^{-N^2/2} \int_{\mathfrak{h}_N} \exp\left(-\frac{\operatorname{Tr} U(A)}{\hbar}\right) dA.$$

Let $\widehat{Z}_N(\hbar) = Z_N(\hbar/N)$. We have seen that

$$\lim_{N\to\infty} \frac{\log \widehat{Z}_N}{N^2} = W_\infty,$$

where W_∞ is given by summation over planar fat graphs:

$$W_\infty = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\tilde{\Gamma}\in \widetilde{G}_c(\mathbf{n})[0]} \frac{1}{|\operatorname{Aut}(\tilde{\Gamma})|}.$$

In particular, the coefficient of $\prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i}$ is the number of (orientation-preserving) gluings of a fat graph of genus zero out of a collection of fat flowers containing n_i i-valent flowers for each i, divided by $\prod_i i^{n_i} n_i!$.

On the other hand, one can compute W_∞ explicitly as a function of g_i by reducing the matrix integral to an integral over eigenvalues, and then using a fundamental fact from the theory of random matrices: the existence of an asymptotic distribution of eigenvalues in the limit $N \to \infty$. This approach allows one to obtain simple closed formulas for the numbers of planar gluings, which are quite nontrivial and for which direct combinatorial proofs were discovered much later.

To illustrate this method, we will restrict ourselves to the case of the potential $U(x) = \frac{x^2}{2} + gx^4$ (so $g_4 = -4g$ and other $g_i = 0$), and set $\hbar = 1$. Then

$$W_\infty = \sum_{n\geq 1} c_n \frac{(-1)^n g^n}{n!},$$

where c_n is a number of connected planar gluings of a set of n 4-valent flowers. In other words, c_n is the number of ways (up to isotopy) to connect n "crosses" in the 2-sphere so that all crosses are connected with each other, all the arms are used, and the connecting lines do not intersect.

Exercise 4.16. Check by drawing pictures that $c_1 = 2$, $c_2 = 36$.

Theorem 4.17 (Brézin, Itzykson, Parisi, and Zuber in 1978 [**BIPZ**])**.** *One has*

$$c_n = (12)^n \frac{(2n-1)!}{(n+2)!}.$$

The proof of this theorem (with some omissions) is given in Section 4.8.

4.8. Proof of Theorem 4.17

We follow the paper [**BIPZ**]. We will assume that g is a positive real number and compute the function $W_\infty(g)$ explicitly. The relevant matrix integral has the form

$$\widehat{Z}_N = \int_{\mathfrak{h}_N} e^{-N \operatorname{Tr}(\frac{1}{2}A^2 + gA^4)} dA.$$

Passing to eigenvalues, we get

$$\widehat{Z}_N = \frac{J_N(g)}{J_N(0)},$$

where

$$J_N(g) = \int_{\mathbb{R}^N} e^{-N(\frac{1}{2}\sum_i \lambda_i^2 + g\sum_i \lambda_i^4)} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda. \tag{4.4}$$

Thus, $W_\infty(g) = E(g) - E(0)$, where $E(g) = \lim_{N\to\infty} N^{-2} \log J_N(g)$.

Proposition 4.18 (Steepest descent principle)**.** *$E(g)$ equals the leading coefficient of the asymptotics as $N \to \infty$ of the maximal value of the logarithm of the integrand in* (4.4)*.*

Proposition 4.18 says, essentially, that the integrand has a sufficiently sharp maximum, so that the leading behavior of the integral can be computed by the steepest descent formula. We note that we cannot apply the steepest descent formula without explanation, since the integral is over a space whose dimension grows as the perturbation parameter $1/N$ goes to 0. In other words, it is necessary to do some estimates, which we will omit. We will just mention that for $g = 0$, this result can be derived from the explicit evaluation of the integral using Hermite polynomials; see Section 4.6. For the general case, we refer the reader to the book [**De**].

The logarithm of the integrand

$$K(\lambda_1, \dots, \lambda_N) := -N(\frac{1}{2}\sum_i \lambda_i^2 + g\sum_i \lambda_i^4) + 2\sum_{i<j} \log|\lambda_i - \lambda_j|$$

has a unique critical point (the global maximum), because it is concave (check it!). This critical point is found by equating the partial derivatives to zero. This yields

$$\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} = N(\tfrac{1}{2}\lambda_i + 2g\lambda_i^3). \tag{4.5}$$

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ be the unique (up to permutations) solution of this system of equations.

Proposition 4.19. *The normalized counting measures $\frac{1}{N}\sum_i \delta(x-\lambda_i)$ converge weakly to a measure $\mu(x) = f(x,g)dx$, where $f(x,g)$ is a continuous function supported on a finite interval $[-2a, 2a]$ and is differentiable on the interior of this interval.*

For the proof we again refer the reader to [**De**, p. 132 and later]. We note that for $g = 0$, by Wigner's semicircle law, $a = 1$ and $f(x,0) = \frac{1}{2\pi}\sqrt{4-x^2}$; so $f(x,g) = \frac{1}{2\pi}\sqrt{4-x^2} + O(g)$.

Now our job will be to find the function $f(x,g)$. Passing to the limit in equation (4.5) (which requires justification that we will omit), we get

$$\int_{-2a}^{2a} \frac{1}{y-x} f(x,g)dx = \frac{1}{2}y + 2gy^3,\ |y| \le 2a,$$

where the integral is understood in the sense of principal value.

This is a linear integral equation on $f(x,g)$, which can be solved in a standard way. Namely, one considers the analytic function

$$F(y) = \int_{-2a}^{2a} \frac{1}{y-x} f(x,g)dx$$

for y in the complex plane but outside of the interval $[-2a, 2a]$. For $y \in [-2a, 2a]$, let $F_+(y)$, $F_-(y)$ denote the limits of $F(y)$ from above and below. Then by the Plemelj formula, the integral equation implies

$$\frac{1}{2}(F_+(y) + F_-(y)) = \frac{1}{2}y + 2gy^3.$$

On the other hand, $F_-(y) = \overline{F_+(y)}$. Hence,

$$\operatorname{Re} F_+(y) = \operatorname{Re} F_-(y) = \frac{1}{2}y + 2gy^3.$$

Now set $y := a(z + z^{-1})$. Then, as y runs through the exterior of $[-2a, 2a]$, z runs through the exterior of the unit circle. So the function $G(z) := F(y)$ is analytic on the outside of the unit circle, with decay at infinity, and

$$\operatorname{Re} G(z) = \frac{1}{2}a(z + z^{-1}) + 2ga^3(z + z^{-1})^3$$

when $|z| = 1$. This implies that $G(z)$ is twice the sum of all negative degree terms of this Laurent polynomial. In other words, we have

$$G(z) = 4ga^3z^{-3} + (a + 12ga^3)z^{-1}.$$

This yields

$$F(y) = \frac{1}{2}y + 2gy^3 - \left(\frac{1}{2} + 4ga^2 + 2gy^2\right)\sqrt{y^2 - 4a^2}.$$

Now $f(y,g)$ is found as the jump of F:

$$f(y,g)=\frac{1}{\pi}\left(\frac{1}{2}+4ga^2+2gy^2\right)\sqrt{4a^2-y^2}.$$

It remains to find a in terms of g. We have $yF(y)\to 1$, $y\to\infty$ (as $\int f(x,g)dx=1$), hence $zG(z)\to 1/a$, $z\to\infty$. This yields

$$\frac{1}{a}=a+12ga^3$$

or

$$12ga^4+a^2-1=0.$$

This allows one to determine a uniquely:

$$a=\left(\frac{(1+48g)^{1/2}-1}{24g}\right)^{1/2}.$$

Now let us calculate $E(g)$. It follows from the above that

$$E(g)=\int_{-2a}^{2a}\int_{-2a}^{2a}\log|x-y|f(x,g)f(y,g)dxdy-\int_{-2a}^{2a}(\tfrac{1}{2}x^2+gx^4)f(x,g)dx.$$

On the other hand, let us integrate the integral equation defining $f(x,g)$ with respect to y (from 0 to u). Then we get

$$2\int_{-2a}^{2a}(\log|x-u|-\log|x|)f(x,g)dx=\tfrac{1}{2}u^2+gu^4.$$

Substituting this into the expression for $E(g)$, we get

$$E(g)=\int_{-2a}^{2a}(\log|u|-\tfrac{1}{4}u^2-\tfrac{1}{2}gu^4)f(u,g)du.$$

Since $f(u,g)$ is known, this integral can be computed. In fact, it can be expressed via elementary functions, and after calculations we get

$$E(g)-E(0)=\log a-\frac{1}{24}(a^2-1)(9-a^2).$$

Substituting here the expression for a, after a calculation one finally gets

$$E(g)-E(0)=\sum_{k=1}^{\infty}(-12g)^k\frac{(2k-1)!}{k!\,(k+2)!}.$$

This implies the required formula for c_n.

Chapter 5

The Euler characteristic of the moduli space of curves

Matrix integrals (in particular, the computation of the polynomial $P_m(x)$) can be used to calculate the orbifold Euler characteristic of the moduli space of curves. This was done by Harer and Zagier in 1986. Here we will give a review of this result (with some omissions).

5.1. Euler characteristics of groups

We start with recalling some basic notions from algebraic topology.

Let Γ be a discrete group, and let Y be a contractible finite-dimensional CW complex, on which Γ acts cellularly. This means that Γ acts by homeomorphisms of Y that map each cell homeomorphically to another cell. We will assume that the stabilizer of each cell is a finite group (i.e., Y is a proper Γ-complex).

Suppose first that the action of Γ is free (i.e., the stabilizers of cells are trivial). This is equivalent to saying that Γ is torsion free (i.e., has no nontrivial finite subgroups), since a finite group cannot act without fixed points on a contractible finite-dimensional cell complex (as it has infinite cohomological dimension).

In this case we can define a cell complex Y/Γ (a classifying space for Γ), and we have $H^i(Y/\Gamma, A) = H^i(\Gamma, A)$ for any coefficient group A. In particular, if Y/Γ

is finite, then Γ has finite cohomological dimension, and the Euler characteristic $\chi(\Gamma) := \sum_i (-1)^i \dim H^i(\Gamma, \mathbb{Q})$ is equal to $\sum_i (-1)^i n_i(Y/\Gamma)$, where $n_i(Y/\Gamma)$ denotes the number of cells in Y/Γ of dimension i.

This setting, however, is very restrictive, since it allows only groups of finite cohomological dimension and, in particular, excludes all nontrivial finite groups. So let us consider a more general setting: assume that some finite index subgroup $\Gamma' \subset \Gamma$, rather than Γ itself, satisfies the above conditions. In this case, one may define the Euler characteristic of Γ in the sense of Wall, which is the rational number $[\Gamma : \Gamma']^{-1}\chi(\Gamma')$.

It is easy to check that the Euler characteristic in the sense of Wall can be computed using the following *Quillen's formula*

$$\chi(\Gamma) = \sum_{\sigma \in \text{cells}(Y)/\Gamma} \frac{(-1)^{\dim \sigma}}{|\operatorname{Stab} \sigma|}.$$

In particular, this number is independent of Γ' (which is also easy to check directly).

Example 5.1. If G is a finite group, then $\chi(G) = |G|^{-1}$ (one takes the trivial group as the subgroup of finite index).

Example 5.2. $G = \mathrm{SL}_2(\mathbb{Z})$. This group contains a subgroup F of index 12, which is free in two generators (check it!). The group F has Euler characteristic -1, since its classifying space Y/F is ∞ (i.e., Y is the universal cover of ∞). Thus, the Euler characteristic of $\mathrm{SL}_2(\mathbb{Z})$ is $-\frac{1}{12}$.

The Euler characteristic in the sense of Wall has a geometric interpretation in terms of orbifolds. Namely, suppose that Γ is as above (i.e. $\chi(\Gamma)$ is a well-defined rational number) and M is a contractible manifold on which Γ acts properly discontinuously. In this case, stabilizers of points are finite, thus M/Γ is an orbifold. This means, in particular, that to every point $x \in M/\Gamma$ is attached a finite group $\mathrm{Aut}(x)$, of size $\leq [\Gamma : \Gamma']$. Let X_m be the subset of M/Γ, consisting of points x such that $\mathrm{Aut}(x)$ has order m. It often happens that X_m has the homotopy type of a finite cell complex. In this case, the *orbifold Euler characteristic* of M/Γ is defined to be

$$\chi_{\text{orb}}(M/\Gamma) = \sum_m \frac{\chi(X_m)}{m}.$$

Now, we claim that $\chi_{\text{orb}}(M/\Gamma) = \chi(\Gamma)$. Indeed, looking at the projection $M/\Gamma' \to M/\Gamma$, it is easy to see that $\chi_{\text{orb}}(M/\Gamma) = \frac{1}{[\Gamma:\Gamma']}\chi(M/\Gamma')$. But M/Γ' is a classifying space for Γ', so $\chi(M/\Gamma') = \chi(\Gamma')$, which implies the claim.

Example 5.3. Consider the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acting on the upper half-plane H. Then H/Γ is the moduli space of elliptic curves. So as a topological space it

is $\mathbb{C}$, where all points have automorphism group $\mathbb{Z}/2$, except the point i having automorphism group $\mathbb{Z}/4$, and $\rho = \frac{-1+i\sqrt{3}}{2}$, which has automorphism group $\mathbb{Z}/6$. Thus, the orbifold Euler characteristic of H/Γ is $(-1)\frac{1}{2}+\frac{1}{4}+\frac{1}{6} = -\frac{1}{12}$. This is not surprising since we proved that $\chi_{\mathrm{orb}}(H/\Gamma) = \chi(\Gamma)$, which was computed to be $-\frac{1}{12}$.

5.2. The mapping class group

Now let $g \geq 1$ be an integer, and let Σ be a closed oriented surface of genus g. Let $p \in \Sigma$, and let Γ_g^1 be the group of isotopy classes of diffeomorphisms of Σ which preserve p. We will recall without proof some standard facts about this group, following the paper of Harer and Zagier, [**HZ**].

The group Γ_g^1 is not torsion free, but it has a torsion free subgroup of finite index. Namely, consider the homomorphism $\Gamma_g^1 \to \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ given by the action of Γ_g^1 on $H_1(\Sigma, \mathbb{Z}/n\mathbb{Z})$. Then for large enough n (in fact, $n \geq 3$), the kernel K_n of this map is torsion free.

It turns out that there exists a contractible finite-dimensional cell complex Y_g, to be constructed below, on which Γ_g^1 acts cellularly with finitely many cell orbits. Thus, the Euler characteristic of Γ_g^1 in the sense of Wall is well defined.

5.3. The Harer-Zagier theorem

The Euler characteristic of Γ_g^1 is given by the following theorem.

Theorem 5.4 (Harer-Zagier [**HZ**])**.** *One has*

$$\chi(\Gamma_g^1) = -\frac{B_{2g}}{2g},$$

where B_n are the Bernoulli numbers.

Remark 5.5. The group Γ_g^1 acts on the Teichmüller space $\mathcal{T}_g^1$, which is, by definition, the space of pairs $((C, z), f)$, where (C, z) is a compact Riemann surface of genus g with a marked point z, and f is an isotopy class of diffeomorphisms $C \to \Sigma$ that map z to p. One may show that $\mathcal{T}_g^1$ is a contractible manifold of dimension $6g - 4$, and that the action of Γ_g^1 on $\mathcal{T}_g^1$ is properly discontinuous. In particular, we may define an orbifold $M_g^1 = \mathcal{T}_g^1/\Gamma_g^1$. This orbifold parametrizes pairs (C, z) as above; therefore, it is called the moduli space of Riemann surfaces (=smooth complex projective algebraic curves) of genus g with one marked point. Thus, Theorem 5.4 gives the orbifold Euler characteristic of the moduli space of curves of genus g with one marked point.

Remark 5.6. If $g > 1$, one may define the analogues of the above objects without marked points, namely the mapping class group Γ_g, the Teichmüller space $\mathcal{T}_g$, and the moduli space of curves $M_g = \mathcal{T}_g/\Gamma_g$. (One can do it for $g = 1$ as well,

but in this case there is no difference with the case of one marked point, since the translation group allows one to identify any two points on Σ.) It is easy to see that for $g > 1$, we have an exact sequence $1 \to \pi_1(\Sigma) \to \Gamma_g^1 \to \Gamma_g \to 1$, which implies that $\chi(\Gamma_g) = \chi(\Gamma_g^1)/\chi(\Sigma)$. Thus, the Harer-Zagier theorem implies that $\chi(\Gamma_g) = \chi_{\text{orb}}(M_g) = \frac{B_{2g}}{4g(g-1)}$.

5.4. Construction of the complex Y_g

We begin the proof of Theorem 5.4 with the construction of the complex Y_g, following [**HZ**]. We will first construct a simplicial complex with a Γ_g^1 action and then use it to construct Y_g.

Let $(\alpha_1, \dots, \alpha_n)$ be a collection of closed simple unoriented curves on Σ, which begin and end at p, and which do not intersect other than at p. Such a collection is called an *arc system* if two conditions are satisfied:

(A) none of the curves is contractible to a point;

(B) none of the curves is contractible to another.

Define a simplicial complex A, whose $n-1$-simplices are isotopy classes of arc systems consisting of $n \geq 1$ arcs, and the boundary of a simplex corresponding to $(\alpha_1, \dots, \alpha_n)$ is the union of simplices corresponding to the arc system $(\alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n)$ (α_i is omitted).

It is clear that the group Γ_g^1 acts simplicially on A.

Example 5.7. Let $g = 1$, i.e., $\Sigma = S^1 \times S^1$. Then $\Gamma_g^1 = \mathrm{SL}_2(\mathbb{Z})$. Up to its action, there are only three arc systems (Figure 5.1). Namely, viewing S^1 as the unit circle in the complex plane and representing arcs parametrically, we may write these three systems as follows:

$$B_0 = \{(1, e^{i\theta})\};\ B_1 = \{(e^{i\theta}, 1), (1, e^{i\theta})\};\ B_2 = \{(e^{i\theta}, 1), (1, e^{i\theta}), (e^{i\theta}, e^{i\theta})\}.$$

From this it is easy to find the simplicial complex A. Namely, let T be the tree with root t_0 connected to three vertices t_1, t_2, t_3, with each t_i connected to two vertices t_{i1}, t_{i2}, each t_{ij} connected to t_{ij1}, t_{ij2}, etc.; see Figure 5.2. Put at every vertex of T a triangle, with sides transversal to the three edges going out of this vertex, and glue the triangles along the sides. This yields the complex A depicted at Figure 5.3 (check it!). The action of $\mathrm{SL}_2(\mathbb{Z})$ (or rather $\mathrm{PSL}_2(\mathbb{Z})$) on this complex is easy to describe. Namely, recall that $\mathrm{PSL}_2(\mathbb{Z})$ is generated by S, U with defining relations $S^2 = U^3 = 1$. The action of S, U on T is defined as follows: S is the reflection with flip with respect to a side of the triangle Δ_0 centered at t_0 (Figure 5.4), and U is the counterclockwise rotation by $2\pi/3$ around t_0.

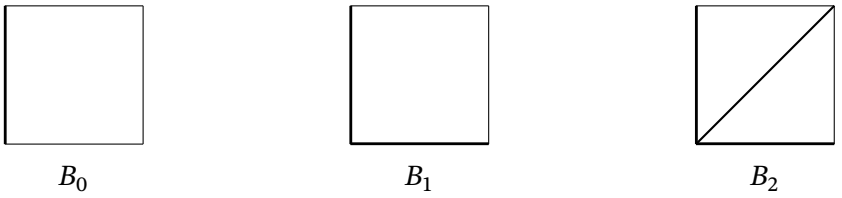


Figure 5.1. Three arc systems.

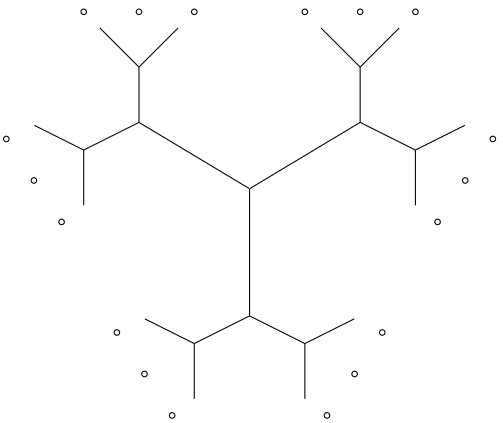


Figure 5.2. The tree T

This example shows that the action of Γ_g^1 on A is not properly discontinuous, as some simplices have infinite stabilizers (in the example, it is the 0-dimensional simplices). Thus, we would like to throw away the "bad" simplices. To do so, let us say that an arc system $(\alpha_1, \ldots, \alpha_n)$ *fills up* Σ if it cuts Σ into a union of regions diffeomorphic to the open disk. Let A_∞ be the union

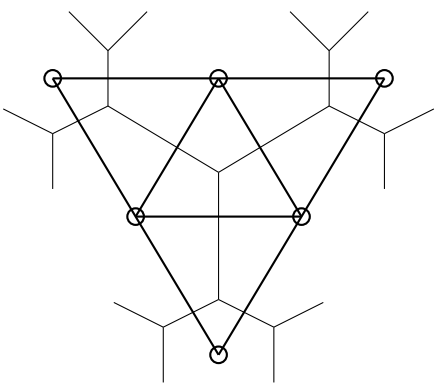


Figure 5.3. The complex A

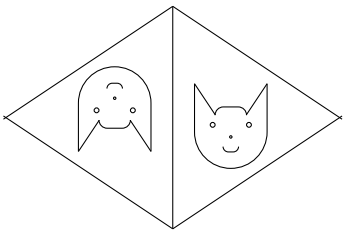


Figure 5.4. Reflection with a flip.

of the simplices in A corresponding to arc systems that do not fill up Σ. This is a closed subset, since the property of not filling up Σ is obviously stable under taking an arc subsystem. Thus, $A \setminus A_\infty$ is an open subset of A. In the example above, it is the complex A with 0-dimensional simplices removed.

The following theorem shows that $A \setminus A_\infty$ is in fact a combinatorial model for the Teichmüller space $\mathcal{T}_g^1$, with the action of Γ_g^1.

Theorem 5.8 (Mumford)**.**

(a) *The action of* Γ_g^1 *on* $A \setminus A_\infty$ *is properly discontinuous.*

(b) $A \setminus A_\infty$ *is topologically a manifold, which is* Γ_g^1*-equivariantly homeomorphic to the Teichmüller space* $\mathcal{T}_g^1$*; in particular, it is contractible.*

Remark 5.9. Theorem 5.8 exhibits the significance of conditions (A) and (B). Indeed, if either of these conditions were dropped, then one could consider arc systems $(\alpha_1, \dots, \alpha_n)$ with arbitrarily large n, while with conditions (A) and (B), as seen from Theorem 5.8, the largest value of n is $6g - 3$.

Remark 5.10. If $g = 1$, Theorem 5.8 is clear from the explicit description of A (convince yourself of this!).

Theorem 5.8 is rather deep, and we will not give its proof, which is beyond the scope of this text. Rather, we will use it to define the "Poincaré dual" CW complex Y_g of $A \setminus A_\infty$. Namely, to each filling arc system $(\alpha_1, \dots, \alpha_n)$ we will assign a $6g - 3 - n$-dimensional cell, and the boundary relation is opposite to the one before. The existence of this CW complex follows from the fact that $A \setminus A_\infty$ is a manifold. For instance, in the case $g = 1$, the complex Y_g is the tree T.

Now, the complex Y_g is contractible (since so is $A \setminus A_\infty$) and admits a cellular action of Γ_g^1 with finitely many cell orbits and finite stabilizers. This means that the Euler characteristic of Γ_g^1 is given by Quillen's formula

$$\chi(\Gamma_g^1) = \sum_{\sigma \in \mathrm{cells}(Y_g)/\Gamma_g^1} (-1)^{\dim \sigma} \frac{1}{|\operatorname{Stab} \sigma|}.$$

Example 5.11. In the $g = 1$ case, T has one orbit of 0-cells and one orbit of 1-cells. The stabilizer of a 0-cell in $SL_2(\mathbb{Z})$ is $\mathbb{Z}/6$, and of a 1-cell is $\mathbb{Z}/4$. Hence, $\chi(SL_2(\mathbb{Z})) = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12}$, which was already computed before by other methods.

5.5. Enumeration of cells in Y_g/Γ_g^1

Now it remains to count the cells in Y_g/Γ_g^1, i.e., to enumerate arc systems which fill Σ (taking into account signs and stabilizers). To do this, we note that by the definition of "filling", any filling arc system S defines a cellular decomposition of Σ. Thus, let S^* be the Poincare dual of this cellular decomposition. Since S has a unique zero cell, S^* has a unique 2-cell. Let n be the number of 1-cells in S (or S^*). Then (Σ, S^*) is obtained by gluing a $2n$-gon (=the unique 2-cell) according to a matching of its sides, preserving orientation. (Note that S can be reconstructed as $(S^*)^*$.)

This allows us to link the problem of enumerating filling arc systems with the problem of counting such gluings, which was solved using matrix integrals. Namely, the problem of enumerating filling arc systems is essentially solved modulo one complication: because of conditions (A) and (B) on an arc system, the gluings we will get are subject to some constraints. Namely, we have

Lemma 5.12. *Let $(\alpha_1, \dots, \alpha_n)$ be a system of curves, satisfying the axioms of a filling arc system except maybe conditions* (A) *and* (B)*. Then*

(i) $(\alpha_1, \dots, \alpha_n)$ *satisfies condition* (A) *iff no edge in the corresponding gluing is glued to a neighboring edge.*

(ii) $(\alpha_1, \dots, \alpha_n)$ *satisfies condition* (B) *iff no two consecutive edges are glued to another pair of consecutive edges in the opposite order.*

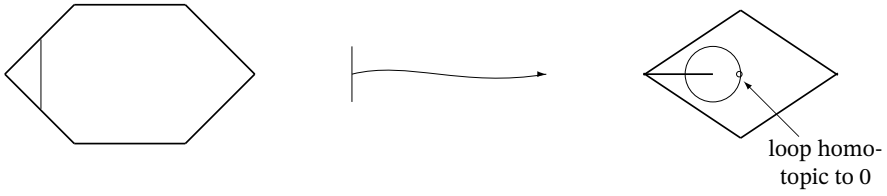


Figure 5.5

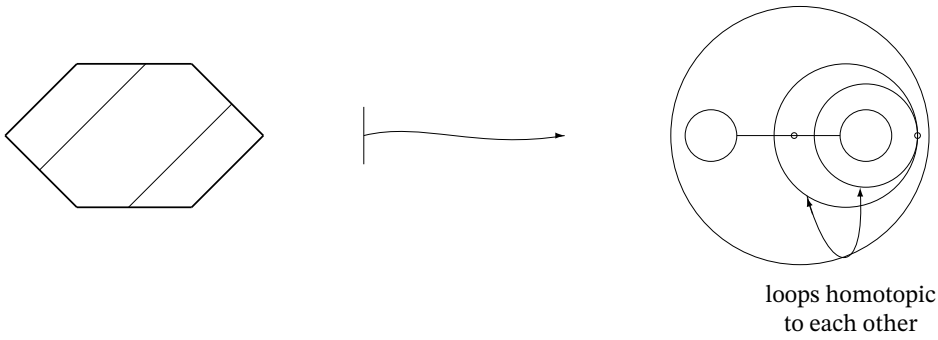


Figure 5.6

The lemma is geometrically evident, and its proof is obtained by drawing a picture; see Figure 5.5 for (i), and Figure 5.6 for (ii). Motivated by the lemma, we will refer to the conditions on a gluing in (i) and (ii) also as conditions (A) and (B).

Denote by $\varepsilon_g(n), \mu_g(n), \lambda_g(n)$ the numbers of gluings of a (labeled) $2n$-gon into a surface of genus g, with no conditions, condition (A), and conditions (A),(B), respectively (so $\varepsilon_g(n)$ is the quantity we already studied).

Proposition 5.13. *One has*[13]

$$\chi(\Gamma_g^1) = \sum_n (-1)^{n-1} \frac{\lambda_g(n)}{2n}.$$

Proof. Each filling arc system σ arises from $2n/|\mathrm{Stab}(\sigma)|$ gluings (since the labeling of the polygon does not matter for the resulting surface with an arc system). Thus, the result follows from Quillen's formula. □

5.6. Computation of $\sum_n (-1)^{n-1} \frac{\lambda_g(n)}{2n}$

Now it remains to compute the sum on the right-hand side. To do this, we will need to link $\lambda_g(n)$ with $\varepsilon_g(n)$, which has already been computed. This is accomplished by the following lemma.

Lemma 5.14.

(i) *One has*

$$\varepsilon_g(n) = \sum_i \binom{2n}{i} \mu_g(n-i).$$

(ii) *One has*

$$\mu_g(n) = \sum_i \binom{n}{i} \lambda_g(n-i).$$

[13]Note that $\lambda_g(n) = 0$ for almost all n, so this sum is finite.

Proof.

(i) Let σ be a matching of the sides of a $2n$-gon Δ with labeled vertices. If there is a pair of consecutive edges that are matched, we can glue them to each other to obtain a $2n-2$-gon. Proceeding like this as long as we can, we will arrive at a $2n-2i$-gon Δ_σ, with a matching σ' of its sides which satisfies condition (A). Note that Δ_σ and σ' do not depend on the order in which neighboring edges were glued to each other, and Δ_σ has a canonical labeling by $1,\dots,2n-2i$, in the increasing order of the "old" labels. Now, we claim that each (Δ_σ,σ') is obtained in exactly $\binom{2n}{i}$ ways; this implies the required statement.

Indeed, let us consider the vertices of Δ that ended up in the interior of Δ_σ. They have mapped to i points in the interior (each gluing of a pair of edges produces a new point). Let us call these points $w_1,\dots,w_i$, and let ν_j be the smallest label of a vertex of Δ that goes to w_j (where we label the vertices so that the kth edge connects vertex k with vertex $k+1$). Then $\nu_1,\dots,\nu_i$ is a subset of $\{1,\dots,2n\}$. This subset completely determines the matching σ if (Δ_σ,σ') are given: namely, we should choose a ν_j such that $\nu_j+1\neq\nu_k$ for any k, and glue the two edges adjacent to ν_j; then relabel by $1,\dots,2n-2$ the remaining vertices (in increasing order of "old" labels), and continue the step again, and so on. From this it is also seen that any set of ν_j may arise. This proves (i).

(ii) Let σ be a matching of Δ (with labeled edges) which satisfies condition (A) but not necessarily (B). If a_1,a_2 are consecutive edges that are glued to consecutive edges b_2,b_1 in the opposite order, then we may unite a_1,a_2 into a single edge a, and b_2,b_1 into b, and obtain a $2n-2$-gon with a matching. Continuing so as long as we can, we will arrive at a $2n-2i$-gon Δ_σ with a new matching σ', which satisfies conditions (A) and (B). In Δ_σ, each (jth) pair of edges is obtained for m_j+1 pairs of edges in Δ. Thus, $\sum_{j=1}^{n-i} m_j = i$. Furthermore, for any (Δ_σ,σ') the collection of numbers $m_1,\dots,m_{n-i}$ defines (Δ,σ) uniquely, up to deciding which of the m_1+1 edges constituting the first edge of Δ_σ should be labeled by 1. Thus, each (Δ_σ,σ') arises in the number of ways given by the formula

$$\sum_{m_1,\dots,m_{n-i}:\sum_{j=1}^{n-i} m_j=i} (m_1+1).$$

It is easy to show (check this!) that this number is equal to $\binom{n}{i}$. This proves (ii).

□

The completion of the proof of Theorem 5.4 depends now on the following computational lemma.

Lemma 5.15. *Let $\varepsilon(n), \mu(n), \lambda(n)$, $n \geq 0$, be sequences satisfying the equations*

$$\varepsilon(n) = \sum_i \binom{2n}{i} \mu(n-i),$$

$$\mu(n) = \sum_i \binom{n}{i} \lambda(n-i).$$

Assume also that $\varepsilon(n) = \binom{2n}{n} f(n)$, where f is a polynomial such that $f(0) = 0$. Then $\lambda(0) = 0$, $\lambda(n)$ has finitely many nonzero values, and

$$\sum_{n\geq 1} (-1)^{n-1} \frac{\lambda(n)}{2n} = f'(0).$$

Proof. Let us first consider any sequences $\varepsilon(n)$, $\mu(n)$, and $\lambda(n)$ linked by the equations of the lemma. Let $E(z), M(z)$, and $L(z)$ be their generating functions (i.e., $E(z) = \sum_{n\geq 0} \varepsilon(n) z^n$ etc.). We claim that

$$E(z) = \frac{1+\sqrt{1-4z}}{2(1-4z)} L\left(\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}}\right).$$

To see this, it suffices to consider the case $\lambda_i = \delta_{ki}$ for some k. In this case,

$$E(z) = \sum_{i,n} \binom{2n}{i}\binom{n-i}{k} z^n = \sum_{p,q\geq 0} \binom{2p+2q}{p}\binom{q}{k} z^{p+q}.$$

But the function

$$F_r(z) := \sum_{p\geq 0} \binom{2p+r}{p} z^p$$

equals

$$F_r(z) = \frac{1}{\sqrt{1-4z}} \left(\frac{1-\sqrt{1-4z}}{2z}\right)^r,$$

as may be easily seen by induction from the recursion

$$F_r = z^{-1}(F_{r-1} - F_{r-2}),$$

where $r \geq 2$. Substituting this into the formula for $E(z)$, one gets (after trivial simplifications)

$$E(z) = \frac{1+\sqrt{1-4z}}{2(1-4z)} \left(\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}}\right)^k,$$

as desired.

Now assume that $\varepsilon(n)$ satisfies the polynomiality condition. This means that $E(z) = P(z\partial)|_{z=0} \frac{1}{\sqrt{1-4z}}$, where P is a polynomial with vanishing constant term. To prove our claim, it suffices to consider the case $P(z) = (1+a)^z - 1$, where a is a formal parameter (so $P'(0) = \log(1+a)$); indeed, the coefficients of

this formal series are $\binom{z}{j}$, $j \geq 1$, which form a basis in the space of polynomials of z with vanishing constant term. In this case we get

$$E(z) = \frac{1}{\sqrt{1-4(1+a)z}} - \frac{1}{\sqrt{1-4z}}.$$

Hence,

$$L(u) = \frac{1}{1+u}\left(\frac{1}{\sqrt{1-4au(1+u)}} - 1\right).$$

Therefore,

$$\sum_k (-1)^{k-1}\frac{\lambda_k}{2k} = \frac{1}{2}\int_{-1}^{0} L(u)\frac{du}{u} = \frac{1}{2}\sum_{p\geq 1}\binom{2p}{p}(-1)^{p-1}a^p \int_0^1 x^{p-1}(1-x)^{p-1}dx.$$

But $\int_0^1 x^{p-1}(1-x)^{p-1}dx$ is an Euler Beta integral, and it equals $\frac{(p-1)!^2}{(2p-1)!}$. Thus,

$$\sum_k (-1)^{k-1}\frac{\lambda_k}{2k} = \sum_{p\geq 1}(-1)^{p-1}\frac{a^p}{p} = \log(1+a),$$

as desired. □

5.7. End of proof of Theorem 5.4

Now we finish the proof of the Harer-Zagier theorem. Recall that using matrix integrals we have proved the formula

$$P_n(x) := \sum_{\mathrm{g}} \varepsilon_{\mathrm{g}}(n)x^{n+1-2\mathrm{g}} = \frac{(2n)!}{2^n n!}\sum_{p\geq 0}\binom{n}{p}2^p\binom{x}{p+1}. \tag{5.1}$$

Let us set $q := n - p$. Then expression (5.1) takes the form

$$P_n(x) = \binom{2n}{n}\sum_{q\geq 0} 2^{-q}\binom{n}{q}\frac{n!}{(n-q+1)!}x(x-1)\dots(x-n+q). \tag{5.2}$$

We claim now that the coefficient of $x^{-2\mathrm{g}}$ ($\mathrm{g} \geq 1$) in the polynomial $\frac{P_n(x)}{x^{n+1}}$ is of the form $\binom{2n}{n}f_{\mathrm{g}}(n)$, where f_{g} is a polynomial. Indeed, contributions to the coefficient of $x^{-2\mathrm{g}}$ come from terms with $q \leq 2\mathrm{g}$ only, so it suffices to check that each of these contributions is as stated. This reduces our task to checking that the coefficients of the Laurent polynomial $Q(x,n) = (1-\frac{1}{x})\dots(1-\frac{n}{x})$ are polynomials in n, which vanish at -1 (except, of course, the leading coefficient). To see this, let $Q(x,a) = \frac{\Gamma(x)}{\Gamma(x-a)x^a}$ (this equals to $Q(x,n)$ if $a = n$). This function has an asymptotic Taylor expansion in $\frac{1}{x}$ as $x \to +\infty$ which is obtained from the Stirling asymptotic expansion of $\Gamma(x)$ given by (2.9), and it is easy to show that the coefficients are polynomials in a. Moreover, $Q(x,-1) = 1$, which implies the required statement.

Furthermore, we claim that $f_g(0) = 0$: again, this follows from the fact that the nonleading coefficients of the expansion of $Q(x, a)$ vanish at $a = 0$. But this is clear, since $Q(x, 0) = 1$.

Thus, we are in a situation where Lemma 5.15 can be applied. So it remains to compute $\sum_{g\geq 1} f_g'(0)x^{-2g}$. To do this, observe that the terms with $q > 1$ do not contribute to $f_g'(0)$, as they are given by polynomials of n that are divisible by n^2. So we only need to consider $q = 0$ and $q = 1$. For $q = 1$, the contribution is the value of

$$\tfrac{1}{2x}(1 - \tfrac{1}{x})\dots(1 - \tfrac{n}{x})$$

at $n = 0$, i.e., it is $\frac{1}{2x}$. For $q = 0$, the contribution is the derivative at 0 with respect to n of $\frac{1}{n+1}(1 - \frac{1}{x})\dots(1 - \frac{n}{x})$, i.e., it is

$$\tfrac{d}{da}|_{a=0}\tfrac{Q(x,a)}{a+1} = -1 + \tfrac{d}{da}|_{a=0}Q(x, a).$$

Thus, we have (asymptotically)

$$\sum_{g\geq 1} f_g'(0)x^{-2g} = \tfrac{1}{2x} + \tfrac{d}{da}|_{a=0}Q(x, a) = \tfrac{1}{2x} + \tfrac{\Gamma'(x)}{\Gamma(x)} - \log x.$$

Now, the asymptotic expansion for Γ'/Γ given by (2.10) implies that $f_g'(0) = -\frac{B_{2g}}{2g}$. This completes the proof.

Exercise 5.16. Prove Theorem 5.8 for $g = 1$.

Exercise 5.17. Let $\Gamma(N)$ be the congruence subgroup of $SL_2(\mathbb{Z})$ which consists of matrices equal to 1 modulo N.

(a) Show that $\Gamma(N)$ is free for $N \geq 3$. (*Hint.* Consider the action of $\Gamma(N)$ on the upper half-plane.) Show that $\Gamma(2)$ is the direct product of a free group $\Gamma_+(2)$ on two generators with $\mathbb{Z}/2\mathbb{Z}$.

(b) Find the number of generators of $\Gamma(N)$, $N \geq 3$, which generate it without relations. (*Hint.* Compute $\chi(\Gamma(N))$).

Exercise 5.18. Let Γ be the group defined by the generators a, b, c with defining relation $ab = ba$. Find the Euler characteristic of Γ.

Exercise 5.19. Consider a triangle Δ in the hyperbolic plane $H = \mathbb{C}_+$ with angles $\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{3}, \gamma = \frac{\pi}{7}$, and let Γ be the subgroup of $PSL_2(\mathbb{R})$ generated by rotations a, b, c around the vertices of Δ by angles $2\alpha, 2\beta, 2\gamma$, respectively.

(i) Show that H/Γ is naturally homeomorphic to a sphere glued out of two copies of Δ, which can be viewed as an orbifold with three points with nontrivial stabilizers (orders 2,3,7).

(ii) Show that the Euler characteristic $\chi(\Gamma)$ is $-\frac{1}{42}$.

(iii) Show that the defining relations for Γ are

$$a^2 = 1,\ b^3 = 1,\ c^7 = 1,\ abc = 1$$

(use an orbifold version of van Kampen's theorem).

(iv) Construct a surjective homomorphism $\phi : \Gamma \to \mathrm{PSL}_2(\mathbb{F}_7)$.

(v) Show that $\mathrm{Ker}\,\phi$ is torsion free and $H/\mathrm{Ker}\,\phi$ is a compact Riemann surface X of genus 3 with an action of $\mathrm{PSL}_2(\mathbb{F}_7)$. Identify X with the Klein quartic $x^3y + y^3z + z^3x = 0$ in $\mathbb{CP}^2$.

Chapter 6

Quantum mechanics

So far we have considered quantum field theory with 0-dimensional spacetime (to make a joke, one may say that the dimension of the space is -1). In this chapter, we will move closer to actual physics: we will consider 1-dimensional spacetime, i.e., the dimension of the space is 0. This does not mean that we will study motion in a 0-dimensional space (which would be really a pity), but it just means that we will consider only point-like quantum objects (particles) and not extended quantum objects (fields). In other words, we will be in the realm of quantum mechanics.

6.1. The path integral in quantum mechanics

Let $U(q)$ be a smooth function on the real line (the potential). We will assume that $U(0) = 0$, $U'(0) = 0$, and $U''(0) = m^2$, where $m > 0$.

Remark 6.1. In quantum field theory the parameter m in the potential is called the *mass* parameter. To be more precise, in classical mechanics it has the meaning of frequency ω of oscillations. However, in quantum theory, thanks to Planck and Einstein, frequency is identified with energy ($E = \hbar\omega$), while in relativistic theory energy is identified with mass (again thanks to Einstein, $E = mc^2$).

We want to construct the theory of a quantum particle moving in the potential field $U(q)$. According to what we discussed before, this means that we want to give sense to and evaluate the normalized correlation functions

$$\langle q(t_1) \dots q(t_n) \rangle := \frac{\int q(t_1) \dots q(t_n) e^{\frac{iS(q)}{\hbar}} Dq}{\int e^{\frac{iS(q)}{\hbar}} Dq},$$

where $S(q) = \int \mathcal{L}(q) dt$, and $\mathcal{L}(q) = \frac{\dot{q}^2}{2} - U(q)$.

As we discussed, such integrals cannot be handled rigorously by means of measure theory if $\hbar$ is a positive number; so we will only define these path integrals "in perturbation theory", i.e. as formal series in $\hbar$.

Before giving this (fully rigorous) definition, we will explain the motivation behind it. We warn the reader that this explanation is heuristic and involves steps which are mathematically nonrigorous (or "formal" in the language of physicists).

6.2. Wick rotation

In Chapter 1 we discussed path integrals with imaginary exponential (quantum mechanics), as well as real exponential (Brownian motion). If $\hbar$ is a number, then the integrals with imaginary exponential cannot be defined measure theoretically. Therefore, people study integrals with real exponential (which can be rigorously defined), and then perform a special analytic continuation procedure called the *Wick rotation*.

In our formal setting ($\hbar$ is a formal parameter), one can actually define the integrals in both the real and the imaginary case. Still, the real case is a bit easier, and thus the Wick rotation is still useful. Besides, the Wick rotation is very important conceptually. Therefore, while it is not technically necessary, we start with introducing the Wick rotation here.

Namely, let us denote $\langle q(t_1)\dots q(t_n)\rangle$ by $\mathcal{G}_n^M(t_1,\dots,t_n)$, and "formally" make a change of variable $\tau = it$ in the formula for $\mathcal{G}_n^M(t_1,\dots,t_n)$. Let $q(t) := q_*(\tau)$. Then, taking into account that $d\tau = idt$, $\frac{dq}{dt} = i\frac{dq_*}{d\tau}$, we get

$$\mathcal{G}_n^M(t_1,\dots,t_n) = \frac{\int q_*(\tau_1)\dots q_*(\tau_n)e^{-\frac{1}{\hbar}\int(\frac{1}{2}(\frac{dq_*}{d\tau})^2+U(q_*))d\tau}Dq_*}{\int e^{-\frac{1}{\hbar}\int(\frac{1}{2}(\frac{dq_*}{d\tau})^2+U(q_*))d\tau}Dq_*}.$$

This shows that

$$\mathcal{G}_n^M(t_1,\dots,t_n) = \mathcal{G}_n^E(it_1,\dots,it_n),$$

where

$$\mathcal{G}_n^E(t_1,\dots,t_n) := \frac{\int q(t_1)\dots q(t_n)e^{-\frac{S_E(q)}{\hbar}}Dq}{\int e^{-\frac{S_E(q)}{\hbar}}Dq}.$$

with $S_E(q) = \int \mathcal{L}_E(q)d\tau$, and $\mathcal{L}_E(q) = \frac{\dot{q}^2}{2} + U(q)$ (i.e., $\mathcal{L}_E$ is obtained from $\mathcal{L}$ by replacing U with $-U$).

This manipulation certainly does not make rigorous sense, but it motivates the following definition.

Definition 6.2. The function $\mathcal{G}_n^M(t_1,\dots,t_n)$ ($t_i \in \mathbb{R}$) is the analytic continuation of the function $\mathcal{G}_n^E(\tau_1,\dots,\tau_n)$ from the point $(t_1,\dots,t_n)$ to the point $(it_1,\dots,it_n)$ along the path $\theta \mapsto e^{i\theta}(t_1,\dots,t_n)$, $0 \le \theta \le \pi/2$.

Of course, this definition will only make sense if we define the function $\mathcal{G}_n^E(t_1,\dots,t_n)$ and show that it admits the required analytic continuation. This will be done below.

Remark 6.3 (On the terminology)**.** The function $\mathcal{G}_n^M(t_1,\dots,t_n)$ is called the *Minkowskian* (time ordered) correlation function, while the function $\mathcal{G}_n^E(t_1,\dots,t_n)$ is called the *Euclidean* correlation function (hence the notation). This terminology will be explained later, when we consider relativistic field theory.

From now on, we will mostly deal with Euclidean correlation functions, and therefore will omit the superscript E when there is no danger of confusion.

6.3. Definition of Euclidean correlation functions

Now our job is to define the Euclidean correlation functions $\mathcal{G}_n(t_1,\dots,t_n)$ (we drop the superscript E to lighten the notation). Our strategy (which will also be used in field theory) is as follows. Recall that if our integrals were finite dimensional, then by Feynman's theorem the expansion of the correlation functions in $\hbar$ would be given by a sum of amplitudes of Feynman diagrams. So, in the infinite-dimensional case, we will use the sum over Feynman diagrams as a *definition* of correlation functions.

More specifically, because of the conditions on U we have an action functional without constant and linear terms in q, so that the correlation function $\mathcal{G}_n(t_1,\dots,t_n)$ should be given by the sum

$$\mathcal{G}_n(t_1,\dots,t_n)=\sum_{\Gamma\in G_{\geq 3}^*(n)}\frac{\hbar^{b(\Gamma)}}{|\operatorname{Aut}(\Gamma)|}F_\Gamma(\ell_1,\dots,\ell_n), \tag{6.1}$$

where $G_{\geq 3}^*(n)$ is defined in Remark 3.7. Thus, we should make sense of (=define) the amplitudes F_Γ in our situation. For this purpose, we need to define the following objects.

1. The space V.
2. The form B on V which defines B^{-1} on V^*.
3. The tensors corresponding to nonquadratic terms in the action.
4. The covectors ℓ_i.

It is clear how to define these objects naturally. Namely, V should be a space of functions on $\mathbb{R}$ with some decay conditions. There are many choices for V, which do not affect the final result. For instance, a good choice (which we will make) is the space $C_0^\infty(\mathbb{R})$ of compactly supported smooth functions on $\mathbb{R}$. Thus V^* is the space of generalized functions on $\mathbb{R}$. Note that V is equipped with the inner product $(f,g)=\int_{\mathbb{R}} f(t)g(t)dt$.

The form B, by analogy with the finite-dimensional case, should be twice the quadratic part of the action. In other words,

$$B(q,q) = \int (\dot{q}^2 + m^2 q^2) dt = (Aq, q),$$

where A is the operator

$$A = -\frac{d^2}{dt^2} + m^2.$$

This means that $B^{-1}(f,f) = (A^{-1}f, f)$.

The operator A^{-1} is an integral operator, with kernel

$$K(s,t) = G(t-s),$$

where $G(t)$ is the Green's function of A, i.e. the fundamental (decaying at infinity) solution of the differential equation

$$(AG)(t) = \delta(t).$$

It is straightforward to find that

$$G(t) = \frac{e^{-m|t|}}{2m} \tag{6.2}$$

(thus B^{-1} is actually defined not on the whole V^* but on a dense subspace of V^*).

Remark 6.4. Here we already see the usefulness of the Wick rotation. Namely, the spectrum of A (interpreted as usual as a self-adjoint unbounded operator on $L^2(\mathbb{R})$) is $[m^2, +\infty)$, so it is invertible and the inverse is bounded. However, if we did not make a Wick rotation, we would deal with the operator $A' = -\frac{d^2}{dt^2} - m^2$, whose spectrum is $[-m^2, +\infty)$, i.e., contains 0, so this operator does not have a bounded inverse.

To make sense of the cubic and higher terms in the action as tensors, consider the decomposition of U in the (asymptotic) Taylor series at $q = 0$:

$$U(q) = \frac{m^2 q^2}{2} - \sum_{n \geq 3} \frac{g_n q^n}{n!}.$$

This shows that cubic and higher terms in the action have the form

$$B_r(q, q, \dots, q) = \int q^r(t) dt.$$

Thus $B_r(q_1, \dots, q_r)$ is an element of $(S^r V)^*$ given by the generalized function $\delta_{t_1 = \dots = t_r}$ (the delta function of the diagonal).

Finally, the functionals ℓ_i are given by $\ell_i(q) = q(t_i)$, so $\ell_i = \delta(t - t_i)$.

This leads to the following Feynman rules of defining the amplitude of a diagram Γ.

1. To the ith external vertex of Γ, assign the number t_i.

2. To each internal vertex j of Γ, assign a variable s_j.

3. On each internal edge that connects vertices j and j', write the Green's function $G(s_j - s_{j'})$.

4. On each external edge that connects i and j, write $G(t_i - s_j)$.

5. On each external edge that connects i and i', write $G(t_i - t_{i'})$.

6. Let $G_\Gamma(\mathbf{t}, \mathbf{s})$ be the product of all these functions.

7. Let $F_\Gamma(\ell_1, \dots, \ell_n) := \prod_j g_{v(j)} \int G_\Gamma(\mathbf{t}, \mathbf{s}) d\mathbf{s}$, where $v(j)$ is the valency of j.

We are finally able to give the following definition.

Definition 6.5. The function $\mathcal{G}_n(t_1, \dots, t_n)$ is defined by formula (6.1).

Remark 6.6. Note that the integrals defining F_Γ are convergent since the integrand always decays exponentially at infinity. It is, however, crucial that we consider only graphs without components having no external vertices. For example, if Γ has a single 4-valent vertex connected to itself by two loops (Figure 6.1), then the amplitude integral involves $\int_{\mathbb{R}} G(0)^2 ds$, which is obviously divergent.

With this definition, the function $\mathcal{G}_n(t_1, \dots, t_n)$ is a Laurent series in $\hbar$ whose coefficients are symmetric functions of $t_1, \dots, t_n$ given by linear combinations of explicit (and convergent) finite-dimensional integrals. Furthermore, it is easy to see that these integrals are in fact computable in elementary functions, i.e., they are (in the region $t_1 \geq \dots \geq t_n$) linear combinations of products of functions of the form $t_i^r e^{at_i}$. This implies the existence of the analytic continuation required in the Wick rotation procedure.

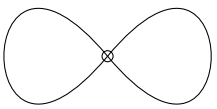


Figure 6.1

Remark 6.7. As in the finite-dimensional case, an alternative setting for making this definition is to assume that g_i are formal parameters. In this case, $\hbar$ can be given a numerical value (e.g., $\hbar = 1$) and the function $\mathcal{G}_n$ will be a well-defined power series in $g_3, g_4, \dots$.

As an example consider a free massive theory, i.e., a *harmonic oscillator*: $U(q) = \frac{m^2q^2}{2}$. In this case, there are no internal vertices, hence we get

Proposition 6.8 (Wick's theorem in quantum mechanics)**.** *One has*

$$\mathcal{G}_n(t_1, \dots, t_n) = 0$$

if n is odd, and

$$\mathcal{G}_{2k}(t_1, \dots, t_{2k}) = \hbar^k \sum_{\sigma \in \Pi_k} \prod_{i \in \{1,\dots,2k\}/\sigma} G(t_i - t_{\sigma(i)}).$$

In particular, $\mathcal{G}_2(t_1, t_2) = \hbar G(t_1 - t_2)$. In other words, $\mathcal{G}_2(t_1, t_2)$ is (proportional to) the Green's function. Motivated by this, physicists often refer to all correlation functions of a quantum field theory as *Green's functions*.

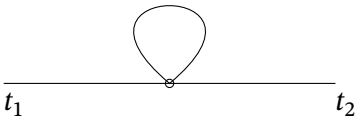


Figure 6.2

Example 6.9. Consider the potential $U(q) = \frac{m^2q^2}{2} - \frac{gq^4}{24}$, and set $\hbar = 1$. In this case, let us calculate the 2-point correlation function modulo g^2. In other words, we have to compute the coefficient of g in this function. Thus we have to consider Feynman diagrams with two external edges and one internal vertex. Such a diagram Γ is unique: it consists of one edge with a loop attached in the middle; see Figure 6.2. This diagram has automorphism group $\mathbb{Z}/2$. The amplitude of this diagram is

$$F_\Gamma = g \int_{\mathbb{R}} G(s, t_1)G(s, t_2)G(s, s)ds = \frac{g}{8m^3} \int_{\mathbb{R}} e^{-m(|s-t_1|+|s-t_2|)}ds.$$

Because of symmetry in t_1 and t_2, we may assume that $t_1 \geq t_2$. Splitting the integral in a sum of three integrals, over $(-\infty, t_2]$, $[t_2, t_1]$, and $[t_1, \infty)$ respectively, we get:

$$\begin{aligned} F_\Gamma &= \frac{g}{8m^3}\left(2\int_0^\infty e^{-m(2s+|t_1-t_2|)}ds + |t_1 - t_2|e^{-m|t_1-t_2|}\right) \\ &= \frac{g}{8m^4}e^{-m|t_1-t_2|}(1 + m|t_1 - t_2|). \end{aligned}$$

Thus

$$\mathcal{G}_2(t_1, t_2) = \widetilde{G}(t_1 - t_2),$$

where

$$\widetilde{G}(t) := \tfrac{1}{2m}e^{-m|t|} + \tfrac{g}{16m^4}e^{-m|t|}(1 + m|t|) + O(g^2).$$

This expression is called the 1-*loop approximation* to the 2-point function, because it comes from 0-loop and 1-loop Feynman diagrams.

Remark 6.10. Here we are considering quantum mechanics of a single 1-dimensional particle. However, everything generalizes without difficulty to the case of an n-dimensional particle or system of particles (i.e., to path integrals over the space of vector-valued, rather than scalar-valued, functions of one variable). Indeed, if q takes values in a Euclidean space V, then the quadratic part of the Lagrangian is of the form $\frac{1}{2}(\dot{q}^2 - M(q))$, where M is a positive-definite quadratic form on V. Diagonalizing M, we may assume that the quadratic part of the Lagrangian looks like $\frac{1}{2}\sum_i(\dot{q_i}^2 - m_i^2 q_i^2)$, which corresponds to a system of independent harmonic oscillators. Thus in quantum theory the propagator will be the diagonal matrix with diagonal entries $\frac{e^{-m_i|t-s|}}{2m_i}$, and the correlation functions can be defined by the usual Feynman diagram procedure.

6.4. Connected correlation functions

Let $\mathcal{G}_n^c(t_1, \ldots, t_n)$ be the *connected correlation* (*or Green*) *functions*, defined by the sum of the same amplitudes as $\mathcal{G}_n(t_1, \ldots, t_n)$ but taken over connected Feynman diagrams only. It is clear that

$$\mathcal{G}_n(t_1, \ldots, t_n) = \sum_{\{1,\ldots,n\}=S_1 \sqcup \cdots \sqcup S_k} \prod_i \mathcal{G}^c_{|S_i|}(t_j; j \in S_i).$$

For example, $\mathcal{G}_2(t_1, t_2) = \mathcal{G}_2^c(t_1, t_2) + \mathcal{G}_1^c(t_1)\mathcal{G}_1^c(t_2)$, etc. Thus, to know the correlation functions, it is sufficient to know the connected correlation functions.

Example 6.11. In a free theory ($U = \frac{m^2q^2}{2}$, the harmonic oscillator), all connected Green's functions except $\mathcal{G}_2$ vanish.

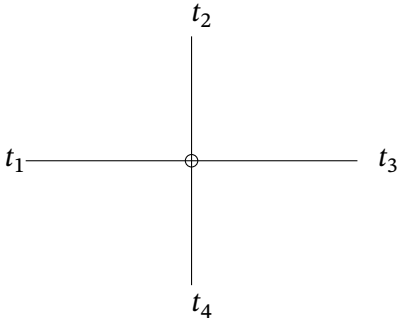


Figure 6.3

Example 6.12. Let us compute the connected 4-point function in the theory associated to the quartic potential $U = \frac{m^2q^2}{2} - \frac{gq^4}{4!}$ as above, modulo g^2. This means, we should compute the contribution of connected Feynman diagrams with one internal vertex and four external edges. Such a diagram Γ is unique—it is the cross (with one internal vertex); see Figure 6.3. This diagram has no nontrivial automorphisms. Thus,

$$\mathcal{G}_4^c(t_1,t_2,t_3,t_4) = g\int_{\mathbb{R}} G(t_1-s)G(t_2-s)G(t_3-s)G(t_4-s)ds + O(g^2).$$

It is elementary to compute this integral; we leave it as an exercise.

6.5. The clustering property

Note that the Green's function $G(t)$ goes to zero at infinity. This implies the following *clustering property* of the correlation functions of the free theory:

$$\lim_{z\to\infty} \mathcal{G}_n(t_1,\dots,t_r,t_{r+1}+z,\dots,t_n+z) = \mathcal{G}_r(t_1,\dots,t_r)\mathcal{G}_{n-r}(t_{r+1}\dots t_n).$$

Moreover, it is easy to show that the same is true in the interacting theory (i.e., with potential) in each degree with respect to $\hbar$ (check it!). The clustering property can be more simply expressed by the equation

$$\lim_{z\to\infty} \mathcal{G}_n^c(t_1,\dots,t_r,t_{r+1}+z,\dots,t_n+z) = 0.$$

This property has a physical interpretation: processes distant from each other are almost statistically independent. Thus it can be viewed as a necessary condition of a quantum field theory to be "physically meaningful".

Remark 6.13. Nevertheless, there exist theories (e.g., so-called *topological quantum field theories*) which do not satisfy the clustering property but are interesting both form a physical and mathematical point of view; see Section 9.2.

6.6. The partition function

Let $J(t)dt$ be a compactly supported measure on the real line. Consider the "partition function with external current J", which is the formal expression

$$Z(J) = \int e^{\frac{-S_E(q)+(J,q)}{\hbar}} Dq.$$

Then we have a formal equality

$$\frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!}\int_{\mathbb{R}^n} \mathcal{G}_n(t_1,\dots,t_n)J(t_1)\dots J(t_n)dt_1\dots dt_n,$$

which, as before, we will use as the definition of $\frac{Z(J)}{Z(0)}$ (here we treat J as a formal variable dependent of t). So the knowledge of $\frac{Z(J)}{Z(0)}$ is equivalent to the

knowledge of all the Green's functions (in other words, $\frac{Z(J)}{Z(0)}$ is their generating function). Furthermore, as in the finite-dimensional case, we have

Proposition 6.14.

$$W(J) := \log \frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^c(t_1, \dots, t_n) J(t_1) \dots J(t_n) dt_1 \dots dt_n$$

(*i.e., W is the generating function of connected Green's functions*).

The proof of this proposition is the same as in the finite-dimensional case.

Remark 6.15. The statement of Proposition 6.14 is equivalent to the relation between usual and connected Green's functions given in Section 6.5.

Remark 6.16. The fact that we can only define amplitudes of graphs all of whose components have at least one 1-valent vertex (see above) means that we actually cannot define either $Z(0)$ or $Z(J)$ but can only define their ratio $\frac{Z(J)}{Z(0)}$.

As in the finite-dimensional case, we have an expansion

$$W(J) = \hbar^{-1} W_0(J) + W_1(J) + \hbar W_2(J) + \dots,$$

where W_j are the j-loop contributions (in particular, W_0 is given by a sum over trees). Furthermore, we have explicit formulas for W_0 and W_1, analogous to the finite-dimensional case.

Proposition 6.17. *One has*

$$W_0(J) = -S_E(q_J) + (q_J, J),$$

where q_J is the extremal of the functional $S_E^J(q) := S_E(q) - (q, J)$ which decays at infinity. Furthermore,

$$W_1(J) = -\frac{1}{2} \log \det L_J,$$

where L_J is the linear operator on V such that

$$d^2 S_E^J(q_J)(f_1, f_2) = d^2 S_E^0(0)(L_J f_1, f_2).$$

The proof of this proposition, in particular, involves showing that q_J is well defined and making sense of $\det L_J$ (as L_J is an operator on an infinite-dimensional space). It is analogous to the proof of the same result in the finite-dimensional case which is given in Section 3.7. To be precise, we gave a proof only in the 0-loop case; but in the 1-loop case, the proof is similar. Therefore we will not give this proof; rather, we will illustrate the statement by an example.

Example 6.18. Let U be the above quartic potential $\frac{m^2 q^2}{2} + \frac{g q^4}{2}$ (in which for convenience we change the sign and normalization of the quartic term) and $J(t) = a\delta(t)$. In this case,

$$S_E^J(q) = \int (\tfrac{\dot{q}^2}{2} + U(q)) dt - aq(0).$$

The Euler-Lagrange equation has the form

$$\ddot{q} = m^2 q + 2gq^3 - a\delta(t).$$

Thus, the function q_J is continuously glued from two solutions q_+, q_- of the nonlinear differential equation

$$\ddot{q} = m^2 q + 2gq^3$$

on $(-\infty, 0]$ and $[0, \infty)$, with jump of derivative at 0 equal to $-a$.

The solutions q_+, q_- decay at infinity together with their first derivatives, so they must be solutions of zero energy:

$$E = \frac{\dot{q_\pm}^2}{2} - U(q_\pm) = 0.$$

Indeed, E must go to zero at infinity and by energy conservation (as in Section 1.1, but with U replaced by $-U$) is independent of t. Thus, by (1.3), $q_\pm$ are defined by the equality

$$t = \pm \int_q^\infty \frac{dx}{\sqrt{2U(x)}} = \pm \int_q^\infty \frac{dx}{mx\sqrt{1 + \frac{gx^2}{m^2}}} = \pm \frac{1}{2m} \log \frac{\sqrt{1 + \frac{gq^2}{m^2}} + 1}{\sqrt{1 + \frac{gq^2}{m^2}} - 1}.$$

After a calculation one gets

$$q_J(t) = \frac{2mg^{-\frac{1}{2}}}{C^{-1}e^{m|t|} - Ce^{-m|t|}},$$

where C is the solution of the equation

$$\frac{C(1 + C^2)}{(1 - C^2)^2} = \frac{ag^{\frac{1}{2}}}{4m^2}$$

which is given by a power series in a with zero constant term. From this it is elementary (but somewhat lengthy) to compute $W_0 = -S_E^J(q_J)$.

Now, the operator L_J is given by the formula

$$L_J = 1 + 6gA^{-1} \circ q_J(t)^2$$

where $A = -\frac{d^2}{dt^2} + m^2$. Thus $\det L_J$ makes sense. Indeed, the operator $A^{-1} \circ q_J(t)^2$ is an integral operator given by the kernel

$$K_J(s,t) := \frac{e^{-m|s-t|} q_J(t)^2}{2m},$$

which decays exponentially at infinity; hence the determinant of the operator $1 + 6gA^{-1} \circ q_J(t)^2$ is well defined.

Remark 6.19. In these computations, g, a were formal variables, but the above computations in fact make sense for real numerical values of these variables as long as $ga^2 + m^2 > 0$.

6.7. 1-particle irreducible Green's functions

Let $\mathcal{G}_n^{1PI}(t_1, \dots, t_n)$ denote 1-*particle irreducible Green's functions*, i.e. those defined by the sum of the same amplitudes as the usual Green's functions, but taken only over 1-particle irreducible Feynman graphs. Define also the *amputated* 1-particle irreducible Green's function: $\mathcal{G}_n^{1PIa} := A^{\otimes n}\mathcal{G}_n^{1PI}$. It is defined by the same sum of amplitudes, except that instead of $G(t_i - s_j)$ for external edges, we write $\delta(t_i - s_j)$.

Let $S_{\text{eff}}(q)$ be the generating function of $\mathcal{G}_n^{1PIa}$, i.e., the formal sum

$$S_{\text{eff}}(q) = S_{\text{quad}}(q) - \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^{1PIa}(t_1, \dots, t_n) q(t_1) \dots q(t_n) dt_1 \dots dt_n,$$

where $S_{\text{quad}}(q)$ is the quadratic part of the action.

Proposition 6.20. *The function $W(J) = \log(\frac{Z(J)}{Z(0)})$ is the Legendre transform of $S_{\text{eff}}(q)$, i.e., it equals $-S_{\text{eff}}(\tilde{q}_J)+(J, \tilde{q}_J)$, where $\tilde{q}_J$ is the extremal of $-S_{\text{eff}}(q)+(J, q)$ decaying at infinity.*

The proof of this proposition is the same as in the finite-dimensional case. The proposition shows that in order to know all Green's functions, it "suffices" to know the amputated 1-particle irreducible Green's functions (namely, the generating function of usual Green's functions can be reconstructed from that for 1PI Green's functions by taking the Legendre transform and exponentiation). Which, as before, is good news, since there are a lot fewer 1PI diagrams than general connected diagrams.

6.8. Momentum space integration

We saw that the amplitude of a Feynman diagram is given by an integral over the space of dimension equal to the number of internal vertices. This is sometimes inconvenient, since even for tree diagrams such integrals can be rather complicated. However, it turns out that if one passes to Fourier transforms, then Feynman integrals simplify and in particular the number of integrations for a connected diagram becomes equal to the number of loops (so for tree diagrams we have no integrations at all).

We will proceed as follows. Instead of the time variable t we will consider the dual energy variable E. A function $q(t)$ with compact support will be replaced by its Fourier transform

$$\hat{q}(E) = \int_{-\infty}^{\infty} e^{-iEt} q(t) dt,$$

where we drop the pre-factor $(2\pi)^{-1/2}$. Then, by Plancherel's theorem, for real functions q_1, q_2, we have

$$(q_1, q_2) = \int_{\mathbb{R}} q_1(t)q_2(t)dt = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{q}_1(E)\overline{\hat{q}_2(E)}dE = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{q}_1(E)\hat{q}_2(-E)dE.$$

Since the Fourier transform of $G(x) = \frac{e^{-m|t|}}{2m}$ is $\frac{1}{E^2+m^2}$, this implies that the propagator is given by

$$B^{-1}(f,f) = \int_{\mathbb{R}^2} G(t-s)f(t)f(s)dtds = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{E^2+m^2}\widehat{f}(E)\widehat{f}(-E)dE.$$

The vertex tensors standing at k-valent vertices were $\delta_{s_1=\cdots=s_k}$, so they will be replaced by $2\pi\delta_{Q_1+\cdots+Q_k=0}$, where Q_i are dual variables to s_i.

Remark 6.21 (On terminology)**.** Physicists refer to the time variables t_i, s_j as *position variables*, and to energy variables E_i, Q_k as *momentum variables*, since in relativistic mechanics (which is the setting we will deal with when we study field theory) there is no distinction between time and position and between energy and momentum (due to the action of the Lorentz group).

This shows that the Feynman rules "in momentum space" for a given connected Feynman diagram Γ with n external vertices are as follows.

1. Orient the diagram Γ so that all external edges are oriented inwards.

2. Assign variables E_i to external edges, and variables Q_j to internal ones. These variables are subject to the linear equations of "the first Kirchhoff law": at every internal vertex, the sum of the variables corresponding to the incoming edges equals the sum of those corresponding to the outgoing edges. Let $Y(\mathbf{E})$ be the space of solutions $\mathbf{Q}$ of these equations (it depends on Γ, but we will not write the dependence explicitly). It is easy to show that this space is nonempty only if $\sum_i E_i = 0$, and in that case $\dim Y(\mathbf{E})$ equals the number of loops of Γ (show this!).

3. On each external edge write $\frac{1}{E_i^2+m^2}$, and on each internal edge write $\frac{1}{Q_k^2+m^2}$. Let $\phi_\Gamma(\mathbf{E}, \mathbf{Q})$ be the product of all these functions.

4. Define the *momentum space amplitude* of Γ to be the distribution $\widehat{F}_\Gamma(\mathbf{E})$:

$$\widehat{F}_\Gamma(E_1, \ldots, E_n) = (2\pi)^{1-\dim Y(\mathbf{E})}\prod_j g_{v(j)}\int_{Y(\mathbf{E})} \phi_\Gamma(\mathbf{E}, \mathbf{Q})d\mathbf{Q}\cdot\delta(E_1+\cdots+E_n)d\mathbf{E},$$

supported on the hyperplane $\sum_i E_i = 0$. It is clear that this distribution is independent of the orientation of Γ.

Remark 6.22. Here we must specify the normalization of the (translation-invariant) Lebesgue measure $d\mathbf{Q}$ on the space $Y(\mathbf{E})$. It is defined in such a way that the volume of $Y(\mathbf{E})/Y_{\mathbb{Z}}(0)$ is 1, where $Y_{\mathbb{Z}}(0)$ is the set of integer elements

in $Y(0)$. So if $T \subset \Gamma$ is a spanning tree, then in the coordinates $\{Q_e, e \notin T\}$ on $Y(\mathbf{E})$, we have $d\mathbf{Q} = \prod_{e \notin T} dQ_e$.

Now we have

Proposition 6.23. *The Fourier transform of the function $F_\Gamma(\delta_{t_1}, \dots, \delta_{t_n})$ is $\widehat{F}_\Gamma(E_1, \dots, E_n)$. Hence, the Fourier transform of the connected Green's function is*

$$\widehat{\mathcal{G}}_n^c(E_1, \dots, E_n) = \sum_{\Gamma \in G^*_{\geq 3}(n)} \frac{\hbar^{b(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} \widehat{F}_\Gamma(E_1, \dots, E_n). \tag{6.3}$$

The proof of the proposition is straightforward.

To illustrate the proposition, consider two examples (assuming $\hbar = 1$).

Example 6.24. The connected 4-point function for the quartic potential modulo g^2 in momentum space looks like

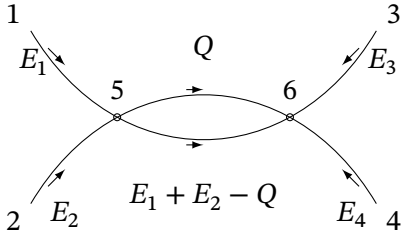
$$\widehat{\mathcal{G}}_4^c(E_1, E_2, E_3, E_4) = 2\pi g \prod_{i=1}^4 \frac{1}{E_i^2 + m^2} \delta(\sum_i E_i) d\mathbf{E} + O(g^2).$$

Example 6.25. Let us compute the 1PI 4-point function in the same problem, but now modulo g^3. Thus, in addition to the above, we need to compute the g^2 coefficient, which comes from 1-loop diagrams. There are three such diagrams, differing by permutation of external edges. One of these diagrams is as follows: it has external vertices 1, 2, 3, 4 and internal ones 5, 6 such that vertices 1, 2 are connected to vertex 5, vertices 3, 4 to vertex 6, and vertices 5 and 6 are connected by two edges; see Figure 6.4. This diagram has the symmetry group $\mathbb{Z}/2$, so its contribution is

$$\frac{g^2}{2} \left(\int_{\mathbb{R}} \frac{dQ}{(Q^2 + m^2)((E_1 + E_2 - Q)^2 + m^2)} \right) \prod_{i=1}^4 \frac{1}{E_i^2 + m^2} \delta(\sum_i E_i) d\mathbf{E}.$$

The integral inside is easy to compute, for example, by residues. This yields

$$\widehat{\mathcal{G}}_4^{1PI}(E_1, E_2, E_3, E_4)$$
$$= 2\pi g \prod_{i=1}^4 \frac{1}{E_i^2 + m^2} \left(1 + \frac{g}{2m} \sum_{i=2}^4 \frac{1}{(E_1 + E_i)^2 + 4m^2} \right) \delta(\sum_i E_i) d\mathbf{E} + O(g^3)$$



Γ.

Figure 6.4

(this is symmetric in E_1, E_2, E_3, E_4 since when $\sum_i E_i = 0$ then for distinct i, j, k, ℓ one has $(E_i + E_j)^2 = (E_k + E_\ell)^2$).

Exercise 6.26. Calculate the 1-particle irreducible 2-point function for a quantum particle with potential $U(q) := \frac{m^2q^2}{2} - \frac{gq^4}{4!}$ modulo g^3 in momentum space, for $\hbar = 1$. (We have done this modulo g^2 in position space in Example 6.9).

Exercise 6.27. Let $U(q) := \frac{m^2q^2}{2} - \frac{gq^3}{3!}$.

(i) Calculate the leading term of the 1-point function $\mathcal{G}_1(t)$ (with respect to g).

(ii) Calculate the connected 2-point function modulo g^3.

Exercise 6.28. Consider the potential $U(x) := \frac{m^2 \sinh^2(gx)}{2g^2}$. Find a formula for $W_0(J)$ (the tree part of $\log(\frac{Z(J)}{Z(0)})$ as explicitly as you can, when $J(t) = a\delta(t)$.

6.9. The Wick rotation in momentum space

To obtain the correlation functions of quantum mechanics, we should, after computing them in the Euclidean setting, Wick rotate them back to the Minkowski setting. Let us do it at the level of Feynman integrals in momentum space. (We could do it in position space as well, but it is instructive for the future to do it in momentum space, since in higher-dimensional field theory, which we will discuss later, the momentum space representation is more convenient.)

Consider the Euclidean propagator

$$\frac{1}{E^2 + m^2} = \int_{\mathbb{R}} G(t) e^{-iEt} dt,$$

where G is the Green's function (6.2). When we do analytic continuation back to the Minkowski setting, we must replace in the correlation functions the time variable t with $e^{i\theta}t$, where θ varies from 0 to $\frac{\pi}{2}$. In particular, the Green's function $G(t)$ must be replaced by $G(e^{i\theta}t)$. So we must consider

$$\int_{\mathbb{R}} G(e^{i\theta}t) e^{iEt} dt = e^{-i\theta} \int_{\mathbb{R}} G(t) e^{ie^{-i\theta}Et} dt = \frac{e^{-i\theta}}{e^{-2i\theta}E^2 + m^2}.$$

As $\theta \to \frac{\pi}{2}$, this function tends (as a distribution) to the function

$$\lim_{\varepsilon \to 0+} \frac{i}{E^2 - m^2 + i\varepsilon}.$$

For brevity the limit sign is usually dropped and this distribution is written as $\frac{i}{E^2 - m^2 + i\varepsilon}$.

We see that in order to compute the correlation functions in momentum space in the Minkowski setting, we should use the same Feynman rules as in the Euclidean setting except that the propagator put on the edges should be

$$\frac{i}{E^2 - m^2 + i\varepsilon}.$$

For instance, the contribution of the diagram in Figure 6.4 is

$$-\frac{g^2}{4\pi}\left(\int_{\mathbb{R}} \frac{dQ}{(Q^2 - m^2 + i\varepsilon)((E_1 + E_2 - Q)^2 - m^2 + i\varepsilon)}\right)\prod_{j=1}^{4} \frac{1}{E_j^2 - m^2 + i\varepsilon}\delta(\sum_i E_j)d\mathbf{E}.$$

6.10. Quantum mechanics on the circle

It is reasonable (at least mathematically) to consider Euclidean quantum mechanical path integrals in the case when the time axis has been replaced with a circle of length L, i.e., $t \in \mathbb{R}/L\mathbb{Z}$ (this corresponds upon Wick rotation to a Brownian particle in a potential field conditioned to return to the original position in a certain time L). This means that we take the path integral over L-periodic functions $q(t)$, i.e., $q(t+L) = q(t)$. In this case, the theory is the same, except the Green's function $G(t)$ is replaced by the periodic solution $G_L(t)$ of the equation $(-\frac{d^2}{dt^2} + m^2)f = \delta(t)$ on the circle. This solution has the form

$$G_L(t) = \sum_{k\in\mathbb{Z}} G(t - kL) = \frac{e^{-m(t-\frac{L}{2})} + e^{-m(\frac{L}{2}-t)}}{2m(e^{\frac{mL}{2}} - e^{-\frac{mL}{2}})},\ 0 \le t \le L. \tag{6.4}$$

We note that in the case of a circle, there is no problem with graphs without external edges (as integral over the circle of a constant function is convergent), and hence one may define not only correlation functions (i.e., $\frac{Z(J)}{Z(0)}$), but also $Z(0)$ itself. Namely, let

$$U(q) = \frac{m^2q^2}{2} + \sum_{n\ge 3} \frac{g_n q^n}{n!},$$

and let $m^2 = m_0^2 + g_2$ (where g_i are formal parameters). Then we can make sense of the ratio $\frac{Z(m_0,\mathbf{g},L,0)}{Z(m_0,0,L,0)}$ (where $Z(m,\mathbf{g},L,0)$ denotes the partition function for the specified values of parameters; from now on the argument 0 will be dropped). Indeed, this ratio is defined by the formula

$$\frac{Z(m_0,\mathbf{g},L)}{Z(m_0,0,L)} = \sum_{\Gamma\in G_{\ge 2}(0)} \frac{\hbar^{b(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} F_\Gamma$$

(where $G_{\ge 2}(0)$ is the set of Feynman graphs without external vertices and all vertices of valency ≥ 2), which is a well-defined expression.

It is instructive to compute this expression in the case

$$g_2 = a,\ g_3 = g_4 = \cdots = 0.$$

In this case, we have only 2-valent vertices, so the only connected Feynman diagrams are N-gons, which are 1-loop. Hence,

$$\log \frac{Z(m_0, \mathbf{g}, L)}{Z(m_0, 0, L)} = W_1 = -\frac{1}{2} \log \det D,$$

where

$$D = 1 + a(-\tfrac{d^2}{dt^2} + m_0^2)^{-1}.$$

This determinant may be computed by looking at the eigenvalues. Namely, the eigenfunctions of $-\frac{d^2}{dt^2} + m_0^2$ in the space $C^\infty(\mathbb{R}/L\mathbb{Z})$ are $e^{\frac{2\pi i n t}{L}}$, with eigenvalues $\frac{4\pi^2 n^2}{L^2} + m_0^2$. So,

$$\det D = \prod_{n \in \mathbb{Z}} \left(1 + \frac{a}{\frac{4\pi^2 n^2}{L^2} + m_0^2}\right).$$

Hence, using the Euler product formula

$$\sinh(z) = z \prod_{n \geq 1} \left(1 + \frac{z^2}{\pi^2 n^2}\right),$$

we get

$$\frac{Z(m_0, \mathbf{g}, L)}{Z(m_0, 0, L)} = \frac{\sinh(\frac{m_0 L}{2})}{\sinh(\frac{mL}{2})}.$$

(Double-check this using summation over Feynman diagrams!)

Remark 6.29. More informally speaking, we see that the partition function $Z(m, L) := Z(m, 0, L)$ for the theory with $U = \frac{m^2 q^2}{2}$ has the form $Z(m, L) = \frac{C}{\sinh(\frac{mL}{2})}$, where C is a constant of our choice. This choice from now on will be $C = \frac{1}{2}$, i.e.,

$$Z(m, L) = \frac{1}{e^{\frac{mL}{2}} - e^{-\frac{mL}{2}}}; \tag{6.5}$$

we will see later (in Example 7.27) why such a choice is preferable.

Note that the function (6.5) has a pole when $m = 0$ (the massless case). This has to do with the fact that the theory of a massless particle is somewhat ill defined, as explained in Section 6.13.

More generally, we may consider a quantum particle moving in a Euclidean space V with potential $\frac{1}{2}(q, Mq)$, where $M : V \to V$ is a positive symmetric operator. By diagonalizing M in an orthonormal basis, we may assume that

$V = \mathbb{R}^d$ and $M = \mathrm{diag}(m_1^2, \dots, m_d^2)$. In this basis, the coordinates are decoupled from each other, so for the partition function we get

$$Z(M,L) = \prod_{j=1}^{d} \frac{1}{e^{\frac{m_j L}{2}} - e^{-\frac{m_j L}{2}}} = \frac{1}{\det(e^{\frac{\sqrt{M}L}{2}} - e^{-\frac{\sqrt{M}L}{2}})}. \tag{6.6}$$

Still more generally, given an orthogonal automorphism $g : V \to V$ commuting with M, we may consider the corresponding *twisted partition and correlation functions*, obtained by taking the path integral over functions $q(t)$ that are quasiperiodic with holonomy g, rather than periodic: $q(t+L) = gq(t)$. For example, suppose $V = \mathbb{C} = \mathbb{R}^2$ and $M = m^2$, i.e., $U(q) = \frac{1}{2}m^2|q|^2$, and let $g = e^{i\theta}$ for $\theta \in [0, 2\pi)$. Then it is easy to compute that the twisted partition function is

$$Z_{\mathrm{tw}}(M,L,g) = \frac{1}{|e^{\frac{mL}{2}} - e^{-\frac{mL}{2}+i\theta}|^2}.$$

On the other hand, if $V = \mathbb{R}$, $M = m^2$, and $g = -1$, then

$$Z_{\mathrm{tw}}(M,L,g) = \frac{1}{e^{\frac{mL}{2}} + e^{-\frac{mL}{2}}}.$$

The general twisted case is a direct product of such cases and the untwisted case, so in generality we get

$$Z_{\mathrm{tw}}(M,L,g) = \frac{1}{\det(e^{\frac{\sqrt{M}L}{2}} - ge^{-\frac{\sqrt{M}L}{2}})}. \tag{6.7}$$

Note that if g does not have eigenvalue 1 (for example, $g = -1$), this function is meaningful even in the massless case $M = 0$.

6.11. The massless case

Consider now the massless case, $m = 0$. In this case the propagator should be obtained by inverting the operator $-\frac{d^2}{dt^2}$, i.e., it should be the integral operator with kernel $G(t-s)$, where $G(t)$ is an even function satisfying the differential equation

$$-G''(t) = \delta(t).$$

There is a 1-parameter family of such solutions,

$$G(t) = -\frac{1}{2}|t| + C.$$

Using this function (for any choice of C), one may define the correlation functions of the free theory by the Wick formula.

Note that the function G does not decay at infinity. Therefore, this theory will not satisfy the clustering property (i.e., is not "physically meaningful").

We will also have difficulties in defining the corresponding interacting theory (i.e., one with a nonquadratic potential), as the integrals defining the amplitudes of Feynman diagrams will diverge. Such divergences are called *infrared divergences*, since they are caused by the failure of the integrand to decay at large times (or, in momentum space, its failure to be regular at low frequencies).

6.12. Circle-valued quantum mechanics

Consider now the theory with the same Lagrangian in which $q(t)$ takes values in the circle of radius r, $\mathbb{R}/2\pi r\mathbb{Z}$ (the "sigma-model"). We can do this at least classically, since the Lagrangian $\frac{\dot{q}^2}{2}$ makes sense in this case.

Let us define the corresponding quantum theory. The main difference from the line-valued case is that since $q(t)$ is circle valued, we should consider not the usual correlators $\langle q(t_1)\dots q(t_n)\rangle$ (which do not make sense since $q(t)$ is not a number but a point on the circle), but rather correlation functions of exponentials $\langle e^{\frac{ip_1q(t_1)}{r}}\dots e^{\frac{ip_nq(t_n)}{r}}\rangle$, where p_j are integers. They should be defined by the path integral

$$\int e^{\frac{ip_1q(t_1)}{r}}\dots e^{\frac{ip_nq(t_n)}{r}}e^{-\frac{S(q)}{\hbar}}Dq, \tag{6.8}$$

where $S(q) := \frac{1}{2}\int \dot{q}^2dt$ and $\int e^{-\frac{S(q)}{\hbar}}Dq$ is agreed to be 1. Note that it suffices to consider only the case $\sum_j p_j = 0$, otherwise the group of translations along the circle acts nontrivially on the integrand, hence under any reasonable definition the integral should be zero.

Now let us define the integral (6.8). Since the integral is invariant under shifts along the target circle, we may as well imagine that we are integrating over $q : \mathbb{R} \to \mathbb{R}$ with $q(0) = 0$. Now let us use the finite-dimensional analogy. Following this analogy, by completing the square, we would get

$$\int e^{\frac{ip_1q(t_1)}{r}}\dots e^{\frac{ip_nq(t_n)}{r}}e^{-\frac{S(q)}{\hbar}}Dq = e^{-\frac{\hbar}{2r^2}B^{-1}(\sum_j p_jq(t_j),\sum_j p_jq(t_j))}$$
$$= e^{-\frac{\hbar}{2r^2}\sum_{j,\ell}p_\ell p_jG(t_\ell-t_j)} = e^{\frac{\hbar}{2r^2}\sum_{\ell<j}p_\ell p_j|t_\ell-t_j|},$$

where $B(q,q) := \int \dot{q}^2dt$. Thus, it is natural to define the correlators by the formula

$$\langle e^{\frac{ip_1q(t_1)}{r}}\dots e^{\frac{ip_nq(t_k)}{r}}\rangle = e^{\frac{\hbar}{2r^2}\sum_{\ell<j}p_\ell p_j|t_\ell-t_j|}.$$

We note that this theory, unlike the line-valued one, *does* satisfy the clustering property. Indeed, if $\sum p_j = 0$ (as we assumed), then (assuming $t_1 \geq t_2 \geq \dots \geq$

t_n), we have

$$\sum_{\ell<j} p_\ell p_j (t_\ell - t_j) = \sum_{j=1}^{n-1} (t_j - t_{j+1})(p_{j+1} + \cdots + p_n)(p_1 + \cdots + p_j)$$
$$= -\sum_j (t_j - t_{j+1})(p_1 + \cdots + p_j)^2,$$

so the clustering property follows from the fact that $(p_1 + \cdots + p_j)^2 \geq 0$.

6.13. Massless quantum mechanics on the circle

Consider now the theory with Lagrangian $\frac{\dot{q}^2}{2}$, where q is a function on the circle of length L. In this case, according to the Feynman yoga, we must invert the operator $-\frac{d^2}{dt^2}$ on the circle $\mathbb{R}/L\mathbb{Z}$, or equivalently solve the differential equation $-G''(t) = \delta(t)$. And here we run into trouble: the operator $-\frac{d^2}{dt^2}$ is not invertible, since it has an eigenfunction 1 with eigenvalue 0; correspondingly, the differential equation in question has no solutions, as $\int G''dt$ must be zero, so $-G''(t)$ cannot equal $\delta(t)$ (one may say that the quadratic form in the exponential is degenerate, and therefore the Gaussian integral turns out to be meaningless). This problem can be resolved by the following technique of "killing the zero mode". Namely, let us invert the operator $-\frac{d^2}{dt^2}$ on the space $C_0^\infty(\mathbb{R}/L\mathbb{Z}) := \{q \in C^\infty(\mathbb{R}/L\mathbb{Z}) : \int q dt = 0\}$ (this may be interpreted as integration over this codimension-1 subspace, on which the quadratic form is nondegenerate). This means that we must find the solution of the differential equation

$$-G''(t) = \delta(t) - \frac{1}{L}$$

such that $\int G dt = 0$ (the constant $\frac{1}{L}$ is forced by the condition that the integral of the right-hand side over the circle must vanish); indeed, in this case for $f \in C_0^\infty(\mathbb{R}/L\mathbb{Z})$, we have

$$-\int_0^L G(t-s)f''(s)ds = -\frac{d^2}{dt^2}\int_0^L G(t-s)f(s)ds$$
$$= \int_0^L (\delta(t-s) - \frac{1}{L})ds = f(t) - \frac{1}{L}\int_0^L f(s)ds = f(t),$$

as needed. Such a solution is indeed unique, and it equals

$$G(t) = \frac{(t - \frac{L}{2})^2}{2L} - \frac{L}{24}, \tag{6.9}$$

$t \in [0, L]$. Thus, for example, $\langle q(0)^2 \rangle = \frac{L}{12}$.

Higher correlation functions are defined in the usual way. Moreover, by taking a direct product of several copies of this theory, one can define the theory

of a massless quantum particle moving L-periodically in a Euclidean space of any dimension.

In the twisted case, this theory can be better behaved. Namely, consider the theory of a free massless particle in a Euclidean space V twisted by the automorphism $g \in O(V)$ having no eigenvalue 1 (for example, $g = -1$). Then the path integral is taken over the space of functions $q : \mathbb{R} \to V$ such that $q(t + L) = gq(t)$, so the propagator is obtained by inverting the operator $-\frac{d^2}{dt^2}$ on the space $L^2_g(\mathbb{R}/L\mathbb{Z})$ of square integrable functions satisfying this condition. This operator has eigenvalues $(\lambda_k + n)^2$, where $e^{2\pi i\lambda_k}$ are the eigenvalues of g. So if g has no eigenvalue 1 (i.e., $\lambda_k \notin \mathbb{Z}$), then these numbers are nonzero, so the operator $-\frac{d^2}{dt^2}$ is invertible (there is no zero mode), and we obtain a well-defined theory. It is easy to see that the twisted partition function of this theory is exactly the function (6.7) for $M = 0$, i.e., $Z_{\mathrm{tw}} = \frac{1}{\det(I-g)}$.

6.14. Circle-valued quantum mechanics on the circle

Finally, let us consider the circle-valued version of the same theory. Thus, our integration variable is a map $q : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}/2\pi r\mathbb{Z}$. So we have a new feature—there are different homotopy classes of maps labeled by degree. Let us first consider integration over degree 0 maps. Then we should argue in the same way as in the case $t \in \mathbb{R}$, and make the definition

$$\langle e^{\frac{ip_1 q(t_1)}{r}} \dots e^{\frac{ip_n q(t_n)}{r}} \rangle_0 := e^{-\frac{\hbar}{2r^2} \sum_{\ell,j} p_\ell p_j G(t_\ell - t_j)},$$

where $\sum_j p_j = 0$. (Here subscript 0 stands for degree 0 maps). Assuming that $0 \le t_1, \dots, t_n \le L$, we find after a short calculation using (6.9) that

$$\langle e^{\frac{ip_1 q(t_1)}{r}} \dots e^{\frac{ip_n q(t_n)}{r}} \rangle_0 = e^{\frac{\hbar}{2r^2}(\sum_{\ell<j} p_\ell p_j |t_\ell - t_j| + \frac{(\sum_j p_j t_j)^2}{L})}$$

(the second summand disappears as $L \to \infty$, and we recover the answer on the line).

It is, however, more natural (as we will see later) to integrate over all maps q, not only degree 0. Namely, let N be an integer. Then all maps of degree N have the form $q(t) + \frac{2\pi r N t}{L}$, where q is a map of degree 0. Thus, if we want to integrate over maps of degree N, we should compute the same integral as in degree 0, but with shift $q \mapsto q + \frac{2\pi r N t}{L}$. But it is easy to see that this shift results simply in rescaling of the integrand by the factor $e^{\frac{2\pi i N}{L} \sum_j p_j t_j - \frac{2\pi^2 r^2 N^2}{\hbar L}}$. Thus, the

integral over all maps should be defined by the formula

$$\langle e^{\frac{ip_1q(t_1)}{r}} \dots e^{\frac{ip_nq(t_n)}{r}} \rangle$$

$$= e^{\frac{\hbar}{2r^2}(\sum_{l<j} p_\ell p_j |t_\ell - t_j| + \frac{(\sum p_j t_j)^2}{L})} \frac{\sum_{N\in\mathbb{Z}} e^{\frac{2\pi i N}{L}\sum_j p_j t_j - \frac{2\pi^2 r^2 N^2}{\hbar L}}}{\sum_{N\in\mathbb{Z}} e^{-\frac{2\pi^2 r^2 N^2}{\hbar L}}}. \tag{6.10}$$

Introduce the elliptic theta-function

$$\vartheta(u,T) := \sum_{N\in\mathbb{Z}} e^{2\pi i u N - \pi T N^2}. \tag{6.11}$$

Then for $L \geq t_1 \geq \dots \geq t_n \geq 0$ formula (6.10) can be rewritten in the form (6.12)

$$\langle e^{\frac{ip_1q(t_1)}{r}} \dots e^{\frac{ip_nq(t_n)}{r}} \rangle = e^{\frac{\hbar}{2r^2}(-\sum_j (t_j - t_{j+1})(p_1+\dots+p_j)^2 + \frac{(\sum_j p_j t_j)^2}{L})} \frac{\vartheta(\frac{\sum_j p_j t_j}{L}, \frac{2\pi r^2}{\hbar L})}{\vartheta(0, \frac{2\pi r^2}{\hbar L})}.$$

Chapter 7

Operator approach to quantum mechanics

In mechanics and field theory (both classical and quantum), there are two main languages—Lagrangian and Hamiltonian. In the classical setting, the Lagrangian language is the language of variational calculus (i.e., one studies extremals of the action functional), while the Hamiltonian language is that of symplectic geometry and Hamilton equations. Correspondingly, in the quantum setting, the Lagrangian language is the language of path integrals, while the Hamiltonian language is the language of operators and Schrödinger equations. We have now studied the first one (at least in perturbation expansion) and are passing to the second one.

7.1. Hamilton's equations in classical mechanics

We start with recalling the Hamiltonian formalism of classical mechanics. For more details, we refer the reader to the excellent book [**A**].

Recall first the Lagrangian description of the motion of a classical particle or system of particles (Section 1.1). The position of a particle is described by a point q of the configuration space X, which we will assume to be a manifold. The Lagrangian of the system is a (smooth) function $\mathcal{L} : TX \to \mathbb{R}$ on the total space of the tangent bundle of X. Then the action functional is $S(q) = \int \mathcal{L}(q, \dot{q})dt$. The trajectories of the particle are the extremals of S. The condition for $q(t)$ to be an extremal of S is equivalent to the Euler-Lagrange equation (=the equation of motion), which in local coordinates has the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}.$$

For example, if X is a Riemannian manifold and $\mathcal{L}(q,v) = \frac{v^2}{2} - U(q)$, where $U : X \to \mathbb{R}$ is a potential function, then the Euler-Lagrange equation is the Newton equation

$$\ddot{q} = -\nabla U(q),$$

where $\ddot{q} = \nabla_{\dot{q}}\dot{q}$ is the covariant derivative with respect to the Levi-Civita connection.

Consider now a system with Lagrangian $\mathcal{L}(q,v)$, whose differential with respect to v (for fixed q) is a diffeomorphism $T_qX \to T_q^*X$. This is definitely true in the above special case of Riemannian X.

Definition 7.1. The *Hamiltonian* (*or energy function*) of the system with Lagrangian $\mathcal{L}$ is the function $H : T^*X \to \mathbb{R}$, which is the Legendre transform of $\mathcal{L}$ along fibers; that is, $H(q,p) = pv_0 - \mathcal{L}(q,v_0)$, where v_0 is the (unique) critical point of $pv - \mathcal{L}(q,v)$. The manifold T^*X is called the *phase space* (*or space of states*). The variable p is called the *momentum variable*.

For example, if $\mathcal{L} = \frac{v^2}{2} - U(q)$, then $H(q,p) = \frac{p^2}{2} + U(q)$.

Remark 7.2. Since the Legendre transform is involutive, we also have that the Lagrangian is the fiberwise Legendre transform of the Hamiltonian.

Let q_i be local coordinates on X. This coordinate system defines a coordinate system (q_i, p_i) on T^*X. We obtain

Proposition 7.3. *The equations of motion are equivalent to Hamilton's equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

in the sense that they are obtained from Hamilton's equations by elimination of p_i.

For example, if $H = \frac{p^2}{2} + U(q)$, then Hamilton's equations take the form

$$\dot{q} = p,\ \dot{p} = -\nabla U(q),$$

or in coordinates

$$\dot{q}_i = p_i,\ \dot{p}_i = -\frac{\partial U}{\partial q_i},$$

which leads to Newton's equations $\ddot{q}_i = -\frac{\partial U}{\partial q_i}$. Thus $H = \frac{\dot{q}^2}{2} + U(q)$, the energy function defined in Section 1.1. Note that $\dot{q} \in T_qX$ while $p \in T_q^*X$, but the equation $\dot{q} = p$ makes sense since X is a Riemannian manifold, and we have a natural identification $T_qX \cong T_q^*X$ using the Riemannian metric.

It is useful to write Hamilton's equations in terms of Poisson brackets. Recall that the manifold T^*X has a canonical symplectic structure

$$\omega = d\alpha,$$

where α is the canonical 1-form on T^*M (called the *Liouville form*) constructed as follows: for any $z \in T_{(q,p)}(T^*X)$,

$$\alpha(z) = (p, d\pi(q,p)z),$$

where $\pi : T^*X \to X$ is the projection and $(,)$ is the pairing between the tangent and cotangent space. In local coordinates, we have

$$\alpha = \sum_i p_i dq_i,\ \omega = \sum_i dp_i \wedge dq_i.$$

Now let (M, ω) be a symplectic manifold (in our case $M = T^*X$). Since ω is nondegenerate, one can define the *Poisson bivector* ω^{-1}, which is a section of the bundle $\wedge^2 TM$. Now, given any two smooth functions f, g on M, one can define a third function—their *Poisson bracket*

$$\{f, g\} = (df \otimes dg, \omega^{-1}). \tag{7.1}$$

This operation is skew-symmetric and satisfies the Jacobi identity, i.e., it is a Lie bracket on $C^\infty(M)$ (this follows from the fact that ω is closed).

Recall that a commutative algebra A is said to be a *Poisson algebra* if it is equipped with a Lie bracket $\{,\}$ such that

$$\{a, bc\} = \{a, b\}c + b\{a, c\},$$

for $a, b, c \in A$. Thus the operation $\{,\}$ defined by (7.1) endows $C^\infty(M)$ with a structure of a Poisson algebra.

For $M = T^*X$, in local coordinates we have

$$\{f, g\} = \sum_i \Big(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \Big).$$

This shows that Hamilton's equations can be written in the following manner in terms of Poisson brackets:

$$\dot{q}_i = \{q_i, H\},\ \dot{p}_i = \{p_i, H\}.$$

Equivalently,

$$\frac{d}{dt} f(q(t), p(t)) = \{f, H\}(q(t), p(t)) \tag{7.2}$$

for any smooth function ("classical observable") $f \in C^\infty(T^*X)$, or, for shorthand

$$\frac{df}{dt} = \{f, H\}.$$

In other words, Hamilton's equations say that the rate of change of the observed value of f equals the observed value of $\{f, H\}$.

7.2. Unbounded self-adjoint operators

The rigorous mathematical treatment of quantum mechanics in the Hamiltonian setting is based on von Neumann's theory of unbounded self-adjoint operators in Hilbert space. Let us recall the basics of this theory.

7.2.1. Spectral theorem for bounded self-adjoint operators. Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle\ ,\ \rangle$ (antilinear in the first argument, as is traditional in quantum physics). We first recall the *spectral theorem* for bounded self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$, which generalizes the diagonalization theorem for a Hermitian matrix.

Theorem 7.4 (von Neumann)**.** *Let A be a bounded self-adjoint operator. There exists a measure space (X, μ), an essentially bounded measurable function $h : X \to \mathbb{R}$, and an isometry $\mathcal{H} \to L^2(X, \mu)$ under which A maps to the operator of multiplication by h. Moreover, the spectrum $\sigma(A)$ is the set of $\lambda \in \mathbb{R}$ for which $h^{-1}(\lambda - \varepsilon, \lambda + \varepsilon)$ is positive for each $\varepsilon > 0$, and the eigenvalues of A (if they exist) are $\lambda \in \mathbb{R}$ such that $\mu(h^{-1}(\lambda)) > 0$, with eigenspace $L^2(h^{-1}(\lambda), \mu) \subset L^2(X, \mu)$.*

7.2.2. Closable and closed operators. Now we pass to not necessarily bounded operators. Let $\mathcal{H}'$ be another separable Hilbert space. A *densely defined linear operator* on $\mathcal{H}$ is a pair (A, V), where $V \subset \mathcal{H}$ is a dense subspace and A is a (possibly unbounded) linear operator $V \to \mathcal{H}'$. The space V is called the *domain* of A; in the notation, we will often suppress it and denote the operator just by A. Such an operator A has a graph $\Gamma_A \subset V \times \mathcal{H}' \subset \mathcal{H} \times \mathcal{H}'$. Let $\overline{\Gamma}_A$ be the closure of Γ_A in $\mathcal{H} \times \mathcal{H}'$. The operator A is said to be *closable* if $(0, u) \in \overline{\Gamma}_A$ for $u \in \mathcal{H}'$ implies $u = 0$, i.e., if the first projection $p_1 : \overline{\Gamma}_A \to \mathcal{H}$ is injective. In this case, setting $V' := p_1(\overline{\Gamma}_A) \subset \mathcal{H}$, we have $V \subset V'$ and obtain an extension of the operator $A : V \to \mathcal{H}'$ to a densely defined operator $\overline{A} : V' \to \mathcal{H}'$ which is called the *closure* of A. If A is closable and $\overline{A} = A$, we will say that A is *closed*; in other words, A is closed iff it has closed graph in $\mathcal{H} \times \mathcal{H}'$. Obviously, the closure $\overline{A}$ is closed for any closable A. Also, if A is bounded, then it is closable, $V' = \mathcal{H}$, and $\overline{A} : \mathcal{H} \to \mathcal{H}'$ is just the continuous (=bounded) extension of A.

In general, however, a densely defined operator need not be closable. For example, if $\mathcal{H}'$ is finite dimensional and $A : V \to \mathcal{H}'$ is unbounded, then there exists a sequence $v_n \in V$ such that $v_n \to 0$ but $\|Av_n\| \geq 1$. Then the sequence $w_n := \frac{v_n}{\|Av_n\|}$ goes to 0, while $\|Aw_n\| = 1$, so, as the unit sphere in $\mathcal{H}'$ is compact, passing to a subsequence if needed, we may assume that $Aw_n \to u$ for some $u \in \mathcal{H}'$ with $\|u\| = 1$. Then $(0, u) \in \overline{\Gamma}_A$ and A is not closable. So we see that A is closable iff it is bounded.

On the other hand, if $\mathcal{H}'$ is infinite dimensional, then there are important classes of unbounded closable operators. For example, consider the case $\mathcal{H} =$

$\mathcal{H}'$. Let us say that an operator $A : V \to \mathcal{H}$ is *symmetric* if $\langle v, Aw \rangle = \langle Av, w \rangle$ for all $v, w \in V$. We claim that every symmetric operator is closable and its closure is symmetric. Indeed, suppose $(v_n, Av_n) \to (0, u)$ for $u \in \mathcal{H}$. Fix a sequence $u_k \in V$ such that $u_k \to u$. Then

$$\langle Au_k, v_n \rangle = \langle u_k, Av_n \rangle \to \langle u_k, u \rangle,\ n \to \infty.$$

But the left-most expression goes to zero, so $\langle u_k, u \rangle = 0$ for all k, hence $\|u\|^2 = 0$ which gives $u = 0$, i.e., A is closable. Moreover, for $v, w \in V'$, there exist sequences $v_n \to v, w_n \to w$ in V such that $Av_n \to \overline{A}v, Aw_n \to \overline{A}w$, thus

$$\langle v, \overline{A}w \rangle = \lim_{n\to\infty} \langle v_n, Aw_n \rangle = \lim_{n\to\infty} \langle Av_n, w_n \rangle = \langle \overline{A}v, w \rangle,$$

so $\overline{A}$ is symmetric.

7.2.3. Adjoint operator. Closed symmetric operators by themselves are not sufficient for quantum mechanics, however, since such operators cannot, in general, be diagonalized. Instead we need (not necessarily bounded) *self-adjoint* operators, which are closed symmetric operators satisfying an important additional property. To formulate this property, we first need to define the notion of an adjoint operator.

Let (A, V) be a closed symmetric operator. Denote by $V^\vee$ the space of $u \in \mathcal{H}$ such that the linear functional $v \mapsto \langle u, Av \rangle$ is bounded on V. In this case by the Riesz representation theorem there exists a unique vector $w \in \mathcal{H}$ such that $\langle u, Av \rangle = \langle w, v \rangle$, which depends linearly on u. Thus we obtain an operator $A^\dagger : V^\vee \to \mathcal{H}$ such that $A^\dagger u = w$. Note that $V^\vee \supset V$ and $A^\dagger$ is an extension of A to $V^\vee$, so $(A^\dagger, V^\vee)$ is a densely defined operator called the *adjoint operator* of (A, V). Furthermore, this operator is closed: if $(u_n, A^\dagger u_n) \to (u, w)$, then for $v \in V$,

$$\langle A^\dagger u_n, v \rangle = \langle u_n, Av \rangle \to \langle u, Av \rangle,\ n \to \infty,$$

and at the same time $\langle A^\dagger u_n, v \rangle \to \langle w, v \rangle$, so $\langle u, Av \rangle = \langle w, v \rangle$, hence $u \in V^\vee$ and $w = A^\dagger u$.

However, we will see that the operator $A^\dagger$ fails to be symmetric, in general. So we may consider the skew-Hermitian form

$$B(v, w) := (A^\dagger v, w) - (v, A^\dagger w)$$

on $V^\vee$ that measures its failure to be symmetric, called the *boundary form* (it is called this because in examples it corresponds to boundary terms arising from integration by parts). By definition, $V \subset \operatorname{Ker} B$ (in fact, one can show that $V = \operatorname{Ker} B$, but we do not need this fact). It is easy to see that closed symmetric extensions of A correspond to isotropic closed subspaces $V \subset L \subset V^\vee$ with respect to the form B; namely, the extension of A to L is defined to be the restriction of $A^\dagger|_L$. Moreover, the adjoint operator to such an extension $(A^\dagger, L)$ is $(A^\dagger, L^\perp)$, where $L^\perp$ is the orthogonal complement of L in $V^\vee$ with respect to B.

7.2.4. Self-adjoint operators. Let us say that a closed symmetric operator (A, V) is *self-adjoint* if $V^\vee = V$, i.e., $A^\dagger = A$. We see that *self-adjoint extensions* of A correspond to *Lagrangian subspaces* L, i.e., those for which $L = L^\perp$. Note that such extensions/subspaces may or may not exist: the necessary and sufficient condition for existence of self-adjoint extensions (or Lagrangian subspaces) is that the signature (n_+, n_-) of the Hermitian form iB satisfies the equation $n_+ = n_-$ (i.e., the so-called *deficiency indices* $n_\pm \in \mathbb{Z}_{\geq 0} \cup \infty$ of A are equal). However, in quantum mechanical models they usually exist and correspond to various spatial boundary conditions.

We say that a symmetric operator (A, V) is *essentially self-adjoint* if the closure $(\overline{A}, V')$ is self-adjoint. Thus an essentially self-adjoint operator has a unique self-adjoint extension, so having such an operator is basically as good as having a self-adjoint one. This notion is convenient, for instance, when we do not want to describe explicitly the space V' (which could be rather ugly).

The importance of unbounded self-adjoint operators consists in the fact that von Neumann's spectral theorem extends naturally to them. Namely, define the *spectrum* $\sigma(A, V)$ of a self-adjoint operator (A, V) to be the subset of $\lambda \in \mathbb{C}$ for which the operator $A - \lambda \mathrm{I} : V \to \mathcal{H}$ fails to be surjective. Then we have

Theorem 7.5. *Theorem* 7.4 *holds for not necessarily bounded self-adjoint operators, except for the statement that h is essentially bounded. Moreover, the domain V of A in its spectral theorem realization is the space of functions $g \in L^2(X, \mu)$ such that $hg \in L^2(X, \mu)$.*

If the measure μ is concentrated on a countable set (i.e., we may take $X = \mathbb{N}$ with $\mu(j) = 1$ for $j \in \mathbb{N}$), then $\mathcal{H}$ has a basis consisting of eigenfunctions, and vice versa; in this case one says that A has *purely point spectrum*. This happens, for example, when A is compact (the Hilbert-Schmidt theorem). The other extreme is *purely continuous spectrum*, when there are no eigenvalues (i.e., in the spectral theorem realization, all points of X have zero measure). The spectral theorem implies that any self-adjoint operator can be uniquely written as an orthogonal direct sum of two self-adjoint operators with purely point and purely continuous spectrum, respectively.

The spectral theorem also implies the following corollary.

Corollary 7.6. *Let (A, V) be a self-adjoint operator. Then there exists a unique 1-parameter group of unitary operators $U(t) = e^{\mathrm{i}At} : \mathcal{H} \to \mathcal{H}$ strongly continuous in t which preserves V and commutes with A, such that for all $v \in V$ the function $t \mapsto U(t)v$ is differentiable and*

$$\frac{d}{dt}(U(t)v) = iAU(t)v.$$

Proof. Using the spectral theorem realization, where A is the operator of multiplication by $h : X \to \mathbb{R}$, we may define $U(t)$ as the operator of multiplication by e^{ith}. □

In fact, the converse also holds: every strongly continuous 1-parameter group $U(t)$ (i.e., a unitary representation of the Lie group $\mathbb{R}$ on $\mathcal{H}$) arises in this way from a unique self-adjoint operator.

Remark 7.7. The spectral theorem implies that if (A, V) is a self-adjoint operator and $Av = \lambda v$ for some nonzero $v \in V$, then $\lambda \in \mathbb{R}$. On the contrary, if (A, V) is only symmetric and not self-adjoint, von Neumann showed that the set of eigenvalues of $A^\dagger$ on its domain is either the (open) upper half-plane $\mathbb{C}_+$ (if $n_+ > 0, n_- = 0$), or the lower half-plane $\mathbb{C}_-$, (if $n_- > 0, n_+ = 0$), or contains both (if $n_+, n_- > 0$).

7.2.5. Examples.

Example 7.8. Consider the symmetric operator $P := -i\frac{d}{dx}$ on $\mathcal{H} = L^2(S)$, where $S := \mathbb{R}/2\pi\mathbb{Z}$ (the momentum operator on the circle). This operator is symmetric on the space $V := C^\infty(S)$, and one can show that it is moreover essentially self-adjoint on this space (check it!). The corresponding space V' is the *Sobolev space* $H^1(S)$ of functions $f \in L^2(S)$ with $f' \in L^2(S)$ in the sense of distributions (note that such functions are continuous). The spectrum of the corresponding self-adjoint operator is purely point and equals $\mathbb{Z}$, with eigenfunctions e^{inx}, i.e., $Pe^{inx} = ne^{inx}$. Thus the spectral realization of A is on $\ell_2(\mathbb{Z})$ with counting measure on which P acts by multiplication by the function n (i.e., this realization reduces simply to the Fourier expansion of functions on S). Similarly, the energy operator $P^2 = -\frac{d^2}{dx^2}$ is essentially self-adjoint on the same space but with smaller domain of the closure—the Sobolev space $H^2(S)$ of functions $f \in L^2(S)$ such that $f'' \in L^2(S)$. Its spectrum is in $(n^2 \mid n \in Z)$ with the same eigenfunctions: $P^2e^{inx} = n^2e^{inx}$.

Example 7.9. The next example is more interesting and prototypical for the theory of self-adjoint extensions. Namely consider the same momentum operator $P := -i\frac{d}{dx}$, but now acting on the dense subspace $V \subset L^2[0, 2\pi]$ of smooth functions with vanishing derivatives of all orders on both ends of the interval. In this case, P is **not** essentially self-adjoint: the space V' is the space of functions $f \in H^1[0, 2\pi]$ with $f(0) = f(2\pi) = 0$, while $V^\vee = H^1[0, 2\pi]$ with

$$B(f, g) = i(\overline{f(2\pi)}g(2\pi) - \overline{f(0)}g(0)).$$

So on the quotient $V^\vee/V' = \mathbb{C}^2$ we have

$$B((a, b), (a, b)) = i(|b|^2 - |a|^2),$$

where $a = f(0)$, $b = f(2\pi)$. Thus a Lagrangian subspace of $V^\vee$ is given by points $b \in \mathbb{C}$ with $|b| = 1$; namely, it is the space L_b of functions $f \in H^1[0, 2\pi]$

with $f(2\pi) = bf(0)$. The spectrum of the corresponding self-adjoint operator is again purely point, so we should look for eigenfunctions in the space L_b. Thus we get eigenfunctions $e^{i(n+s)x}$ where $b = e^{2\pi is}$. So the set of eigenvalues is $\mathbb{Z} + s$, and we see that the spectrum depends on the choice of the self-adjoint extension.

Observe also that any complex number λ is the eigenvalue of the (non-self-adjoint!) operator $P^\dagger$ on $V^\vee$, with eigenvector $e^{i\lambda x}$.

Example 7.10. Now consider the same momentum operator $P := -i\frac{d}{dx}$ but acting on the space $V = C_0^\infty(\mathbb{R})$ of compactly supported smooth functions, a subspace of $\mathcal{H} = L^2(\mathbb{R})$. In this case P is essentially self-adjoint, with $V' = V^\vee$ being the Sobolev space $H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R})\}$. The spectral theorem realization of P is on $L^2(\mathbb{R})$ as the operator of multiplication by x, which is given by the Fourier transform. Thus the spectrum of this operator is purely continuous and constitutes the whole real line $\mathbb{R}$. Similarly, the operator $P^2 = -\frac{d^2}{dx^2}$ is also essentially self-adjoint on V, with domain the Sobolev space $H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R})\}$, and its self-adjoint extension has purely continuous spectrum $\mathbb{R}_{\geq 0}$.

Example 7.11. And yet again, take $P := -i\frac{d}{dx}$, but now on the subspace V of $\mathcal{H} = L^2(\mathbb{R}_{\geq 0})$ of compactly supported smooth functions with vanishing derivatives at 0. This operator is not essentially self-adjoint: the space V' is the space of $f \in H^1(\mathbb{R}_{\geq 0})$ with $f(0) = 0$, while $V^\vee$ is the whole $H^1(\mathbb{R}_{\geq 0})$. Thus the space $V^\vee/V'$ is 1-dimensional with form B given by $B(f,g) = -i\overline{f(0)}g(0)$, and so there are no self-adjoint extensions (the deficiency indices are not equal: $n_+ = 1, n_- = 0$).

Let us find the eigenvalues of $P^\dagger$ on $V^\vee$. The eigenvector with eigenvalue λ is $e^{i\lambda x}$, and it belongs to $V^\vee$ iff $\lambda \in \mathbb{C}_+$. Thus the set of eigenvalues of $P^\dagger$ on $V^\vee$ is the upper half-plane $\mathbb{C}_+$.

Example 7.12. Let $A = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2$ with $V = C_0^\infty(\mathbb{R})$ (quantum harmonic oscillator). Then A is essentially self-adjoint and $\overline{A}$ has pure point spectrum $n + \frac{1}{2}$, $n \in \mathbb{N}$, with eigenvectors $H_n(x)e^{-\frac{x^2}{2}}$, where H_n are Hermite polynomials (Theorem 4.13).

Remark 7.13. More generally, it is known that if $U(x)$ is a piecewise continuous potential on $\mathbb{R}$ which tends to $+\infty$ at $\pm\infty$, then the operator $A := -\frac{1}{2}\frac{d^2}{dx^2} + U(x)$ is essentially self-adjoint on $V = C_0^\infty(\mathbb{R})$ and has pure point spectrum, with eigenvalues $E_0 < E_1 \leq E_2 \leq \ldots$ (it is shown in Proposition 7.21 below that E_0 is always a simple eigenvalue and the corresponding eigenvector is a positive function).

Example 7.14. Let M be a compact Riemannian manifold with boundary ∂M, $\mathcal{H} = L^2(M)$, and let $A = \Delta$ be the Laplace operator on M acting on the space V of smooth functions on M vanishing with all derivatives on the boundary. In this case V' is the Sobolev space $H^2(M)$ (functions $f \in L^2(M)$ such that $\Delta f \in L^2(M)$) which vanish with the first normal derivative on ∂M, and $V^\vee = H^2(M)$. By Stokes' formula

$$\int_M (u\Delta v - v\Delta u)dx = \int_{\partial M} (u\partial_{\mathbf{n}} v - v\partial_{\mathbf{n}} u)d\sigma,$$

where $\mathbf{n}$ denotes the normal derivative to ∂M, so we have

$$B(f,g) = i\int_{\partial M} (\overline{f}\partial_{\mathbf{n}} g - g\partial_{\mathbf{n}}\overline{f})d\sigma.$$

So if $\partial M = 0$, the operator A is essentially self-adjoint and has a unique self-adjoint extension, while if $\partial M \neq 0$, it is not and there are many self-adjoint extensions corresponding to various boundary conditions on ∂M. The most common ones are the Dirichlet boundary condition $f = 0$ and Neumann boundary condition $\partial_{\mathbf{n}} f = 0$. Of course, the spectra associated to these conditions (which are always purely point) are completely different.

The simplest example with nontrivial boundary is $M = [0, \pi]$, in which case we have $\dim V^\vee/V' = 4$ and

$$B(f,g) = i(\overline{f}g' - \overline{f'}g)|_0^\pi.$$

For the Dirichlet boundary conditions $f(0) = f(\pi) = 0$ we get eigenbasis $\sin nx$ with eigenvalues $-n^2$, $n \in \mathbb{Z}_{\geq 1}$, while for the Neumann boundary conditions $f'(0) = f'(\pi) = 0$ we get eigenbasis $\cos nx$ also with eigenvalues $-n^2$ but now for $n \in \mathbb{Z}_{\geq 0}$.

Let us consider the mixed boundary condition,

$$f(0) = 0,\ f'(\pi) - af(\pi) = 0$$

for some real number a. Then the eigenfunctions are $\sin \lambda x$, where

$$\lambda \cos \pi\lambda = a \sin \pi\lambda.$$

Thus the eigenvalues are $-\lambda^2$, where λ runs over solutions of the equation

$$\lambda\ \operatorname{cotan} \pi\lambda = a.$$

For example, in the limit $a \to \infty$ we recover the answer for the Dirichlet boundary condition.

Exercise 7.15. Let $H = -\frac{1}{2}\frac{d^2}{dx^2} + a\chi_{[-1,1]}(x)$, where χ is the indicator function and $a \in \mathbb{R}$, and let it be defined on $V = C_0^\infty(\mathbb{R}) \subset \mathcal{H} = L^2(\mathbb{R})$. Show that H is essentially self-adjoint and find the spectrum and eigenvalues of its self-adjoint extension (consider separately the cases $a \geq 0$ and $a < 0$).

Hint. As explained above, the spectrum consists of $E \in \mathbb{R}$ for which the operator $H - E : V' \to \mathcal{H}$ is not surjective. So try to solve the equation

$$(H - E)u = f$$

for $f \in \mathcal{H}$ as

$$u(x) = \int_{\mathbb{R}} G(x, y) f(y) dy,$$

where $G(x, y)$ is the fundamental solution of the equation

$$(H - E)u = \delta(x - y).$$

You should get that there are no eigenfunctions for $a \geq 0$ (purely continuous spectrum), while for $a < 0$ the spectrum is mixed: there is continuous spectrum and also some eigenfunctions with negative eigenvalues; they are called *bound states*.

7.3. Hamiltonians in quantum mechanics

The "yoga" of quantization says that to quantize classical mechanics on a manifold X, we need to replace the classical space of states T^*X by the quantum space of states—the Hilbert space $\mathcal{H} = L^2(X)$ on square integrable complex half-densities on X (or, more precisely, the corresponding projective space). Further, we need to replace classical observables, i.e., (sufficiently nice) real functions $f \in C^\infty(T^*X)$, by quantum observables $\widehat{f}$, which are (unbounded, densely defined) operators on $\mathcal{H}$, not commuting with each other in general. Then the (expected) value of an observable A in a state $\psi \in \mathcal{H}$ of unit norm is, by definition, $\langle \psi, A\psi \rangle$ (provided that it is well defined).

The operators $\widehat{f}$ should linearly depend on f. Moreover, they should depend on a positive real parameter $\hbar$ called the *Planck constant*, and satisfy the relation

$$[\widehat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}} + O(\hbar^2), \ \hbar \to 0. \tag{7.3}$$

Since the role of Poisson brackets of functions is played in quantum mechanics by commutators of operators, this relation expresses the condition that classical mechanics should be the limit of quantum mechanics as $\hbar \to 0$.[14]

We must immediately disappoint the reader by confessing that there is no canonical choice of the quantization map $f \mapsto \widehat{f}$. Nevertheless, there are some standard choices of $\widehat{f}$ for particular f, which we will now discuss.

[14]Note that the assignment $f \mapsto \widehat{f}$ cannot possibly satisfy the identity $\widehat{fg} = \widehat{f}\hat{g}$ since the product of functions is commutative but the product of operators is not.

Let us restrict ourselves to the situation $X = \mathbb{R}$, so on the phase space we have coordinates q (position) and p (momentum). In this case we can naturally think of half-densities as functions, and there are the following standard conventions.

1. $\widehat{f} = f(q)$ (multiplication operator by $f(q)$) when f is independent of the momentum p.
2. $\widehat{p^m} \to (-i\hbar\frac{d}{dq})^m$.

(Note that these conventions satisfy (1.9) since $[\hat{q}, \hat{p}] = i\hbar$, while $\{q, p\} = 1$.)

Example 7.16. For the classical Hamiltonian $H = \frac{p^2}{2} + U(q)$ the quantization is the *Schrödinger operator*

$$\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2} + U(q).$$

Remark 7.17. The extension of these conventions to other functions is not unique. However, such an extension will not be used, so we will not specify it.

Now let us see what the quantum analogue of Hamilton's equations should be. In accordance with the outlined quantization yoga, Poisson brackets should be replaced in quantum theory by commutators (with coefficient $-i\hbar^{-1}$). Thus, Hamilton's equations should be replaced by the equation

$$\frac{d}{dt}\langle\psi(t), A\psi(t)\rangle = -\frac{i}{\hbar}\langle\psi(t), [A, \widehat{H}]\psi(t)\rangle,$$

where $\langle , \rangle$ is the Hermitian form on $\mathcal{H}$ and $\widehat{H}$ is some quantization of the classical Hamiltonian H. Since this equation must hold for any A, it is equivalent to the *Schrödinger equation*

$$\dot{\psi} = -\frac{i}{\hbar}\widehat{H}\psi$$

up to changing ψ by a time-dependent phase factor (check it!). Thus, the quantum analogue of the Hamilton equations is the Schrödinger equation.

Remark 7.18. This "derivation" of the Schrödinger equation is definitely not a mathematical argument. It is merely a reasoning aimed to motivate a definition.

To solve the initial value problem for the Schrödinger equation, we need to make sense of the Hamiltonian $\widehat{H}$ as an unbounded self-adjoint operator on $\mathcal{H}$ in the sense of von Neumann, which in practice boils down to giving spatial boundary conditions for ψ, in addition to the initial value. The general solution of the Schrödinger equation then has the form

$$\psi(t) = e^{-\frac{it\widehat{H}}{\hbar}}\psi(0),$$

where $e^{-\frac{it\widehat{H}}{\hbar}}$ is the 1-parameter group of unitary operators attached to the self-adjoint operator $\widehat{H}$, which exists thanks to von Neumann's spectral theorem. Therefore, for any quantum observable A it is reasonable to define a new observable

$$A(t) := e^{\frac{it\widehat{H}}{\hbar}} A(0) e^{-\frac{it\widehat{H}}{\hbar}}$$

(such that to observe $A(t)$ is the same as to evolve for time t and then observe $A = A(0)$). The observable $A(t)$ satisfies the equation

$$A'(t) = -\frac{i}{\hbar}[A(t), \widehat{H}]$$

called the *operator Schrödinger equation*, and we have

$$\langle \psi(t), A\psi(t) \rangle = \langle \psi(0), A(t)\psi(0) \rangle. \tag{7.4}$$

The two sides of this equation represent two pictures of quantum mechanics: Schrödinger's (states change in time, observables do not) and Heisenberg's (observables change in time, states do not). Equation (7.4) expresses the equivalence of the two pictures.

7.4. Decay of eigenfunctions of a Schrödinger operator with potential growing at infinity

Consider a 1-dimensional particle with potential $U \in C(\mathbb{R})$. Assume that $U(q) \to +\infty$ as $|q| \to \infty$. In this case, it is known from basic analysis (the theory of Sturm-Liouville operators) that the Schrödinger operator $\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2} + U(q)$ is essentially self-adjoint on the space $C_0^\infty(\mathbb{R})$ of compactly supported smooth functions and its spectrum is purely point—a sequence of real eigenvalues (with multiplicities 1 or 2) going to $+\infty$. Note also that every eigenspace of $\widehat{H}$ clearly has a basis consisting of real-valued functions.

Suppose $\Psi \in L^2(\mathbb{R})$ is a real-valued eigenfunction of $\widehat{H}$, so $\widehat{H}\Psi = E\Psi$ (thus $\Psi \in C^2(\mathbb{R})$).

Proposition 7.19. *Let $q_0 \geq 0$ be such that $U(q) - E > 0$ for $|q| \geq q_0$. Then:*

(i) *Ψ does not change sign on $[q_0, +\infty)$ and $(-\infty, -q_0]$.*

(ii) *For $q \geq q_0$ we have*

$$|\Psi(q)| \leq |\Psi(q_0)| \exp\left(-\frac{1}{\hbar}\int_{q_0}^{q} \min_{[x,+\infty)} \sqrt{2(U-E)}\,dx\right),$$

$$|\Psi(-q)| \leq |\Psi(-q_0)| \exp\left(-\frac{1}{\hbar}\int_{-q}^{-q_0} \min_{(-\infty,x]} \sqrt{2(U-E)}\,dx\right).$$

(iii) *If* $\Psi > 0$ *on* $[q_0, +\infty)$ *or* $(-\infty, -q_0]$*, then it is strictly convex and* Ψ*,* Ψ' *monotonically approach* 0 *on this interval as* $|q| \to \infty$*. Also for* $q > q_0$ *we have*

$$|\Psi'(q)| < \frac{|\Psi(q_0)|}{q-r} \exp\left(-\frac{1}{\hbar}\int_{q_0}^{r} \min_{[x,+\infty)} \sqrt{2(U-E)}dx\right),$$

$$|\Psi'(-q)| < \frac{|\Psi(-q_0)|}{q-r} \exp\left(-\frac{1}{\hbar}\int_{-q_0}^{-r} \min_{(-\infty,x]} \sqrt{2(U-E)}dx\right)$$

for any $q > r \geq q_0$*.*

Proof.

(i) Suppose $\Psi(x) = 0$ for some $x \geq q_0$. Then $\Psi'(x) \neq 0$, so replacing Ψ by $-\Psi$ if needed, we may assume that $\Psi'(x) > 0$. Thus for $a > x$ sufficiently close to x, $\Psi(a) > 0$ and $\Psi'(a) > 0$. Since $\Psi \in L^2(\mathbb{R})$, we have $\liminf_{q\to+\infty} |\Psi(q)| = 0$, so there exists $b > a$ such that $\Psi(b) < \Psi(a)$. Let c be a global maximum point of Ψ on $[a, b]$. Then $\Psi(c) > \Psi(a), \Psi(b)$, so $c \neq a, b$, thus $\Psi'(c) = 0$, $\Psi''(c) \leq 0$. But this is a contradiction, as

$$\Psi''(c) = \frac{2}{\hbar^2}(U(c) - E)\Psi(c)$$

and $U(c) - E > 0$, $\Psi(c) > 0$. Thus Ψ cannot change sign on $[q_0, +\infty)$. The proof that Ψ cannot change sign on $[-\infty, -q_0]$ is similar.

(ii) Let $q_* \geq q_0$ and $C := \frac{1}{\hbar}\min_{[q_*,+\infty)} \sqrt{2(U-E)}$. By (i) we may assume that $\Psi(q) > 0$ for $q \geq q_*$ (replacing Ψ by $-\Psi$ if needed). Consider the function $\Phi(q) := \Psi(q) - \Psi(q_*)e^{-C(q-q_*)}$. Then $\Phi(q_*) = 0$ and $\liminf_{q\to+\infty} |\Phi(q)| = 0$. Suppose that $\Phi(y) > 0$ for some $y \in (q_*, +\infty)$. Pick $b > y$ with $\Phi(b) < \Phi(y)$, and let a be a global maximum point of Φ on $[q_*, b]$. Then $\Phi(a) \geq \Phi(y)$, hence $\Phi(a) > \max(0, \Phi(b))$. So $\Phi'(a) = 0$, $\Phi''(a) \leq 0$. But

$$\begin{aligned}\Phi''(a) &= \frac{2}{\hbar^2}(U(a) - E)\Psi(a) - C^2\Psi(q_*)e^{-C(a-q_*)} \\ &= \left(\frac{2}{\hbar^2}(U(a) - E) - C^2\right)\Psi(a) + C^2\Phi(a) > 0,\end{aligned}$$

which is a contradiction. Thus $\Phi(q) \leq 0$ on $[q_*, +\infty)$. This implies that for $q > q_*$ we have

$$\frac{\log\Psi(q) - \log\Psi(q_*)}{q - q_*} \leq -C = -\frac{1}{\hbar}\min_{[q_*,+\infty)} \sqrt{2(U-E)}.$$

Taking the limit $q \to q_*$, we get

$$\frac{\Psi'(q_*)}{\Psi(q_*)} \leq -\frac{1}{\hbar}\min_{[q_*,+\infty)} \sqrt{2(U-E)}.$$

Integrating this with respect to q_* from q_0 to q, we obtain

$$\log \frac{\Psi(q)}{\Psi(q_0)} \le -\frac{1}{\hbar} \int_{q_0}^{q} \min_{[q_*,+\infty)} \sqrt{2(U-E)} dq_*,$$

which after exponentiation yields the first inequality of (ii). The proof of the second inequality is similar.

(iii) Suppose $\Psi > 0$ on $[q_0, +\infty)$. Then $\Psi''(q) = \frac{2}{\hbar^2}(U(q) - E)\Psi(q) > 0$ on this interval, so Ψ is strictly convex. Hence Ψ' is strictly increasing and thus has a limit $L \in \mathbb{R} \cup \{+\infty\}$ as $q \to +\infty$. Also by (ii) $\Psi(q) \to 0$ as $q \to \infty$, so $L = 0$, thus $\Psi'(q) < 0$ for $q \in [q_0, +\infty)$. Thus Ψ is strictly decreasing on $[q_0, +\infty)$.

Finally, by monotonicity of Ψ', for $q > r \ge q_0$,

$$|\Psi'(q)|(q-r) < \int_r^q |\Psi'(x)| dx = -\int_r^q \Psi'(x) dx = \Psi(r) - \Psi(q) < \Psi(r),$$

so $|\Psi'(q)| < \frac{\Psi(r)}{q-r}$. Applying (ii) with q replaced by r to estimate $\Psi(r)$, we get the first bound of (iii). This proves (iii) for the interval $[q_0, +\infty)$. The proof for the other interval is the same. □

Proposition 7.19 items (ii) and (iii) imply that eigenfunctions ψ of $\widehat{H}$ and their first derivatives Ψ' decay very rapidly at infinity.[15] The physical meaning of Proposition 7.19(ii) is that since $U(q) \to +\infty$ when $|q| \to \infty$, the particle is trapped in the low energy region and is extremely unlikely to travel far in either direction. Note that classically this is not just unlikely but plainly impossible by the energy conservation law, while quantum mechanically it is possible (an effect called *tunneling*), even though very unlikely.

Remark 7.20. Proposition 7.19 with the same proof holds more generally when U is bounded below, for $E < E_\infty := \liminf_{|q|\to\infty} U(q)$;[16] in particular, we get $|\Psi(q)|, |\Psi'(q)| = o(e^{-K|q|})$ as $|q| \to \infty$ for any $0 < K < \frac{\sqrt{2(E_\infty - E)}}{\hbar}$. Moreover, eigenfunctions for a quantum particle in $\mathbb{R}^d$ and more generally on a connected Riemannian manifold with potential bounded below have similar decay properties.

[15] In contrast, Ψ'' may be quite irregular if the potential U has irregular behavior. This can be seen by considering a singular potential, say $U(x) = \frac{x^2}{2} + \sum_{n\in\mathbb{Z}} \delta(x-n)$. Eigenfunctions of $\widehat{H}$ with this potential make sense and have $\Psi''(n) = \pm\infty$ for $n \in \mathbb{Z}$, $|n| \gg 0$. So approximating $U(x)$ by continuous potentials, e.g., $\frac{x^2}{2} + \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{\pi t_n}} e^{-\frac{(x-n)^2}{t_n}}$, and tuning the parameters $t_n > 0$, we can get examples where $|\Psi''(n)|$ grows faster than any function of n.

[16] Eigenvalues $E \ge E_\infty$ can also exist, but the corresponding eigenfunctions may decay at a slower rate. For example, the L^2-function $\Psi(q) = \frac{1}{(1+q^2)^s}$, $s > \frac{1}{4}$ satisfies $\frac{1}{2}\Psi'' = U(q)\Psi$, where $U(q) = \frac{s((2s+1)q^2-1)}{(1+q^2)^2}$. In this case $E = E_\infty = 0$, and Ψ has only power decay.

7.5. The ground state

As before, assume that $U(q) \to +\infty$ as $q \to \pm\infty$, and let E_0 be the smallest eigenvalue of $\widehat{H}$.

Proposition 7.21. *E_0 is a simple eigenvalue, and the corresponding eigenfunction Ω is positive (after suitable normalization).*

Proof. By shifting U, we may assume that $U \geq 0$. For a piecewise C^1 function $\phi : \mathbb{R} \to \mathbb{R}$ (with ϕ' having finitely many discontinuities with continuous one-sided derivatives) define the energy of ϕ by

$$E(\phi) := \int_{\mathbb{R}} (\tfrac{\hbar^2}{2}|\phi'(q)|^2 + U(q)|\phi(q)|^2)dq \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

For instance, if $\Psi \in L^2(\mathbb{R})$ is an eigenfunction of $\widehat{H}$ with eigenvalue E, then by Proposition 7.19(ii),(iii), Ψ, Ψ' decay rapidly at ∞, so using integration by parts, we have $E(\Psi) = (\Psi, \widehat{H}\Psi) = E||\Psi||^2$.

If $\phi \in C_0^\infty(\mathbb{R})$, then by assumption

$$E(\phi) = (\phi, \widehat{H}\phi) \geq E_0||\phi||^2.$$

Thus by taking limits we have

$$E(\phi) \geq E_0||\phi||^2$$

for piecewise C^1 functions ϕ (even though $E(\phi)$ may now be infinite). Hence if ϕ is such a function of unit norm and $E(\phi) = E_0$, then ϕ is a global minimum of E under the constraint $||\phi||^2 = 1$. So by Lagrange multipliers and integration by parts, $(\widehat{H} - E)\phi = 0$ for some $E \in \mathbb{R}$, and moreover $E = (\phi, \widehat{H}\phi) = E(\phi) = E_0$. So

$$(\widehat{H} - E_0)\phi = 0 \tag{7.5}$$

as distributions.

Now let Ω be a real-valued eigenfunction of $\widehat{H}$ of unit norm with eigenvalue E_0, so $E(\Omega) = E_0$. Suppose that $\Omega(a) = 0$ for some $a \in \mathbb{R}$. Then (7.5) for $\phi = \Omega$ implies that $\Omega'(a) \neq 0$. So $(\widehat{H} - E_0)|\Omega|$ is a nonzero distribution (singular at a). Yet $E(|\Omega|) = E(\Omega) = E_0$, a contradiction with (7.5) for $\phi = |\Omega|$.

Thus by replacing Ω by $-\Omega$ if needed, we can arrange that $\Omega > 0$. This also implies that E_0 is a simple eigenvalue (if Ω_1, Ω_2 are two linearly independent real eigenfunctions of $\widehat{H}$ with eigenvalue E_0 and $\Omega_1(a) \neq 0$ for some $a \in \mathbb{R}$, then the eigenfunction $\Omega = \Omega_2 - \frac{\Omega_2(a)}{\Omega_1(a)}\Omega_1$ vanishes at a). □

Remark 7.22. Proposition 7.21 extends with a similar proof to the case of quantum particles on $\mathbb{R}^d$ and more generally on a connected Riemannian manifold with potential growing at infinity.

The function Ω, scaled to be positive and have unit norm, is called the *ground state*, *vacuum state*, or *vacuum vector*, since it has lowest energy. Physicists often shift the Hamiltonian by a constant so that the energy of this state is zero (i.e., "there is no matter").

7.6. Feynman-Kac formula

The correlation functions in the Hamiltonian setting are defined by the formula

$$\mathcal{G}_n^{\mathrm{Ham}}(t_1, \dots, t_n) := \langle \Omega, q(t_1) \dots q(t_n)\Omega \rangle,$$

where $q(t)$ is the operator quantizing the observable "coordinate of the particle at the time t".

Remark 7.23. Physicists usually write the inner product $\langle v, Aw \rangle$ as $\langle v|A|w\rangle$. In particular, Ω is written as $\langle 0|$ or $|0\rangle$ (the so-called Dirac *bra-ket notation*).

Theorem 7.24 (Feynman-Kac formula)**.** *If $t_1 \geq \dots \geq t_n$, then the function $\mathcal{G}_n^{\mathrm{Ham}}$ admits an asymptotic expansion in $\hbar$ (near $\hbar = 0$), which coincides with the path integral correlation function $\mathcal{G}_n^M$ constructed above. Equivalently, the Wick rotated function $\mathcal{G}_n^{\mathrm{Ham}}(-it_1, \dots, -it_n)$ equals $\mathcal{G}_n^E(t_1, \dots, t_n)$.*

This theorem plays a central role in quantum mechanics, and we will prove it below. Before we do so, let us formulate an analogue of this theorem for "quantum mechanics on the circle".

Let $\mathcal{G}_{n,L}(t_1, \dots, t_n)$ denote the correlation function on the circle of length L (for $0 \leq t_n \leq \dots \leq t_1 \leq L$), and let Z_L be the partition function on the circle of length L, defined from (Euclidean) path integrals. Also, let

$$Z_{\mathrm{Ham}}(L) := \mathrm{Tr}(e^{-\frac{L\widehat{H}}{\hbar}})$$

be the Hamiltonian partition function, and let

$$\mathcal{G}_{n,L}^{\mathrm{Ham}}(-it_1, \dots, -it_n) := \frac{\mathrm{Tr}(q(-it_n) \dots q(-it_1) e^{-\frac{L\widehat{H}}{\hbar}})}{\mathrm{Tr}(e^{-\frac{L\widehat{H}}{\hbar}})}$$

be Hamiltonian correlation functions.

Theorem 7.25 (Feynman-Kac formula on the circle)**.** *The functions $Z_{\mathrm{Ham}}(L)$, $\mathcal{G}_{n,L}^{\mathrm{Ham}}$ admit asymptotic expansions in $\hbar$, which coincide with the functions $Z(L) = Z_{\mathrm{Lagr}}(L)$ and $\mathcal{G}_{n,L} = \mathcal{G}_{n,L}^{\mathrm{Lagr}}$ computed in the Lagrangian setting from path integrals (Section 6.10).*

Note that Theorem 7.24 is obtained from Theorem 7.25 by sending L to infinity. Thus, it is sufficient to prove Theorem 7.25.

Remark 7.26. As we mentioned before, the function $\mathcal{G}_n^E$ can be defined by means of the Wiener integral, and the equality

$$\mathcal{G}_n^{\mathrm{Ham}}(-it_1,\dots,-it_n) = \mathcal{G}_n^E(t_1,\dots,t_n)$$

actually holds for numerical values of $\hbar$, and not just in the sense of power series expansions. The same holds for the equalities $Z_{\mathrm{Ham}}(L) = Z(L)$, $\mathcal{G}_{n,L}^{\mathrm{Ham}} = \mathcal{G}_{n,L}$. However, these results are technically more complicated (as they require nontrivial analytic input), and thus are beyond the scope of this book.

Example 7.27. Consider the case of the quadratic potential. By renormalizing variables, we can assume that $\hbar = m = 1$, so $U = \frac{q^2}{2}$. In this case we know that $Z_{\mathrm{Lagr}}(L) = \frac{1}{e^{\frac{L}{2}} - e^{-\frac{L}{2}}}$ (formula (6.5)). On the other hand, $\widehat{H}$ is the Hamiltonian of the quantum harmonic oscillator,

$$\widehat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{q^2}{2}.$$

The eigenvectors of this operator are $H_n(x)e^{-\frac{x^2}{2}}$, where H_n are the Hermite polynomials ($n \geq 0$), and the eigenvalues are $n + \frac{1}{2}$ (see Theorem 4.13). Hence,

$$Z_{\mathrm{Ham}}(L) = e^{-\frac{L}{2}} + e^{-\frac{3L}{2}} + \dots = \frac{1}{e^{\frac{L}{2}} - e^{-\frac{L}{2}}} = Z_{\mathrm{Lagr}}(L),$$

as expected from the Feynman-Kac formula. This shows the benefit of the choice $C = \frac{1}{2}$ in the normalization of $Z_{\mathrm{Lagr}}(L)$, as in Remark 6.29: a different choice would necessitate adding a constant to $\widehat{H}$ to preserve the validity of the Feynman-Kac formula.

More generally, as in Section 6.10, we may also consider a quantum particle in a Euclidean space V with potential $\frac{1}{2}(q, Mq)$, where $M : V \to V$ is a positive self-adjoint operator. Then, splitting V into eigenspaces of M, we get

$$Z_{\mathrm{Ham}}(L) = \frac{1}{\det(e^{\frac{\sqrt{M}L}{2}} - e^{-\frac{\sqrt{M}L}{2}})},$$

which is again equal to $Z_{\mathrm{Lagr}}(L)$ (formula (6.6)).

Furthermore, again as in Section 6.10, we can twist the theory by an orthogonal automorphism $g : V \to V$ commuting with M. In this case, g also acts on $\mathcal{H}$, and the Hamiltonian partition function by definition is

$$Z_{\mathrm{Ham}}(L) := \mathrm{Tr}(ge^{-L\widehat{H}}). \tag{7.6}$$

Computing this trace, we get

$$Z_{\mathrm{Ham}}(L) = \frac{1}{\det(e^{\frac{\sqrt{M}L}{2}} - ge^{-\frac{\sqrt{M}L}{2}})}, \tag{7.7}$$

which once again coincides with $Z_{\mathrm{Lagr}}(L)$ (formula (6.7)), showing that the Feynman-Kac formula (with definition (7.7)) still holds in the twisted case.

7.7. Proof of the Feynman-Kac formula in the free case (harmonic oscillator)

Consider again the quadratic Hamiltonian $\widehat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{q^2}{2}$ of the quantum Harmonic oscillator. Note that it can be written in the form

$$\widehat{H} = a^\dagger a + \tfrac{1}{2},$$

where $a = \frac{1}{\sqrt{2}}(\frac{d}{dq} + q)$, $a^\dagger = \frac{1}{\sqrt{2}}(-\frac{d}{dq} + q)$. The operators $a, a^\dagger$ define a representation of the Heisenberg Lie algebra on (a dense subspace of) the Hilbert space $\mathcal{H}$,

$$[a, a^\dagger] = 1.$$

Thus the eigenvectors of $\widehat{H}$ are $(a^\dagger)^n\Omega$, where $\Omega = e^{-\frac{q^2}{2}}$ is the lowest eigenvector and the corresponding eigenvalues are $n + \frac{1}{2}$, $n \in \mathbb{Z}_{\geq 0}$ (as we already saw before in Theorem 4.13).

Remark 7.28. The operators a and $a^\dagger$ are called the *annihilation and creation operators*, since $a\Omega = 0$, while all eigenvectors of $\widehat{H}$ can be "created" from Ω by action of powers of $a^\dagger$.

Now, we have

$$q(0) = q = \frac{1}{\sqrt{2}}(a + a^\dagger).$$

Since $[a^\dagger a, a] = -a$, $[a^\dagger a, a^\dagger] = a^\dagger$, we have

$$q(t) = \frac{1}{\sqrt{2}} e^{ita^\dagger a}(a + a^\dagger)e^{-ita^\dagger a} = \frac{1}{\sqrt{2}}(e^{-it}a + e^{it}a^\dagger).$$

This shows that

$$\mathcal{G}^{\mathrm{Ham}}_{n,L}(-it_1, \dots, -it_n) = 2^{-\frac{n}{2}} \frac{\mathrm{Tr}(\prod_{j=1}^n (e^{t_j}a^\dagger + e^{-t_j}a)e^{-L(a^\dagger a + \frac{1}{2})})}{\mathrm{Tr}(e^{-L(a^\dagger a + \frac{1}{2})})}.$$

Now we can easily prove Theorem 7.25. Indeed, let us move the terms $e^{t_1}a^\dagger$ and $e^{-t_1}a$ around the trace (using the cyclic property of the trace). This will

yield, after a short calculation, using (6.4) :

$$\begin{aligned}\mathcal{G}^{\mathrm{Ham}}_{n,L}(-it_1,\dots,-it_n)\\ &= \sum_{j=2}^{n} \tfrac{1}{2}\mathcal{G}^{\mathrm{Ham}}_{n-2,L}(-it_2,\dots,-it_{j-1},-it_{j+1},\dots,-it_n)\left(\frac{e^{t_1-t_j}}{e^L-1}-\frac{e^{t_j-t_1}}{e^{-L}-1}\right)\\ &= \sum_{j=2}^{n} \mathcal{G}^{\mathrm{Ham}}_{n-2,L}(-it_2,\dots,-it_{j-1},-it_{j+1},\dots,-it_n)G_L(t_1-t_j).\end{aligned}$$

This implies the theorem by induction in n.

Remark 7.29.

1. In the quadratic case the Feynman-Kac formula holds as an equality not just between formal expansions, but also between usual functions.

2. Note that the equality $\frac{e^{t-s}}{e^L-1}-\frac{e^{s-t}}{e^{-L}-1}=G_L(t-s)$ used above holds only if $t\geq s$. In fact, the matrix coefficient $\langle\Omega, q(t_1)\dots q(t_n)\Omega\rangle$ is not symmetric in t_j, as the operators $q(t_j)$ do not commute. Thus the Feynman-Kac formula only holds if $t_1\geq\ldots\geq t_n$. For this reason the correlation function $\mathcal{G}^M_n$ is called *time-ordered*—it corresponds to the matrix coefficient where the operators $q(t_j)$ are ordered chronologically.

7.8. Proof of the Feynman-Kac formula (general case)

Now we consider an arbitrary potential $U(q):=\frac{m^2q^2}{2}-V(q)$, where

$$V(q)=\sum_{k\geq 3}\frac{g_k q^k}{k!}.$$

For simplicity we will assume that the coefficients g_j are formal parameters and $\hbar=1$ (the latter condition does not cause a loss of generality, as this situation can be achieved by rescaling). Let us first consider the case of partition function. We have

$$Z_{\mathrm{Ham}}(L)=\mathrm{Tr}(e^{-L\widehat{H}})=\mathrm{Tr}(e^{-L(\widehat{H}_0-V)}),$$

where $\widehat{H}_0=-\frac{1}{2}\frac{d^2}{dq^2}+\frac{1}{2}m^2q^2$ is the free (=quadratic) part of the Hamiltonian. Since g_j are formal parameters, we have a series expansion

$$\begin{aligned}(7.8)\quad e^{-L(\widehat{H}_0-V)}=e^{-L\widehat{H}_0}+\sum_{N\geq 1}\int_{L\geq s_1\geq\ldots\geq s_N\geq 0}\\ \times e^{-(L-s_1)\widehat{H}_0}Ve^{-(s_1-s_2)\widehat{H}_0}V\dots e^{-(s_{n-1}-s_n)\widehat{H}_0}Ve^{-s_n\widehat{H}_0}d\mathbf{s}\end{aligned}$$

This follows from the general fact that in the (completed) free algebra with generators A, B, one has

(7.9)
$$e^{A+B} = e^A + \sum_{N\geq 1} \int_{1\geq s_1\geq \ldots \geq s_N \geq 0} e^{(1-s_1)A} B e^{(s_1-s_2)A} B \ldots e^{(s_{N-1}-s_N)A} B e^{s_N A} d\mathbf{s}$$

(check this identity!).

Equation (7.8) shows that

$$Z_{\mathrm{Ham}}(L) = \sum_{N\geq 0} \sum_{j_1,\ldots,j_N=3}^{\infty} \frac{g_{j_1}\cdots g_{j_N}}{j_1!\ldots j_N!} \int_{1\geq s_1\geq \ldots \geq s_N\geq 0} \times \mathrm{Tr}(q_0(-is_1)^{j_1} \ldots q_0(-is_N)^{j_N} e^{-L\widehat{H}_0}) d\mathbf{s},$$

where $q_0(t)$ is the operator $q(t)$ in the free theory associated to the potential $\frac{m^2q^2}{2}$.

Since the Feynman-Kac formula for free theory has already been proved, we know that the trace on the right-hand side can be evaluated as a sum over matchings. To see what exactly is obtained, let us collect the terms corresponding to all permutations of $j_1, \ldots, j_N$ together. This means that the summation variables will be the numbers $i_3, i_4, \ldots$ of occurrences of $3, 4, \ldots$ among $j_1, \ldots, j_N$. Further, to every factor $q_0(-is)^j$ there will be assigned a j-valent vertex, with a variable s attached to it, and it is easy to see that $Z_{\mathrm{Ham}}(L)$ equals the sum over all ways of connecting the vertices (i.e., Feynman diagrams Γ) of integrals

$$\int_{0\leq s_1,\ldots,s_N\leq L} \prod_{v-w} G_L(s_v - s_w) d\mathbf{s}$$

multiplied by the coefficients $\frac{\prod_k g_k^{i_k}}{|\mathrm{Aut}\,\Gamma|}$. Thus, $Z_{\mathrm{Ham}}(L) = Z(L)$, as desired.

Now let us consider correlation functions. Thus we have to compute

$$\mathrm{Tr}(e^{-(L-t_1)\widehat{H}} q e^{-(t_1-t_2)\widehat{H}} q \ldots q e^{-t_n\widehat{H}}).$$

Expanding each exponential inside the trace as above, we will clearly get the same Feynman diagram sum, except that the Feynman diagrams will contain n external vertices marked by variables $t_1, \ldots, t_n$. This implies that $\mathcal{G}_{n,L}^{\mathrm{Ham}} = \mathcal{G}_{n,L}$, and we are done.

7.9. The massless case

Consider now the massless case, $m = 0$, in the Hamiltonian setting. For maps $q : \mathbb{R} \to \mathbb{R}$, we have $\mathcal{H} = L^2(\mathbb{R})$, and $\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2}$. This operator has continuous spectrum, and there is no lowest eigenvector Ω (in fact, there are no eigenvectors in L^2 at all), which means that we cannot define the correlation

functions in the usual way, i.e., as $\langle \Omega, q(t_1) \dots q(t_n)\Omega \rangle$. (This is a reflection, in the Hamiltonian setting, of the difficulties related to the growth of the Green's function at infinity, i.e., infrared divergences, which we encountered in the Lagrangian setting, cf. Section 6.11).

Consider now the case $q : \mathbb{R} \to S^1 = \mathbb{R}/2\pi r\mathbb{Z}$. In this case, we have the same Hamiltonian but acting in the space $\mathcal{H} := L^2(S^1)$. The eigenvectors of this operator are $e^{\frac{iNq}{r}}$, with eigenvalues $\hbar^2 \frac{N^2}{2r^2}$. In particular, the lowest eigenvector is $\Omega = 1$. Thus the Hamiltonian correlation functions (in the Euclidean setting, for $t_1 \geq \dots \geq t_n$) are

$$\langle \Omega, e^{\frac{t_1 \widehat{H}}{\hbar}} e^{\frac{ip_1 q}{r}} e^{\frac{(t_2 - t_1)\widehat{H}}{\hbar}} \dots e^{\frac{ip_n q}{r}} e^{-\frac{t_n \widehat{H}}{\hbar}} \Omega \rangle$$
$$= e^{-\frac{\hbar}{2r^2} \sum_j (t_j - t_{j+1})(p_1 + \dots + p_j)^2},$$

which is equal to the correlation function in the Lagrangian setting (cf. Section 6.12), in full agreement with the Feynman-Kac formula.

Now we pass to the case of circle-valued quantum mechanics on the circle (Section 6.14). In this case, we have

$$\mathrm{Tr}(e^{-\frac{L\widehat{H}}{\hbar}}) = \sum_{N \in \mathbb{Z}} e^{-\frac{N^2 \hbar L}{2r^2}}$$

and

$$\mathrm{Tr}(e^{\frac{t_1 \widehat{H}}{\hbar}} e^{\frac{ip_1 q}{r}} e^{\frac{(t_2 - t_1)\widehat{H}}{\hbar}} \dots e^{\frac{ip_n q}{r}} e^{\frac{(L - t_n)\widehat{H}}{\hbar}}) = \sum_{N \in \mathbb{Z}} e^{-\frac{\hbar}{2r^2} \sum_{j=0}^{n} (t_j - t_{j+1})(N - p_1 - \dots - p_j)^2},$$

where $t_0 := L$, $t_{n+1} := 0$. Simplifying this expression, we obtain

$$\mathrm{Tr}(e^{\frac{t_1 \widehat{H}}{\hbar}} e^{\frac{ip_1 q}{r}} e^{\frac{(t_2 - t_1)\widehat{H}}{\hbar}} \dots e^{\frac{ip_n q}{r}} e^{\frac{(L - t_n)\widehat{H}}{\hbar}})$$
$$= e^{-\frac{\hbar}{2r^2} \sum_j (t_j - t_{j+1})(p_1 + \dots + p_j)^2} \sum_{N \in \mathbb{Z}} e^{-\frac{\hbar}{2r^2}(LN^2 - 2N \sum_j p_j t_j)}$$
$$= e^{-\frac{\hbar}{2r^2} \sum_j (t_j - t_{j+1})(p_1 + \dots + p_j)^2} \vartheta\left(\frac{\hbar}{2\pi i r^2} \sum_j p_j t_j, \frac{\hbar L}{2\pi r^2}\right).$$

Comparing with (6.12), we see that the Feynman-Kac formula reduces to the modular invariance of the theta-function:

$$\vartheta\left(\frac{u}{iT}, \frac{1}{T}\right) = \sqrt{T} e^{\frac{\pi u^2}{T}} \vartheta(u, T) \tag{7.10}$$

with $T = \frac{2\pi r^2}{\hbar L}$ (which follows from the Poisson summation formula applied to the Gaussian).

Note that the Feynman-Kac formula in this example would have been false had we ignored the topologically nontrivial maps in the Lagrangian setting (Section 6.14). Thus we may say that the Feynman-Kac formula "sees topology". This ability of the Feynman-Kac formula to "see topology" (in much more

complex situations) lies at the foundation of many interrelations between geometry and quantum field theory.

Remark 7.30. It should be noted that the contributions of topologically nontrivial maps from the source circle to the target circle are, strictly speaking, beyond our usual setting of perturbation theory, since they are exponentially small in $\hbar$. To be specific, the contribution from maps of degree N mostly comes from those maps which are close to the minimal action map $q_N(t) = \frac{2\pi t N r}{L}$, so it is of the order $e^{-\frac{2\pi^2 N^2 r^2}{L\hbar}}$. The maps $q_N(t)$ are the simplest examples of "instantons"—nonconstant solutions of the classical equations of motion, which have finite action (and are nontrivial in the topological sense). Exponentially small contributions to the path integral coming from integration over neighborhoods of instantons are called "instanton corrections to the perturbation series".

Remark 7.31. This calculation allows us to give sense to the partition function $Z(L)$ of the line-valued massless quantum mechanics on the circle (Section 6.13). To this end, we just need to look at the asymptotics $r \to \infty$ of the partition function,

$$Z(r, L) = \vartheta(0, \tfrac{\hbar L}{2\pi r^2}) = r\sqrt{\tfrac{2\pi}{\hbar L}}\vartheta(0, \tfrac{2\pi r^2}{\hbar L}).$$

Since $\vartheta(0, T) \to 1$ as $T \to \infty$, for the leading coefficient of the asymptotics we have (up to numerical scaling, which we are free to choose):

$$Z(L) \sim \frac{1}{\sqrt{\hbar L}}.$$

Note however that in this case we cannot write $Z(L) = \mathrm{Tr}(e^{-\frac{L\widehat{H}}{\hbar}})$ since this operator is not trace class. Also the vector $\Omega = 1$ is not normalizable. Thus this theory is somewhat ill defined, as already mentioned in Section 6.13.

7.10. Spectrum of the Schrödinger operator for a piecewise constant periodic potential

In this section we demonstrate the behavior of the spectrum of a 1-dimensional Schrödinger operator on the example of a piecewise constant periodic potential, when the eigenvalues and eigenfunctions can be computed fairly explicitly.

We consider the Schrödinger operator on the circle $\mathbb{R}/2\pi\mathbb{Z}$ given by $H := -\frac{\hbar^2}{2}\partial^2 + U(x)$, where U is a piecewise continuous 2π-periodic potential. Clearly, without loss of generality we may assume that $\int_0^{2\pi} U(x)dx = 0$; otherwise we can shift $U(x)$ by a constant. As discussed in Sections 7.4 and 7.5, by the theory

of Sturm-Liouville operators, the operator H has discrete spectrum, i.e., eigenvalues $E_0 < E_1 \le E_2 \le \dots$ going to $+\infty$ with the corresponding eigenfunctions $\Psi_0, \Psi_1, \Psi_2, \dots$. For example, if $U = 0$, then $E_0 = 0$ and $E_{2m-1} = E_{2m} = \frac{\hbar^2 m^2}{2}$ for $m > 0$, with eigenfunctions $\Psi_0 = 1, \Psi_{2m-1} = \sin mx, \Psi_{2m} = \cos mx$.

Consider now the simplest nontrivial example—the piecewise constant potential

$$U(x) = \begin{cases} Mb, & 0 \le x < a, \\ -Ma, & a \le x < 2\pi, \end{cases} \tag{7.11}$$

where $a, b, M > 0$, $a + b = 2\pi$.

Eigenfunctions of H with eigenvalue E are nonzero periodic solutions of the differential equation $H\Psi = E\Psi$. Thus eigenvalues are the numbers E for which the monodromy matrix of this equation has eigenvalue 1. To find such E, for every $p \in \mathbb{R}$ introduce the basis f_p, g_p of the space of solutions of the equation $H\Psi = E\Psi$ such that $f_p(p) = g_p'(p) = 1, g_p(p) = f_p'(p) = 0$. For example,

$$f_0(x) = \cos \beta x,\ g_0(x) = \frac{\sin \beta x}{\beta}$$

for $0 \le x < a$ and

$$f_a(x) = \cos \alpha(x-a),\ g_a(x) = \frac{\sin \alpha(x-a)}{\alpha(x-a)}$$

for $a \le x < 2\pi$, where $\alpha := \sqrt{\frac{2}{\hbar^2}(E + Ma)}$, $\beta := \sqrt{\frac{2}{\hbar^2}(E - Mb)}$. Thus the monodromy matrices along the intervals $[0, a]$, $[a, 2\pi]$ in these bases are

$$A := \begin{pmatrix} \cos \beta a & \beta^{-1} \sin \beta a \\ -\beta \sin \beta a & \cos \beta a \end{pmatrix},$$

$$B := \begin{pmatrix} \cos \alpha b & \alpha^{-1} \sin \alpha b \\ -\alpha \sin \alpha b & \cos \alpha b \end{pmatrix}.$$

Thus the condition for a periodic solution is that the matrix AB (monodromy around the circle) has an eigenvalue 1. Since $\det A = \det B = 1$, in this case the second eigenvalue of AB is also 1 (generically this matrix is conjugate to a unipotent Jordan block), so the condition is $\mathrm{Tr}(AB) = 2$, which gives

$$\cos \beta a \cos \alpha b - \tfrac{1}{2}(\tfrac{\alpha}{\beta} + \tfrac{\beta}{\alpha}) \sin \beta a \sin \alpha b = 1. \tag{7.12}$$

Thus the eigenvalues of H are the solutions E of (7.12).

If $a < E < b$, then $\beta = i\beta_*$ is imaginary, so (7.12) can be written in terms of real parameters as

$$\cosh \beta_* a \cos \alpha b - \tfrac{1}{2}(\tfrac{\alpha}{\beta_*} - \tfrac{\beta_*}{\alpha}) \sin \beta_* a \sin \alpha b = 1. \tag{7.13}$$

As mentioned above, if $M = 0$, then for each $n \geq 1$ the operator H has double eigenvalue $\frac{1}{2}\hbar^2 n^2$. We would like to see what happens to this eigenvalue for large n as we turn on M and keep the product $\hbar n$ in a bounded interval $[C^{-1}, C]$ (so $\hbar \to 0$).

Let us rewrite (7.12) in the form

$$1 - \cos(\beta a + \alpha b) = \left(1 - \tfrac{1}{2}(\tfrac{\alpha}{\beta} + \tfrac{\beta}{\alpha})\right) \sin \beta a \sin \alpha b \tag{7.14}$$

and look for solutions

$$E = \frac{1}{2}(\hbar^2 n^2 + \varepsilon),$$

where $|\varepsilon| \ll \frac{1}{n}$. We have

$$\beta = \sqrt{n^2 + \tfrac{\varepsilon - 2Mb}{\hbar^2}} = n\left(1 + \tfrac{\varepsilon - 2Mb}{2\hbar^2 n^2} - \tfrac{(\varepsilon - 2Mb)^2}{8\hbar^4 n^4} \dots\right),$$

$$\alpha = \sqrt{n^2 + \tfrac{\varepsilon + 2Ma}{\hbar^2}} = n\left(1 + \tfrac{\varepsilon + 2Ma}{2\hbar^2 n^2} - \tfrac{(\varepsilon + 2Ma)^2}{8\hbar^4 n^4} \dots\right),$$

so

$$\beta a + \alpha b = 2\pi n\left(1 + \tfrac{\varepsilon}{2\hbar^2 n^2} - \tfrac{M^2 ab}{2\hbar^4 n^4} + \dots\right).$$

Thus the left-hand side of (7.14) has the form

$$1 - \cos(\beta a + \alpha b) = \frac{\pi^2}{2}\left(\frac{\varepsilon}{\hbar^2 n} - \frac{M^2 ab}{2\hbar^4 n^3}\right)^2 + \dots.$$

We also have

$$1 - \tfrac{1}{2}(\tfrac{\alpha}{\beta} + \tfrac{\beta}{\alpha}) = -\frac{\pi^2 M^2}{2\hbar^4 n^4} + \dots.$$

So using that $b = 2\pi - a$ and hence $\sin nb = -\sin na$, we get

$$\left(1 - \tfrac{1}{2}(\tfrac{\alpha}{\beta} + \tfrac{\beta}{\alpha})\right) \sin \beta a \sin \alpha b = \frac{\pi^2 M^2}{2\hbar^4 n^4} \sin^2 na + \dots.$$

Thus we obtain

$$\frac{\varepsilon}{\hbar^2 n} - \frac{M^2 a(2\pi - a)}{2\hbar^4 n^3} = \pm \frac{M|\sin na|}{\hbar^2 n^2},$$

which yields

$$\varepsilon = \frac{M^2 a(2\pi - a)}{2\hbar^2 n^2} \pm \frac{M|\sin na|}{n}.$$

We see that the double eigenvalue $\Lambda_n = \frac{\hbar^2 n^2}{2}$, $n > 0$ for $M = 0$ bifurcates into two eigenvalues

$$\Lambda_n^{\pm}(M) = \Lambda_n + \frac{M^2 a(2\pi - a)}{8\Lambda_n} \pm \frac{M|\sin na|}{2n} + o((M + \tfrac{1}{n})^2),\ M \to 0, n \to \infty. \tag{7.15}$$

7.11. WKB approximation and the Weyl law

The goal of this section is to explain how to compute semiclassical asymptotics of eigenvalues and eigenfunctions of quantum Hamiltonians. This method is called the *Wentzel-Kramers-Brillouin* (*WKB*) *approximation*, named after the authors of three separate papers which introduced it independently in 1926.

We start with a general discussion of WKB approximation for linear ODE. Suppose we have an equation

$$\hbar \frac{dF}{dx} = AF \tag{7.16}$$

for a vector-function of one variable $F(x) \in \mathbb{C}^n$, where $A(x) \in \mathrm{Mat}_n(\mathbb{C})$ is a matrix-valued function (smooth on a certain interval $I \subset \mathbb{R}$). We would like to understand the asymptotic behavior of solutions of this equation as $\hbar \to 0$. To this end, assume for simplicity that $A(x)$ has simple spectrum for generic x, and let $v_1(x), \dots, v_n(x)$ be its column eigenvectors with eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$, and let $v_1^*(x), \dots, v_n^*(x)$ be the dual basis of row eigenvectors (i.e., $(v_j^*, v_j) = 1$). Let us now look for solutions of (7.16) in the form

$$F(x) = e^{\frac{\phi(x)}{\hbar}}(\psi_0(x) + \hbar\psi_1(x) + \hbar^2\psi_2(x)\dots),$$

where $\psi_0(x) \neq 0$ and the series in parentheses is formal (here ϕ takes values in $\mathbb{C}$ and ψ_j in $\mathbb{C}^n$). Substituting, we get

$$(\hbar\partial_x + \phi'\mathrm{I} - A)(\psi_0 + \hbar\psi_1 + \hbar^2\psi_2 + \dots) = 0,$$

which in degree 0 with respect to $\hbar$ yields the equation

$$A\psi_0 = \phi'\psi_0.$$

Thus $\phi' = \lambda_j$ is an eigenvalue of A, so

$$\phi(x) = \int \lambda_j(x)dx,\ \psi_0(x) = f(x)v_j(x),$$

where f is a scalar function.

Further, in degree 1 in $\hbar$ we obtain the equation

$$\psi_0' = (A - \lambda\mathrm{I})\psi_1,$$

i.e.,

$$f'v_j + fv_j' = (A - \lambda\mathrm{I})\psi_1.$$

For this to have a solution ψ_1, we need $(v_j^*, f'v_j + fv_j') = 0$, i.e.,

$$f' = -(v_j^*, v_j')f.$$

Thus

$$f(x) = \exp\Big(-\int (v_j^*, v_j')dx\Big).$$

Now we can recursively solve for $\psi_1, \psi_2, \dots$. This leads to the following result.

Theorem 7.32. *There is a unique, up to scaling, basis of formal solutions of equation* (7.16) *of the form*

$$F_j(x) = \exp\left(\frac{\int \lambda_j(x)dx}{\hbar}\right)\left(\exp\left(-\int (v_j^*(x), v_j'(x))dx\right)v_j(x) + O(\hbar)\right).$$

Let us now apply this theorem to the stationary Schrödinger equation

$$(-\tfrac{\hbar^2}{2}\partial_x^2 + U(x))\Psi = E\Psi. \tag{7.17}$$

Set $p(x) := \sqrt{2(E - U(x))}$, then (7.17) takes the form

$$\hbar^2\partial_x^2\Psi = -p^2\Psi.$$

This can be written as the system of equations

$$\hbar\partial_x\begin{pmatrix}\Psi\\ \hbar\Psi'\end{pmatrix} = \begin{pmatrix}0 & 1\\ -p^2 & 0\end{pmatrix}\begin{pmatrix}\Psi\\ \hbar\Psi'\end{pmatrix}.$$

Thus we have equation (7.16) with $A = \begin{pmatrix}0 & 1\\ -p^2 & 0\end{pmatrix}$. So we have

$$\lambda_1 = ip,\ \lambda_2 = -ip,$$

and we may take

$$v_1 = \begin{pmatrix}1\\ ip\end{pmatrix},\ v_2 = \begin{pmatrix}1\\ -ip\end{pmatrix},$$

so that

$$v_1^* = \frac{1}{2}(1, -ip^{-1}),\ v_2^* = \frac{1}{2}(1, ip^{-1}).$$

Thus we obtain the following formal solutions of (7.17):

$$\begin{aligned}\Psi_\pm &= \exp\left(\pm\frac{i\int pdx}{\hbar}\right)\left(\exp\left(-\frac{1}{2}\int p^{-1}p'dx\right) + O(\hbar)\right)\\ &= p^{-\frac{1}{2}}\exp\left(\pm\frac{i\int pdx}{\hbar}\right)(1 + O(\hbar)).\end{aligned}$$

We get

Theorem 7.33 (Local WKB approximation)**.** *Equation* (7.17) *has a basis of formal solutions*

$$\Psi_\pm(x) = (2(E - U(x))^{-\frac{1}{4}}\exp\left(\pm\frac{i\int\sqrt{2(E - U(x))}dx}{\hbar}\right)(1 + O(\hbar)).$$

With some analytic work, one can prove that these formal solutions are, in fact, asymptotic expansions of actual solutions when $\hbar \to 0$.

The WKB approximation can also be used to find asymptotic distribution of eigenvalues of a Schrödinger operator when it has discrete spectrum. Let us explain, somewhat informally, how this works.

As an example, consider the stationary Schrödinger equation (7.17) on the circle $\mathbb{R}/2\pi\mathbb{Z}$ with piecewise continuous 2π-periodic potential $U(x)$. We would like to write an asymptotic formula for the nth eigenvalue $E_n(\hbar)$ of the operator $H = -\frac{1}{2}\hbar^2\partial^2 + U(x)$ when $n \sim \frac{A}{\hbar}$ for a given constant A. This is equivalent to determining the number $\nu(E)$ of eigenvalues of H satisfying the inequality $\Lambda \leq E$ for a given constant E.

To this end, we will use Theorem 7.33. Assume first that

$$E > \sup U(x).$$

The periodicity condition for the solutions $\Psi_\pm$ in Theorem 7.33 (called the *quantization condition* in quantum mechanics) in the zeroth approximation is that

$$\int_0^{2\pi} \sqrt{2(E - U(x))}dx = 2\pi n\hbar,\ n \in \mathbb{Z}_{\geq 0}. \tag{7.18}$$

It follows that if (7.18) holds, then the number of eigenvalues of H which are $\leq E$ is about $2n$. So we get

Proposition 7.34.

$$\nu(E) \sim \frac{A(E)}{\hbar},\ \hbar \to 0,\ \textit{where}\ A(E) := \frac{1}{\pi}\int_0^{2\pi} \sqrt{2(E - U(x))}dx.$$

Thus for sufficiently large A, we have

$$E_{[\frac{A}{\hbar}]}(\hbar) \sim E(A),$$

where $E(A)$ is the solution of the equation

$$A = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2(E - U(x))}dx.$$

Note that $A(E)$ is the area of the region in the classical phase space $T^*S^1 = S^1 \times \mathbb{R}$ defined by the inequality

$$H_{\text{cl}} \leq E,$$

where $H_{\text{cl}} := \frac{1}{2}p^2 + U(x)$ is the corresponding classical Hamiltonian. Moreover, one can show that with this definition of $A(E)$, the formula

$$\nu(E) \sim \frac{A(E)}{\hbar}$$

in fact holds in a much larger generality, whenever H has discrete spectrum (namely, for the operator $-\frac{1}{2}\hbar^2\Delta + U(x)$ on any compact Riemannian manifold, or even on a noncompact one when one has $U(x) \to +\infty$ as $x \to \infty$). This formula is known as the *Weyl law*.

Exercise 7.35. Prove the Weyl law on the circle for $E \leq \sup U(x)$.

Finally, let $U(x) := MU_0(x)$, where U_0 is a fixed potential, and consider the asymptotics of eigenvalues for small M, assuming that $\hbar \ll M$ (i.e., $\frac{1}{n} \ll M$). In this case we can write equation (7.18) as

$$\sqrt{2E}\int_0^{2\pi}\left(1-\frac{MU_0(x)}{2E}-\frac{M^2U_0(x)^2}{8E^2}+o(M^2)\right)dx = 2\pi n\hbar. \tag{7.19}$$

As before, we assume without loss of generality that $\int_0^{2\pi} U_0(x)dx = 0$. Let $I := \frac{1}{2\pi}\int_0^{2\pi} U_0(x)^2dx$. Then we obtain

$$\sqrt{2E} = n\hbar + \frac{M^2I}{2(2E)^{\frac{3}{2}}} + o(M^2) = n\hbar(1+\frac{M^2I}{2n^4\hbar^4}+\dots). \tag{7.20}$$

It follows that

$$E = \tfrac{1}{2}n^2\hbar^2(1+\tfrac{M^2I}{n^4\hbar^4}+\dots) = \Lambda_n + \frac{M^2I}{8\Lambda_n}+\dots.$$

This gives the first correction of the eigenvalue $\Lambda_n := \frac{1}{2}n^2\hbar^2$ as we turn on M.

For example, if $U(x)$ is given by (7.11), then $I = a(2\pi - a)$, and we recover the asymptotics (7.15) without the last (bifurcation) term (which is negligible compared to $\frac{M^2a(2\pi-a)}{8\Lambda_n}$ in the range $\frac{1}{n} \ll M$).

Chapter 8

Fermionic integrals

8.1. Bosons and fermions

In physics there exist two kinds of particles—bosons and fermions. So far we have dealt with bosons only, but many important particles are fermions, e.g., electron, proton, etc. Thus it is important to adapt our techniques to the fermionic case.

In quantum theory, the difference between bosons and fermions is as follows: if the space of states of a single particle is $\mathcal{H}$, then the space of states of the system of k such (indistinguishable) particles is the symmetric power $S^k\mathcal{H}$ for bosons and the exterior power $\Lambda^k\mathcal{H}$ for fermions. In particular, in the fermionic case, if $\dim \mathcal{H} = n$, then the space of states of $\geq n+1$ identical particles is zero—the *Pauli exclusion principle* (leading, for instance, to the fact that the number of electrons in an atom at the mth energy level is bounded above by $2m^2$). In classical theory, this means that the space of states of a bosonic particle is a usual real vector space (or, more generally, a manifold), while for a fermionic particle it is an *odd vector space*. Mathematically "odd" means that the algebra of smooth functions on this space (i.e., the algebra of classical observables) is an *exterior* algebra (unlike the case of a usual, *even* space, for which the algebra of polynomial functions is a *symmetric* algebra).

More generally, one may consider systems of classical particles or fields, some of which are bosonic and some fermionic. In this case, the space of states will be a *supervector space*, i.e., the direct sum of an even and an odd space (or, more generally, a *supermanifold*—a notion we will define below).

When such a theory is quantized using the path integral approach, one has to integrate functions over supermanifolds. Thus, we should learn to integrate

over supermanifolds and then generalize to this case our Feynman diagram techniques. This is what we do in this chapter.

8.2. Supervector spaces

2 Let k be a field of characteristic zero.[17] A *supervector space* (or briefly, superspace) over k is just a $\mathbb{Z}/2$-graded vector space: $V = V_0 \oplus V_1$. If $V_0 = k^n$ and $V_1 = k^m$, then V is denoted by $k^{n|m}$. The notions of an (even) linear map (=operator), direct sum, tensor product, and dual space for supervector spaces are defined in the same way as for $\mathbb{Z}/2$-graded vector spaces. In other words, the tensor category of supervector spaces is the same as that of $\mathbb{Z}/2$-graded vector spaces.

However, the notions of a supervector space and a $\mathbb{Z}/2$-graded vector space are **not** the same. The difference is as follows. The category of vector (and hence $\mathbb{Z}/2$-graded vector) spaces has a *symmetric structure*, which is the standard isomorphism $V \otimes W \to W \otimes V$ (given by $v \otimes w \to w \otimes v$). This isomorphism allows one to define symmetric powers $S^i V$, exterior powers $\Lambda^i V$, etc. For supervector spaces, there is also a symmetry $V \otimes W \to W \otimes V$, but it is defined differently. Namely, $v \otimes w$ goes to $(-1)^{ij} w \otimes v$, $v \in V_i, w \in V_j$ $(i, j \in \{0, 1\})$. In other words, it is the same as usual except that if v, w are both odd, then $v \otimes w \mapsto -w \otimes v$. As a result, we can define the superspaces $S^i V$ and $\Lambda^i V$ for a superspace V, but they are not the same as the symmetric and exterior powers in the usual sense. For example, if V is purely odd ($V = V_1$), then $S^i V$ is the ith exterior power of the usual space V, and $\Lambda^i V$ is the ith symmetric power of the usual space V (purely even for even i and purely odd for odd i). Thus in general for $V = V_0 \oplus V_1$, we have the following expressions for the symmetric algebra $SV := \oplus_{i \geq 0} S^i V$ and exterior algebra $\Lambda V := \oplus_{i \geq 0} \Lambda^i V$:

$$SV = SV_0 \otimes \Lambda V_1,\ \Lambda V = \Lambda V_0 \otimes SV_1.$$

For a superspace V, let ΠV be the same space with opposite parity, i.e., $(\Pi V)_j = V_{1-j}$, $j = 0, 1$. An *odd linear map* of superspaces $V \to W$ is a usual (even, i.e., degree-preserving) linear map $V \to \Pi W$ (or $\Pi V \to W$). We have

$$S^i V = \Pi^i(\Lambda^i \Pi V),\ \Lambda^i V = \Pi^i(S^i \Pi V).$$

Let $V = V_0 \oplus V_1$ be a finite-dimensional superspace. Define the algebra of polynomial functions on V, $\mathcal{O}(V)$, to be the algebra SV^* (where symmetric powers are taken in the supersense). Thus, $\mathcal{O}(V) = SV_0^* \otimes \Lambda V_1^*$, where V_0 and V_1 are regarded as usual spaces. More explicitly, if $x_1, \ldots, x_n$ are linear coordinates on V_0, and $\xi_1, \ldots, \xi_m$ are linear coordinates on V_1, then $\mathcal{O}(V) =$

[17] In the physical context, $k = \mathbb{R}$ or $\mathbb{C}$, so we restrict ourselves to fields of characteristic zero. However, the theory extends mutatis mutandis to fields of any characteristic $\neq 2$.

$k[x_1, \dots, x_n, \xi_1, \dots, \xi_m]$, with defining relations

$$x_i x_j = x_j x_i,\ x_i \xi_r = \xi_r x_i,\ \xi_r \xi_s = -\xi_s \xi_r$$

(in particular, $\xi_r^2 = 0$). Note that this algebra is itself a (generally, infinite-dimensional) supervector space, and it is commutative in the supersense. Also, if V, W are two superspaces, then $\mathcal{O}(V \oplus W) = \mathcal{O}(V) \otimes \mathcal{O}(W)$, where the tensor product of algebras is understood in the supersense, i.e.

$$(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd),$$

where $p(x)$ is the parity of x.

8.3. Supermanifolds

Now assume that $k = \mathbb{R}$. Then by analogy with the above for any supervector space V we can define the algebra of smooth functions, $C^\infty(V) := C^\infty(V_0) \otimes \Lambda V_1^*$. In fact, this is a special case of the following more general setting.

Definition 8.1. A *supermanifold* M is a usual manifold M_0 with a sheaf C^∞_M of $\mathbb{Z}/2\mathbb{Z}$ graded algebras (called the *structure sheaf*), which is locally isomorphic to $C^\infty_{M_0} \otimes \Lambda(\xi_1, \dots, \xi_m)$. The algebra of global sections of the sheaf C^∞_M is denoted by $C^\infty(M)$ and is called the algebra of smooth functions on M.

The manifold M_0 is called the *reduced manifold* of M. The dimension of M is the pair of integers $n|m$, where $n = \dim M_0$.

For example, a supervector space V is a supermanifold of dimension $\dim V_0 | \dim V_1$. Another (more general) example of a supermanifold is a superdomain $U := U_0 \times V_1$, i.e., a domain $U_0 \subset V_0$ together with the sheaf $C^\infty_{U_0} \otimes \Lambda V_1^*$. Moreover, the definition of a supermanifold implies that any supermanifold is "locally isomorphic" to a superdomain.

Let M be a supermanifold. An *open set* U in M is the supermanifold $(U_0, C^\infty_M|_{U_0})$, where U_0 is an open set in M_0.

By the definition, supermanifolds form a category $\mathcal{S}$. Let us describe explicitly morphisms in this category, i.e., maps $F : M \to N$ between supermanifolds M and N. By the definition, it suffices to assume that M, N are superdomains, with global coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_m$, and $y_1, \dots, y_p, \eta_1, \dots, \eta_q$, respectively (here x_i, y_i are even variables, and ξ_i, η_i are odd variables). Then the map F is defined by the formulas

$$y_i = f_{0,i}(x_1, \dots, x_n) + f^{j_1 j_2}_{2,i}(x_1, \dots, x_n)\xi_{j_1}\xi_{j_2} + \dots,$$

$$\eta_i = a^j_{1,i}(x_1, \dots, x_n)\xi_j + a^{j_1 j_2 j_3}_{3,i}(x_1, \dots, x_n)\xi_{j_1}\xi_{j_2}\xi_{j_3} + \dots,$$

where $f_{0,i}, f^{j_1 j_2}_{2,i}, \dots, a^j_{1,i}, a^{j_1 j_2 j_3}_{3,i}, \dots$ are usual smooth functions, and we assume summation over repeated indices. These formulas determine F completely since for any $g \in C^\infty(N)$ one can find $g \circ F \in C^\infty(M)$ by Taylor's formula.

For example, if $M = N = \mathbb{R}^{1|2}$, $F(x, \xi_1, \xi_2) = (x + \xi_1\xi_2, \xi_1, \xi_2)$, and $g = g(x)$, then

$$g \circ F(x, \xi_1, \xi_2) = g(x + \xi_1\xi_2) = g(x) + g'(x)\xi_1\xi_2.$$

Finally, we have a notion of a Cartesian product of supermanifolds, which is completely analogous to the notion of a Cartesian product of manifolds, so we omit the definition.

8.4. Supermanifolds and vector bundles

Let M_0 be a manifold, and let E be a real vector bundle on M_0. Then we can define the supermanifold $M := \mathrm{Tot}(\Pi E)$, the total space of E with changed parity. Namely, the reduced manifold of M is M_0, and the structure sheaf C^∞_M is the sheaf of sections of ΛE^*, where E^* is the dual vector bundle to E. This gives rise to a functor $S : \mathcal{B} \to \mathcal{S}$, from the category of manifolds with vector bundles to the category of supermanifolds. We also have a functor S_* in the opposite direction; namely, $S_*(M)$ is the manifold M_0 with the vector bundle corresponding to the $C^\infty_{M_0}$-module dual to I/I^2, where I is the nilpotent radical of C^∞_M.

Proposition 8.2 (whose proof we leave as an exercise) gives a classification of supermanifolds, showing that they are all obtained from vector bundles on ordinary manifolds.

Proposition 8.2.

(i) $S_* \circ S = \mathrm{Id}$;

(ii) $S \circ S_* = \mathrm{Id}$ *on isomorphism classes of objects.*

The usefulness of this proposition is limited by the fact that, as one can see from the above description of maps between supermanifolds, $S \circ S_*$ is **not** the identity on morphisms (e.g., it maps the automorphism $x \to x + \xi_1\xi_2$ of $\mathbb{R}^{1|2}$ to Id), and it could not possibly have been so since S, S_* are not equivalences of categories (namely, S is not full and S_* is not faithful). In fact, the category of supermanifolds is not equivalent to the category of manifolds with vector bundles (namely, the category of supermanifolds "has more morphisms").

Remark 8.3.

1. The relationship between these two categories is quite similar to the relationship between the categories of (finite-dimensional) filtered and graded vector spaces, respectively. For them we also have functors S, S_* with the same properties; namely S is the functor of viewing a graded space as a filtered space, and S_* is the functor of associated graded (check it!). Therefore in supergeometry, it is better to avoid using realizations of supermanifolds as $S(M_0, E)$, similarly to how in linear algebra it is better to avoid choosing a splitting of a filtered vector space.

2. In the definition of a supermanifold one can replace the real exterior algebra $\Lambda(\xi_1,\dots,\xi_m)$ with the complexified exterior algebra $\Lambda_{\mathbb{C}}(\xi_1,\dots,\xi_m)$. This gives a notion of a $\mathbb{C}$-supermanifold, which generalizes the notion of an ordinary smooth manifold with the sheaf of complex-valued (as opposed to real-valued) smooth functions. Similarly to Proposition 8.2, isomorphism classes of $\mathbb{C}$-supermanifolds with reduced submanifolds M_0 are in bijection with isomorphism classes of *complex* vector bundles on M_0, so they are more general (as not every complex vector bundle is the complexification of a real one). Otherwise, the theory of $\mathbb{C}$-supermanifolds (which does actually arise in quantum field theory, see Remark 10.4) is completely parallel to the theory of usual supermanifolds.

One may also similarly define complex analytic and algebraic supermanifolds.

8.5. Supertrace and superdeterminant (Berezinian)

Before proceeding further, we need to generalize to the supercase the basic notions of linear algebra, such as trace and determinant of a matrix.

Let $R := R_0 \oplus R_1$ be a supercommutative $\mathbb{C}$-algebra. Fix two nonnegative integers m, n. Let $\mathrm{Mat}_{n|m}(R)$ be the algebra of $n+m$ by $n+m$ matrices over R which have the block decomposition

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

so that A_{00} is n by n, A_{11} is m by m, and A_{00}, A_{11} have even entries (i.e., in R_0), while A_{01}, A_{10} have odd entries (i.e., in R_1). We would like to define the *supertrace* of A as a linear function

$$\mathrm{sTr}(A) = \sum_{i,j=1}^{n+m} \lambda_{ij} a_{ij},\ \lambda_{ij} \in \mathbb{Z},$$

so that $\mathrm{sTr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = n$ and $\mathrm{sTr}(AB) = \mathrm{sTr}(BA)$ for any R and $A, B \in \mathrm{Mat}_{n|m}(R)$. Thus we must have $\mathrm{sTr}(A) = \mathrm{Tr}(A_{00}) + \varepsilon \mathrm{Tr}(A_{11})$ for some $\varepsilon \in \mathbb{Z}$, and taking all blocks of A, B except A_{01}, B_{10} to be zero, we get $\varepsilon = -1$. So the supertrace of A has to be defined by the formula[18]

$$\mathrm{sTr}(A) = \mathrm{Tr}(A_{00}) - \mathrm{Tr}(A_{11}).$$

Now let us generalize to the supercase the definition of determinant. For a finite-dimensional algebra R and $C \in \mathrm{Mat}_{n|m}(\mathbb{R})$, we would like to have

$$\mathrm{sdet}(e^C) = e^{\mathrm{sTr}\, C} = e^{\mathrm{Tr}(C_{00}) - \mathrm{Tr}(C_{11})}, \tag{8.1}$$

[18] Physicists write the supertrace of A as $\mathrm{Tr}((-1)^F A)$, where F denotes the parity operator.

which generalizes the usual property of trace and determinant. So in the case of a block-diagonal matrix $C = C_{00} \oplus C_{11}$ we get

$$\operatorname{sdet}(e^C) = \frac{\det(e^{C_{00}})}{\det(e^{C_{11}})}.$$

Thus if $A = A_{00} \oplus A_{11}$ is block-diagonal, we must have

$$\operatorname{sdet} A = \frac{\det A_{00}}{\det A_{11}}.$$

This shows that we cannot hope that the superdeterminant will be a polynomial in the entries of A—it has to be a rational function defined only on some open subset. In fact, if we want to have the usual property $\operatorname{sdet}(AB) = \operatorname{sdet}(A)\operatorname{sdet}(B)$, then there is just one possibility. Indeed, suppose that

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} a_+ + ba_-c & ba_- \\ a_-c & a_- \end{pmatrix}.$$

By (8.1), we must have

$$\operatorname{sdet} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \operatorname{sdet} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = 1,$$

hence

$$\operatorname{sdet}(A) = \frac{\det a_+}{\det a_-}.$$

In other words, the superdeterminant has to be defined by the formula

$$\operatorname{sdet}(A) = \frac{\det(A_{00} - A_{01}A_{11}^{-1}A_{10})}{\det(A_{11})},$$

provided that A_{11} is invertible; otherwise the superdeterminant is not defined.

This function is also called the *Berezinian* of A and denoted $\operatorname{Ber}(A)$. So for $m = 0$ one has $\operatorname{Ber}(A) = \det(A)$, and for $n = 0$ one has $\operatorname{Ber}(A) = (\det A)^{-1}$.

Remark 8.4. Recall for comparison that if A is a purely even block matrix, then

$$\det(A) = \det(A_{00} - A_{01}A_{11}^{-1}A_{10}) \det(A_{11}).$$

Proposition 8.5.

(i) *For any $A, B \in \operatorname{Mat}_{n|m}(R)$ with A_{11}, B_{11} invertible, we have*

$$\operatorname{Ber}(AB) = \operatorname{Ber}(A) \operatorname{Ber}(B).$$

(ii) *If R is finite dimensional and $A(t) \in \operatorname{Mat}_{n|m}(R)$ is a C^1-function near 0 with $A(0)$ invertible, then*

$$\tfrac{d}{dt}|_{t=0} \operatorname{Ber}(A(t)) = \operatorname{sTr}(A'(0)A(0)^{-1}) \operatorname{Ber}(A(0)).$$

(iii) *If R is finite dimensional, then for any $C \in \operatorname{Mat}_{n|m}(R)$ we have*

$$\operatorname{Ber}(e^C) = e^{\operatorname{sTr} C}.$$

Proof. (i) From the triangular factorization, it is clear that it suffices to consider the case

$$A = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix},$$

where X, Y are matrices with odd elements, so that

$$AB = \begin{pmatrix} 1 & Y \\ X & 1+XY \end{pmatrix}.$$

Then the required identity is

$$\det(1 - Y(1+XY)^{-1}X) = \det(1+XY).$$

To prove this identity, recall that $X : V_0 \to V_1 \otimes R$ and $Y : V_1 \to V_0 \otimes R$. We have

$$\begin{aligned}
\det(1 - Y(1+XY)^{-1}X) &= \sum_{k\geq 0} (-1)^k \operatorname{Tr}(Y(1+XY)^{-1}X|_{\Lambda^k V_0}) \\
&= \sum_{k\geq 0} (-1)^k \operatorname{sTr}(Y(1+XY)^{-1}|_{\Lambda^k V_1} \circ X|_{\Lambda^k V_0}) \\
&= \sum_{k\geq 0} (-1)^k \operatorname{sTr}(XY(1+XY)^{-1}|_{\Lambda^k V_1}) \\
&= \sum_{k\geq 0} \operatorname{Tr}(XY(1+XY)^{-1}|_{S^k \Pi V_1}) \\
&= \det(1 - XY(1+XY)^{-1})^{-1} = \det(1+XY).
\end{aligned}$$

(ii) By (i) we may replace $A(t)$ by $A(t)A(0)^{-1}$, so it suffices to consider the case $A(0) = 1$, where the statement easily follows from the definition.

(iii) Consider the function $f(t) := \operatorname{Ber}(e^{Ct})$. By (ii) it satisfies the differential equation $f'(t) = \operatorname{sTr}(C)f(t)$ with $f(0) = 1$. Thus $f(t) = e^{\operatorname{sTr}(C)t}$, and the statement follows by setting $t = 1$. □

8.6. Lie supergroups and superalgebras

Recall that symmetries of manifolds are described by Lie groups and the corresponding infinitesimal symmetries by their Lie algebras. Thus to understand symmetries of supermanifolds, we must generalize these notions to the supercase, which we do in this section.

The notion of a Lie supergroup and of its action on a supermanifold is defined completely analogously to that in usual geometry.

Definition 8.6. A (real or complex) *Lie supergroup* is a group object in the category of (real or complex) supermanifolds, i.e., a supermanifold equipped with a group structure such that the multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are regular maps of supermanifolds.

An *action* of a Lie supergroup G on a supermanifold M is a smooth map $a : G \times M \to M$ such that

$$a \circ (\mathrm{id}_G \times a) = a \circ (m \times \mathrm{id}_M) : G \times G \times M \to M.$$

Likewise, the definition of a Lie superalgebra is analogous to that of a Lie algebra.

Definition 8.7. A *Lie superalgebra* over a field k of characteristic zero is a supervector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a bilinear operation $[\cdot, \cdot]$ called the *super Lie bracket* such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j \bmod 2}$, satisfying the following two properties:

1. *Super skew-symmetry*: for homogeneous elements $X, Y \in \mathfrak{g}$,

$$[X, Y] = -(-1)^{p(X)p(Y)}[Y, X];$$

2. *Super Jacobi identity*: for homogeneous elements $X, Y, Z \in \mathfrak{g}$,

$$(-1)^{p(X)p(Z)}[X, [Y, Z]] + (-1)^{p(Y)p(X)}[Y, [Z, X]] + (-1)^{p(Z)p(Y)}[Z, [X, Y]] = 0.$$

In particular, we see that if $\mathfrak{g}$ is a Lie superalgebra, then the even part $\mathfrak{g}_0$ is a Lie algebra in the classical sense, while the odd part $\mathfrak{g}_1$ forms a module over $\mathfrak{g}_0$.

Similar to usual Lie algebras, natural examples of Lie superalgebras arise from $\mathbb{Z}_2$-graded algebras and their derivations. Namely, let $A = A_0 \oplus A_1$ be such an algebra. Then, first of all, A has a natural structure of a Lie superalgebra defined by $[a, b] = ab - (-1)^{p(a)p(b)}ba$ for homogeneous $a, b \in A$ (the super-commutator). In particular, if V is a supervector space, then we have the Lie superalgebra $\mathfrak{gl}(V) = \mathrm{End}(V)$ (with the grading induced by the grading on V).

Next, define an *even derivation* of A to be an ordinary degree-preserving derivation, i.e., an even linear map $Q : A \to A$ such that one has $Q(ab) = Q(a)b + aQ(b)$. On the other hand, define an *odd derivation* of A to be an odd linear map $Q : A \to A$ such that for homogeneous a, b one has $Q(ab) = Q(a)b + (-1)^{p(a)}aQ(b)$ (for example, the de Rham differential on an ordinary manifold is an odd derivation of its algebra of differential forms, graded by degree modulo 2). Let $\mathfrak{g}_0, \mathfrak{g}_1$ be the spaces of even and odd derivations of A, respectively. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \subset \mathrm{End}(A)$ is closed under the super-commutator, i.e., it is a Lie superalgebra with

$$[X, Y] := XY - (-1)^{p(X)p(Y)}YX$$

for homogeneous X, Y (check it!). The Lie superalgebra $\mathfrak{g}$ is called the *Lie algebra of super-derivations* of A, denoted $\mathrm{sDer}(A)$.

For example, if $A = C^\infty(M)$ for a supermanifold M then a super-derivation of A is called a vector field on M, and such vector fields form a Lie superalgebra

$\mathrm{Vect}(M)$. As in usual geometry, in local coordinates vector fields are written as

$$v = \sum_i f_i \partial_{x_i} + \sum_j g_j \partial_{\xi_j},$$

where $f_i, g_j \in C^\infty(M)$.

By analogy with the theory of usual Lie groups, one can define the Lie algebra of a Lie supergroup.

Definition 8.8. The *Lie algebra* $\mathrm{Lie}(G)$ of a Lie supergroup G is the Lie superalgebra $\mathrm{sDer}(C^\infty(G))^{G_{\text{left}}}$ of left-invariant superderivations of $C^\infty(G)$.[19]

As in classical theory, if $\mathfrak{m} \subset C^\infty(G)$ is the maximal ideal of $1 \in G$, then an element $X \in \mathfrak{g} := \mathrm{Lie}(G)$ defines a linear functional on $\mathfrak{m}/\mathfrak{m}^2$, i.e., it can be viewed an element of the tangent space $T_1 G$, which defines an isomorphism of supervector spaces $\mathfrak{g} \cong T_1 G$. Also, as in the even case, we have the exponential map $\exp : \mathfrak{g} \to G$ invertible near $0 \in \mathfrak{g}$, and the inverse is the logarithm map $\log : U \to \mathfrak{g}$ where U is a neighborhood of 1 in G.

The three fundamental theorems of Lie theory generalize *mutatis mutandis* to Lie supergroups, resulting in an equivalence between categories of simply connected Lie supergroups (i.e., ones with simply connected reduced part) and finite-dimensional Lie superalgebras (both in the real and the complex case). Namely, they follow in a fairly straightforward way from the corresponding classical theorems.

Here are some examples of Lie supergroups and superalgebras.

Example 8.9.

1. A finite-dimensional supervector space V is an abelian Lie supergroup, with Lie superalgebra being the same space V with zero bracket.

2. The general linear supergroup $\mathrm{GL}(m|n)(\mathbb{R})$, where $m, n \in \mathbb{Z}_{\geq 0}$. To construct it, consider the algebra $A := \mathrm{End}(\mathbb{R}^{m|n})$ of (not necessarily homogeneous) endomorphisms of the supervector space $\mathbb{R}^{m|n}$. This is just the matrix algebra $\mathrm{Mat}_{m+n}(\mathbb{R})$, but it carries a $\mathbb{Z}_2$-grading in which $\deg(E_{ij}) = 0$ if $i, j \leq m$ or $i, j > m$ and $\deg(E_{ij}) = 1$ if $i \leq m < j$ or $j \leq m < i$. Thus A is a supermanifold (of superdimension $(m^2 + n^2|2mn)$, with reduced part $A_0 = \mathrm{Mat}_m(\mathbb{R}) \times \mathrm{Mat}_n(\mathbb{R})$) equipped with an associative unital product $A \times A \to A$. The inversion map i, however, is only defined on the open subset $A^\times \subset A$ whose reduced part is the open subset $A_0^\times = \mathrm{GL}(m, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ (and maps this subset to itself). Luckily, the multiplication map $A \times A \to A$ restricts to a map $A^\times \times A^\times \to A^\times$, which implies that $A^\times$ is a real Lie supergroup, denoted $\mathrm{GL}(m|n)(\mathbb{R})$. The Lie algebra $\mathrm{Lie}\,\mathrm{GL}(m|n)(\mathbb{R}) = \mathfrak{gl}(m|n)(\mathbb{R})$ is simply

[19] Here if G is a real Lie group, then $C^\infty(G)$ stands for the algebra of real-valued smooth functions, while if G is a complex Lie group, then it denotes the *sheaf* of holomorphic functions on G.

the algebra A equipped with the super-commutator. The complex supergroup $\mathrm{GL}(m|n)(\mathbb{C})$ and its Lie algebra $\mathfrak{gl}(m|n)(\mathbb{C})$ are defined similarly.

3. The supersymmetry group $\mathcal{Q}$ in dimension 1. This supergroup is the supermanifold $\mathbb{R}^{1|1}$ with the operation given in coordinates by

$$(t_1, \theta_1)(t_2, \theta_2) = (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)$$

(it is easy to check that this operation is associative, although noncommutative). The Lie algebra $\mathrm{Lie}\,\mathcal{Q}$ is then $\mathbb{R}^{1|1}$ with basis $H = (1, 0)$ (even) and $Q = (0, 1)$ (odd) with $[Q, H] = 0$, $[Q, Q] = 2H$. The right translation action of $\mathcal{Q}$ on functions is given by

$$((x, \xi) \circ f)(t, \theta) = f((t, \theta)(x, \xi)) = f(t + x + \theta\xi, \theta + \xi),$$

which means that the corresponding action of the Lie algebra is given by

$$H \mapsto \frac{\partial}{\partial t},\ Q \mapsto \frac{\partial}{\partial \theta} + \theta\frac{\partial}{\partial t}.$$

Similarly, the left translation action is

$$((x, \xi) * f)(t, \theta) = f((x, \xi)^{-1}(t, \theta)) = f(t - x + \theta\xi, \theta - \xi),$$

so infinitesimally

$$H \mapsto -\frac{\partial}{\partial t},\ Q \mapsto -\frac{\partial}{\partial \theta} + \theta\frac{\partial}{\partial t}.$$

Clearly, these two actions commute.

Again, the complex versions are defined similarly.

8.7. Integration on superdomains

We would now like to develop integration theory on supermanifolds. Before doing so, let us recall how it is done for usual manifolds. In this case, one proceeds as follows.

1. Define integration of compactly supported (say, smooth) functions on a domain in $\mathbb{R}^n$.

2. Find the transformation formula for the integral under change of coordinates (i.e., discover the factor $|J|$, where J is the Jacobian).

3. Define a *density* on a manifold to be a quantity which is locally the same as a function, but that multiplies by $|J|$ under coordinate change (unlike true functions, which do not multiply by anything). Then define the integral of compactly supported densities on the manifold using partitions of unity. The independence of the integral on the choices is guaranteed by the change of variable formula and the definition of a density.

We will now realize this program for supermanifolds. We start with defining integration over superdomains.

Let $V = V_0 \oplus V_1$ be a supervector space. The *Berezinian* of V is the line $\Lambda^{\mathrm{top}}V_0^* \otimes \Lambda^{\mathrm{top}}V_1$ (where V_0, V_1 are treated as usual spaces). Suppose that V is

equipped with a nonzero element dv of its Berezinian (called a *supervolume element*).

Let U_0 be an open set in V_0, and let $f \in C^\infty(U_0) \otimes \Lambda V_1^*$ be a compactly supported smooth function on the superdomain $U := U_0 \times V_1$, i.e.

$$f = \sum f_i \otimes \omega_i,\ f_i \in C^\infty(U_0),\ \omega_i \in \Lambda V_1^*,$$

and f_i are compactly supported. Let dv_0, dv_1 be volume forms on V_0, V_1 such that $dv = dv_0/dv_1$.

Definition 8.10. The integral $\int_U f(v)dv$ is $\int_{U_0}(f(v), (dv_1)^{-1})dv_0$.

It is clear that this quantity depends only on dv and not on dv_0 and dv_1 separately.

Thus, $\int_U f(v)dv$ is defined as the integral of the suitably normalized top coefficient of f (expanded with respect to some homogeneous basis of ΛV_1^*). To write it in coordinates, let $x_1, \dots, x_n, \xi_1, \dots, \xi_m$ be a linear system of coordinates on V such that $dv = \frac{dx_1 \dots dx_n}{d\xi_1 \dots d\xi_m}$ (such coordinate systems will be called unimodular with respect to dv). Then $\int_U f(v)dv$ equals $\int_{U_0} f_{\text{top}}(x_1, \dots, x_n)dx_1 \dots dx_n$, where f_{top} is the coefficient of $\xi_1 \dots \xi_m$ in the expansion of f.

In coordinates this reduces to the convention

$$\int_{\mathbb{R}^{0|1}} \xi(d\xi)^{-1} = 1, \tag{8.2}$$

and more generally

$$\int_{\mathbb{R}^{0|m}} \xi_1 \dots \xi_m (d\xi_1 \dots d\xi_m)^{-1} = \int_{\mathbb{R}^{0|m}} \xi_1 \dots \xi_m (d\xi_m)^{-1} \dots (d\xi_1)^{-1} = 1 \tag{8.3}$$

(applying (8.2) repeatedly).[20]

8.8. Berezin's change of variable formula

Let V be a vector space, and let $f \in \Lambda V^*$ and $v \in V$. Denote by $\frac{\partial f}{\partial v}$ the result of contraction of f with v.

Let U, U' be superdomains, and let $F : U \to U'$ be a morphism. As explained above, given linear coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_m$ on U and $y_1, \dots, y_p, \eta_1, \dots, \eta_q$ on U', we can describe F by expressing y_i and η_j as functions of x_i and ξ_j. Define the *Berezin matrix* of $F, A := DF(x, \xi)$ by the formulas,

$$A_{00} = (\frac{\partial y_i}{\partial x_k}),\ A_{01} = (\frac{\partial y_i}{\partial \xi_\ell}),\ A_{10} = (\frac{\partial \eta_j}{\partial x_k}),\ A_{11} = (\frac{\partial \eta_j}{\partial \xi_\ell}).$$

[20] In the notation of [**QFS**], as is conventional in physics literature, integral (8.2) would be written as $\int \xi d\xi$; see [**QFS**, v. 1, p. 363]. However, it seems that the notation involving the inverse of $d\xi$ (not used in physics literature) is more suggestive: it reminds us that $d\xi$ scales reciprocally to ξ, and explains why (8.3) does not violate the sign rule for odd variables (inversion reverses the order of factors).

Clearly, this is a super-analogue of the Jacobi matrix.

The main theorem of supercalculus is the following theorem.

Theorem 8.11 (Berezin's change of variable formula)**.** *Let g be a smooth function with compact support on U', and let $F : U \to U'$ be an isomorphism. Let dv, dv' be supervolume elements on U, U'. Then*

$$\int_{U'} g(v')dv' = \int_U g(F(v))|\operatorname{Ber}(DF(v))|dv,$$

where the Berezinian is computed with respect to unimodular coordinate systems.

Here if $f(\xi) = a+$terms containing ξ_j, $a \in \mathbb{R}$, $a \neq 0$, then by definition $|f(\xi)| := f(\xi)$ if $a > 0$ and $|f(\xi)| := -f(\xi)$ if $a < 0$.

Proof. The chain rule of the usual calculus extends verbatim to supercalculus. Thus, by Proposition 8.5(i), if we know the statement for two isomorphisms $F_1 : U_2 \to U_1$ and $F_2 : U_3 \to U_2$, then we know it for the composition $F_1 \circ F_2$.

Let $F(x_1, \dots, x_n, \xi_1, \dots, \xi_m) = (x_1', \dots, x_n', \xi_1', \dots, \xi_m')$. We see that it suffices to consider the following cases.

1. x_i' depend only on x_k, $k = 1, \dots, n$, and $\xi_j' = \xi_j$.
2. $x_i' = x_i + z_i$, where z_i lie in the ideal generated by ξ_j, and $\xi_j' = \xi_j$.
3. $x_i' = x_i$.

Indeed, it is clear that any isomorphism F is a composition of isomorphisms of types 1, 2, 3.

In case 1, the statement of the theorem follows from the usual change of variable formula. Thus it suffices to consider cases 2 and 3.

In case 2, it is sufficient to consider the case when only one coordinate is changed by F, i.e. $x_1' = x_1 + z$, and $x_i' = x_i$ for $i \geq 2$. In this case we have to show that the integral of

$$g(x_1 + z, x_2, \dots, x_n, \xi)(1 + \tfrac{\partial z}{\partial x_1}) - g(x_1, x_2, \dots, x_n, \xi)$$

is zero. But this follows easily upon expansion in powers of z, since all the terms are manifestly total derivatives with respect to x_1.

In case 3, we can also assume $\xi_j' = \xi_j$, $j \geq 2$, and a similar (actually, even simpler) argument proves the result. □

8.9. Integration on supermanifolds

Now we will define densities on supermanifolds. Let M be a supermanifold, and let $\{U_\alpha\}$ be an open cover of M together with isomorphisms $f_\alpha : U_\alpha \to U_\alpha'$, where U_α' is a superdomain in $\mathbb{R}^{n|m}$. Let $g_{\alpha\beta} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ be the transition map $f_\alpha f_\beta^{-1}$. Then a density s on M is a choice of an

element $s_\alpha \in C^\infty_M(U_\alpha)$ for each α, such that on $U_\alpha \cap U_\beta$ one has $s_\beta(z) = s_\alpha(z)|\operatorname{Ber}(g_{\alpha\beta})(f_\beta(z))|$.

Remark 8.12. It is clear that a density on M is a global section of a certain sheaf on M, called the sheaf of densities.

Now, for any (compactly supported) density ω on M, the integral $\int_M \omega$ is well defined. Namely, it is defined as in usual calculus: one uses a partition of unity ϕ_α such that $\operatorname{Supp}\phi_\alpha \subset (U_\alpha)_0$ are compact subsets, and sets $\int_M \omega := \sum_\alpha \int_M \phi_\alpha\omega$ (where the summands can be defined using f_α). Then Berezin's theorem guarantees that the final answer will be independent of the choices made.

This also suggests how diffeomorphisms and vector fields on M should act on densities. Namely, for a diffeomorphism g of M we should set

$$(g \circ s(z)dz) := s(g(z))|\operatorname{Ber}(g)(z)|dz. \tag{8.4}$$

Indeed, this formula is forced by the requirement that for a compactly supported density ω on M, we have

$$\int_M \omega = \int_M g \circ \omega.$$

Differentiating (8.4) then yields a formula for the action on densities of the Lie superalgebra $\operatorname{Vect}(M)$, such that for $v \in \operatorname{Vect}(M)$ and compactly supported density ω on M we have

$$\int_M v \circ \omega = 0.$$

Exercise 8.13. Compute this action of $\operatorname{Vect}(M)$ on densities in local coordinates.

Exercise 8.14. Let X be an n-dimensional manifold, and let ΠTX be the supermanifold of superdimension (n, n) obtained by changing parity of fibers of the tangent bundle TX. Show that there is a canonical isomorphism $C^\infty(\Pi TX) = \Omega^\bullet(X)$, where $\Omega^\bullet(X) = \oplus_{j=0}^n \Omega^j(X)$ is the space of mixed degree differential forms on X (here $\Omega^j(X)$ is the space of differential forms of degree j which has parity j mod 2). Deduce that ΠTX carries a canonical odd vector field d given by the de Rham differential. Also show that if X is oriented, then ΠTX carries a canonical density dv such that for compactly supported $F \in C^\infty(\Pi TX)$,

$$\int_{\Pi TX} F dv = \int_X F_n,$$

where F_n is the degree n component of F as a mixed differential form.

8.10. Gaussian integrals in an odd space

Now let us generalize to the odd case the theory of Gaussian integrals, which was, in the even case, the basis for the path integral approach to quantum mechanics and field theory.

Recall first the notion of *Pfaffian*. Let A be a skew-symmetric matrix of even size. Then the determinant of A is the square of a polynomial in the entries of A. This polynomial is determined by this condition up to sign. The sign is usually fixed by requiring that the polynomial should be 1 for the direct sum of matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. With this convention, this polynomial is called the Pfaffian of A and denoted $\mathrm{Pf}A$. The Pfaffian obviously has the property $\mathrm{Pf}(X^TAX) = \mathrm{Pf}(A)\det(X)$ for any matrix X.

Now let V be a $2m$-dimensional vector space with a volume element dv, and let B be a skew-symmetric bilinear form on V. We define the Pfaffian $\mathrm{Pf}B$ of B to be the Pfaffian of the matrix of B in any unimodular basis (by the above transformation formula, it does not depend on the choice of the basis). It is easy to see (by reducing B to the canonical form) that

$$\frac{\Lambda^m B}{m!} = \mathrm{Pf}(B)dv.$$

In terms of matrices, this translates into the following (well-known) formula for the Pfaffian of a skew-symmetric matrix of size $2m$:

$$\mathrm{Pf}(A) = \sum_{\sigma\in\Pi_m} \varepsilon_\sigma \prod_{i\in\{1,\dots,2m\}, i<\sigma(i)} a_{i\sigma(i)},$$

where Π_m is the set of matchings of $\{1,\dots,2m\}$, and ε_σ is the sign of the permutation sending $1,\dots,2m$ to $i_1,\sigma(i_1),\dots,i_m,\sigma(i_m)$ (where $i_r < \sigma(i_r)$ for all r). For example, for $m = 2$ (i.e., a 4-by-4 matrix),

$$\mathrm{Pf}(A) = a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}.$$

Now consider an odd vector space V of dimension $2m$ with a volume element $d\xi$. Let B be a symmetric bilinear form on V (i.e. a skew-symmetric form on ΠV). Let $\xi_1,\dots,\xi_{2m}$ be unimodular linear coordinates on V (i.e. $d\xi = d\xi_1\wedge\dots\wedge d\xi_{2m}$). So if $\xi = (\xi_1,\dots,\xi_{2m})$, then $B(\xi,\xi) = \sum_{i,j} b_{ij}\xi_i\xi_j$, where (b_{ij}) is a skew-symmetric matrix.

Proposition 8.15.

$$\int_V e^{\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} = \mathrm{Pf}(B).$$

Proof. The integral equals $\frac{1}{m!}\frac{\wedge^m B}{d\xi}$, which is precisely $\mathrm{Pf}(B)$. □

This formula has the following important special case. Let Y be a finite-dimensional odd vector space, and let $V = Y \oplus Y^*$. The space V has a canonical volume element $dv = dydy^*$, defined as follows: if $e_1, \dots, e_m$ is a basis of Y and $e_1^*, \dots, e_m^*$ is the dual basis of Y^*, then

$$dydy^* = e_1 \wedge e_1^* \wedge \dots \wedge e_m \wedge e_m^* = e_1 \wedge \dots \wedge e_m \wedge e_m^* \wedge \dots \wedge e_1^*.$$

Let $A : Y \to Y$ be a linear operator. Then we can define an even smooth function S on the odd space $V = Y \oplus Y^*$ as follows: $S(y, y^*) = (Ay, y^*)$. More explicitly, if ξ_i are coordinates on Y corresponding to the basis e_i, and η_i the dual system of coordinates on Y^*, then

$$S(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_m) = \sum_{i,j} a_{ij}\xi_j\eta_i,$$

where (a_{ij}) is the matrix of A in the basis e_i.

Proposition 8.16.

$$\int_V e^S (dv)^{-1} = \det A.$$

Proof. We have $S(y, y_*) = \frac{1}{2}B((y, y_*), (y, y_*))$, where B is the symmetric form on V given by the formula

$$B((y, y^*), (w, w^*)) = (Ay, w^*) + (Aw, y^*).$$

It is easy to see that $\mathrm{Pf}(B) = (-1)^{\frac{m(m-1)}{2}} \det(A)$. On the other hand,

$$d\xi_1 d\eta_1 \dots d\xi_m d\eta_m = (-1)^{\frac{m(m-1)}{2}} d\xi_1 \dots d\xi_m d\eta_1 \dots d\eta_m.$$

Hence Proposition 8.16 follows from Proposition 8.15.

Another proof can be obtained by direct evaluation of the top coefficient. □

8.11. The Wick formula in the odd case

Let V be a $2m$-dimensional odd space with a volume form $d\xi$, and let $B \in S^2V^*$ be a nondegenerate form (symmetric in the supersense and antisymmetric in the usual sense). Let $\lambda_1, \dots, \lambda_n$ be linear functions on V. Then $\lambda_1, \dots, \lambda_n$ can be regarded as odd smooth functions on the superspace V.

Theorem 8.17.

$$\int_V \lambda_1(\xi) \dots \lambda_n(\xi) e^{-\frac{1}{2}B(\xi,\xi)} (d\xi)^{-1} = \mathrm{Pf}(-B)\,\mathrm{Pf}(B^{-1}(\lambda_i, \lambda_j)).$$

(*By definition, this is zero if n is odd.*) *In other words, we have*

$$\int_V \lambda_1(\xi)\dots\lambda_n(\xi)e^{-\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} = \mathrm{Pf}(-B)\sum_{\sigma\in\Pi_m}\varepsilon_\sigma\prod_{i\in\{1,\dots,2m\},i<\sigma(i)}B^{-1}(\lambda_i,\lambda_{\sigma(i)}).$$

Proof. We prove the second formula. Choose a basis e_i of V with respect to which the form B is standard: $B(e_j,e_l)=1$ if $j=2i-1, l=2i$, and $B(e_j,e_l)=0$ for other pairs $j<l$. Since both sides of the formula are polylinear with respect to $\lambda_1,\dots,\lambda_n$, it suffices to check it if $\lambda_1=e^*_{i_1},\dots,\lambda_n=e^*_{i_n}$. This is easily done by direct computation (in the sum on the right-hand side, only one term may be nonzero). □

Exercise 8.18. Let $Y=\mathbb{R}^{\frac{n(n+1)}{2}+\frac{m(m-1)}{2}|mn}$ be the real superspace of matrices

$$A=\begin{pmatrix}A_{00} & A_{01}\\ A_{10} & A_{11}\end{pmatrix}$$

(where A_{00} is n by n and A_{11} is m by m) which are symmetric in the supersense, i.e., A_{00} is symmetric, A_{11} is skew-symmetric, and $A_{01}^T=A_{10}$. Let $Y_+\subset Y$ be the superdomain of those matrices for which $A_{00}>0$. Let dA be a supervolume element on Y. Let f be a compactly supported smooth function on Y_+. Show that

$$\int_{Y_+\times\mathbb{R}^{n|m}} f(A)e^{-x^TA_{00}x-2x^TA_{01}\xi-\xi^TA_{11}\xi}dAdx(d\xi)^{-1} = C\int_{Y_+} f(A)\,\mathrm{Ber}(A)^{-1/2}dA$$

(C is a constant). What is C?

Exercise 8.19. Prove the Amitsur-Levitzki identity: if $X_1,\dots,X_{2n}$ are n-by-n matrices over a commutative ring, then

$$\sum_{\sigma\in S_{2n}}(-1)^\sigma X_{\sigma(1)}\dots X_{\sigma(2n)}=0.$$

Hint.

(a) Show that for any n-by-n matrix X with anticommuting entries, $X^{2n}=0$ (namely, show that traces of X^{2k} vanish for all positive k, then use the Cayley-Hamilton theorem for X^2).

(b) Apply this to $X=\sum_{i=1}^{2n}X_i\xi_i$, where ξ_i are anticommuting variables.

Chapter 9

Quantum mechanics for fermions

9.1. Feynman calculus in the supercase

Wick's theorem allows us to extend Feynman calculus to the supercase. Namely, let

$$V = V_0 \oplus V_1$$

be a finite-dimensional real superspace with a supervolume element $dv = dv_0(dv_1)^{-1}$ equipped with a symmetric nondegenerate form $B = B_0 \oplus B_1$ ($B_0 > 0$). Let

$$S(v) = \frac{1}{2}B(v,v) - \sum_{r\geq 3} \frac{B_r(v,v,\dots,v)}{r!}$$

be an even function on V (the action). Note that B_r, $r \geq 3$, can contain mixed terms involving both odd and even variables, e.g., $x\xi_1\xi_2$ (the so-called "Yukawa term"). We will consider the integral

$$I(\hbar) = \int_V \ell_1(v_0)\dots\ell_n(v_0)\lambda_1(v_1)\dots\lambda_p(v_1)e^{-\frac{S(v)}{\hbar}}dv,$$

where v_0, v_1 are the even and odd components of v. Then this integral has an expansion in $\hbar$ written in terms of Feynman diagrams. Since v has both odd and even part, these diagrams will contain "odd" and "even" edges (which are usually depicted by straight and wiggly lines, respectively). More precisely, let us write

$$B_r(v,v,\dots,v) = \sum_{s=0}^{r} \binom{r}{s} B_{s,r-s}(v_1,\dots,v_1,v_0,\dots,v_0),$$

where $B_{s,r-s}$ has homogeneity degree s with respect to v_1 and $r-s$ with respect to v_0 (i.e., it will be nonzero only for even s). Then to each term $B_{s,r-s}$ we assign an $(s, r-s)$-valent flower, i.e., a flower with s odd and $r-s$ even outgoing edges, and for the set of odd outgoing edges, specify which orderings are even. Then, given an arrangement of flowers, for every matching σ of outgoing edges, we can define an amplitude $\mathbb{F}(\sigma)$ by contracting the tensors $B_{s,r-s}$ (and being careful with the signs). It is easy to check that all matchings giving the same graph will contribute to $I(\hbar)$ with the same sign, and thus we have almost the same formula as in the bosonic case:

$$I(\hbar) = (2\pi)^{\frac{\dim V_0}{2}} \hbar^{\frac{\dim V_0 - \dim V_1}{2}} \frac{\operatorname{Pf}(-B_1)}{\sqrt{\det B_0}} \sum_{\Gamma} \frac{\hbar^{b(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} \mathbb{F}_\Gamma(\ell_1, \dots, \ell_n, \lambda_1, \dots, \lambda_p),$$

where the summation is taken over graphs with n even and p odd outgoing edges.

Remark 9.1. More precisely, we can define the sign ε_σ of a matching σ as follows: label outgoing edges by $1, 2, \dots$, starting from the first flower, then the second, etc., so that the labeling is even on each flower. Then write the labels in a sequence, enumerating (in any order) the pairs defined by σ (the element with the smaller of the two labels goes first). The sign ε_σ is by definition the sign of this ordering (as a permutation of $1, 2, \dots$). Then $\mathbb{F}_\Gamma$ is $\mathbb{F}(\sigma)$ for any matching σ yielding Γ which is *positive*, i.e., such that $\varepsilon_\sigma = 1$. For a negative matching, $\mathbb{F}_\Gamma = -\mathbb{F}(\sigma)$.

In most (but not all) situations considered in physics, the action is quadratic in the fermionic variables, i.e.

$$S(v) = S_b(v_0) - \tfrac{1}{2} S_f(v_0)(v_1, v_1),$$

where $S_f(v_0)$ is a skew-symmetric bilinear form on ΠV_1. In this case, using fermionic Wick's theorem, we can perform exact integration with respect to v_1, and reduce $I(\hbar)$ to a purely bosonic integral. For example, if we have only ℓ_i and no λ_i, then

$$I(\hbar) = \hbar^{-\frac{\dim V_1}{2}} \int_{V_0} \ell_1(v_0) \dots \ell_n(v_0) e^{-\frac{S_b(v_0)}{\hbar}} \operatorname{Pf}(S_f(v_0)) dv_0.$$

In this situation, all vertices which have odd outgoing edges, will have only two of them, and therefore in any Feynman diagram with even outgoing edges, odd lines form nonintersecting simple curves, called *fermionic loops*. (In fact, the last formula is nothing but the result of regarding these loops as a new kind of vertices—convince yourself of this!) In this case, there is the following simple way of assigning signs to Feynman diagrams. For each vertex with two odd outgoing edges, we orient the first edge inward and the second one outward. We allow only connections (matchings) that preserve orientations (so the fermionic loops become oriented). Then the sign is $(-1)^r$, where r is the

number of fermionic loops (i.e., each fermionic loop contributes a minus sign). This follows from the fact that an even cycle is an odd permutation.

9.2. Fermionic quantum mechanics

Let us now pass from finite-dimensional fermionic integrals to quantum mechanics, i.e. integrals over fermionic functions of one (even) real variable t.

Let us first discuss fermionic classical mechanics in the Lagrangian setting. Its difference with the bosonic case is that the "trajectory" of the particle is described by an *odd-valued* function of one variable, i.e. $\psi : \mathbb{R} \to \Pi V$, where V is a vector space. Mathematically, this means that the space of fields (=trajectories) is an odd vector space $\Pi C^\infty(\mathbb{R}, V)$. A Lagrangian $\mathcal{L}(\psi)$ is a local expression in such a field (i.e., a polynomial in $\psi, \dot{\psi}, \dots$), and an action is the integral $S = \int_{\mathbb{R}} \mathcal{L} dt$. This means that the action is an element of the space $\Lambda(C_0^\infty(\mathbb{R}, V)^*)$.

Consider for example the theory of a single real-valued free fermion $\psi(t)$. By definition, the Lagrangian for such a theory is

$$\mathcal{L} = \frac{1}{2}\psi\dot{\psi},$$

i.e., the action is

$$S = \frac{1}{2}\int \psi\dot{\psi} dt.$$

This Lagrangian is the odd analogue of the Lagrangian of a free particle, $\frac{\dot{q}^2}{2}$.

Remark 9.2. Note that $\psi\dot{\psi} \neq \frac{d}{dt}(\frac{\psi^2}{2}) = 0$, since $\psi\dot{\psi} = -\dot{\psi}\psi$, so this Lagrangian is "reasonable". On the other hand, the same Lagrangian would be unreasonable in the bosonic case, as it would be a total derivative, and hence the action for ψ decaying at infinity would be zero. Finally, note that it would be equally unreasonable to use in the fermionic case the usual bosonic Lagrangian $\frac{1}{2}(\dot{q}^2 - m^2q^2)$; it would identically vanish if q were odd valued.

The Lagrangian $\mathcal{L}$ is invariant under the group of reparametrizations $\mathrm{Diff}_+(\mathbb{R})$, and the Euler-Lagrange equation for this Lagrangian is

$$\dot{\psi} = 0$$

(i.e., no dynamics). Theories with such properties are called *topological quantum field theories*.

Let us now turn to quantum theory in the Lagrangian setting, i.e., the theory given by the Feynman integral $\int \psi(t_1)\dots\psi(t_n)e^{\frac{iS(\psi)}{\hbar}} D\psi$. In the bosonic case, we "integrated" such expressions over the space $C_0^\infty(\mathbb{R})$. This integration did not make immediate sense because of difficulties with measure theory in infinite dimensions. So we had to make sense of this integration in terms of

$\hbar$-expansion, using Wick's formula and Feynman diagrams. In the fermionic case, the situation is analogous. Namely, now we must integrate functions over $\Pi C_0^\infty(\mathbb{R})$, which are elements of $\Lambda\mathcal{D}(\mathbb{R})$, where $\mathcal{D}(\mathbb{R})$ is the space of distributions on $\mathbb{R}$. Although in the fermionic case we do not need measure theory (as integration is completely algebraic), we still have trouble defining the integral: recall that by definition the integral should be the top coefficient of the integrand as the element of $\Lambda\mathcal{D}(\mathbb{R})$, which makes no sense since in the exterior algebra of an infinite-dimensional space there is no top component. Thus we have to use the same strategy as in the bosonic case, i.e. Feynman diagrams.

Let us, for instance, define the quantum theory for a free real-valued fermion, i.e., one described by the Lagrangian $\mathcal{L} = \frac{1}{2}\psi\dot{\psi}$. According to the yoga we used in the bosonic case, the two-point function of this theory $\langle\psi(t_1)\psi(t_2)\rangle$ should be the function $G(t_1 - t_2)$, where G is the solution of the differential equation

$$\frac{dG}{dt} = i\delta(t)$$

(the factor i comes from the exponent in the Feynman integral; note that in the fermionic case it does **not** go away under Wick rotation, since the Lagrangian density $\mathcal{L}dt$ is invariant under reparametrizations of t, including $t \mapsto it$).

The general solution of this equation has the form

$$G(t) = \frac{i}{2}\,\mathrm{sign}(t) + C.$$

Because of the fermionic nature of the field $\psi(t)$, it is natural to impose the requirement that $G(-t) = -G(t)$, i.e., that the correlation functions are antisymmetric; this singles out the solution $G(t) = \frac{i}{2}\,\mathrm{sign}(t)$ (we also see from this condition that we should set $G(0) = 0$). As usual, the $2n$-point correlation functions are defined by the Wick formula. That is, for distinct t_j,

$$\langle\psi(t_1)\dots\psi(t_{2n})\rangle = (-1)^\sigma(2n-1)!\,!\,(\tfrac{i}{2})^n,$$

where σ is the permutation that orders t_j in the decreasing order. If at least two points coincide, the correlation function is zero.

Thus we see that the correlation functions are invariant under $\mathrm{Diff}_+(\mathbb{R})$. In other words, we have a *topological quantum field theory*.

Note that the correlation functions in the Euclidean setting for this model are the same as in the Minkowski setting, since they are (piecewise) constant in t_j. In particular, they do not decay at infinity, and hence our theory does not have the clustering property.

We have considered the theory of a massless fermionic field. Consider now the massive case. This means, we want to add to the Lagrangian a quadratic term in ψ which does not contain derivatives. If we have only one field ψ, the only choice for such term is ψ^2, which is zero. So in the massive case we must

have at least two fields. Let us therefore consider the theory of two fermionic fields ψ_1, ψ_2 with (Euclidean) Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2) - m\psi_1\psi_2,$$

where $m > 0$ is a mass parameter. Using integration by parts, this can also be written in terms of a complex-valued fermion $\psi = \psi_1 + i\psi_2$,

$$\mathcal{L} = \frac{1}{2}(\psi^\dagger\dot{\psi} + im\psi^\dagger\psi),$$

where $\psi^\dagger := \psi_1 - i\psi_2$. The Green's function for this model satisfies the differential equation

$$\frac{dG}{dt} - MG = i\delta(t),$$

where $M = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$ and G is a 2-by-2 matrix-valued function. The general solution of this equation is

$$G(t) = \begin{cases} e^{Mt}Q_-, & t < 0, \\ e^{Mt}Q_+, & t > 0, \end{cases}$$

where $Q_+ - Q_- = i$. Now, we want the Wick rotated Green's function $G(-it)$ to have the clustering property. Thus we want

$$\lim_{t\to+\infty} e^{-iMt}Q_+ = 0, \quad \lim_{t\to-\infty} e^{-iMt}Q_- = 0.$$

This implies that $Q_+ = iP_+$, $Q_- = -iP_-$, where $P_\pm$ are the orthogonal projectors to the eigenspaces of iM with eigenvalues $\pm m$ (and $G(0) = 0$).

Remark 9.3. It is easy to generalize this analysis to the situation when ψ takes values in a positive-definite inner product space V, and $M : V \to V$ is a skew-symmetric operator, since such a theory is a direct product of several theories of the above two types (massless and massive).

In the case when M is nondegenerate (the fully massive case), one can define the corresponding theory with interactions, i.e. with higher than quadratic terms in ψ. Namely, one defines the correlators as sums of amplitudes of appropriate Feynman diagrams. We leave it to the reader to work out this definition, by analogy with the finite-dimensional case, which we have discussed above.

9.3. Super Hilbert spaces

The space of states of a quantum system is a Hilbert space. As we plan to do Hamiltonian quantum mechanics for fermions, we must define a super-analogue of this notion.

Suppose $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a $\mathbb{Z}/2$-graded complex vector space.

Definition 9.4.

(1) A Hermitian form on $\mathcal{H}$ is an even sesquilinear form $\langle\ ,\ \rangle$, such that $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for even x, y, and $\langle x, y\rangle = -\overline{\langle y, x\rangle}$ for odd x, y.

(2) A Hermitian form is positive definite if $\langle x, x\rangle > 0$ for even $x \neq 0$, and $-i\langle x, x\rangle > 0$ for odd $x \neq 0$. A super Hilbert space is a superspace with a positive definite Hermitian form $\langle\ ,\ \rangle$, which is complete in the corresponding norm.

(3) Let $\mathcal{H}, \mathcal{H}'$ be super Hilbert spaces, and let $T : \mathcal{H} \to \mathcal{H}'$ be a homogeneous (i.e., either purely odd or purely even) linear operator. The Hermitian adjoint operator $T^\dagger : \mathcal{H}' \to \mathcal{H}$ is defined by the equation

$$\langle x, T^\dagger y\rangle = (-1)^{p(x)p(T)}\langle Tx, y\rangle,$$

where p denotes the parity.[21]

9.4. The Hamiltonian setting for fermionic quantum mechanics

Let us now discuss what should be the Hamiltonian picture for the theory of a free fermion. Let V be a finite-dimensional real vector space with a positive definite inner product $(,)$ which we identify with V^* using $(,)$, and consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}((\psi, \dot{\psi}) - (\psi, M\psi)),$$

where $\psi : \mathbb{R} \to \Pi V$, and $M : V \to V$ is a skew-symmetric operator.

To understand what the Hamiltonian picture should be, let us compare with the bosonic case. Namely, consider the Lagrangian

$$\mathcal{L}_b = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2),$$

where $q : \mathbb{R} \to V$. In this case, the classical space of states is

$$Y := T^*V = V \oplus V^* \cong V \oplus V.$$

The equations of motion are Newton's equations

$$\ddot{q} = -\omega^2 q,$$

which can be reduced to Hamilton's equations

$$\dot{q} = p,\ \dot{p} = -\omega^2 q.$$

The algebra of classical observables is $C^\infty(Y)$, with Poisson bracket defined by $\{a, b\} = \pi(a, b)$, $a, b \in Y^*$, where π is the form on Y^* inverse to the natural symplectic form on Y (the *Poisson bivector*). The Hamiltonian H is determined

[21] Note that if $T_1 : \mathcal{H}' \to \mathcal{H}'', T_2 : \mathcal{H} \to \mathcal{H}'$ are two homogeneous operators, then $(T_1T_2)^\dagger = (-1)^{p(T_1)p(T_2)}T_2^\dagger T_1^\dagger$.

(up to adding a constant) by the condition that the equations of motion are $\dot{f} = \{f, H\}$; in this case it is $H = \frac{1}{2}(p^2 + \omega^2 q^2)$.

The situation in the fermionic case is analogous, with some important differences which we will explain below. Namely, it is easy to compute that the equation of motion (i.e., the Euler-Lagrange equation) is

$$\dot{\psi} = M\psi.$$

The main difference with the bosonic case is that this equation is of first and not of second order, so the space of classical states is just ΠV (no momentum or velocity variables are introduced). Hence, the algebra of classical observables is $C^\infty(\Pi V) = \Lambda V^* \cong \Lambda V$. To define a Poisson bracket on this algebra, recall that ΠV has a natural "symplectic structure", defined by the *symmetric* form $(,)$ on V. Thus we can define a Poisson bracket on ΛV by the same formula as above: $\{a, b\} = (a, b)$ when $a, b \in V$. More precisely, $\{,\}$ is a unique skew-symmetric (in the supersense) bilinear operation on ΛV which restricts to (a, b) for $a, b \in V$, and it is a derivation with respect to each variable:

$$\{a, bc\} = \{a, b\}c + (-1)^{p(a)p(b)} b\{a, c\},$$

where $p(a)$ denotes the parity of a.

Now it is easy to see what should play the role of the Hamiltonian. More precisely, the definition with Legendre transform is not valid in our situation, since the Legendre transform was done with respect to the velocity variables, which we do not have in the fermionic case. On the other hand, as we discussed in Chapter 8, in the bosonic case the equation of motion

$$\dot{f} = \{f, H\}$$

determines H uniquely, up to a constant. The situation is the same in the fermionic case. Namely, by looking at the equation of motion $\dot{\psi} = M\psi$, it is easy to see that the Hamiltonian equals

$$H = \frac{1}{2}(\psi, M\psi).$$

In particular, if $M = 0$, then the Hamiltonian is zero (a characteristic property of topological field theories).

Now let us turn to quantum theory. In the bosonic case the algebra of quantum observables is a noncommutative deformation of the algebra $C^\infty(Y)$ in which the relation $\{a, b\} = (a, b)$ is replaced with its quantum analogue

$$ab - ba = i(a, b)$$

(up to the Planck constant factor, which here we will set to 1). In particular, the subalgebra of polynomial observables is the Weyl algebra $W(Y)$, generated by Y^* with this defining relation. By analogy with this, we should define the

algebra of quantum observables in the fermionic case to be generated by V with the relation

$$ab + ba = i(a, b)$$

(it deforms the relation $ab + ba = 0$ which defines ΛV). So we recall the following definition.

Definition 9.5. Let V be a vector space over a field k with a symmetric bilinear form Q. The *Clifford algebra* $\mathrm{Cl}(V, Q)$ is generated by V with defining relations $ab + ba = Q(a, b)$, $a, b \in V$.

We see that the algebra of quantum observables should be $\mathrm{Cl}(V_{\mathbb{C}}, i(,))$. Note that as in the classical case, this algebra is naturally $\mathbb{Z}/2$ graded, so that we have even and odd quantum observables.

Now let us see what should be the Hilbert space of quantum states. In the bosonic case it was $L^2(V)$, which is, by the well-known Stone-von Neumann theorem, the unique irreducible unitary representation of $W(Y)$. By analogy, in the fermionic case the Hilbert space of states should be an irreducible unitary representation of $\mathrm{Cl}(V_{\mathbb{C}})$ on a supervector space $\mathcal{H}$.

For simplicity, let us restrict ourselves to the case when $\dim V = 2r$ is even. In this case, it is well known that the algebra $\mathrm{Cl}(V_{\mathbb{C}})$ is simple (i.e., isomorphic to a matrix algebra) and has a unique irreducible representation $\mathcal{H}$, of dimension 2^r. This representation is constructed as follows: Choose a decomposition $V_{\mathbb{C}} = V_+ \oplus V_-$, where V_+, V_- are Lagrangian subspaces under $(,)$ (naturally dual to each other). Then $\mathcal{H} = \Lambda V_+$, where $V_+ \subset V_{\mathbb{C}} \cong V_{\mathbb{C}}$ acts by multiplication and V_- by differentiation multiplied by i. The structure of the superspace on $\mathcal{H}$ is the standard one on the exterior algebra.

It is easy to show that the space $\mathcal{H}$ carries a unique up-to-scaling Hermitian form, such that $V \subset V_{\mathbb{C}}$ acts by self-adjoint operators. This form is positive definite. So the situation is similar to the bosonic case.[22]

Let us now see which operator on $\mathcal{H}$ should play the role of the Hamiltonian of the system. The most natural choice is to define the quantum Hamiltonian to be the obvious quantization of the classical Hamiltonian $H = \frac{1}{2}(\psi, M\psi)$. Namely, if ε_j is an orthonormal basis of V (so $2\varepsilon_j^2 = i$, $\varepsilon_j\varepsilon_k = -\varepsilon_k\varepsilon_j$) and (a_{kj}) is the (skew-symmetric) matrix of M in this basis, then one sets

$$\widehat{H} = \frac{1}{2}\sum_{k,j} a_{kj}\varepsilon_k\varepsilon_j = \sum_{k<j} a_{kj}\varepsilon_k\varepsilon_j,$$

[22]For example, let $\dim V = 2$ with basis vectors u, v of V such that $(u, u) = (v, v) = 2$, $(u, v) = 0$. Then $V_{\mathbb{C}}$ has a basis $\xi = \frac{1}{2}(u - iv), \eta = \frac{1}{2}(u + iv)$ such that $(\xi, \xi) = (\eta, \eta) = 0$ and $(\xi, \eta) = 1$, so $u = \xi + \eta$ and $v = i(\xi - \eta)$. The space $\mathcal{H}$ has a basis $1, \xi$ (with 1 even and ξ odd), and ξ acts by multiplication (where $\xi^2 = 0$) while η acts by $i\partial_\xi$. The inner product on $\mathcal{H}$ is defined by $\langle 1, 1\rangle = 1$, $\langle \xi, \xi\rangle = i$, so it is positive definite. The action of u, v is given by $u \mapsto \xi + i\partial_\xi$, $v \mapsto i\xi + \partial_\xi$, so that $u^2 = v^2 = i$, as required by the relations of the Clifford algebra. Note that $\langle \xi, \xi\rangle = \langle 1, i\partial_\xi \xi\rangle$, so $\xi^\dagger = i\partial_\xi$. Thus V indeed acts on $\mathcal{H}$ by self-adjoint operators.

where ε_j are now viewed as elements of the Clifford algebra (or, more precisely, as the corresponding self-adjoint operators on $\mathcal{H}$). Note that this operator is self-adjoint since $p(\varepsilon_j) = 1$, so $(\varepsilon_j\varepsilon_k)^\dagger = -\varepsilon_k^\dagger\varepsilon_j^\dagger = -\varepsilon_k\varepsilon_j = \varepsilon_j\varepsilon_k$ for $j \neq k$. To compute this operator more explicitly, we will assume (without loss of generality) that the decomposition of $V_{\mathbb{C}}$ that we chose is stable under M. Let ξ_j be an eigenbasis of M in V_+ with eigenvalues im_j where $m_j \geq 0$, and let ∂_j be differentiations along the vectors of this basis. Then

$$\widehat{H} = \frac{1}{2}\sum_j m_j(\xi_j\partial_j - \partial_j\xi_j) = \sum_j m_j\left(\xi_j\partial_j - \frac{1}{2}\right).$$

This shows that the partition function on the circle of length L for our theory is

$$Z = \mathrm{sTr}(e^{-L\widehat{H}}) = \prod_j (e^{\frac{m_jL}{2}} - e^{-\frac{m_jL}{2}}) = \sqrt{\det(e^{-\frac{iML}{2}} - e^{\frac{iML}{2}})}. \tag{9.1}$$

Also, similar to the bosonic case (see Section 6.10 and Example 7.27), we may consider the same theory twisted by an orthogonal automorphism $g : V \to V$ which commutes with M. In this case we have

$$Z_{\mathrm{tw}} = \mathrm{sTr}(ge^{-L\widehat{H}}) = \sqrt{\det(e^{-\frac{iML}{2}} - ge^{\frac{iML}{2}})}.$$

Note that this can be nonzero even in the massless case $M = 0$ (namely, if g has no eigenvalue 1).

Now we would like to consider the fermionic analogue of the Feynman-Kac formula. For simplicity, consider the fully massive case, when $\dim V$ is even and $m_j \neq 0$ (i.e., M is nondegenerate). In this case, we have a unique up-to-scaling lowest eigenvector of $\widehat{H}$, namely $\Omega = 1$.

Let $\psi(0) \in V \otimes \mathrm{End}(\mathcal{H})$ be the element corresponding to the action map $V^* \to \mathrm{End}(\mathcal{H})$ (the *Clifford multiplication*), and let $\psi(t) = e^{it\widehat{H}}\psi(0)e^{-it\widehat{H}}$. Also, denote by $\langle\psi(t_1)\dots\psi(t_n)\rangle$, $t_1 \geq \dots \geq t_n$, the correlation function for the free theory in the Lagrangian setting, taking values in $V^{\otimes n}$ (so in this expression $\psi(t_j)$ is a formal symbol and not an operator).

Theorem 9.6 (Feynman-Kac formula)**.**

(1) *For the free theory on the line we have*

$$\langle\psi(t_1)\dots\psi(t_n)\rangle = \langle\,\Omega, \psi(t_1)\dots\psi(t_n)\Omega\,\rangle.$$

(2) *For the free theory on the circle of length L we have*

$$\langle\psi(t_1)\dots\psi(t_n)\rangle = \frac{\mathrm{sTr}(\psi(t_1)\dots\psi(t_n)e^{-L\widehat{H}})}{\mathrm{sTr}(e^{-L\widehat{H}})}.$$

Exercise 9.7. Prove this theorem. (The proof is analogous to Theorem 7.24 in the free case).

It should now be straightforward for the reader to formulate and prove the Feynman-Kac formula for an interacting (i.e., not necessarily free) quantum-mechanical model which includes both bosonic and fermionic massive fields. We leave this as an instructive exercise.

Exercise 9.8.

(i) Consider quantum mechanics with Yukawa coupling. That is, we have a scalar boson $\phi(t)$ and two fermions $\psi_1(t), \psi_2(t)$, and the Euclidean Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 + m^2\phi^2 + \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2) - \mu\psi_1\psi_2 + g\phi\psi_1\psi_2.$$

Compute the 2-point function $\langle\phi(t)\phi(0)\rangle$ modulo g^3 (in the Euclidean setting).

Hint. The correction to the free theory answer is given by one Feynman diagram. Remember about automorphism groups and the minus sign corresponding to fermionic loops.

(ii) In the same theory, compute the 2-point function $\langle\psi_1(t)\psi_1(0)\rangle$ modulo g^3 (in the Euclidean setting). Does the corresponding diagram have nontrivial automorphisms?

Chapter 10

Field theories in higher dimensions

10.1. Minkowski and Euclidean space

Now we pass from quantum mechanics to quantum field theory in dimensions $d > 1$. As we have already mentioned, we have two main settings.

1. Minkowski space. In the Minkowski setting fields are functions on a spacetime $V = V_M$, which is a real-inner product space of signature $(1, d-1)$. This is where physical processes actually "take place".

Recall some standard facts and definitions. The *light cone* in V is the cone described by the equation $|\mathbf{v}|^2 = 0$, where $|\mathbf{v}|^2 := (\mathbf{v}, \mathbf{v})$. Vectors belonging to the light cone are called *lightlike*. The light cone divides the space V into *spacelike vectors* $|\mathbf{v}|^2 < 0$ (outside the cone), and *timelike vectors* $|\mathbf{v}|^2 > 0$ (inside the cone). We will choose one of the two components of the interior of the cone and call it positive; it will be denoted by V_+. The opposite (negative) component is denoted by V_-.

The symmetry group of V is the pseudo-orthogonal group $O(V) \cong O(1, d-1)$. Often (e.g., when doing Hamiltonian field theory) it is necessary to split V in an orthogonal direct sum $V = \mathbb{R} \oplus V_s$ of space and time. In this decomposition, the space V_s is required to be spacelike (i.e., negative definite), which implies that the time axis $\mathbb{R}$ has to be timelike (positive definite). Note that such a splitting is not unique, and that fixing it breaks the symmetry group $O(1, d-1)$ down to $O(d-1) \times \mathbb{Z}/2$, where $\mathbb{Z}/2$ is generated by the *time reversal symmetry* T sending $(t, \mathrm{x}) \in \mathbb{R}^d$ to $(-t, \mathrm{x})$.

To do explicit calculations, one further chooses Cartesian coordinates[23] $x^1, \dots, x^{d-1}$ on V_s and t on the time axis $\mathbb{R}$, so that $\mathbf{v} = (t, x^1, \dots, x^{d-1})$. In these coordinates the inner product takes the form

$$|\mathbf{v}|^2 = c^2t^2 - \sum_{j=1}^{d-1} (x^j)^2,$$

where c is the speed of light. This explains the origin of the term "light cone"—it consists of worldlines of free photons (particles of light) traveling in space in some direction at speed c. To simplify notation, we will choose units of measurement so that $c = 1$.

The component group of $O(1, d-1)$ is $\mathbb{Z}/2 \times \mathbb{Z}/2$, whose generators are represented by the time reversal symmetry T and the *parity symmetry* P, sending $(t, x^1, x^2, \dots, x^{d-1}) \in \mathbb{R}^d$ to $(t, -x^1, x^2, \dots, x^{d-1})$. The subgroup $\mathrm{SO}(1, d-1)$ of orientation-preserving orthogonal transformations is called the *Lorentz group*; it is the group of transformations of spacetime in special relativity. Therefore, field theories in Minkowski space, which are in an appropriate sense "compatible" with the action of $\mathrm{SO}(1, d-1)$, are called *relativistic*. Note that the Lorentz group has two connected components—its component group $\mathbb{Z}/2$ is generated by PT and its identity component is $\mathrm{SO}_+(1, d-1) \subset \mathrm{SO}(1, d-1)$, the subgroup of elements that preserve each of the two components $V_\pm$ of the light cone.

2. Euclidean space. Fields are functions on a spacetime V_E, which is a positive-definite inner-product space. It plays an auxiliary role and has no direct physical meaning, although path integrals computed in this space are similar to expectation values in statistical mechanics. In this case the symmetry group is $O(d)$ and its identity component is $\mathrm{SO}(d)$, the Euclidean version of the Lorentz group.

The two settings are related by the "Wick rotation". Namely, the Euclidean space V_E corresponding to the Minkowski space V_M is the real subspace in $(V_M)_{\mathbb{C}}$ consisting of vectors $(it, x^1, \dots, x^{d-1})$, where t and x^j are real. In other words, to pass to the Euclidean space, one needs to make a change of variable $t \mapsto it$. Note that under this change, the standard metric on the Minkowski space, $dt^2 - \sum_j (dx^j)^2$ goes into a negative definite metric $-dt^2 - \sum_j (dx^j)^2$. However, the minus sign is traditionally dropped, and one considers instead the positive metric $dt^2 + \sum_j (dx^j)^2$ on V_E.

[23] In accordance with traditional notation in field theory, for coordinates on spacetime we will use upper indices.

10.2. Free scalar boson

Consider the theory of a free real-valued scalar bosonic field ϕ of mass m.[24] The procedure of quantization of this theory in the Lagrangian setting is a straightforward generalization from the case of quantum mechanics. Namely, the Lagrangian for this theory in Minkowski space is

$$\mathcal{L} = \frac{1}{2}(|\nabla\phi|^2 - m^2\phi^2),$$

where $\nabla\phi$ is the gradient of ϕ, and the Euler-Lagrange equation is the *Klein-Gordon equation*

$$(\Box + m^2)\phi = 0,$$

where $\Box$ is the d'Alembertian (wave operator),

$$\Box := (\tfrac{\partial}{\partial t})^2 - \sum_{j=1}^{d-1}(\tfrac{\partial}{\partial x^j})^2.$$

Thus to define the corresponding quantum theory, we should invert the operator $\Box + m^2$. This operator is essentially self-adjoint on compactly supported smooth functions and thus defines a self-adjoint operator, but as in the quantum mechanics case, it is not invertible—its spectrum is the whole $\mathbb{R}$, as can be easily seen by taking the Fourier transform. So as before, it is best to proceed using the Wick rotation.

After the Wick rotation (i.e., the transformation $t \mapsto it$), we arrive at the Euclidean Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(|\nabla\phi|^2 + m^2\phi^2),$$

and the Euler-Lagrange equation is the Poisson equation

$$(-\Delta + m^2)\phi = 0$$

(the Euclidean counterpart of the Klein-Gordon equation). So to define the quantum theory, i.e., the path integral

$$\int \phi(x_1)\dots\phi(x_n)e^{-S(\phi)}D\phi,$$

where $S = \int_{\mathbb{R}^d} \mathcal{L}dx$, we now need to invert the self-adjoint operator $A = -\Delta + m^2$ (initially defined as an essentially self-adjoint operator on smooth compactly supported functions), whose spectrum is $[m^2, \infty)$, so it is invertible when $m > 0$. (Here we set $\hbar = 1$, which in a free quantum theory can always be achieved by renormalizing the fields.) The operator A^{-1} is an integral operator

[24] Here the word "scalar" means not that the field takes values in scalars, but rather that it transforms under changes of coordinates as a function and not a more complicated tensor field. In particular, a scalar field can be complex valued, or more generally valued in a Euclidean space of any dimension.

whose Schwartz kernel is $G(x-y)$, where $G(x)$ is the Green's function, i.e., the fundamental solution of the Poisson equation:

$$-\Delta G + m^2 G = \delta.$$

To solve this equation, note that the solution is rotationally invariant. Therefore, outside of the origin,

$$G(x) = g(|x|),$$

where g is a function on $(0,\infty)$ such that

$$-g'' - \frac{d-1}{r}g' + m^2 g = 0$$

(where the left-hand side is the radial part of the operator A). This is a version of the Bessel equation. If $m > 0$, the two basic solutions are $r^{\frac{2-d}{2}} J_{\pm\frac{2-d}{2}}(imr)$, where J is the Bessel function. (Actually, these functions are elementary for odd d.) Since we want G to decay at infinity (clustering property), we should pick the unique up-to-scaling linear combination which decays at infinity, namely,

$$g(r) = Cr^{\frac{2-d}{2}}(J_{\frac{2-d}{2}}(imr) + i^d J_{-\frac{2-d}{2}}(imr)),\ d \neq 2. \tag{10.1}$$

For $d = 2$, this expression is zero, and one should instead take the limit of the right-hand side divided by $d-2$ as $d \to 2$. The normalizing constant can be found from the condition that $AG = \delta$.

Remark 10.1. It is easy to check that for $d = 1$ this function equals the familiar Green's function for quantum mechanics, $\frac{e^{-mr}}{2m}$.

If $m = 0$ (the massless case), the basis of solutions is $\{1, r\}$ for $d = 1$, $\{1, \log r\}$ for $d = 2$, and $\{1, r^{2-d}\}$ for $d > 2$. Thus, if $d \leq 2$, we do not have a decaying solution, and thus the corresponding quantum theory will be deficient: it will not satisfy the clustering property.[25] On the other hand, for $d > 2$ we have a unique up-to-scaling decaying solution $g = Cr^{2-d}$. The normalizing constant is found as in the massive case.

Exercise 10.2. Compute $g(r)$ completely (including normalization) for all $d \geq 2$ and $m \geq 0$.

The higher correlation functions are found from the 2-point function via the Wick formula, as usual.

We should now note a fundamental difference between quantum mechanics and quantum field theory in $d > 1$ dimensions. This difference comes from the fact that while for $d = 1$, the Green's function $G(x)$ is continuous at $x = 0$, for $d > 1$ it is singular at $x = 0$. Namely, $G(x)$ behaves like $|x|^{2-d}$ as $x \to 0$ for

[25] Nevertheless, for $d = 2$ such theories are very important; see Chapter 12.

$d > 2$, and as $\log|x|$ for $d = 2$. Thus for $d > 1$, unlike the case $d = 1$, the path integral

$$\int \phi(x_1)\dots\phi(x_n)e^{-S(\phi)}D\phi$$

(as defined above) makes sense only if $x_i \neq x_j$. In other words, this path integral should be regarded not as a function but rather as a distribution. Luckily, there is a canonical way to do it, since the Green's function $G(x)$ is locally L^1.

Now we can Wick-rotate this theory back into the Minkowski space. It is clear that the Green's function will then turn into

$$G_M(x) = g(i|x|), \text{ where } |x| := \sqrt{(x,x)} = \sqrt{t^2 - \sum_{i=1}^{d-1}(x^j)^2},$$

which involves Bessel functions of both real and imaginary argument (depending on whether x is timelike or spacelike) and has a singularity on the light cone $|x|^2 = 0$. In particular, it is easy to check that $G_M(x)$ is real-valued for spacelike x, while for timelike x it is not. The function $G_M(x)$ satisfies the equation

$$(\Box + m^2)G_M = i\delta.$$

The higher correlation functions, as before, are determined from this by the Wick formula.

Actually, it is more convenient to describe this theory "in momentum space", where the Green's function can be written more explicitly. Namely, the Fourier transform $\widehat{G}(p)$ of the distribution $G(x)$ is a solution of the equation

$$p^2\widehat{G} + m^2\widehat{G} = 1,$$

obtained by Fourier transforming the differential equation for G. Thus,

$$\widehat{G}(p) = \frac{1}{p^2 + m^2},$$

as in the quantum mechanics case. Therefore, as in quantum mechanics, the Wick rotation produces the distribution

$$\widehat{G}_M(p) = \frac{i}{p^2 - m^2 + i\varepsilon},$$

which is the Fourier transform of $G_M(x)$.

10.3. Spinors

To consider field theory for fermions, we must generalize to the case of $d > 1$ the basic fermionic Lagrangian $\frac{1}{2}\psi\frac{d\psi}{dt}$. To do this, we must replace $\frac{d}{dt}$ by some differential operator on V. This operator should be of first order, since in fermionic quantum mechanics it was important that the equations of motion are first-order equations. Also the action should be relativistically invariant,

i.e., stable under the Poincaré group $\mathbf{P} := \mathrm{SO}_+(V) \ltimes V$. Clearly, it is impossible to define such an operator if ψ is a scalar-valued (odd) function on V. Thus, a fermionic field in field theory of dimension $d > 1$ cannot be scalar valued, but rather must take values in a real representation S of $\mathrm{SO}_+(V)$, such that there exists a nonzero intertwining operator $V \to \mathrm{Sym}^2 S^*$. The simplest representations for which this property is satisfied are the *spin representations* (except that they aren't quite representations of $\mathrm{SO}_+(V)$ but rather are representations of its double cover). They are indeed basic in fermionic field theory, and we will now briefly discuss them (for more detail see "Notes on spinors" by P. Deligne, in [**QFS**, vol. 1, p. 99]).

First consider the complex case. Let V be a complex inner-product space of dimension $d > 1$. Let $\mathrm{Cl}(V)$ be the Clifford algebra of V, defined by the relation $\xi\eta + \eta\xi = 2(\xi, \eta)$, $\xi, \eta \in V$. As we discussed, for even d it is simple and has a unique irreducible representation S of dimension $2^{\frac{d}{2}}$, while for odd d it has two such representations S', S'' of dimension $2^{\frac{d-1}{2}}$. It is easy to show that the space $\mathrm{Cl}_2(V)$ of quadratic elements of $\mathrm{Cl}(V)$ (i.e., the subspace spanned by elements of the form $\xi\eta - \eta\xi$, $\xi, \eta \in V$) is closed under bracket and constitutes the Lie algebra $\mathfrak{o}(V)$. Thus $\mathfrak{o}(V)$ acts on S (respectively, S', S''). This action does not integrate to an action of $\mathrm{SO}(V)$, but integrates to an action of its double cover $\mathrm{Spin}(V)$.

If d is even, the representation S of $\mathrm{Spin}(V)$ is not irreducible (even though it is irreducible over $\mathrm{Cl}(V)$). Namely, recall that S is the exterior algebra of a Lagrangian subspace of V. Thus it splits in a direct sum $S = S_+ \oplus S_-$ (odd and even elements). The subspaces S_+, S_- are subrepresentations of S, which are irreducible. They are called the *half-spin representations*. The half-spin representations are interchanged by the adjoint action of $O(V)$ on $\mathrm{Spin}(V)$ ($\mathrm{SO}(V)$ clearly acts trivially, so this is, in fact, an action of $O(V)/\mathrm{SO}(V) = \mathbb{Z}/2$ on the set of irreducible representations of $\mathrm{SO}(V)$). Note that in contrast, for odd d we have $O(V) = \mathrm{SO}(V) \times \mathbb{Z}/2$, so the $\mathbb{Z}/2$ acts on representations of $\mathrm{Spin}(V)$ trivially.

If d is odd, the representations S' and S'' of $\mathrm{Spin}(V)$ are irreducible and isomorphic. Any of them will be denoted by S and called the *spin representation*. Thus, we have the spin representation S for both odd and even d, but for even d it is reducible.

An important structure attached to the spin representation S is the intertwining operator $\Gamma : V \to \mathrm{End}\, S$ called *Clifford multiplication*, given by the action of $V \subset \mathrm{Cl}(V)$ in S, which we already encountered above. This intertwiner allows us to define the *Dirac operator*

$$\mathbf{D} = \sum_i \Gamma^i \frac{\partial}{\partial x^i}, \tag{10.2}$$

where x^i are coordinates on V associated to an orthonormal basis e_i, and $\Gamma^i = \Gamma(e_i)$. This operator acts on functions from V to S, and $\mathbf{D}^2 = -\Delta$, so $\mathbf{D}$ is a square root of minus the Laplacian. The matrices Γ^i are called Γ-*matrices*.

Note that for even d, one has $\Gamma(\mathbf{v}) : S_\pm \to S_\mp$, so $\mathbf{D}$ acts from functions with values in $S_\pm$ to functions with values in $S_\mp$.

By a *polyspin representation* of $\mathrm{Spin}(V)$ we will mean any linear combination of S_+, S_- for even d, and any multiple of S for odd d. For even d and a polyspin representation

$$Y = Y_+ \otimes S_+ \oplus Y_- \otimes S_-$$

(i.e., $Y_\pm = \mathrm{Hom}(S_\pm, Y)$), where Y_+, Y_- are vector spaces, set

$$Y' := Y_+ \otimes S_- \oplus Y_- \otimes S_+,$$

while for odd d and

$$Y = Y_0 \otimes S,$$

where Y_0 is a vector space, we set $Y' := Y$; thus $Y \mapsto Y'$ is an endofunctor on the category of polyspin representations. Then for every polyspin representation Y and $\mathbf{v} \in V$, we have the Clifford multiplication operator $\Gamma(\mathbf{v}) : Y \to Y'$.

Now assume that V is a real inner-product space with Minkowski metric. In this case we can define the group $\mathrm{Spin}_+(V)$ to be the preimage of $\mathrm{SO}_+(V)$ under the map $\mathrm{Spin}(V_\mathbb{C}) \to \mathrm{SO}(V_\mathbb{C})$. It is a double cover of $\mathrm{SO}_+(V)$ (if $d = 2$, this double cover is disconnected and actually a direct product by $\mathbb{Z}/2$).

By a *real polyspin representation* of $\mathrm{Spin}_+(V)$ we will mean a real representation Y of this group such that $Y_\mathbb{C}$ is a polyspin representation of $\mathrm{Spin}(V_\mathbb{C})$.

Remark 10.3. Recall that in quantum mechanics of an electron in the hydrogen atom, the rotational symmetry of $\mathbb{R}^3$ induces an action of the Lie algebra $\mathfrak{o}(3)$ on the space of states $\mathcal{H}$. Thus $\mathcal{H}$ falls into a direct sum of irreducible representations V_ℓ of $\mathfrak{o}(3)$ with highest weight $\ell = 0, 1, 2, \dots$ (of dimension $\ell + 1$). In physics language V_ℓ is called the representation of *spin* (i.e., angular momentum) $\frac{\ell}{2}$. Namely, the generator M_z of the subalgebra $\mathfrak{o}(2) \subset \mathfrak{o}(3)$ corresponding to rotation around the z-axis which acts in V_ℓ with eigenvalues $\frac{\ell}{2}, \frac{\ell-1}{2}, \dots, -\frac{\ell}{2}$ is the angular momentum operator (for the z-axis), and if $\psi \in \mathcal{H}$ is a state with $M_z\psi = m\psi$, $m \in \frac{1}{2}\mathbb{Z}$, then one says that ψ has spin m. So the spin of an irreducible $\mathfrak{o}(3)$-subrepresentation $V \subset \mathcal{H}$ is the largest spin of a state $\psi \in V$ (i.e., the largest eigenvalue of M_z on V). Note that representations V_ℓ of half-integer spin (i.e., $\ell \in 1 + 2\mathbb{Z}_{\geq 0}$) do not integrate to the group $\mathrm{SO}(3)$—only to its universal (double) cover $\widetilde{\mathrm{SO}(3)} = \mathrm{Spin}(3) = \mathrm{SU}(2)$.

By analogy, there is also a notion of *spin of a field* in field theory. Namely, in Euclidean classical or quantum field theory in $\mathbb{R}^d$, types of fields correspond to irreducible finite-dimensional unitary representations R of the compact Lie

algebra $\mathfrak{o}(d) = \operatorname{Lie} \mathrm{SO}(d)$ (or, equivalently, unitary representations of the universal cover $\widetilde{\mathrm{SO}(d)}$ of the compact Lie group $\mathrm{SO}(d)$). For example, one can have *tensor fields* corresponding to representations of $\mathrm{SO}(d)$, or *spinor fields* corresponding to the spin representations S or $S_\pm$. In general, to any R one can attach a real number $s(R)$ called the *spin* of R. Namely, consider the embedding $\rho : \mathrm{SO}(2) \hookrightarrow \mathrm{SO}(d)$ using the first two coordinates, let ρ_* be the corresponding map of complexified Lie algebras, and let $M := \rho_* \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \in \mathfrak{o}(d)$. Then $s(R)$ is the largest eigenvalue of M on R. In dimensions $d \geq 3$, the universal cover of $\mathrm{SO}(d)$ is its double cover $\mathrm{Spin}(d)$, which means that spins of all fields belong to $\frac{1}{2}\mathbb{Z}_{\geq 0}$. For example, for a scalar the spin is 0, for a spinor it is $\frac{1}{2}$, for a vector it is 1, for a metric it is 2, for a j-form it is j, etc. On the other hand, the universal cover of $\mathrm{SO}(2)$ is not $\mathrm{Spin}(2)$, but rather $\mathbb{R}$. This creates in two dimensions a possibility of fields whose spin is any nonnegative real number. Such fields are called *anyons* (fields of any spin), and we will see how they appear in 2-dimensional conformal field theory in Section 12.9.

10.4. Fermionic Lagrangians

Now let us consider Lagrangians for a *spinor* field, i.e., a field $\psi : \mathbb{R}^d \to Y$ with values in a polyspin representation Y. Note that in even dimensions such fields are split into fields valued in S_+ and S_-, respectively. Such spinors are called *chiral*, or *Weyl* spinors. A *Dirac spinor* is a combination of a chiral and anti-chiral Weyl spinor (i.e., $Y = S_+ \oplus S_-$).

As the Lagrangian is supposed to be real in the Minkowski setting, we will require in that case that Y be real: $Y = Y_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Thus $Y_{\mathbb{R}}$ is a direct sum of irreducible real representations of $\mathrm{Spin}(1, d-1)$. If such a representation W remains irreducible after tensoring with $\mathbb{C}$ (i.e., $\mathrm{End}_{\mathbb{R}} W = \mathbb{R}$), one says that the corresponding component of ψ is a *Majorana spinor*. If d is even, then in this case $W_{\mathbb{C}}$ is chiral (S_+ or S_-), so we say that ψ is a *Majorana-Weyl spinor*. Also, if $W_{\mathbb{C}}$ is a sum of two copies of an irreducible representation (so $\mathrm{End}_{\mathbb{R}} W = \mathbb{H}$), then ψ is said to be a *symplectic Majorana spinor*. If d is even, then in this case $W_{\mathbb{C}}$ is chiral ($2S_+$ or $2S_-$), so we say that ψ is a *symplectic Majorana-Weyl spinor*. The table in [**QFS**, p. 129] shows that Majorana spinors exist in dimensions $d = 1, 2, 3$ modulo 8, so Majorana-Weyl spinors for $d = 2$ mod 8, and symplectic Majorana spinors exist in dimensions $d = 5, 6, 7$ modulo 8, so symplectic Majorana-Weyl spinors in dimensions $d = 6$ modulo 8. There are no Majorana spinors in dimensions $d = 0, 4$ modulo 8 (as $\mathrm{End}_{\mathbb{R}}(W) = \mathbb{C}$).

Now let us see what we need in order to write the "kinetic term" $(\psi, \mathbf{D}\psi)$. Clearly, to define such a term (so that the corresponding term in the action does not reduce to zero via integration by parts), we need an invariant nondegenerate pairing $(,)$ between Y and Y' (i.e., an isomorphism of representations

$Y' \cong Y^*$) such that for any $v \in V$, the bilinear form $(x, \Gamma(\mathbf{v})y)$ on Y is symmetric.

Let us find for which Y this is possible. The behavior of Spin groups depends on d modulo 8 (*real Bott periodicity*). Thus we will list the answers labeling them by d mod 8 (they are easily extracted from the tables given in Deligne's text in [**QFS**, vol. 1, p. 99]). First we summarize properties of spin representations.

0. $S_\pm$ orthogonal.
1. S orthogonal, $S \otimes S \to V$ symmetric.
2. $S_+^* = S_-$, $S_\pm \otimes S_\pm \to V$ symmetric.
3. S symplectic, $S \otimes S \to V$ symmetric.
4. $S_\pm$ symplectic.
5. S symplectic, $S \otimes S \to V$ antisymmetric.
6. $S_+ = S_-^*$, $S_\pm \otimes S_\pm \to V$ antisymmetric.
7. S orthogonal, $S \otimes S \to V$ antisymmetric.

Thus the possibilities for the kinetic term are as follows:

0. $Y = n(S_+ \oplus S_-)$; (,) gives a perfect pairing between Y_+ and Y_-.
1. $Y = nS$; (,) gives a symmetric inner product on Y_0.
2. $Y = nS_+ \oplus kS_-$; (,) gives symmetric inner products on $Y_\pm$.
3. $Y = nS$; (,) gives a symmetric inner product on Y_0.
4. $Y = n(S_+ \oplus S_-)$; (,) gives a perfect pairing between Y_+ and Y_-.
5. $Y = 2nS$; (,) gives a skew-symmetric inner product on Y_0.
6. $Y = 2nS_+ \oplus 2kS_-$; (,) gives skew-symmetric inner products on $Y_\pm$.
7. $Y = 2nS$; (,) gives a skew-symmetric inner product on Y_0.

Let us now find when we can also add a mass term. Recall that the mass term has the form $(\psi, M\psi)$, so it corresponds to an invariant real skew-symmetric operator $M : Y \to Y^* \cong Y'$ (note that by definition, Γ^i commute with M). Let us list those Y from the above list for which such a nondegenerate operator exists.

0. $Y = 2n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \to Y_\mp$ are skew-symmetric under (,).
1. $Y = 2nS$; $M : Y_0 \to Y_0$ is skew-symmetric under (,).
2. $Y = n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \cong Y_\mp$ satisfy $M_+^* = -M_-$ under (,).
3. $Y = nS$; $M : Y_0 \to Y_0$ is symmetric under (,).
4. $Y = n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \to Y_\mp$ are symmetric under (,).
5. $Y = 2nS$; $M : Y_0 \to Y_0$ is symmetric under (,).

6. $Y = 2n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \cong Y_\mp$ satisfy $M_+^* = -M_-$ under $(,)$.
7. $Y = 2nS$; $M : Y_0 \to Y_0$ is skew-symmetric under $(,)$.

We note that upon Wick rotation from Minkowski to Euclidean space, it may turn out that a real spin representation Y will turn into a complex representation which has no real structure. Namely, this happens for massless spinors that take values in $S_\pm$ if $d = 2 \bmod 8$. These representations have a real structure for Minkowskian V (i.e., for $\mathrm{Spin}_+(1, d-1)$), but no real structure for Euclidean V (i.e., for $\mathrm{Spin}(d)$). This is quite obvious, for example, when $d = 2$ (check it!).

Remark 10.4. One may think that this causes a problem in the path integral approach to quantum field theory, where we would be unsure whether to integrate over real or complex odd space. However, this problem does not actually arise, since odd integration is purely algebraic and does not distinguish between real and complex spaces. This shows that in the presence of such fields to correctly count fermionic degrees of freedom, one should do it in Minkowskian signature.

10.5. Free fermions

Let us now consider a free theory for a spinor field $\psi : V \to \Pi Y$, where $V \cong \mathbb{R}^d$ is the Minkowski space and Y is a real polyspin representation, defined by a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi, (\mathbf{D} - M)\psi),$$

where M is allowed to be degenerate (we assume that Y is such that this expression makes sense). The equation of motion is

$$\mathbf{D}\psi = M\psi.$$

Thus, to define the corresponding quantum theory, we need to invert the operator $\mathbf{D} - M$. As usual, this cannot be done because of a singularity, and it is best to use the Wick rotation.

The Wick rotation produces the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi, (\mathbf{D}_E + M)\psi)$$

(note that the i in the kinetic term is hidden in the definition of the Euclidean Dirac operator). We invert $\mathbf{D}_E + M$ to obtain the Euclidean Green's function. To do this, it is convenient to go to momentum space, i.e., perform a Fourier transform. Namely, after Fourier transform $\mathbf{D}_E$ turns into the operator $i\mathbf{p}$, where $\mathbf{p} = \sum_j p_j \Gamma^j$ and p_j are the operators of multiplication by the

momentum coordinates p_j. Thus, the Green's function (i.e., the 2-point function) $G(x) \in \mathrm{Hom}(Y^*, Y)$ is the inverse Fourier transform of the matrix-valued function $\frac{1}{i\mathbf{p}+M}$.

In the Euclidean case the group $\mathrm{Spin}(V)$ is compact and the spin representations carry natural positive invariant Hermitian forms. So in this case without loss of generality we may consider polyspin representations equipped with such positive forms, and on every polyspin representation such a form is unique up to isomorphism. Let

$$M^\dagger : Y^* \to Y$$

be the Hermitian adjoint operator to M. Then the reality condition is that M is Hermitian: $M^\dagger = M$. Thus

$$(-i\mathbf{p} + M)(i\mathbf{p} + M) = p^2 + M^2$$

with $M^2 \geq 0$, so that

$$\widehat{G}(p) = (p^2 + M^2)^{-1}(-i\mathbf{p} + M).$$

This shows that $G(x)$ is expressed through the Green's function in the bosonic case by differentiations (Exercise: how?). After Wick rotation back to the Minkowski space, we get

$$\widehat{G}_M(p) = (p^2 - M^2 + i\varepsilon)^{-1}(\mathbf{p} + iM).$$

Finally, the higher correlation functions, as usual, are found from the Wick formula.

10.6. Hamiltonian formalism of classical field theory

Let us now develop the Hamiltonian approach to QFT, extending the Hamiltonian formalism of quantum mechanics. We start with classical field theory, extending the Hamiltonian formalism of classical mechanics. As in the Lagrangian setting, this can be done by formalizing the idea that field theory is mechanics of a continuum of particles occupying each point of the space $\mathbb{R}^{d-1}$.

Namely, consider a free scalar bosonic field $\phi(x)$ on a Minkowski space $\mathbb{R}^d$. As we have discussed, its Lagrangian is $\mathcal{L} = \frac{1}{2}(|\nabla\phi|^2 - m^2\phi^2)$ and the equation of motion is the Klein-Gordon equation

$$\phi_{tt} - \Delta_s\phi + m^2\phi = 0,$$

where Δ_s is the spatial Laplacian. This is a second-order equation with respect to t, so the initial value problem for this equation has the form

$$\phi(0, \mathrm{x}) = q(\mathrm{x}),\ \phi_t(0, \mathrm{x}) = q_*(\mathrm{x})$$

(there is a standard explicit formula for the solution of this problem, expressing it via the fundamental solution of the Klein-Gordon equation). Thus it is natural to introduce the phase space

$$\mathcal{Y} := T^*C_0^\infty(\mathbb{R}^{d-1}) := C_0^\infty(\mathbb{R}^{d-1}) \oplus C_0^\infty(\mathbb{R}^{d-1})$$

of pairs (q, q_*) of smooth functions with compact support, on which the dynamics of the Klein-Gordon equation takes place (note that the space $C_0^\infty(\mathbb{R}^{d-1})$ is invariant under this dynamics since the speed of wave propagation is finite, namely equals 1). Note that the phase space is an infinite-dimensional symplectic space with constant symplectic form

$$\omega((q_1, q_{*1}), (q_2, q_{*2})) = \int_{\mathbb{R}^{d-1}} (q_{*1}(\mathrm{x})q_2(\mathrm{x}) - q_{*2}(\mathrm{x})q_1(\mathrm{x}))d\mathrm{x}.$$

Also for any point $\mathrm{x} \in \mathbb{R}^{d-1}$, we have the local linear functionals (distributions)

$$(q, q_*) \mapsto q(\mathrm{x}),\ (q, q_*) \mapsto q_*(\mathrm{x})$$

which we will denote by $\phi(\mathrm{x})$ and $\phi_t(\mathrm{x})$, respectively. From these functionals we can make other linear functionals: for example, given $\rho \in C_0^\infty(\mathbb{R}^{d-1})$, we can define the functionals

$$\phi(\rho)(q, q_*) := \int_{\mathbb{R}^{d-1}} q(\mathrm{x})\rho(\mathrm{x})d\mathrm{x},\ \phi_t(\rho)(q, q_*) := \int_{\mathbb{R}^{d-1}} q_*(\mathrm{x})\rho(\mathrm{x})d\mathrm{x}.$$

The Poisson bracket between such functionals can be computed by the formulas

$$\{\phi(\rho_1), \phi(\rho_2)\} = 0,\ \{\phi_t(\rho_1), \phi_t(\rho_2)\} = 0,$$

$$\{\phi(\rho_1), \phi_t(\rho_2)\} = \int_{\mathbb{R}^{d-1}} \rho_1(\mathrm{x})\rho_2(\mathrm{x})d\mathrm{x}.$$

This can be written as a *field-theoretic Poisson bracket*:

$$\{\phi(\mathrm{x}), \phi(\mathrm{y})\} = 0,\ \{\phi_t(\mathrm{x}), \phi_t(\mathrm{y})\} = 0,\ \{\phi(\mathrm{x}), \phi_t(\mathrm{y})\} = \delta(\mathrm{x} - \mathrm{y}); \tag{10.3}$$

then the previous formulas can be recovered by integrating both sides against $\rho_1(\mathrm{x})\rho_2(\mathrm{y})$.

Similarly, one may consider nonlinear polynomial local functionals, given by differential polynomials $P(\phi, \phi_t)$ evaluated at a point x, such as ϕ^n, ϕ_t^2, $(\nabla_s\phi)^2$, $\phi^2(\nabla_s\phi)^2$ (where ∇_s is the spatial gradient), etc., and even nonpolynomial ones depending on finitely many derivatives of ϕ, such as $e^\phi(\nabla_s\phi)^2$, $\cos\phi$, and so on. They are called local because they depend only on the derivatives of ϕ at a single point x. Note that such local functionals are also allowed to explicitly depend on x; e.g., $f(\mathrm{x})\phi(\mathrm{x})$, where f is a smooth function.

Poisson brackets of local functionals are computed using the chain rule, the Leibniz rule, and the fact that taking Poisson brackets commutes with differentiation by x. So given two local functionals P and Q, we obtain

$$\{P(\phi)(\mathrm{x}), Q(\phi)(\mathrm{y})\} = \sum_\alpha \{P, Q\}_\alpha(\phi)(\mathrm{x})\partial_{\mathrm{x}}^\alpha \delta(\mathrm{x}-\mathrm{y})$$

for some local functionals $\{P, Q\}_\alpha$, where ∂^α are monomials in the derivatives. For example, for $d = 2$ with coordinates t, x we have

$$\{\phi_t'(\mathrm{x})\phi_t(\mathrm{x}), \tfrac{1}{3}\phi^3(\mathrm{y})\}$$
$$-\big(\phi_t'(\mathrm{x})\phi(\mathrm{x})^2 + 2\phi_t(\mathrm{x})\phi'(\mathrm{x})\phi(\mathrm{x})\big)\delta(\mathrm{x}-\mathrm{y}) - \phi_t(\mathrm{x})\phi(\mathrm{x})^2\delta'(\mathrm{x}-\mathrm{y}).$$

This Poisson bracket can of course be extended to products of local functionals at different points using the Leibniz rule.

The Hamiltonian of the theory is then given by integrating a local functional against the constant density,

$$H(\phi) = \frac{1}{2}\int_{\mathbb{R}^{d-1}} (\phi_t^2 + (\nabla_s\phi)^2 + m^2\phi^2)d\mathrm{x}.$$

Namely, it is determined (up to a constant) by the condition that the Hamilton equation

$$F_t = \{F, H\}$$

for local functionals ϕ, ϕ_t is equivalent to the Klein-Gordon equation.

Thus, the Hamiltonian dynamics allows us to define the local functionals not just at a point $\mathrm{x} \in \mathbb{R}^{d-1}$ but actually at any point $(t, \mathrm{x}) \in \mathbb{R}^d$, so that $\phi(0, \mathrm{x}) = \phi(\mathrm{x})$ and similarly for ϕ_t. When we do, by definition we get $\phi_t(t, \mathrm{x}) = \frac{d}{dt}\phi(t, \mathrm{x})$ and the local functional $\phi(t, \mathrm{x})$ becomes a solution of the Klein-Gordon equation,

$$\phi_{tt} - \Delta_s\phi + m^2\phi = 0.$$

This can be used to compute the Poisson brackets: for example, we see that

$$\{\phi(t_1, \mathrm{x}_1), \phi(t_2, \mathrm{x}_2)\} = \mathbf{G}(t_2 - t_1, \mathrm{x}_2 - \mathrm{x}_1), \tag{10.4}$$

where $\mathbf{G}(t, \mathrm{x})$ solves the Klein-Gordon equation with initial conditions

$$\mathbf{G}(0, \mathrm{x}) = 0,\ \mathbf{G}_t(0, \mathrm{x}) = \delta(\mathrm{x}).$$

To find it, take the Fourier transform. Then we get a distribution $\widehat{\mathbf{G}}$ supported on the two-sheeted hyperboloid X_m given by the equation

$$E^2 = \mathrm{p}^2 + m^2,$$

of the form

$$\widehat{\mathbf{G}}(E, \mathrm{p}) = f_+(\mathrm{p})\delta_{X_m^+} + f_-(\mathrm{p})\delta_{X_m^-},$$

where $X_m^{\pm}$ are the sheets of X_m. Moreover, the initial conditions give (up to appropriate normalization)

$$\int_{\mathbb{R}} \widehat{\mathbf{G}}(E,\mathrm{p})dE = 0,\ \int_{\mathbb{R}} \widehat{\mathbf{G}}(E,\mathrm{p})EdE = 1,$$

which yields

$$f_+(\mathrm{p}) + f_-(\mathrm{p}) = 0,\ \sqrt{\mathrm{p}^2+m^2}(f_+(\mathrm{p}) - f_-(\mathrm{p})) = 1.$$

Thus $f_+ = -f_- = \frac{1}{2\sqrt{\mathrm{p}^2+m^2}}$, and we have

$$\widehat{\mathbf{G}}(E,\mathrm{p}) = \frac{1}{2\sqrt{\mathrm{p}^2+m^2}}(\delta_{X_m^+} - \delta_{X_m^-}).$$

Now $\mathbf{G}$ can be found by taking the inverse Fourier transform (it expresses via Bessel functions).

Note that since the speed of light is 1 and nothing can travel faster than that, this distribution $\mathbf{G}$ is supported on the *solid* light cone $|t| \geq |\mathrm{x}|$, so we have $\{\phi(t_1,\mathrm{x}_1),\phi(t_2,\mathrm{x}_2)\} = 0$ if the points (t_1,x_1) and (t_2,x_2) are *spacelike separated*, i.e., the vector $(t_1-t_2,\mathrm{x}_1-\mathrm{x}_2)$ is spacelike. This property is called *space locality*, a mathematical expression of *causality* in special relativity.

Now, differentiating (10.4) and applying the Leibniz rule, we can compute the Poisson bracket of any two local functionals given by differential polynomials in ϕ. Moreover, since $\mathbf{G}$ satisfies the Klein-Gordon equation, this formula descends to the quotient of the algebra of such differential polynomials by the differential ideal given by the equation of motion.

Remark 10.5.

1. A part of this analysis extends straightforwardly to the case of nonfree theories, for example the ϕ^4-*theory*, having the Lagrangian

$$\mathcal{L} = \frac{1}{2}(|\nabla\phi|^2 - m^2\phi^2) - \frac{g}{4}\phi^4.$$

In this case the Klein-Gordon equation is replaced by its nonlinear deformation

$$\phi_{tt} - \Delta_s\phi + m^2\phi + g\phi^3 = 0,$$

so there is a nontrivial issue of existence of solutions of the initial value problem for this nonlinear PDE. However, this issue is irrelevant if we just want to consider Poisson brackets of local functionals on $\mathbb{R}^{d-1}$ or its formal neighborhood, since then the computations are purely formal (algebraic).

2. Suppose the Lagrangian $\mathcal{L}(\phi,\phi_t)$ is not necessarily of the form

$$\mathcal{L}(\phi,\phi_t) = \frac{1}{2}\phi_t^2 + \mathcal{L}_0(\phi), \tag{10.5}$$

but is a more general (convex) function of ϕ_t. In this case, as in classical mechanics, one passes from Lagrangian to Hamiltonian picture by Legendre transform. In particular, ϕ_t is replaced by the *momentum field* $\phi_* := \mathcal{L}_{\phi_t}$, which is dual to ϕ, i.e.,

$$\{\phi(\mathrm{x}), \phi(\mathrm{y})\} = 0,\ \{\phi_*(\mathrm{x}), \phi_*(\mathrm{y})\} = 0, \{\phi(\mathrm{x}), \phi_*(\mathrm{y})\} = \delta(\mathrm{x}-\mathrm{y}). \tag{10.6}$$

Of course, if $\mathcal{L}$ is given by (10.5) then $\phi_* = \phi_t$ as above.

An important fact is that this structure is invariant under the *Poincaré group* $\mathbf{P} := \mathrm{SO}_+(V) \ltimes V$ generated by Minkowski rotations and translations, where $V = \mathbb{R}^d$ is the spacetime (the semidirect product of the *Lorentz group* $\mathrm{SO}_+(V)$ and the group of translations V).[26] This follows from the fact that the action of the theory is relativistically invariant. Namely, the action of $\mathbf{P}$ on the phase space $\mathcal{Y}$ can be described as follows. For $(q, q_*) \in Y$, denote by $\phi^{q,q_*}(t, \mathrm{x})$ the solution of the Klein-Gordon equation with initial conditions

$$\phi^{q,q_*}(0, \mathrm{x}) = q(\mathrm{x}),\ \phi_t^{q,q_*}(0, \mathrm{x}) = q_*(\mathrm{x}).$$

Then for $g \in \mathbf{P}$ given by

$$g(t, \mathrm{x}) = (at + b \cdot \mathrm{x} + c, \alpha t + \beta \mathrm{x} + \gamma),\ b, \alpha, \gamma \in \mathbb{R}^{d-1},\ a, c, \beta \in \mathbb{R},$$

we have

$$g(q, q_*) = (\tilde{q}, \tilde{q}_*),$$

where

$$\tilde{q}(\mathrm{x}) = \phi^{q,q_*}(b \cdot \mathrm{x} + c, \beta \mathrm{x} + \gamma),$$

$$\tilde{q}_*(\mathrm{x}) = (\partial_\alpha \phi^{q,q_*})(b \cdot \mathrm{x} + c, \beta \mathrm{x} + \gamma) + a\phi_t^{q,q_*}(b \cdot \mathrm{x} + c, \beta \mathrm{x} + \gamma),$$

with ∂_α being the spatial derivative along α.

In particular, note that the *spatial Euclidean group* $\mathrm{SO}(\mathbb{R}^{d-1}) \ltimes \mathbb{R}^{d-1}$ acts by manifest geometric symmetries, while time translations act by the Hamiltonian flow.

Finally, note that this discussion extends in a straightforward way to theories including fermions. In this case, as in fermionic classical mechanics, we get a field theoretic *super*-Poisson bracket on classical fields, which is symmetric rather than skew symmetric if both fields are odd. Also, since odd fields take values in polyspin representations, the Poincaré group should be replaced by its double cover $\widetilde{\mathbf{P}} := \mathrm{Spin}_+(V) \ltimes V$.

For example, consider the theory of a free Dirac spinor ψ on the Minkowski space $\mathbb{R}^d$ with values in the spin representation S of $\mathrm{Spin}_+(1, d-1)$, so the Minkowskian Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\psi, \mathbf{D}\psi)$$

[26] This group depends on d, but for brevity we will not indicate this dependence in the notation.

and the equation of motion is the Dirac equation $\mathbf{D}\psi = 0$. The space of solutions of this equation may be identified with the space of initial conditions $\psi(0, \mathrm{x})$, and the Hamiltonian is

$$H = -\frac{1}{2}\int_{\mathbb{R}^{d-1}} (\psi, \mathbf{D}_s\psi)d\mathrm{x},$$

where $\mathbf{D}_s$ is the spatial Dirac operator. We leave the details to the reader.

10.7. Classical gauge theory

Classical abelian gauge theory is a classical field theory which describes electromagnetic waves. Namely, recall that electromagnetism in the Minkowski space $\mathbb{R}^4$ is described by *Maxwell's equations*. The unknowns in these equations are the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ which are vector fields on the space $\mathbb{R}^3$ with coordinates x^1, x^2, x^3 depending on time $t = x^0$. In natural units Maxwell's equations in vacuum have the following form.

- The Gauss law for electric field: $\nabla \cdot \mathbf{E} = 0$;
- The Gauss law for magnetic field: $\nabla \cdot \mathbf{B} = 0$;
- The Faraday law: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$;
- The Ampère-Maxwell law: $\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$.

It is convenient to rewrite these equations using differential forms on the Minkowski space $\mathbb{R}^4$. To this end, using the standard volume element on $\mathbb{R}^3$, we can view $\mathbf{B}$ as a differential 2-form $F_{\mathbf{B}}$ on $\mathbb{R}^3$ given by

$$F_{\mathbf{B}} = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2.$$

Likewise, using the metric on $\mathbb{R}^3$, we can view $\mathbf{E}$ as a 1-form on $\mathbb{R}^3$ given by

$$\omega_{\mathbf{E}} = E_1 dx^1 + E_2 dx^2 + E_3 dx^3.$$

Define the 2-form F on $\mathbb{R}^4$ by

$$F := \omega_{\mathbf{E}} \wedge dt + F_{\mathbf{B}}.$$

Then

$$dF = d\omega_{\mathbf{E}} \wedge dt + dF_{\mathbf{B}} = d_s\omega_{\mathbf{E}} \wedge dt + \frac{\partial F_{\mathbf{B}}}{\partial t} \wedge dt + d_s F_{\mathbf{B}},$$

where d_s is the spatial differential. Then the second and third Maxwell equations can be written as

$$dF = 0.$$

To write the remaining equations in terms of F, consider the *Hodge $*$-operator* on the space of exterior i-forms on $V = \mathbb{R}^d$ arising from the identification $\wedge^i V \cong \wedge^{d-i} V^* \cong \wedge^{d-i} V$ using the Minkowski metric on V and the standard volume

element. This operator is linear over functions, and for $d = 4, i = 2$ is defined on the standard basis of 2-forms by

$$*(dx^i \wedge dt) = dx^j \wedge dx^k,\ *(dx^j \wedge dx^k) = -dx^i \wedge dt,$$

where $(ijk) = (123) \in S_3$; in particular, $*^2 = -1$. Thus we get

$$*F = F_{\mathbf{E}} - dt \wedge \omega_{\mathbf{B}}.$$

Hence the first and fourth Maxwell equations can be written as

$$d * F = 0.$$

This way of writing Maxwell's equations makes it manifest that they are invariant under the Poincaré group $\mathbf{P} := \mathrm{SO}_+(V) \ltimes V$.

In the presence of electric charges, Maxwell's equations become inhomogeneous. Namely, the first equation (the Gauss law for the electric field) in suitable units of measurement takes the form

$$\nabla \cdot \mathbf{E} = \rho,$$

where ρ is the charge density, while the fourth equation (the Ampère-Maxwell law) takes the form

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J},$$

where $\mathbf{J} = \mathbf{J}(t, \mathrm{x}) = (J_1, J_2, J_3)$ is the *electric current* (flux of charge)—a time-dependent vector field on $\mathbb{R}^3$. The other two Maxwell equations remain unchanged. Thus in the differential form formulation, the first equation $dF = 0$ remains unchanged, while the second equation $d * F = 0$ is replaced by

$$d * F = \mathbb{J},$$

where

$$\mathbb{J} = \rho dx^1 \wedge dx^2 \wedge dx^3 - J_1 dx^2 \wedge dx^3 \wedge dt - J_2 dx^3 \wedge dx^1 \wedge dt - J_3 dx^1 \wedge dx^2 \wedge dt \tag{10.7}$$

is the 4-dimensional electric current, an external field that we consider fixed (imposed on the system). Clearly, this equation can only have a solution if

$$d\mathbb{J} = 0,$$

i.e.,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

(charge conservation).

Example 10.6 (Electrostatics)**.** Suppose we have a solution of the Maxwell equations with $\mathbf{B} = 0$ on $\mathbb{R} \times \mathcal{U}$, where $\mathcal{U}$ is a charge-free open subset of $\mathbb{R}^3$ (say, simply connected). Then $\mathbf{E}$ is t-independent and satisfies the equations

$$\nabla \cdot \mathbf{E} = 0,\ \nabla \times \mathbf{E} = 0.$$

The second equation implies that the form $\omega_{\mathbf{E}}$ is closed (hence exact) on $\mathcal{U}$, i.e., $\mathbf{E}$ is a potential field:

$$\mathbf{E} = -\nabla U,$$

where U is a potential function on $\mathcal{U}$ defined uniquely up to adding a constant. The first equation then yields

$$\Delta U = 0,$$

where Δ is the Laplace operator, i.e., U is harmonic.

For instance, the potential U created by a positive unit charge at the origin is $U(x) = \frac{1}{4\pi|\mathrm{x}|}$, $\mathrm{x} \in \mathbb{R}^3$, so if $\rho(\mathrm{x})$ is the charge density, then

$$U(\mathrm{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\mathrm{u})}{|\mathrm{x} - \mathrm{u}|} d\mathrm{u}.$$

In this case, on the whole $\mathbb{R}^3$ the potential U satisfies the equation

$$\Delta U = -\rho.$$

For example, for a positive unit charge at the origin $\rho(\mathrm{x}) = \delta(\mathrm{x})$, so we have $\Delta U = -\delta$, satisfied by the function $U(\mathrm{x}) = \frac{1}{4\pi|\mathrm{x}|}$.

We would now like to place Maxwell's equations into the realm of Lagrangian field theory, by interpreting them as Euler-Lagrange equations for a suitable Lagrangian. For instance, in Example 10.6 the corresponding action functional (in the charge-free region) is written in terms of the potential $U(\mathrm{x})$:

$$S = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla U(\mathrm{x})|^2 d\mathrm{x}.$$

So in the general case we should try to generalize the potential U to include the magnetic field while preserving Poincaré invariance. There is an obvious natural way to do so. The equation $dF = 0$ means that

$$F = dA$$

is the differential of a 1-form A on $\mathbb{R}^4$ called the *gauge field*; e.g., in Example 10.6 we may take $A = -U(\mathrm{x})dt$. Then the Lagrangian density whose Euler-Lagrange equations are Maxwell's equations can be written in terms of A:

$$\mathcal{L} = \frac{1}{2}|dA|^2 dx = \frac{1}{2} dA \wedge *dA = \frac{1}{2} F \wedge *F. \tag{10.8}$$

More precisely, now the first equation $dF = 0$ is satisfied tautologically since $F = dA$ (*Bianchi identity*), so the actual Euler-Lagrange equation is the second equation $d * F = 0$, or

$$d * dA = 0.$$

However, we now have an essentially new feature: A is only unique up to *gauge transformations* $A \mapsto A + df$, where f is a smooth function. In other words, the gauge field A is **not** a physical observable—only its differential F

(called the *field strength*) is. Of course, in some sense this was already present in Example 10.6, but only in a mild way: the individual potential value $U(\mathrm{x})$ is not observable, but the difference $U(\mathrm{x}) - U(\mathrm{y})$ is, since U is uniquely defined up to adding a constant. On the other hand, A is nonunique in a much more substantial way.

We can think about this in a more geometric way, as follows. It is natural to interpret the gauge field A as a *connection* $\nabla_A = d + A$ on the trivial $U(1)$-bundle over $\mathbb{R}^4$. In this case, $F = dA$ is the *curvature* of this connection. The *group of gauge transformations* is the group of automorphisms of the trivial $U(1)$-bundle, $\mathcal{G} = C^\infty(\mathbb{R}^4, U(1))$. This group acts on the space Conn of connections on the right by

$$(d + A) \circ g = g^{-1}(d + A)g = d + A + g^{-1}dg.$$

So if $f : \mathbb{R}^4 \to \mathbb{R}$ and $g = e^{if}$, then

$$(d + A) \circ g = d + A + df$$

(the factor i drops out because of the identification $\mathrm{Lie}\, U(1) = i\mathbb{R} \cong \mathbb{R}$). Thus the space of fields in this theory (before imposing equations of motion) is the quotient $\mathrm{Conn}/\mathcal{G}$. It follows that the phase space is $\mathrm{Conn}_{\mathrm{M}}/\mathcal{G}$, where $\mathrm{Conn}_{\mathrm{M}} \subset \mathrm{Conn}$ is the space of connections satisfying Maxwell's equation $d * F = 0$.

A similar analysis applies in presence of boundary conditions for A and g on some region in $\mathbb{R}^4$, or asymptotic conditions at infinity.

Note that Lagrangian (10.8) makes sense in any dimension $d \geq 2$ (the case $d = 1$ is trivial). The theory defined by this Lagrangian is called the *pure* $U(1)$ *gauge theory*.

Remark 10.7. As noted above, in presence of charges the Maxwell equation $d * dA = 0$ is replaced by

$$d * dA = \mathbb{J}.$$

To produce this as the Euler-Lagrange equation, the Lagrangian density $\mathcal{L}(A)$ should be replaced by

$$\mathcal{L}(A, \mathbb{J}) = \mathcal{L}(A) + \mathbb{J} \wedge A.$$

Note that while this density is not gauge invariant, the corresponding action

$$S(A, \mathbb{J}) = \int_{\mathbb{R}^4} \mathcal{L}(A, \mathbb{J}) = S(A) + \int_{\mathbb{R}^4} \mathbb{J} \wedge A$$

is: if we replace A by $A + df$, where f is compactly supported, then $S(A, \mathbb{J})$ is changed by $\int_{\mathbb{R}^4} \mathbb{J} \wedge df$, which vanishes by Stokes' formula since $d\mathbb{J} = 0$.

Note also that this theory has a natural generalization to the case when the group $U(1)$ is replaced by any compact Lie group G (possibly noncommutative). In this case, the field is a connection on the trivial G-bundle on $\mathbb{R}^d$ (the *nonabelian gauge field*), which concretely is a 1-form $A = \sum_i A_i dx^i$ on

$\mathbb{R}^d$ with values in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$; the corresponding connection is $\nabla_A = d + A$. The curvature of this connection is the 2-form on $\mathbb{R}^d$ with values in $\mathfrak{g}$ given by the formula

$$F = dA + \frac{1}{2}[A, A],$$

where for $P, Q : \mathbb{R}^d \to \mathfrak{g}$ and 1-forms α, β on $\mathbb{R}^d$

$$[P\alpha, Q\beta] := [P, Q]\alpha \wedge \beta.$$

Thus in coordinates

$$F = \sum_{i<j} F_{ij} dx^i \wedge dx^j,$$

where

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j].$$

To write the action, we need to fix a positive-definite invariant inner product $(,)$ on $\mathfrak{g}$. Then the action is

$$S = \frac{1}{2}\int_{\mathbb{R}^d} |F|^2 dx = \frac{1}{2}\int_{\mathbb{R}^d} (F, *F),$$

which is often written as $\frac{1}{2}\int_{\mathbb{R}^d} F \wedge *F$, with implied application of the inner product $(,)$ to the two F-factors.

The curvature F satisfies the Bianchi identity

$$d_A F = 0,$$

where d_A is the covariant exterior differential, and the Euler-Lagrange equation is

$$d_A * F = 0.$$

These are the *Yang-Mills equations*, and the corresponding classical field theory is called *pure gauge theory* (*or Yang-Mills theory*) on $\mathbb{R}^d$ with *gauge group* G. We note that if G is nonabelian, then this theory is not free (the Lagrangian contains cubic and quartic terms with respect to A).

The gauge symmetry aspect of abelian gauge theory is also present in the nonabelian case: we have the group of gauge transformations $\mathcal{G} = C^\infty(\mathbb{R}^d, G)$ which acts on the space Conn of connections by

$$(d + A) \circ g = g^{-1}(d + A)g = d + g^{-1}Ag + g^{-1}dg, \tag{10.9}$$

i.e.,

$$A \mapsto g^{-1}Ag + g^{-1}dg,$$

where $g^{-1}Ag$ stands for the (right) adjoint action of g on A. So the space of fields before imposing equations of motion is $\mathrm{Conn}/\mathcal{G}$, while the phase space is

$\mathrm{Conn}_{YM}/\mathcal{G}$, where $\mathrm{Conn}_{YM} \subset \mathrm{Conn}$ is the space of solutions of the Yang-Mills equation[27] $d_A * F = 0$.

So, as in the abelian case, A is not a physical observable; and even the curvature F of A is not any more, because it transforms by $F \mapsto g^{-1}Fg$ under gauge transformations. A way to construct a local observable is to fix a G-invariant polynomial p on $\mathfrak{g}$ and consider $p(F(v,w))$, where v, w are vector fields on the spacetime. For example, if R is a finite-dimensional representation of G, then $\mathrm{Tr}_R(F(v,w)^r)$ is a local observable.

Another kind of physical observable that is often considered in gauge theory is the *Wilson line functional*: given a closed curve (path) γ on $\mathbb{R}^d$ and a finite-dimensional representation R of G, the corresponding Wilson line functional is the functional

$$A \to \mathrm{Tr}(\mathrm{Hol}_\gamma(A), R),$$

where $\mathrm{Hol}_\gamma(A)$ is the holonomy of A along the path γ. This is clearly gauge-invariant, so it defines a function on $\mathrm{Conn}_{YM}/\mathcal{G}$, i.e., an observable.

Let us now discuss the Hamiltonian formulation of classical $U(1)$-gauge theory. To this end, it is convenient to apply the gauge transformation

$$g(t,\mathrm{x}) = \mathrm{Hol}_{[0,t]\times\mathrm{x}}(A) = \exp\left(-\int_0^t A_0(u,\mathrm{x})du\right),$$

which brings the connection into a form in which $A_0 = 0$, i.e.,

$$A = \sum_{j=1}^{d-1} A_j dx^j.$$

This is called *temporal gauge*. In temporal gauge, we have

$$F = \sum_{j=1}^{d-1} \frac{\partial A_j}{\partial t} dt \wedge dx^j + \sum_{1\le i<j\le d-1} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right) dx^i \wedge dx^j,$$

so the action is $S = \int_{\mathbb{R}^d} \mathcal{L} dx$, where the Lagrangian $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L} = \frac{1}{2}|F|^2 &= \frac{1}{2}\left(\sum_{j=1}^{d-1}\left(\frac{\partial A_j}{\partial t}\right)^2 - \sum_{1\le i<j\le d-1}\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)^2\right) \\ &= \frac{1}{2}\left(\left|\frac{\partial A}{\partial t}\right|^2 - |d_s A|^2\right).\end{aligned}$$

[27] Usually gauge transformations are required to tend to the identity and connections to become flat at infinity.

It follows that the Hamiltonian is

$$H = \frac{1}{2}\int_{\mathbb{R}^{d-1}} \left(\sum_{i=1}^{d-1} |\dot{A}_i|^2 + \sum_{1\le i<j\le d-1} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right)^2 \right) d\mathrm{x}$$
$$= \frac{1}{2}\int_{\mathbb{R}^{d-1}} (|\dot{A}|^2 + |d_s A|^2) d\mathrm{x},$$

where A is now a connection on $\mathbb{R}^{d-1}$ and $\dot{A}_i$ are conjugate variables to A_i.

This looks somewhat similar to a free scalar field theory with $d-1$ scalars $A_i, 1 \le i \le d-1$ (except that $A, \dot{A}$ transform as vectors and not scalars under the Lorentz group). We would like to use this analogy to define the phase space of $U(1)$ gauge theory equipped with a symplectic structure. Recall that in scalar field theory the phase space was the cotangent bundle to the space of fields, i.e., the space of all pairs (ϕ, ϕ_t), with its natural symplectic form. However, there is an important distinction: the equation $d * F = 0$ together with the temporal gauge condition leads to a constraint[28] on $\dot{A}_i$, namely,

$$\sum_{i=1}^{d-1} \frac{\partial \dot{A}_i}{\partial x^i} = 0. \tag{10.10}$$

Thus in $U(1)$ gauge theory, unlike scalar field theory, the relevant space $\mathcal{C}$ of pairs $(A, \dot{A})$ is not the full cotangent bundle $T^*\mathrm{Conn}^0$ of the space Conn^0 of $U(1)$-connections on $\mathbb{R}^{d-1}$, but rather its subspace cut out by (10.10). The only sensible thing to do in order to give $\mathcal{C}$ a symplectic structure would be to consider the restriction ω of the natural symplectic form of the cotangent bundle $T^*\mathrm{Conn}^0$ to $\mathcal{C}$. But this restriction is degenerate, with radical $\mathcal{C}^\perp$ defined by the equations

$$\dot{A} = 0,\ d_s A = 0,$$

which are equivalent to the condition that $A = d_s f$ for a smooth function f on $\mathbb{R}^3$. Hence ω does **not** define a symplectic (not even a Poisson) structure on $\mathcal{C}$, and so $\mathcal{C}$ cannot be the correct phase space.

The reason this happens is that by passing to the temporal gauge, we actually have not eliminated all the gauge symmetry: there is still a remaining residual group of gauge transformations

$$\mathcal{G}^0 = C^\infty(\mathbb{R}^{d-1}, U(1)) \subset \mathcal{G}$$

of time-independent gauge transformations, which acts by

$$(d_s + A) \circ e^{if} = d_s + A + d_s f.$$

[28] While this equation follows from the equations of motion, we call it a constraint because it constrains the values of the fields $A, \dot{A}$ at $t = 0$.

So to get to the actual phase space of the theory, we must take the quotient by $\mathcal{G}^0$:

$$\mathcal{M} = \mathcal{C}/\mathcal{G}^0 = \mathcal{C}/\mathcal{C}^\perp = T^*(\mathrm{Conn}^0/\mathcal{G}^0).$$

Then, since the form ω is $\mathcal{G}^0$-invariant, it descends to a symplectic form on $\mathcal{M}$, which is actually the canonical symplectic structure on the cotangent bundle $T^*(\mathrm{Conn}^0/\mathcal{G}^0)$, defining the corresponding Poisson bracket. Likewise, since H is $\mathcal{G}^0$-invariant, it defines a function on $\mathcal{M}$, which is the actual Hamiltonian of the theory.

Remark 10.8.

1. Other gauges commonly used for gauge fixing are the *Lorentz gauge* $d * A = 0$ (a gauge invariant under the Lorentz group), the *Coulomb gauge* $d_s * \mathbf{A} = 0$, where $\mathbf{A}$ is the spatial part of A, and *light cone gauge* $\mathbf{v} \cdot A = 0$, where $\mathbf{v} \in \mathbb{R}^d$ is a nonzero vector in the light cone, i.e., $|\mathbf{v}|^2 = 0$; for example, $A_0 + A_{d-1} = 0$.

2. Note that a point on the space $\mathrm{Conn}^0/\mathcal{G}^0$ is specified by $d-2$ functions on $\mathbb{R}^{d-1}$ (or, as physicists say, $d-2$ physical degrees of freedom). Namely, while a connection on $\mathbb{R}^{d-1}$ has $d-1$ degrees of freedom $A_1, \dots, A_{d-1}$, modding out by the gauge group $\mathcal{G}^0$ eliminates one of them. For instance, for $d = 4$, using the light cone gauge $A_0 + A_3 = 0$, we have a residual action of $\mathrm{SO}(2)$ preserving this gauge, and the two remaining degrees of freedom A_1, A_2 transform in its tautological representation $\mathbb{R}^2$. Thus upon complexification the two degrees of freedom can be chosen to be the eigenlines of the $\mathrm{SO}(2)$ action (corresponding to two polarizations of light in classical electrodynamics), i.e., $A_1 \pm iA_2$. A similar situation occurs in quantum theory, as discussed below in Section 10.12.

This discussion extends *mutatis mutandis* to the case of general (possibly nonabelian) gauge group G. We just need to replace the exponential in the formula for $g(t, \mathrm{x})$ by ordered exponential, the expression $d_s A$ for the spatial curvature by $d_s A + \frac{1}{2}[A, A]$, and include the term $g^{-1}Ag$ in the formula for the gauge action. Also in the equation cutting out $\mathcal{C}$, we need to replace derivatives by covariant derivatives, so it takes the form

$$\sum_{i=1}^{d-1} \left(\frac{\partial \dot{A}_i}{\partial x^i} + [A_i, \dot{A}_i] \right) = 0;$$

thus it is now nonlinear and $\mathcal{C}$ is a submanifold (no longer a linear subspace) of $T^*\mathrm{Conn}^0$. In the end, we get the phase space $\mathcal{M} = \mathcal{C}/\mathcal{G}^0 = T^*(\mathrm{Conn}^0/\mathcal{G}^0)$ with Hamiltonian

$$H = \frac{1}{2} \int_{\mathbb{R}^{d-1}} (|\dot{A}|^2 + |d_s A + \tfrac{1}{2}[A, A]|^2) d\mathrm{x}.$$

Finally, note that we have considered gauge theories in which the gauge field is the only field. As noted above, such gauge theories are called *pure*.

However, to describe interactions of elementary particles, one must consider more general gauge theories where besides the gauge field A there are *matter fields*—scalars and spinors, modeling various types of particles. We will see many examples of such theories below.

10.8. Free quantum electrodynamics (pure $U(1)$ gauge theory)

Free quantum electrodynamics is the pure quantum $U(1)$ gauge theory in which the basic field is the gauge field A (the *photon*), a massless vector on spacetime. However, as noted in the previous section, it is not an observable quantity due to gauge symmetry; instead, the observable quantity is the field strength $F = dA$ with components $F_{ij} = \partial_i A_j - \partial_j A_i$. For simplicity, consider this theory in the Euclidean space $\mathbb{R}^d$. Then, the two-point function for the field strength is the Green's function of the Maxwell equations. So in momentum space the Green's function is

$$\langle \widehat{F}_{ij}(p)\widehat{F}_{k\ell}(-p)\rangle = \frac{1}{p^2}\left(\delta_{j\ell}p_i p_k - \delta_{jk}p_i p_\ell - \delta_{i\ell}p_j p_k + \delta_{ik}p_j p_\ell\right), \tag{10.11}$$

and in position space the corresponding Fourier transformed expression is given by the formula

$$\langle F_{ij}(x)F_{k\ell}(0)\rangle = \frac{-2(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk})|x|^2+d(\delta_{j\ell}x^i x^k-\delta_{jk}x^i x^\ell-\delta_{i\ell}x^j x^k+\delta_{ik}x^j x^\ell)}{\Omega_{d-1}|x|^{d+2}}, \tag{10.12}$$

where $\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of the $(d-1)$-sphere.

Since the theory is free, this determines all the correlation functions using Wick's formula.

Exercise 10.9. Derive formulas (10.11) and (10.12).

10.9. Hamiltonian formalism of QFT: the Wightman axioms

To quantize the picture of Section 10.6, we need to define a Hilbert space $\mathcal{H}$ and lift classical observables (local functionals and their integrals) to (densely defined) operators on $\mathcal{H}$, notably lift the classical Hamiltonian H to a quantum Hamiltonian $\widehat{H}$ depending on the Planck constant $\hbar$ which should be a self-adjoint (in general, unbounded) operator on $\mathcal{H}$. Moreover, this should be done in such a way that commutators vanish at $\hbar = 0$ and in first order in $\hbar$ recover the Poisson bracket. We should also have a unitary representation π of the double cover $\widetilde{\mathbf{P}}$ of the Poincaré group on the space $\mathcal{H}$ such that the 1-parameter subgroup of time translations acts by the quantum dynamics 1-parameter group $e^{-it\widehat{H}}$. This generalization of Hamiltonian quantum mechanics can be accomplished by means of so-called *Wightman axioms*, which we now describe.

First of all, for the quantum theory to have good properties, we want the energy to be bounded below. Thus we introduce the following definition. Let us fix an orthogonal decomposition $V = \mathbb{R} \oplus V_s$ into space and time and consider the self-adjoint operator

$$\widehat{H}_\pi := i\frac{d}{dt}|_{t=0}\pi(t,0),$$

where $(t,0) \in V \subset \widetilde{\mathbf{P}}$.

Definition 10.10. A unitary representation $\pi : \widetilde{\mathbf{P}} \to \mathrm{Aut}\,\mathcal{H}$ is said to be *positive energy* if the spectrum of $\widehat{H}_\pi$ is bounded below.

Note that every unitary representation π of V has a spectrum $\sigma(\pi)$, which is a closed subset of $V^* \cong V$; namely, $\sigma(\pi)$ is the set of characters of V that occur (discretely or continuously) in π (i.e., the smallest set containing the support of the Fourier transform of the distribution $\langle w_1, \pi(v)w_2 \rangle$, $v \in V$, for any $w_1, w_2 \in \mathcal{H}$).

Lemma 10.11. *Suppose* $\dim V \geq 2$. *Then π is positive energy if and only if $\sigma(\pi)$ is contained in the positive part of the solid light cone, $\overline{V}_+$.*

Proof. By definition, π is of positive energy iff the orthogonal projection of $\sigma(\pi)$ onto the dual of the time axis is bounded below. Since $\sigma(\pi)$ is invariant under $\mathrm{SO}_+(V)$, this implies the statement (an $\mathrm{SO}_+(V)$-orbit on V has bounded below projection iff it is contained in $\overline{V}_+$). □

Note that this is false for $d = 1$ (quantum mechanics), where the Hamiltonian can be shifted by a constant without any effect on the theory. But the latter is no longer so in a relativistic quantum field theory on a Minkowski space of dimension > 1.

We are now ready to give Wightman's definition of a QFT. Let $\mathcal{S} = \mathcal{S}(V)$ be the Schwartz space of V.

Definition 10.12. A *Wightman QFT* on a Minkowski space V entails the following data:

1. A finite-dimensional real algebraic super-representation $R = R_0 \oplus R_1$ of $\mathrm{Spin}_+(V)$ (the *field space*).

2. A super Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ carrying a positive energy unitary representation $\pi : \widetilde{\mathbf{P}} \to \mathrm{Aut}\,\mathcal{H}$ of the double cover of the Poincaré group, $\widetilde{\mathbf{P}} = \mathrm{Spin}_+(V) \ltimes V$.

3. A dense $\widetilde{\mathbf{P}}$-stable subspace $\mathcal{D} \subset \mathcal{H}$.

4. A $\widetilde{\mathbf{P}}$-invariant unit vector $\Omega \in \mathcal{D}$ called the *vacuum vector*.

5. A $\widetilde{\mathbf{P}}$-invariant even linear map $\phi : \mathcal{S} \otimes R^* \to \mathrm{End}\,\mathcal{D}$ called the *field map*

This data is subject to the following axioms.

A1. If f is real, then $\phi(f)$ is Hermitian symmetric (in the supersense).

A2. ϕ is weakly continuous, i.e., for every $w_1, w_2 \in \mathcal{D}$, the functional $\mathcal{S} \otimes R^* \to \mathbb{C}$ defined by $f \mapsto \langle w_1, \phi(f) w_2 \rangle$ is continuous.

A3. $\mathcal{D}$ is spanned (algebraically) by vectors $\phi(f_1) \dots \phi(f_n)\Omega$.

A4. *Space locality*: If f_1, f_2 have spacelike separated supports, i.e., for any $v_1 \in \operatorname{supp} f_1$, $v_2 \in \operatorname{supp} f_2$, we have $|v_1 - v_2|^2 < 0$, then

$$[\phi(f_1), \phi(f_2)] = 0$$

(with commutator understood in the supersense).

In addition, if $\mathcal{H}^{\widetilde{\mathbf{P}}} = \mathbb{C}\Omega$, one says that we have a Wightman QFT with a *unique vacuum*.

We will also always assume that our QFT is *nondegenerate*, i.e., for every irreducible subrepresentation $E \subset R_j^*$, $j = 0, 1$, one has $\phi|_{\mathcal{S}\otimes E} \neq 0$; otherwise, we can simply remove this subrepresentation without any effect on the theory.

A fundamental fact about Wightman QFT is the following theorem, which we will not prove here (for the proof see [**QFS**, v. 1, p. 382]). Let ζ be the generator of the kernel of the map $\mathrm{Spin}_+(V) \to \mathrm{SO}_+(V)$, so $\zeta^2 = 1$.

Theorem 10.13. (*The spin-statistics theorem*) *If $E \subset R_j^*$ is a subrepresentation, then $\zeta|_E = (-1)^j$.*

In other words, there is a relationship between the *spin* (mod integers) of a quantum field (essentially, the eigenvalue of ζ) and its *statistics*, i.e., whether it is bosonic (even) or fermionic (odd). Namely, the theorem says that all bosonic fields must have $\zeta = 1$ (integer spin) and all fermionic fields must have $\zeta = -1$ (half-integer spin).

Remark 10.14. We will see that the theory of free bosons and fermions can be naturally formulated as a Wightman QFT. Moreover, this is also the case for a number of nonfree theories, which is the subject of a difficult area of mathematical physics called *constructive field theory*. Still, most theories that physicists really care about are either not known to be Wightman QFT or simply fail to be ones for various reasons (perturbative theories, low energy effective theories, nonunitary theories, Euclidean theories, theories living on compact manifolds, etc.) Thus we will view Wightman axioms just as one (somewhat limited) rigorous model for our mathematical understanding of QFT.

Remark 10.15. Note that our formulation of the spin-statistics theorem holds in particular for $d = 2$. This is a somewhat subtle case. Namely, for $d = 2$, we have $\mathrm{SO}_+(V) = \mathbb{R}_{>0}$, and the group $\mathrm{Spin}_+(V) \cong \mathrm{SO}_+(V) \times \{\pm 1\}$ is disconnected, with the element ζ generating the $\{\pm 1\}$. In particular, we have a 1-dimensional

character ξ of $\mathrm{Spin}_+(V)$ which kills $\mathrm{SO}_+(V)$ and sends ζ to -1. At first sight, twisting a field by this character would lead to a violation of Theorem 10.13. However, this twisting is not allowed by our version of Wightman axioms, since ξ is not an algebraic representation. Indeed, in a Euclidean signature, unlike a Minkowskian one, the short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1$$

does not split (it is a connected double cover of the circle), so the analogue of ξ does not exist. In other words, the twist by ξ would be inconsistent with a possibility of Wick rotation.

10.10. Wightman functions

Proposition 10.16. *In a Wightman QFT on a Minkowski space V, for every $n \geq 1$ there exists a unique tempered distribution W_n on V^n valued in $R^{*\otimes n}$ such that*

$$W_n(f_1 \boxtimes \cdots \boxtimes f_n) = \langle\, \Omega, \phi(f_1) \dots \phi(f_n)\Omega \,\rangle.$$

We leave the proof of this proposition as an exercise.

We will therefore think of W_n as a (generalized) function on V^n valued in $R^{*\otimes n}$, denoted $W_n(x_1, \dots, x_n)$, so that

$$W_n(f_1 \boxtimes \cdots \boxtimes f_n) = \int_{V^n} W_n(x_1, \dots, x_n) f_1(x_1) \dots f_n(x_n) dx_1 \dots dx_n,$$

where the product on the right-hand side involves contraction of corresponding copies of R and R^*. Thus, given $u_1, \dots, u_n \in R$, we have the scalar-valued distribution

$$W_n^{u_1,\dots,u_n}(x_1, \dots, x_n) := (W_n(x_1, \dots, x_n), u_1 \otimes \cdots \otimes u_n).$$

In other words, we may define an operator-valued distribution $\phi(x)$ such that

$$\phi(f) = \int_V \phi(x) f(x) dx;$$

then

$$W_n(x_1, \dots, x_n) = \langle\, \Omega, \phi(x_1) \dots \phi(x_n)\Omega \,\rangle.$$

Definition 10.17. The generalized functions $W_n(x_1, \dots, x_n)$ are called the *Wightman (correlation) functions* of the Wightman QFT.

Note that Wightman functions completely determine the Wightman QFT as follows. Let $\widetilde{\mathcal{D}} := T(\mathcal{S} \otimes R)$ (the tensor algebra), so it is spanned by elements $f_1 \otimes f_2 \otimes \cdots \otimes f_n$, $f_i \in \mathcal{S} \otimes R$. Define the inner product on $\widetilde{\mathcal{D}}$ by

$$\langle\, f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \,\rangle$$
$$:= (-1)^{\sum_{i<j} p(f_i)p(f_j)} W_{n+m}(\overline{f}_n \boxtimes \cdots \boxtimes \overline{f}_1 \boxtimes g_1 \boxtimes \cdots \boxtimes g_m).$$

It is easy to see that this inner product is well defined, and

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle = \langle \phi(f_1) \dots \phi(f_n)\Omega, \phi(g_1) \dots \phi(g_m)\Omega \rangle$$

(where f_i are purely odd or purely even). Thus the inner product $\langle , \rangle$ on $\widetilde{\mathcal{D}}$ is nonnegative definite, the Hilbert space $\mathcal{H}$ can be recovered as the completion of $\widetilde{\mathcal{D}}$ with respect to $\langle , \rangle$, and $\mathcal{D}$ is the image of $\widetilde{\mathcal{D}}$ in $\mathcal{H}$. (Note that the map $\widetilde{\mathcal{D}} \to \mathcal{H}$ need not be injective.) Moreover, the vector Ω is the image of $1 \in \widetilde{\mathcal{D}}$ in $\mathcal{D}$, and the representation π is obtained by extending the action of $\widetilde{\mathbf{P}}$ on $\mathcal{D}$ (which descends from $\widetilde{\mathcal{D}}$) by continuity.

So we can ask: what conditions should Wightman functions satisfy to define a Wightman QFT? Let us list some necessary conditions, which follow from the above discussion. To this end, denote by

$$W : T(\mathcal{S} \otimes R) \to \mathbb{C}$$

the natural linear map and by $* : T(\mathcal{S} \otimes R) \to T(\mathcal{S} \otimes R)$ the antilinear map given by $(f_1 \otimes \cdots \otimes f_n)^* = (-1)^{\sum_{i<j} p(f_i)p(f_j)} \overline{f}_n \otimes \cdots \otimes \overline{f}_1$.

Proposition 10.18. *The Wightman functions W_n of a Wightman QFT satisfy the following properties.*

(1) *W_n are $\widetilde{\mathbf{P}}$-invariant.*

(2) *Positive energy: the Fourier transform of W_n is supported on the set of $(p_1, \dots, p_n) \in V^n$ such that $p_{j+1} + \cdots + p_n \in \overline{V}_+$ for $j = 1, \dots, n-1$.*

(3) $W_n(f^*) = \overline{W_n(f)}$.

(4) *Space locality:*

$$W_n^{u_1,\dots,u_i,u_{i+1},\dots,u_n}(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = W_n^{u_1,\dots,u_{i+1},u_i,\dots,u_n}(x_1, \dots, x_{i+1}, x_i, \dots, x_n)$$

if $|x_i - x_{i+1}|^2 < 0$.

(5) *Positivity: $W(f^* \otimes f) \geq 0$ for any $f \in T(\mathcal{S} \otimes R)$.*

Proof. (1) follows from the invariance of the vacuum vector and the field map. (3) follows from the fact that for real f, $\phi(f)$ is Hermitian symmetric. (4) follows from the space locality axiom. (5) follows from positivity of the inner product on $\mathcal{H}$. So it remains to prove (2). Let us do so for $n = 2$, the general proof is similar (see [**QFS**, pp. 381–382]).

By translation invariance we have

$$W_2(v_1, v_2) = \mathbb{W}(v),$$

where $v = v_2 - v_1$. Thus, our job is to show that the Fourier transform of $\mathbb{W}$ is supported on $\overline{V}_+$. We have

$$\mathbb{W}(v) = \langle \Omega, \phi(0)\phi(v)\Omega \rangle = \langle \Omega, \phi(0)\pi(v)\phi(0)\pi(-v)\Omega \rangle = \langle \phi(0)\Omega, \pi(v)\phi(0)\Omega \rangle.$$

So the statement follows from the fact that every character of V which occurs in $\mathcal{H}$ belongs to $\overline{V}_+$. □

In fact, it turns out that these necessary conditions are also sufficient, and we have the following theorem, which can be proved by following the above reconstruction procedure (but we will not give a proof):

Theorem 10.19. *If a collection of distributions W_n satisfies conditions* (1)–(5) *of Proposition* 10.18*, then they define a Wightman QFT.*

Remark 10.20. The 1-point Wightman function $W_1(x) = \langle \Omega, \phi(x)\Omega \rangle$ is a constant c by translation invariance, i.e., it is an element of R^*, and by invariance under rotations it is in $(R^*)^{\mathrm{Spin}_+(V)}$. Thus we may (and will) assume without loss of generality that $c = 0$ (otherwise, we can replace $\phi(x)$ by $\phi(x) - c$). So we may assume without loss of generality that $W_1 = 0$.

Remark 10.21. The positivity property for the 2-point function can be written as

$$\int_{V^2} \mathbb{W}(x_2 - x_1)\overline{f(x_1)}f(x_2)dx_1dx_2 \geq 0,$$

where $\mathbb{W}(x) = W_2(0, x)$. Thus, taking Fourier transforms, we have

$$\int_V \widehat{\mathbb{W}}(p)\overline{\widehat{f}(p)}\widehat{f}(p)dp \geq 0.$$

This shows that $\widehat{\mathbb{W}}(p)dp$ is a measure concentrated on $\overline{V}_+$ and valued in non-negative Hermitian forms on $R_{\mathbb{C}}$.

Remark 10.22. One can formulate an analogue of Wightman axioms in Euclidean signature, using analytic continuations of Wightman functions to imaginary time called *Schwinger functions*. These are the *Osterwalder-Schrader axioms*, which are discussed in [**QFS**, pp. 387–394].

10.11. The CPT symmetry

All quantum field theories have an important symmetry called the *CPT symmetry*, which follows from the Wightman axioms ([**QFS**, p. 392], [**T3**, Section 2.3]). This symmetry is a composition of three commuting transformations: conjugating the charges (C), changing the orientation of space by an element of $O(d-1)$ with determinant -1 (P), and changing the direction of time (T) (see Section 10.1 for a discussion of P and T). Note that individually these transformations need not be symmetries of the theory, only their composition must. The CPT symmetry must act on the Hilbert space $\mathcal{H}$ of the theory by an involutive *anti-unitary* operator, i.e., an anti-linear operator preserving the norm.[29]

[29] Namely, T acts by an anti-unitary operator and C, P by unitary operators.

In other words, this means that the unitary action of the Lorentz group $\mathrm{Spin}_+(1,d-1)$ on $\mathcal{H}$ extends to an action of $\mathrm{Spin}(1,d-1)$, so that the non-identity component of this Lie group acts anti-unitarily (in particular, it does not matter which orientation reversing element in $O(d-1)$ we choose for P, as the composition of any two such elements belongs to $\mathrm{SO}(d-1)$, hence to the Lorentz group).

In particular, *charge conjugation* means that fields or particles living in complex representations of some symmetry group (subgroup of the Poincaré group, gauge group, etc.) are mapped to their anti-partners living in complex conjugate representations (for compact groups, this is the same as dual representations). So one consequence of the CPT symmetry is that the field representation of $\mathrm{Spin}_+(1,d-1)$ is real in Minkowski signature (which is a part of Wightman axioms).

For example, the CPT symmetry sends chiral (Weyl) spinors to anti-chiral ones in spacetime dimensions $4k$ (but not so for Majorana-Weyl spinors in dimensions $4k+2$), since the chiral spin representations $S_\pm$ of $\mathrm{Spin}(1,4k-1)$, unlike those of $\mathrm{Spin}_+(1,4k+1)$, have no real structure (see [**QFS**, p. 129]). Thus, in a QFT in dimensions $4k$ the number of chiral and anti-chiral spinors should be the same, while in a QFT in dimensions $4k+2$ the CPT symmetry does not prohibit chiral spinors without anti-chiral partners.

10.12. The mass spectrum and particle content of a quantum field theory

Let $\mathcal{H}^{(1)} \subset \mathcal{H}$ be the closure of the span of vectors $\phi(x)\Omega$, $x \in V$. It is called the space of 1-*particle states*, and it is clearly a $\widetilde{\mathbf{P}}$-subrepresentation of $\mathcal{H}$. The *mass spectrum* of the theory is determined by the structure of this representation. So we need to discuss the representation theory of $\widetilde{\mathbf{P}}$.

Since $\widetilde{\mathbf{P}}$ is a semidirect product, its irreducible unitary representations are unitarily induced. Namely, let $\mathcal{O}$ be an orbit of $\mathrm{Spin}_+(V)$ on V. The stabilizer $\widetilde{\mathbf{P}}_0$ of a point $\mathbf{v}_0 \in \mathcal{O}$ is called the *little group*. Let ρ be an irreducible unitary representation of $\widetilde{\mathbf{P}}_0$. Then ρ defines an equivariant unitary bundle on $\mathcal{O}$ with total space $(\widetilde{\mathbf{P}}\times\rho)/\widetilde{\mathbf{P}}_0$ where $\widetilde{\mathbf{P}}_0$ acts diagonally. Thus we can consider the space $\mathcal{H}_{\mathcal{O},\rho}$ of square integrable half-densities on $\mathcal{O}$ with values in this bundle. This space carries a unitary representation of $\widetilde{\mathbf{P}}$. A theorem of Mackey then says that this unitary representation is irreducible, and all irreducible representations of $\widetilde{\mathbf{P}}$ arise uniquely in this way. For example, if $\mathcal{O} = \{0\}$, then $\mathcal{H}_{0,\rho}$ is just a unitary irreducible representation of $\mathrm{Spin}_+(V)$.

Taking Fourier transforms (see Remark 10.21), we see that by the Wightman axioms, if $\mathcal{H}_{\mathcal{O},\rho}$ occurs in $\mathcal{H}^{(1)}$, then ρ needs to be finite dimensional (as

$\mathcal{H}^{(1)}$ is generated by a quotient of $S \otimes R$, and R is finite dimensional). For example, for $d \geq 3$ and $\mathcal{O} = \langle 0 \rangle$ the only choice is the trivial representation, as the group $\mathrm{Spin}_+(V)$ is a connected semisimple noncompact Lie group. Moreover, if the theory has a unique vacuum, then the trivial representation occurs in $\mathcal{H}$ discretely with multiplicity 1 as the span of the vacuum vector Ω. As $\mathcal{H}^{(1)}$ is orthogonal to Ω (since $W_1 = 0$), we see that the trivial representation cannot occur in $\mathcal{H}^{(1)}$.

The irreducible constituents of $\mathcal{H}^{(1)}$ are referred to as *particles*, and their collection is called the *particle content* of the theory. Assume that the theory has *finitely many particle types*, i.e., the space $\mathcal{H}^{(1)}$ is a direct sum of finitely many irreducible representations (this holds in physically relevant examples). For every particle, the dimension of ρ is called the *number of physical degrees of freedom* of this particle. Physical degrees of freedom can be bosonic or fermionic, depending on whether the particle is a boson or a fermion.

Now we are ready to discuss the precise structure of the representation $\mathcal{H}^{(1)}$. By the positive energy condition, the only orbits that can occur are X_m^+ defined by $E = \sqrt{p_s^2 + m^2}$, $E > 0$, where $p = (E, p_s)$ with E denoting the energy and p_s the spatial part of the momentum (here for $d = 2$ the set X_0^+ falls into two orbits X_0^{++} and X_0^{+-} defined by $E = \pm p_s > 0$). For $m > 0$ this is the upper sheet of a 2-sheeted hyperboloid and for $m = 0$ it is the upper part of the light cone (which is a union of two orbits for $d = 2$, otherwise a single orbit).

If $m > 0$, we may take $\mathbf{v}_0 = (m, 0, \dots, 0)$, then $\widetilde{\mathbf{P}}_0 = \mathrm{Spin}(d-1)$, so ρ is a (necessarily finite-dimensional) unitary representation of this compact Lie group. Physicists say that the representation $\mathcal{H}_{\mathcal{O},\rho}$ corresponds to a *massive particle of mass m and type* ρ. Particles arising in physically relevant quantum field theories are usually *scalars* ($\rho = \mathbb{C}$), *spinors* (ρ is a spin representation of $\mathrm{Spin}(d-1)$) and *vectors* ($\rho = \mathbb{C}^{d-1}$ is the vector representation $\mathrm{Spin}(d-1)$). Note that by the spin-statistics theorem, for $d \geq 3$ scalars and vectors are bosons and spinors are fermions. Note also that for $d = 2$, we have $\widetilde{\mathbf{P}}_0 = 1$, so there is no notion of spin for quantum particles (even though there is one for the original fields, i.e., representations of the Lorentz group $\mathrm{Spin}_+(1,1) = \mathbb{R}_{>0}$ which occur in R), and we just have bosons and fermions (both having $\rho = \mathbb{C}$).

If $m = 0$, $d \geq 3$, then we can take $\mathbf{v}_0 = (1, 1, 0, \dots, 0)$. Then the little group is the nonreductive Lie group $\widetilde{\mathbf{P}}_0 = \mathrm{Spin}(d-2) \ltimes \mathbb{R}^{d-2}$. (Note that this $\mathbb{R}^{d-2}$ is not a part of the translation group V, but a subgroup of the Lorentz group $\mathrm{Spin}^+(1, d-1)$!) Since ρ is contained in the restriction of R to the little group, the subgroup $\mathbb{R}^{d-2}$ has to act trivially on ρ, so ρ is an irreducible representation of the compact Lie group $\mathrm{Spin}(d-2)$ (in fact, if $d \geq 4$, any finite-dimensional unitary representation of the little group automatically has this property). Physicists say that the representation $\mathcal{H}_{\mathcal{O},\rho}$ corresponds to a

massless particle of type ρ. The classification of massless particles is the same as for massive ones, and the most common particles are again scalars ($\rho = \mathbb{C}$), spinors (ρ is a spin representation of $\mathrm{Spin}(d-2)$), and vectors ($\rho = \mathbb{C}^{d-2}$ is the vector representation $\mathrm{Spin}(d-2)$). However, note that since for massless particles ρ is a representation of $\mathrm{Spin}(d-2)$ rather than $\mathrm{Spin}(d-1)$, they in general have fewer physical degrees of freedom than massive ones. For example, a massive vector has one more physical degree of freedom than a massless one, and for even $d \geq 4$ a massive spinor has twice as many physical degrees of freedom than a massless one.

Note also that if $d = 3$, then $\mathrm{Spin}(d-2) = 1$, so as in the $d = 2$ case there is no notion of spin of quantum particles (even though there is one for the original fields), and we just have bosons and fermions (both having $\rho = \mathbb{C}$); thus for $d = 3$ there is no distinction between scalar and vector particles.

Finally, if $m = 0, d = 2$, then there are two choices for $\mathbf{v}_0$: $(1,1)$ and $(1,-1)$. They have trivial stabilizer, so the little group is trivial and $\rho = \mathbb{C}$. Thus we have two types of massless particles of each kind (bosons and fermions): *right-moving* and *left-moving*, corresponding to the two choices of $\mathbf{v}_0$. These particles are called this way since the corresponding operators $\phi(x)$ satisfy the conditions $\phi(t,\mathrm{x}) = \phi(0,\mathrm{x}-t)$, $\phi(t,\mathrm{x}) = \phi(0,\mathrm{x}+t)$, respectively, which classically would be right-moving and left-moving waves.

Let us now say a few words about vector particles, which we have not yet seen. In physically relevant theories such particles come from *gauge fields* (see Section 10.7). For example, a $U(1)$ gauge particle is called a *photon*, and it gives rise to a massless vector (see Section 10.14).

We also note that massless vectors for $d = 4$ and massive vectors for $d = 3$ behave in a special way. Namely, in both cases the compact part of the little group is $\mathrm{Spin}(2)$ whose 2-dimensional (complex) vector representation ρ is reducible and splits as $\rho_+ \oplus \rho_-$, where ρ_+ and ρ_- are 1-dimensional characters. This yields the decomposition

$$\mathcal{H}_{\mathcal{O},\rho} = \mathcal{H}_{\mathcal{O},\rho_+} \oplus \mathcal{H}_{\mathcal{O},\rho_-}.$$

This means that vector particles come in two different polarizations corresponding to the eigenvalues $\lambda = \pm 1$ of the suitable generator $\mathbf{h} \in \mathrm{Lie}\,\mathrm{Spin}(2)$ (called *helicities*). However, in a consistent QFT such particles of both helicities have to be present in equal numbers due to the CPT symmetry (Section 10.11), since it changes representations of any group to complex conjugate representations and hence switches particles of helicities λ and $-\lambda$. Thus when talking about a vector particle in these cases, one always means a pair of half-vector particles of opposite helicities ± 1. This way, a massive (respectively, massless) vector particle in any dimension d has $d-1$ (respectively, $d-2$) physical degrees of freedom.

The set M of numbers m corresponding to representations $\mathcal{H}_{X_m^+,\rho}$ (or $m = 0$ for $\mathcal{H}_{X_0^{+\pm},\rho}$ if $d = 2$) occurring in $\mathcal{H}^{(1)}$, i.e., the collection of the masses of all particles, is called the *mass spectrum* of the theory. One says that the theory has a *mass gap* when $\inf M = m > 0$. In this case the spectrum of $\widehat{H}$ is $\{0\} \cup [m, +\infty)$, so there is a gap between 0 and m. To find the mass spectrum, it suffices to look at the function $\widehat{\mathbb{W}}$: the mass spectrum is just the intersection of its support with the energy axis (this follows from Remark 10.21).

Remark 10.23. We would like to stress once again the distinction between the description of a QFT in terms of *fields* (based on the representation theory of $\mathrm{Spin}(d)$) and its description in terms of *quantum particles* (based on the representation theory of $\mathrm{Spin}(d-1)$ for massive particles and $\mathrm{Spin}(d-2)$ for massless ones). The somewhat subtle interplay between these two descriptions is central in understanding the structure and behavior of quantum field theories.

Exercise 10.24. Let $\mathcal{H}_m$ be the unitary representation of $\mathbf{P}$ corresponding to a massive vector particle, and let $\mathcal{H}_0^0$ be the unitary representation of $\mathbf{P}$ corresponding to a massless vector particle.

(i) Show that for $d \geq 4$, $\mathcal{H}_0^0$ cannot be deformed into a massive representation of $\mathbf{P}$ (i.e., one concentrated on X_m^+, $m \neq 0$).

(ii) Show that $\mathcal{H}_0^0 \oplus L^2(X_0^+)$ deforms into $\mathcal{H}_m$ with $m > 0$; i.e. the combination of a massless vector and a massless scalar deforms into a massive vector. This is the *Higgs mechanism*, discussed in more detail in Section 11.8.

10.13. Spin and helicity of particles

To a representation ρ of the group $\mathrm{Spin}(d-1)$ arising from a massive particle in dimension $d \geq 3$ or of the group $\mathrm{Spin}(d-2)$ arising from a massless particle in dimension $d \geq 4$, one can attach an important invariant called *spin*. Namely, let $\mathrm{Spin}(2)$ be a subgroup of $\mathrm{Spin}(d-1)$, $\mathrm{Spin}(d-2)$ corresponding to a plane $\mathbb{R}^2 \subset \mathbb{R}^{d-1}, \mathbb{R}^{d-2}$ (all such subgroups are conjugate, so it does not matter which one we take), and let $\mathbf{h}$ be a generator of Lie $\mathrm{Spin}(2)$ which has eigenvalues ± 1 on its vector representation. Then the spin of a particle of type ρ is defined to be the largest eigenvalue of $\mathbf{h}$ on ρ. It is easy to see that if $\rho = \rho_\lambda$ has the highest weight $\lambda = (\lambda_1, \ldots, \lambda_r)$, then

$$\mathrm{spin}(\rho_\lambda) = \max_{1 \leq j \leq r} \lambda_j.$$

Thus the spin takes values in $\frac{1}{2}\mathbb{Z}$.

For example, for massless particles in dimensions 4, 5 and massive particles in dimensions 3, 4, the compact part of the little group is $\mathrm{Spin}(2)$ or $\mathrm{Spin}(3)$, so it has rank $r = 1$. Thus in these cases the spin coincides with the highest weight.

It is not hard to show that in a free QFT, the spin of a particle can be arbitrarily high, and any type ρ of particles is possible. However, in an interacting QFT, there are strong restrictions on the spin of particles; e.g., the *Weinberg-Witten theorem* ([**QFS**, pp. 1152–1155]) says that in a renormalizable QFT (see Section 13.3), one cannot have interacting massless particles of spin > 1, so the only possible spins are $0, \frac{1}{2}, 1$.

Let us now discuss the notion of *helicity* for massive particles. Let ρ be as above, and let $p_s \in \mathbb{R}^{d-1}$ be the spatial momentum vector for a particle of mass m (i.e., $p = (E, p_s)$ is the relativistic momentum and $p_s^2 = E^2 - m^2$, where E is the energy). Let $G(p_s) \subset \mathrm{Spin}(d-1)$ be the stabilizer of p_s. If $p_s \neq 0$, then $G(p_s) \cong \mathrm{Spin}(d-2)$, and one can decompose ρ into a direct sum of irreducible subrepresentations $\rho_j(p_s)$ of $G(p_s)$ (these representations are independent of p_s when viewed as representations of $\mathrm{Spin}(d-2)$, so in this case we will denote them just by ρ_j). A vector $\psi \in \rho_j(p_s)$ is then said to have (pure) helicity $\mathrm{spin}(\rho_j)$ at p_s. For example, for $d = 4$, if the representation ρ has spin j, then it has $2j+1$ independent states of helicities $j, j-1, \dots, -j$ (at every p_s).

For massless particles, since $G(p_s)$ coincides with the compact part of the little group, there is no difference between the notions of spin and helicity, and in particular helicity of a particle is a single quantum number independent of p_s. In this case, physicists often prefer to use the term "helicity" instead of "spin".[30]

10.14. The Hamiltonian formulation of free QFTs

Let us now construct a Wightman QFT corresponding to a scalar boson of mass $m > 0$. Recall that in the Lagrangian setting we had a 2-point function $G_M(x_2 - x_1)$, where $G_M(x)$ is a distribution satisfying the Klein-Gordon equation

$$(\Box + m^2)G_M = i\delta.$$

So at first sight for the corresponding Wightman QFT, we want to have $\mathbb{W}(x) = G_M(x)$, so that the Lagrangian and Hamiltonian approaches agree. However, the function $G_M(x)$ is even, while for $\mathbb{W}(x)$ we are supposed to have $\mathbb{W}(-x) = \overline{\mathbb{W}(x)}$, so our equality needs to be relaxed. In fact, the correct condition is that the identity $\mathbb{W}(x) = G_M(x)$ only needs to hold when x is spacelike or when $x \in \overline{V}_+$. When $x \in \overline{V}_-$, we should rather have $\mathbb{W}(x) = \overline{G_M(x)}$. In other words,

$$G_M(x_2 - x_1) = W_2^T(x_1, x_2)$$

[30] More precisely, physicists say that spin is measured when the particle is at rest, i.e., at $p_s = 0$, when $G(p_s) = \mathrm{Spin}(d-1)$. At the same time they say that helicity is the projection of spin to the direction of motion when $p_s \neq 0$, and that in the massless case the particle is always moving at the speed of light, so we cannot put it at rest by a Lorentz transformation and measure its spin, and can only measure its helicity.

is the so-called *time ordered* 2-point function, i.e., one obtained from $W_2(x_1, x_2)$ when x_1, x_2 are put in chronological order (where in the spacelike separated case the order does not matter due to space locality). In fact, we have already seen this phenomenon in quantum mechanics when we found that the Feynman-Kac formula only holds when the times in the correlator are ordered chronologically (Remark 7.29(2)).

We claim that with this definition the function $\mathbb{W}(x)$ satisfies the Klein-Gordon equation

$$(\Box + m^2)\mathbb{W} = 0$$

on the nose (without the delta-function on the right-hand side). Indeed, we have $\operatorname{Re}\mathbb{W}(x) = \operatorname{Re} G_M(x)$, which satisfies the Klein-Gordon equation, so it remains to show that $\operatorname{Im}\mathbb{W}(x)$ satisfies it as well. But it is easy to see that $(\Box + m^2)\operatorname{Im}\mathbb{W}(x)$ is a distribution supported at the origin of homogeneity degree $-d$, so it is a multiple of δ. Since $\operatorname{Im}\mathbb{W}(x)$ is an odd function, this distribution must be zero, as claimed.

Also, since $G_M(x)$ is real for spacelike x, we get $\mathbb{W}(-x) = \overline{\mathbb{W}(x)}$. Thus the Fourier transform $\widehat{\mathbb{W}}(p)$ is real valued, supported on the hyperboloid X_m and invariant under $\mathrm{SO}_+(V)$. It follows that

$$\widehat{\mathbb{W}}(p) = c_+\delta_{X_m^+} + c_-\delta_{X_m^-}.$$

where $c_\pm \in \mathbb{R}$, but in fact it can be shown that only $\delta_{X_m^+}$ occurs (this follows from the exponential decay of the Euclidean 2-point correlation function at infinity). Thus

$$\widehat{\mathbb{W}}(p) = c\delta_{X_m^+}.$$

In fact, one can show that $c = 2\pi$.

Similarly, we define higher W_n for $n > 2$ by the Wick formula, and this analysis implies after some work that these functions define a Wightman QFT.

In this case, $\mathcal{H}^{(1)} = L^2(X_m^+)$, so we have a single particle of mass m.

The Hamiltonian theory of a free massless scalar is defined similarly, by setting $m = 0$ in the above formulas.

Analogously, one can define the Hamiltonian theory of a massive and massless spinor using the formula for the Green's function in Section 10.5. Namely, as discussed in Section 10.12, a massive spinor particle corresponds to an irreducible spin representation ρ of $\mathrm{Spin}(d-1)$, while a massless one corresponds to an irreducible spin representation ρ of $\mathrm{Spin}(d-2)$. Here we must remember that the CPT symmetry (Section 10.11) requires that every particle come with its antiparticle, which in some dimensions implies that every chiral spinor particle must come with an anti-chiral partner. For example, this is so for massless spinors in dimension $d = 4$.

Finally, the theory of a gauge (vector) particle is defined similarly, using the formula for the Green's function in Section 10.8; it is a massless particle that corresponds to the vector representation ρ of $\mathrm{Spin}(d-2)$. Note that for $d=4$ this representation is reducible and splits into two parts corresponding to two polarizations of a photon; however, both must be present simultaneously due to CPT symmetry (see Section 10.12).

For more detail, we refer the reader to [**Co**, **PS**, **QFS**].

10.15. Measurable quantities

While Wightman (=correlation) functions of a QFT are physical observables, they are not measurable quantities. What can be measured in a lab for a QFT that describes real-world particle interactions (such as the *standard model*) are *masses* of particles and also *scattering amplitudes* (*S-matrices*), which are nontrivial only in interacting theories. These quantities can also be extracted theoretically from the Wightman functions, which allows one to check QFT answers experimentally.

Namely, as we have seen above, to find the masses of particles, one should consider the Fourier transform $\widehat{W}_2^T(p,-p)$ of the *time ordered* 2-point function $W_2^T(x_1,x_2)$ (see Section 10.14). It is a meromorphic function with poles (in the real domain) located on surfaces X_m^+ defined in Section 10.12 for various values $m=m_j$. These m_j are precisely the masses of the particles ξ_j of the theory (note that there can be several types of particles having the same mass).

Similarly, scattering amplitudes are determined from poles of the Fourier transform $\widehat{W}_n^T(p_1,\dots,p_n)$ of the time ordered n-point correlation functions for $n>2$: namely, these functions have poles at the same surfaces $X_{m_j}^+$ with respect to each variable, and the scattering amplitudes are components of iterated residues of $\widehat{W}_n^T$ on the intersections of the loci

$$\{(p_1,\dots,p_n) : \sum_k p_k = 0,\ p_i \in X_{m_{j(i)}}\},\ 1\le i\le n,$$

for various collections of indices $(j(1),\dots,j(n))$. Namely, such a multi-residue gives the amplitude of interaction of particles $\xi_{j(1)},\dots,\xi_{j(n)}$ coming in with momenta $p_1,\dots,p_n$. In a lab, these momenta must satisfy $p_i^2=m_{j(i)}^2$ (i.e., as physicists say, be on the *mass shell*). This is why one cannot measure $\widehat{W}_n^T(p_1,\dots,p_n)$ at arbitrary points, but can only measure its values (or, rather, residues) on this mass shell.

Scattering theory is beyond the scope of this book; for more detail see [**QFS**, pp. 405–412, pp. 461–469] and references therein.

10.16. Normal ordering, composite operators, and operator product expansion in a free QFT

In classical field theory, given a classical scalar field $\phi(x)$, we may consider arbitrary polynomials and even any smooth functions of ϕ. The same is true for quantum mechanics, where $\phi(t)$ is a self-adjoint (possibly unbounded) operator on the Hilbert space $\mathcal{H}$ of quantum states, so using its spectral decomposition, we may define functions of ϕ. However, in quantum field theory in d dimensions with $d \geq 2$ the situation is more complicated. Indeed, in this case $\phi(x)$ is not a usual operator-valued function of x, but rather a generalized one—an operator-valued distribution, and we know that for singular distributions, such as $\delta(x)$, we cannot even define the square $\delta(x)^2$.

Indeed, let $\phi(x)$ be a free quantum scalar boson. Then the 2-point correlation function

$$\langle \phi(x)\phi(y)\rangle = \langle \Omega, \phi(x)\phi(y)\Omega\rangle = G(x-y)$$

blows up when $|x-y|^2 = 0$ (so in Euclidean signature, when $x = y$), so the operator $\phi^2(x)$ cannot possibly be well defined.

Thus, if we want to quantize the classical field $\phi^2(x)$, we need to regularize the corresponding operator product. This can be done by a standard regularization procedure called the *normally ordered product*.

For example, in Euclidean signature, the operator product $\phi(x)\phi(y)$ is well defined when $x \neq y$; indeed, by Wick's formula,

$$\begin{aligned}\langle \phi(x)\phi(y)\phi(z_1)\dots\phi(z_k)\rangle \\ = G(x-y)\langle \phi(z_1)\dots\phi(z_k)\rangle \\ + \sum_{i\neq j} G(x-z_i)G(y-z_j)\langle \phi(z_1)\dots\widehat{\phi}(z_i)\dots\widehat{\phi}(z_j)\dots\phi(z_k)\rangle,\end{aligned}$$

where the hat indicates omissions (here $x, y, z_1, \dots, z_k$ are distinct). Now, when $x \to y$, the first summand in this formula blows up while the second one does not. So it is natural to define the *normally ordered product* $:\phi(x)\phi(y):$ just by throwing away the singular terms, i.e., by the condition that its correlation function with $\phi(z_1)\dots\phi(z_k)$ is

$$\begin{aligned}\langle :\phi(x)\phi(y): \phi(z_1)\dots\phi(z_k)\rangle \\ = \sum_{i\neq j} G(x-z_i)G(y-z_j)\langle \phi(z_1)\dots\widehat{\phi}(z_i)\dots\widehat{\phi}(z_j)\dots\phi(z_k)\rangle.\end{aligned}$$

This is equivalent to just saying that

$$:\phi(x)\phi(y): = \phi(x)\phi(y) - G(x-y).$$

Note that while $\phi(x)\phi(y)$ blows up when $x = y$, the normally ordered product $:\phi(x)\phi(y):$ does not:

$$\langle :\phi^2(x):\phi(z_1)\dots\phi(z_k)\rangle = \sum_{i\neq j} G(x-z_i)G(x-z_j)\langle\phi(z_1)\dots\widehat{\phi}(z_i)\dots\widehat{\phi}(z_j)\dots\phi(z_k)\rangle.$$

This defines a *composite operator* $:\phi^2(x):$, which is a well defined operator-valued distribution.

Similarly one may define the normally ordered product $:\phi(x_1)\dots\phi(x_m):$ of any number of factors, by removing all the singular terms from the correlators. For example,

$$:\phi(x)\phi(y)\phi(z): = \phi(x)\phi(y)\phi(z) - G(x-y)\phi(z) - G(y-z)\phi(x) - G(z-x)\phi(y).$$

Such a product is well defined for all values of $x_1,\dots,x_m$ and is commutative (independent of ordering of factors) and associative. We can also differentiate by x_j any number of times, to define the normally ordered product of arbitrary derivatives of ϕ. Evaluating such products on the diagonal (when all points are the same), we obtain composite operators attached to any differential monomials (hence polynomials) with respect to ϕ, such as $:\phi^3(x):$, $:\partial_i\phi(x)\partial_j\phi(x):$, etc.

Exercise 10.25. Derive a formula for the correlation function of several composite operators (evaluated at different points) in the theory of the scalar boson.

In particular, we can now consider the product of two composite operators, e.g., $:\phi^2(x):\phi(y)$. Of course, this has a singularity at $x = y$, and an important problem is to understand the nature of this singularity. This is achieved by the procedure called the *operator product expansion*, which replaces the nonexistent multiplication of composite operators.

To explain this procedure, consider first the simplest example of operator product,

$$\phi(x)\phi(y) = G(x-y) + :\phi(x)\phi(y):.$$

Using Taylor's formula, this can be rewritten so that the right-hand side only contains $\phi(y)$ and no $\phi(x)$,

$$\phi(x)\phi(y) = G(x-y) + \sum_{\mathbf{n}} \frac{(x-y)^{\mathbf{n}}}{\mathbf{n}!} :\partial^{\mathbf{n}}\phi(y)\cdot\phi(y):,$$

where $\mathbf{n} := (n_1,\dots,n_d)$, $(x-y)^{\mathbf{n}} := \prod_{i=0}^{d-1}(x^i - y^i)^{n_i}$, $\partial^{\mathbf{n}} := \prod_i \partial_i^{n_i}$, and $\mathbf{n}! := \prod_i n_i!$. In this sum, all terms except the first one are regular (i.e., continuous) at $x = y$.

Let us now try to write down a similar expansion for a more complicated example of an operator product, $:\phi^2(x):\phi(y)$. We have

$$\begin{aligned}\langle :\phi^2(x):\phi(y)\phi(z_1)\dots\phi(z_k)\rangle \\ = 2G(x-y)\langle\phi(x)\phi(z_1)\dots\phi(z_k)\rangle \\ + \sum_{j,m,n \text{ distinct}} G(x-z_j)G(x-z_m)G(y-z_n) \\ \times\langle\phi(z_1)\dots\widehat{\phi(z_j)}\dots\widehat{\phi(z_m)}\dots\widehat{\phi(z_n)}\dots\phi(z_k)\rangle.\end{aligned}$$

where the hats indicate omitted factors. Thus we get

$$:\phi^2(x):\phi(y) = 2G(x-y)\phi(x) + :\phi^2(x)\phi(y):.$$

As before, using Taylor's formula, this can be rewritten so that the right-hand side only contains $\phi(y)$ and no $\phi(x)$:

$$\begin{aligned}:\phi^2(x):\phi(y) = 2G(x-y)\sum_{\mathbf{n}}\frac{(x-y)^{\mathbf{n}}}{\mathbf{n}!}\partial^{\mathbf{n}}\phi(y) \\ + \sum_{\mathbf{n},\mathbf{m}}\frac{(x-y)^{\mathbf{n}+\mathbf{m}}}{\mathbf{n}!\,\mathbf{m}!}:\partial^{\mathbf{n}}\phi(y)\cdot\partial^{\mathbf{m}}\phi(y)\cdot\phi(y):.\end{aligned}$$

We now see that there are many singular terms: $G(x)$ behaves as $|x|^{2-d}$ for $d > 2$ and as $\log|x|$ for $d = 2$, so the singular terms are the ones in the first summand with $|\mathbf{n}| \le d-2$, where $|\mathbf{n}| := \sum_i n_i$. For example, for $d = 3$ for the massless boson we have

$$:\phi^2(x):\cdot:\phi(y): = \frac{2}{|x-y|}(\phi(y) + (x-y)\bullet\nabla\phi(y)) + \text{regular}.$$

As a final example, consider the product $:\phi^2(x):\cdot:\phi^2(y):$. A similar computation yields

$$:\phi^2(x):\cdot:\phi^2(y): = 2G^2(x-y) + 4G(x-y):\phi(x)\phi(y): + :\phi^2(x)\phi^2(y):,$$

and as before we can expand this to remove $\phi(x)$ using Taylor's formula. Namely, expanding the second summand, we get

$$\begin{aligned}:\phi^2(x):\cdot:\phi^2(y): \\ = 2G^2(x-y) + 4G(x-y)\sum_{\mathbf{n}}\frac{(x-y)^{\mathbf{n}}}{\mathbf{n}!}:\partial^{\mathbf{n}}\phi(y)\cdot\phi(y): + :\phi^2(x)\phi^2(y):,\end{aligned}$$

and the last summand can be expanded similarly. For example, for $d = 3$ for the massless boson we have

$$\begin{aligned}:\phi^2(x):\cdot:\phi^2(y): \\ = \frac{2}{|x-y|^2} + \frac{4}{|x-y|}:\phi^2(y): + \frac{4(x-y)}{|x-y|}\bullet:\nabla\phi(y)\cdot\phi(y): + \text{regular}.\end{aligned}$$

Yet we see that the number of singular terms is finite. In fact, it is not hard to prove the following proposition (see [**QFS**, vol. 1, p. 449]).

Proposition 10.26. *Let A, B be two composite operators in the theory of a free scalar boson. Then there exists a unique collection of functions $F_j(y)$ and composite operators $C_j(y)$ such that we have an asymptotic expansion*

$$A(x)B(y) \sim \sum_j F_j(x-y)C_j(y),\ x \to y$$

such that for every N we have $|F_j(z)| = O(|z|^N)$, $z \to 0$, for all but finitely many j. In particular, there are finitely many singular terms (not continuous at $x = y$).

The expansion of Proposition 10.26 is called the *operator product expansion*. It is not hard to show that it exists in any free quantum field theory.

Chapter 11

Symmetries and geometric aspects of field theory

11.1. Field theories on manifolds

As already mentioned above, an important feature of classical and quantum field theory is the possibility of considering field theories not just on a Euclidean or Minkowskian space, but more generally on Riemannian and Lorentzian manifolds. This possibility is key for many deep connections of quantum field theory with geometry and topology. The main examples of this are theories on $X = \mathbb{R} \times X_{\rm s}$, where $X_{\rm s}$ is a Riemannian $(d-1)$-dimensional space manifold with metric $\sum_{i,j=1}^{d-1} g_{ij} dx^i dx^j$, and $\mathbb{R}$ is the time line, with Lorentzian metric

$$|dx|^2 := (dt)^2 - \sum_{i,j=1}^{d-1} g_{ij} dx^i dx^j,$$

and Euclidean theories on a Riemannian d-dimensional spacetime manifold X. This includes the Wick rotation of the previous example: $X = \mathbb{R} \times X_{\rm s}$,

$$|dx|^2 := (dt)^2 + \sum_{i,j=1}^{d-1} g_{ij} dx^i dx^j.$$

Here we will consider mostly classical field theories on manifolds. These theories can then be quantized using either Lagrangian or Hamiltonian approach, but we will not discuss this, except in some examples. The story is parallel to the case of flat space considered above, but we should make sure

that the kinetic term and other terms in the Lagrangian are defined canonically (i.e., do not depend on the choice of coordinates). For simplicity consider the Euclidean case (in the Lorentzian case the story is similar). We restrict ourselves to reviewing the most common types of classical fields in such theories, as well as the corresponding kinetic and other terms in their Lagrangians. A more complete discussion can be found in [**QFS**].

1. **Scalar (bosonic) fields.** As discussed above, in the simplest case a scalar field is just a real function on X (real scalar), but one can also consider scalars valued in a finite-dimensional real vector space with a positive inner product (for example, $\mathbb{C}$, for complex scalars) or, more generally, valued in a real vector bundle on X. The kinetic term[31] for a scalar $\phi : X \to E$ valued in a vector space $E \cong E^*$ with inner product is $|d\phi(x)|^2$, the squared norm of the vector $d\phi \in T_x^*X \otimes E$ with respect to the inner products on T_xX and E. Thus if this vector has components $\frac{\partial \phi_i}{\partial x^j}$ in orthonormal bases, then

$$|d\phi|^2 = \sum_{i,j} (\tfrac{\partial \phi_i}{\partial x^j})^2.$$

More generally, if E is a vector bundle on X, then we need to fix an inner product on E (i.e., E should be an orthogonal bundle) and also a connection A preserving this inner product, which gives rise to the covariant derivative operator ∇_A; if E is trivialized on a local chart $U \subset X$, then A becomes a 1-form on U with values in the Lie algebra $\mathfrak{o}(E)$ and we have $\nabla_A = d + A$. In this case, an E-valued scalar field ϕ is a section of E over X, and the kinetic term is $|\nabla_A \phi|^2$, which in local trivialization has the form $|d\phi + A\phi|^2$. If the bundle E and connection A are trivial, we obtain the usual kinetic term $|\nabla\phi|^2$, where $\nabla\phi = d\phi$.

Note that for a scalar field ϕ, we can always add to the kinetic term a mass term $m^2|\phi|^2$. More generally, we can add a mass term $(\phi, Q\phi)$, where Q is a self-adjoint endomorphism of E. Still more generally, we can add a general potential term $U(\phi)$.

Finally, the theory of scalar fields has an important generalization in which the scalar field takes values in a Riemannian manifold M. This generalization is called the *nonlinear sigma model*. Thus the field is a map $\phi : X \to M$, and the Lagrangian is $\mathcal{L} = \frac{1}{2}|d\phi|^2$, which in local coordinates has the form

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{\dim M} G_{ij}(\phi)(d\phi^i, d\phi^j),$$

where G_{ij} is the Riemannian metric on M. We may also add a potential $U(\phi)$, where U is a smooth function on M.

[31] We drop the factor $\frac{1}{2}$ for brevity, and also omit the factor dx, working with the Lagrangian rather than Lagrangian density.

2. **Spinor (fermionic) fields.** Spinor fields can be defined on a *spin manifold* X, i.e., an oriented manifold equipped with a spin structure (a lift of the tangent bundle from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$). For such a structure to exist, the second Stiefel-Whitney class $w_2(X) \in H^2(X, \mathbb{Z}/2)$ must vanish, and then the possible spin structures on X are parametrized by a torsor over $H^1(X, \mathbb{Z}/2)$. On a spin manifold X, we have the canonically defined *spin bundle* SX, which is the associated complex vector bundle to the above $\mathrm{Spin}(n)$ bundle via the spin representation $\mathrm{Spin}(n) \to \mathrm{Aut}(S)$. This bundle carries a natural inner product and a connection induced by the Levi-Civita connection of X that preserves this inner product. Moreover, as explained in Section 10.3, in even dimensions we have $S = S_+ \oplus S_-$, where S_+, S_- are irreducible representations of $\mathrm{Spin}(n)$, so we have $SX = S_+X \oplus S_-X$, an orthogonal decomposition of SX into two subbundles called the *chiral spin bundles*. These subbundles are switched when the orientation of X is changed to the opposite one.

Spinor fields, in the most basic case, are sections of the spin bundle SX. The sections of S_+X and S_-X, as noted in Section 10.4, are called *chiral spinors*.

The possible kinetic and mass terms for spinors on the flat space are described in Section 10.4, and the story on a curved manifold is similar. The only new feature is that we have to define the Dirac operator $\mathbf{D}$ for a spinor field on an arbitrary spin manifold. To this end, all we have to do is replace ordinary partial derivatives in formula (10.2) by the covariant ones with respect to the Levi-Civita connection,

$$\mathbf{D} = \sum_i \Gamma^i \nabla_i^{LC}. \tag{11.1}$$

More generally, similar to the scalar field case, we may consider spinors valued in a vector bundle E with an inner product and an orthogonal connection A, i.e., sections of the bundle $SX \otimes E$. This bundle carries a tensor product connection $\nabla_A^{\mathrm{total}} = \nabla^{LC} \otimes 1 + 1 \otimes \nabla_A$, and the Dirac operator is defined by the formula

$$\mathbf{D}_A = \sum_i \Gamma^i \nabla_{A,i}^{\mathrm{total}}.$$

3. **Gauge fields.** Let G be a compact Lie group and let $\mathfrak{g} = \mathrm{Lie}\, G$ be equipped with a positive invariant inner product. Analogously to Section 10.7, gauge fields are connections A on principal G-bundles E on X, so in local trivialization A is a 1-form on X with values in $\mathfrak{g}$ and the covariant derivative with respect to A looks like $\nabla_A = d + A$. The connection A has curvature F_A (the *field strength*), which is a 2-form on X with values in the adjoint bundle $\mathrm{ad}\, E$. In a local trivialization the curvature of A is the Maurer-Cartan form

$$F_A = dA + \frac{1}{2}[A, A].$$

In particular, if G is abelian, then we just have $F_A = dA$. The kinetic term for a gauge field A is $|F_A|^2 dx = F_A \wedge *F_A$, where the squared norm is taken with respect to the inner product on $(\wedge^2 T^*X \otimes \operatorname{ad} E)_x$ induced by the inner products on T_xX and $\mathfrak{g}$ (note that this does not depend on the identification of Lie algebras $(\operatorname{ad} E)_x \cong \mathfrak{g}$ since the form on $\mathfrak{g}$ is invariant).

Note that on a 1-dimensional manifold, any two connections are locally isomorphic, and for any connection A we have $F_A = 0$. For this reason, there are no (dynamical) gauge fields in mechanics (i.e., for spacetime $\mathbb{R}^1$)—they are only considered in field theory on $\mathbb{R}^d$ in dimension $d \geq 2$.[32]

It makes sense to fix the topological type of the C^∞-bundle E (which does not change under deformations) and consider the space $\operatorname{Conn}(E)$ of all connections A on E. If $A_1, A_2 \in \operatorname{Conn}(E)$, then $\nabla_{A_1} - \nabla_{A_2} \in \Omega^1(X) \otimes \operatorname{ad} E$, so $\operatorname{Conn}(E)$ is an affine space with underlying vector space $\Omega^1(X) \otimes \operatorname{ad} E$. Moreover, this space carries a natural right affine linear action of the group $\mathcal{G}_E = C^\infty(X, \operatorname{Ad}(E))$ of gauge transformations, which in local trivialization looks like

$$A^g = g^{-1}dg + g^{-1}Ag.$$

The configuration space of a classical gauge theory is then

$$\mathcal{M} := \sqcup_{\text{topological types } E} \operatorname{Conn}(E)/\mathcal{G}_E, \tag{11.2}$$

so the phase space is the cotangent bundle $T^*\mathcal{M}$.

11.2. Energy-momentum tensor

The ability to put a field theory on any Riemannian (or Lorentzian) manifold X implies the existence of a special local functional or, in quantum theory, local operator (on any spacetime manifold, notably the flat space $\mathbb{R}^d$), called the *energy-momentum tensor* or *stress-energy tensor*. Namely, in a classical field theory defined by a Lagrangian density $\mathcal{L}dx$, the energy-momentum tensor is -2 times the variation of the action $S = \int_X \mathcal{L}dx$ with respect to the metric,

$$T^{ij}(x)dx = -2\frac{\delta S}{\delta g_{ij}(x)},$$

where dx is the volume element on X defined by the metric.[33] Thus, the diffeomorphism covariance (i.e., coordinate independence) of the action implies that $T(x)$ is a section of the bundle S^2TX. However, since on a Riemannian (or Lorentzian) manifold, one has a canonical isomorphism of vector bundles $TX \cong T^*X$, we may also consider the energy-momentum tensor with one or

[32] However, on the circle S^1, unlike $\mathbb{R}^1$, there are nontrivial G-connections parametrized by the holonomy around the circle, and such a connection can be one of the integration variables in a quantum-mechanical path integral (taking values in the finite-dimensional space of conjugacy classes in G).

[33] The terms "energy-momentum tensor" or "stress-energy tensor" come from the fact that in various physical theories components of T may be interpreted as density or flux of energy, momentum, or stress.

two indices lowered: the operator version $T^i_j(x)$ (a symmetric endomorphism of TX) and the bilinear form version $T_{ij}(x)$ (a symmetric bilinear form on TX).

In quantum theory we should have a local operator corresponding to this functional. This operator may be defined purely in terms of correlation functions,

$$\begin{aligned}\langle\phi(y_1)\dots\phi(y_n)\Omega, T^{ij}(x)\phi(z_1)\dots\phi(z_m)\Omega\rangle_g dx \\ = -2\frac{\delta\langle\phi(y_1)\dots\phi(y_n)\Omega, \phi(z_1)\dots\phi(z_m)\Omega\rangle_g}{\delta g_{ij}(x)},\end{aligned}$$

or in a Euclidean QFT,

$$\langle\phi(y_1)\dots\phi(y_n)T^{ij}(x)\rangle_g dx = -2\frac{\delta\langle\phi(y_1)\dots\phi(y_n)\rangle_g}{\delta g_{ij}(x)}.$$

Example 11.1. Consider the classical Euclidean scalar field theory with action $S(\phi) = \frac{1}{2}\int_{\mathbb{R}^d}|\nabla\phi(x)|^2 dx$. If we introduce a metric $g(x)$ on $\mathbb{R}^d$, then the action takes the form

$$S_g(\phi) = \frac{1}{2}\int_{\mathbb{R}^d} g(x)(\nabla\phi(x), \nabla\phi(x))\sqrt{\det g}dx.$$

Hence

$$T_{ij} = \partial_i\phi\partial_j\phi - \tfrac{1}{2}|\nabla\phi|^2\delta_{ij}.$$

For example, consider the case $d = 2$ with metric $|du|^2$, where $u = x + it$ is a complex coordinate. Then a direct computation shows that the bilinear form version of T is

$$T = T_+(du)^2 + T_-(d\overline{u})^2,$$

where

$$T_+ = (\partial\phi)^2,\ T_- = (\bar\partial\phi)^2.$$

In particular, since the equation of motion is $\partial\bar\partial\phi = 0$, we see that

$$\bar\partial T_+ = \partial T_- = 0,$$

i.e., T_+ is a holomorphic quadratic differential, while T_- is an antiholomorphic one. These are manifestations of the conformal invariance of this theory (see Section 11.5 below).

11.3. Symmetries in classical field theory

In studying any physical system, it is crucial to find all its symmetries and use them to their full potential. For example, the equations of motion of a particle in a rotationally symmetric potential field can be fully solved by utilizing the rotational symmetry (see [**A**]).

The most fundamental fact about symmetries in classical or quantum mechanics is that for any 1-parameter group of symmetries of the system there

is an (essentially unique) observable responsible for this symmetry, which is conserved in this system; i.e., every 1-parameter symmetry corresponds to a conservation law, and vice versa. This statement is called *Noether's theorem*.

Let us first explain the precise meaning of Noether's theorem in the setting of classical mechanics. Suppose we have a system with phase space being a symplectic manifold (M, ω) (typically $M = T^*X$, where X is the configuration space, and $\omega = d\alpha$ is the differential of the Liouville form) and Hamiltonian $H \in C^\infty(M)$. Let g^t be a 1-parameter group of symmetries of this system, i.e., of symplectic diffeomorphisms of M which preserve H. Let $v := \frac{d}{dt}|_{t=0} g^t$ be the vector field generating the flow g^t. Then we have $L_v H = 0$ and $L_v \omega = 0$ (i.e., v is a symplectic vector field, so $\omega_v := \omega(v, ?)$ is a closed 1-form). The last condition is equivalent to the requirement that L_v is a *Poisson derivation*, i.e.

$$L_v\{a, b\} = \{L_v a, b\} + \{a, L_v b\}.$$

Let us assume that M is simply connected (for example, we can restrict ourselves to a neighborhood of a point in M or X). In this case ω_v is exact, so there exists $Q \in C^\infty(M)$ (unique up to adding a constant) such that $\omega_v = dQ$. Then for any observable $F \in C^\infty(M)$ we have $L_v F = \{Q, F\}$. Moreover $\{Q, H\} = L_v H = 0$. The observable Q is thus conserved under the Hamiltonian flow and is the conservation law corresponding to the 1-parameter group g^t. It is called (especially in the setting of field theory) the *Noether* (*conserved*) *charge* of the symmetry.

A trivial example of this is the Hamiltonian flow h^t defined by the Hamiltonian H itself, i.e., the time translation symmetry acting on observables by $(h^t f)(s) = f(s - t)$; in this case $Q = H$, so the corresponding conserved quantity is H (the energy). Other examples include the momenta $p_1, \dots, p_n$ which correspond to translation symmetry (for $X = \mathbb{R}^n$) and *angular momenta* $M_{kj} := x^k p_j - x^j p_k$ corresponding to rotational symmetries around the codimension 2 hyperplanes $x^k = x^j = 0$.

More generally, suppose G is a Lie group acting by symmetries of the system. Let $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of G. Any element $y \in \mathfrak{g}$ gives rise to a 1-parameter subgroup $e^{ty} \in G$, so defines a conserved quantity Q_y such that

$$\{Q_y, F\} = F \cdot y$$

for each $F \in C^\infty(M)$, where

$$(F \cdot y)(m) := \frac{d}{dt}|_{t=0} F(e^{ty} m),\ m \in M.$$

More precisely, Q_y is defined only up to adding a constant, so let us fix some linear assignment $y \mapsto Q_y$.

Moreover, it is clear that for $y, z \in \mathfrak{g}$,

$$\{Q_y, Q_z\} = Q_{[z,y]} + C(y, z),$$

where $C(y,z)$ is a skew-symmetric bilinear form on $\mathfrak{g}$ which arises because Q_y is uniquely determined by y only up to adding a constant. Furthermore, by the Jacobi identity, the form C is a 2-cocycle,

$$C([x,y],z) + C([y,z],x) + C([z,x],y) = 0.$$

It follows that the assignment[34] $y \mapsto -Q_y$ is almost a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M)$, but not quite: rather, it defines a homomorphism

$$\mu : \hat{\mathfrak{g}} \to C^\infty(M),$$

where $\hat{\mathfrak{g}}$ is a 1-dimensional central extension of $\mathfrak{g}$, i.e., the vector space $\hat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{R}$ with commutator

$$[(y,a),(z,b)] = ([y,z], C(y,z)).$$

Namely, $\mu(y,a) = -Q_y + a$.

The map μ may be viewed as an element of $C^\infty(M) \otimes \hat{\mathfrak{g}}^*$, i.e., geometrically as a C^∞-map

$$\mu : M \to \hat{\mathfrak{g}}^*.$$

This map is called the *moment map* and plays a fundamental role in symplectic geometry.[35]

The following example shows that the cohomology class of C may be nonzero, which means that we may not be able to choose Q_y to make $C = 0$.

Example 11.2. The group $\mathbb{R}^{2n}$ acts on $M = T^*\mathbb{R}^n$ (with trivial Hamiltonian $H = 0$) by translations. So we have $\mathfrak{g} = \mathbb{R}^{2n}$ and $C(y,z) = \omega(y,z)$. Thus $\hat{\mathfrak{g}}$ is the *Heisenberg Lie algebra* $\mathbb{R}^{2n} \oplus \mathbb{R}$ with commutation relations

$$[(y,a),(z,b)] = ([y,z], C(y,z)),$$

which is a nontrivial central extension of $\mathfrak{g}$. In this case, the moment map $\mu : \mathbb{R}^{2n} \to \hat{\mathfrak{g}}^*$ identifies $\mathbb{R}^{2n}$ with the hyperplane of elements of the form $(f,1) \in \hat{\mathfrak{g}}^*$, $f \in \mathfrak{g}^*$, which is an orbit of the coadjoint representation of $\hat{\mathfrak{g}}$ (check it!).

However, in many examples the cohomology class $[C] \in H^2(\mathfrak{g})$ is, in fact, zero, i.e., $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ as Lie algebras. For instance, this is automatically so if G is a compact Lie group. In this case, we may choose Q_y so that $C = 0$, and we have a moment map

$$\mu : M \to \mathfrak{g}^*.$$

A similar discussion applies to classical field theory, using the formalism of Section 10.6. Namely, in this case, the Noether charge is given by the integral over the space of a certain local field called *Noether current*. This terminology is motivated by formula (10.7).

[34] Without the minus sign, it would be (almost) a Lie algebra anti-homomorphism.

[35] For example, for translation symmetries of the free particle, μ is the momentum p of the particle, which explains the terminology "moment map".

For example, consider the free massive boson ϕ on $\mathbb{R} \oplus \mathbb{R}^{d-1}$. The Hamiltonian is

$$H = \frac{1}{2}\int_{\mathbb{R}^{d-1}} (\phi_t^2 + |\nabla_x \phi|^2 + m^2\phi^2) dx.$$

Thus $H = \int_{\mathbb{R}^{d-1}} J dx$, where

$$J = \frac{1}{2}(\phi_t^2 + |\nabla_x \phi|^2 + m^2\phi^2) = \frac{1}{2}(\phi_t^2 + \sum_{j=1}^{d-1} \phi_{x^j}^2 + m^2\phi^2) \tag{11.3}$$

is the Noether current associated to the time translation symmetry.

Similarly, the Noether current for the spatial translation in the kth coordinate is

$$J_k = \phi_t \phi_{x^k}. \tag{11.4}$$

Indeed, using the formulas of Section 10.6, we have

$$\{J_k(x), \phi(y)\} = -\phi_{x_k}(x)\delta(x-y), \quad \{J_k(x), \phi_t(y)\} = \phi_t(x)\partial_k\delta(x-y).$$

Thus defining the charge as

$$P_k = \int_{\mathbb{R}^{d-1}} J_k(x) dx,$$

and using integration by parts, we get

$$\{P_k, \phi(y)\} = -\phi_{x^k}(y), \ \{P_k, \phi_t(y)\} = -\phi_{tx^k}(y),$$

as needed.

Let us now discuss Noether currents from the Lagrangian point of view. For simplicity consider the theory of a single scalar field ϕ on $\mathbb{R}^d$; the general case is similar. Suppose that the theory is defined by a first-order Lagrangian $\mathcal{L}(\phi, \phi_{x^0}, \dots, \phi_{x^{d-1}})$, where x^i are the coordinates in $\mathbb{R}^d$ (i.e., it does not explicitly depend on x). As usual, the action is $S = \int_{\mathbb{R}^d} \mathcal{L} dx$. Thus the Euler-Lagrange equation is

$$\mathcal{L}_\phi - \sum_{i=0}^{d-1} \partial_i \mathcal{L}_{\phi_{x^i}} = 0. \tag{11.5}$$

Recall that an infinitesimal symmetry of the theory is then a derivation δ of the algebra $\mathcal{A}$ of local functionals of ϕ which preserves the action, i.e., $\delta S = 0$. In order for δ to preserve S, it must preserve the Lagrangian density $\mathcal{L}dx$ up to an exact form, i.e.,

$$\delta(\mathcal{L}dx) = dF,$$

where F is a local $d-1$-form. This can be written as

$$(\mathcal{L}_\phi \delta\phi + \sum_{k=0}^{d-1} \mathcal{L}_{\phi_{x^k}} \delta(\phi_{x^k}))dx + \mathcal{L}\delta(dx) = dF. \tag{11.6}$$

We do not assume that δ commutes with ∂_i (this would be too restrictive, e.g., this fails for Lorentz symmetries), but let us assume that the commutator $[\partial_i, \delta]$ is a pure spacetime symmetry, in particular

$$[\partial_k, \delta]\phi = \sum_{j=0}^{d-1} \phi_{x^j} [\partial_k, \delta] x^j. \tag{11.7}$$

This assumption is satisfied in all examples we care about.

Let

$$\delta_* \phi := \delta\phi - \sum_{j=0}^{d-1} \phi_{x^j} \delta x^j$$

be the *active variation* of ϕ under the symmetry δ (to be distinguished from the *passive variation* $\delta\phi$). Define the Noether current attached to the symmetry δ to be the $d-1$-form

$$\mathbb{J} := \sum_{k=0}^{d-1} \mathcal{L}_{\phi_{x^k}} \delta_* \phi \widehat{dx^k} + \mathcal{L} \iota_\delta(dx) - F,$$

where $\widehat{dx}_k := \iota_{\partial_k}(dx)$ is the interior product of ∂_k with the volume form $dx = dx^0 \wedge \cdots \wedge dx^{d-1}$. Thus using (11.6), we get

$$\begin{aligned} d\mathbb{J} &= \sum_{k=0}^{d-1} (\partial_k(\mathcal{L}_{\phi_{x^k}} \delta_* \phi) + \partial_k(\mathcal{L}) \delta x^k) dx + \mathcal{L}\delta(dx) - dF \\ &= \sum_{k=0}^{d-1} (\partial_k(\mathcal{L}_{\phi_{x^k}} \delta_* \phi) + \partial_k(\mathcal{L}) \delta x^k) dx - (\mathcal{L}_\phi \delta\phi + \sum_{k=0}^{d-1} \mathcal{L}_{\phi_{x^k}} \delta(\phi_{x^k})) dx. \end{aligned}$$

If ϕ satisfies the Euler-Lagrange equation (11.5) then this yields

$$\begin{aligned} &\sum_{k=0}^{d-1} (\mathcal{L}_{\phi_{x^k}} \partial_k \delta_* \phi + \partial_k(\mathcal{L}) \delta x^k) dx - \sum_{k=0}^{d-1} (\mathcal{L}_{\phi_{x^k}} \delta(\phi_{x^k}) + \mathcal{L}_\phi \phi_{x^k} \delta x^k) dx \\ &\qquad = \sum_{k=0}^{d-1} \mathcal{L}_{\phi_{x^k}} ([\partial_k, \delta]\phi - \sum_{j=0}^{d-1} \phi_{x^j} [\partial_k, \delta] x^j) dx. \end{aligned}$$

Thus by (11.7),

$$d\mathbb{J} = 0,$$

i.e., the current $\mathbb{J} = \sum_{i=0}^{d-1} J_i \widehat{dx}_i$ is *conserved*. Note that this only holds *on-shell*, i.e. when the equations of motion are satisfied, as opposed to *off-shell* (without equations of motion).

Now assume that ϕ is a solution of the Euler-Lagrange equation compactly supported in the spatial direction. Then the charge attached to the current $\mathbb{J}$ is

$$Q(\phi) = \int_{V_0} \mathbb{J} = \int_{V_0} J_0 dx^1 \dots dx^{d-1}, \tag{11.8}$$

with $V_t \subset \mathbb{R}^d$ defined by the equation $x^0 = t$, where x^0 is the time coordinate. Since $\mathbb{J}$ is closed (conserved), by Stokes' formula for any $t \in \mathbb{R}$

$$Q(\phi) = \int_{V_t} \mathbb{J},$$

which implies that

$$\frac{dQ}{dt} = \{Q, H\} = 0,$$

i.e., Q is an integral of motion, as expected. Also it is easy to check using (10.6) that

$$\{Q, \phi\} = \delta^* \phi, \tag{11.9}$$

i.e., Q realizes the symmetry δ.

Example 11.3. Consider the Minkowskian scalar field theory in two dimensions, with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\phi_t^2 - \phi_x^2) - U(\phi).$$

The Euler-Lagrange equation is

$$\phi_{tt} - \phi_{xx} = -U'(\phi).$$

This theory has a Lorentzian boost symmetry defined by the vector field $t\partial_x + x\partial_t$. For this symmetry, $\delta\phi = 0$, $F = 0$ (pure spacetime symmetry), thus

$$\delta_* \phi = -t\phi_x - x\phi_t,$$

and so

$$\mathbb{J} = -(t\phi_x + x\phi_t)(\phi_t dx + \phi_x dt) + \mathcal{L}(xdx - tdt).$$

Hence we have

$$J_0 = -(t\phi_x + x\phi_t)\phi_t + \mathcal{L}x = -t\phi_t\phi_x - x(\tfrac{\phi_t^2+\phi_x^2}{2} + U(\phi)),$$

so the corresponding charge is

$$Q = -\int_{\mathbb{R}} (t\phi_t\phi_x + x(\tfrac{\phi_t^2+\phi_x^2}{2} + U(\phi)))dx.$$

It is readily checked that the Euler-Lagrange equation implies that $\mathbb{J}$ is conserved ($d\mathbb{J} = 0$) and

$$\{Q, \phi\} = -t\phi_x - x\phi_t = \delta_* \phi.$$

Note that for a pure spacetime symmetry of a classical field theory given by a vector field $v = v^j \partial_{x^j}$ on the spacetime X, the Noether current $\mathbb{J}_v$ can be expressed in terms of the energy-momentum tensor T (Section 11.2). Namely, recall that T can be viewed as a section of $S^2 T^* X$. Then one can show that

$$\mathbb{J}_v = *Tv, \text{ or in coordinates } \mathbb{J}_v = T_{ij} v^j * dx^i, \tag{11.10}$$

where $*$ is the Hodge $*$-operator. In particular, if the vector field v preserves the action, then the current $\mathbb{J}_v$ is conserved, i.e., $d\mathbb{J}_v = 0$.

Example 11.4. Consider the theory of a massless scalar field on $\mathbb{R}^d$ with coordinates $t = x^0, x^1, \dots, x^{d-1}$, and let $v = \partial_k$ be the translation symmetry in the kth spatial coordinate. Then the Noether current on $\mathbb{R}^d$ corresponding to v is the $d-1$-form

$$\mathbb{J}_k = T_{kj} * dx^j = \phi_{x^k} * d\phi - \tfrac{1}{2} |d\phi|^2 * dx^k,$$

and it is easy to check that the equation of motion $d * d\phi = 0$ implies that $d\mathbb{J}_k = 0$ (i.e., $\mathbb{J}_k$ is conserved). Also, since $*dx^k$ has a factor dt, its restriction to $\mathbb{R}^{d-1}$ is zero. So, since $*d\phi|_{\mathbb{R}^{d-1}} = \phi_t d\mathrm{x}$, we get

$$\mathbb{J}_k|_{\mathbb{R}^{d-1}} = J d\mathrm{x},\ J = \phi_t \phi_{x^k},$$

i.e., formula (11.4).

Exercise 11.5. Prove formula (11.10).

Exercise 11.6.

(i) Check equation (11.9).

(ii) Show that in the case of free scalar field theory, formula (11.8) for translation symmetries yields formulas (11.3) and (11.4).

(iii) Consider the theory of Example 11.3 for ϕ valued in a Euclidean space E, and compute the Noether currents and charges for infinitesimal symmetries from the Lie algebra $\mathfrak{so}(E)$.

For a general spacetime manifold X of dimension d, the story is similar. Namely, to every infinitesimal symmetry one can assign a Noether current $\mathbb{J}$, which is a *closed* $d-1$*-form* on X, i.e., $d\mathbb{J} = 0$ (the current $\mathbb{J}$ is *conserved*) provided that equations of motion are satisfied, so that one can obtain the corresponding Noether charge Q as $\int_C \mathbb{J}$, where $C \subset X$ is a *space cycle* of dimension $d-1$. Moreover, by Stokes' theorem, Q does not change when this cycle is deformed (at least under suitable decay conditions at infinity). For example, if $X = \mathbb{R} \times X_{\mathrm{s}}$, where $\mathbb{R}$ is the time axis, then we can take $C = X_t = \{(t, \mathrm{x}) | \mathrm{x} \in X_{\mathrm{s}}\}$. Then the integral $Q = \int_{X_t} \mathbb{J}$ does not depend on t, i.e., it is conserved under the Hamiltonian flow ($\{Q, H\} = 0$). Also, the restriction of $\mathbb{J}$ to the space cycle X_0 is a top degree form on X_0 of the form $J(\mathrm{x}) d\mathrm{x}$, where $J(\mathrm{x})$ is a function on X_0; this is the Noether current from the Hamiltonian point of view as discussed above.

Exercise 11.7.

(i) Let M be a symplectic manifold and let G be a Lie group acting freely and properly on M by symplectic diffeomorphisms. Let $\mathfrak{g} := \operatorname{Lie} G$ and suppose that $\mu : M \to \mathfrak{g}^*$ is a moment map for this action. Show that $\mu^{-1}(0) \subset M$ is a G-invariant closed submanifold of M of codimension $\dim G$ and that the manifold $M /\!\!/ G := \mu^{-1}(0)/G \subset M/G$ of dimension $\dim M - 2\dim G$ has a natural symplectic structure. The symplectic manifold $M /\!\!/ G$ is called the *Hamiltonian reduction* or *Marsden-Weinstein reduction* of M by the action of G.

(ii) Show that the Poisson bracket $\{f, h\}$ of two smooth functions f, h on a small neighborhood U of $x \in M /\!\!/ G$ defined by this symplectic structure can be computed as follows: pick a neighborhood of $\widetilde{U}$ of x in M/G such that $U = \widetilde{U} \cap M /\!\!/ G$; extend f, h to smooth functions $\tilde{f}, \tilde{h}$ on $\widetilde{U}$; compute the Poisson bracket $\{\widetilde{f}, \widetilde{h}\}$ as G-invariant functions on the preimage of $\widetilde{U}$ in M; and then restrict the resulting function on $\widetilde{U}$ to U. (In particular, the result is well defined, i.e., does not depend on the extensions of f, h.)

(iii) Suppose G acts freely and properly on a manifold X (on the right). Then it acts symplectically on T^*X, and a moment map $\mu : T^*X \to \mathfrak{g}^*$ is given by

$$\mu(x, p)(a) = (p, a_x),$$

where $a_x \in T_xX$ is the tangent vector corresponding to $a \in \mathfrak{g}$. Show that there is a canonical symplectic diffeomorphism $T^*(X/G) \cong T^*X /\!\!/ G$.

(iv) Let G be a compact Lie group, let $\mathcal{M} = \operatorname{Conn}(Y)$ be the space of connections on a principal G-bundle P on a manifold Y, and let $\mathcal{G} = C^\infty(Y, \operatorname{Ad} P)$ be the group of gauge transformations acting on $\mathcal{M}$. Apply an infinite-dimensional version of (iii) to this situation to describe the phase space for a classical gauge theory as $T^*\mathcal{M} /\!\!/ \mathcal{G}$ (mind you, the action of $\mathcal{G}$ on $\mathcal{M}$ is not free in general, but it is generically free, so for the purposes of this exercise this issue may be ignored).

11.4. Symmetries in quantum field theory

As usual, the discussion of Section 11.3 extends to quantum theory, with observables replaced by operators. Namely, in this case, we have a unitary *projective representation* $\pi : G \to \operatorname{Aut}(\mathcal{H})$ of the Lie group G of symmetries on the Hilbert space $\mathcal{H}$ of quantum states of the system, so that $[\pi(g), \widehat{H}] = 0$, where $\widehat{H} : \mathcal{H} \to \mathcal{H}$ is the Hamiltonian (an unbounded self-adjoint operator).

Remark 11.8. Note that this representation need not be linear (in general, it is only projective) because actual quantum states are nonzero vectors in $\mathcal{H}$ *up to scaling*, i.e., points of the projective space $\mathbb{P}\mathcal{H}$. In other words, π is a unitary representation of a certain *central extension* of G; this central extension is

a quantum analogue of the one we have seen in the classical case. Such projective, nonlinearizable actions are actually common in quantum mechanics. Prototypical examples of this are the *Heisenberg uncertainty relation* $[\hat{p}, \hat{x}] = -i\hbar$, when the classical 2-dimensional group (or Lie algebra) of translations of the phase plane is replaced in quantum theory by the 3-dimensional Heisenberg group (Lie algebra), and the *phenomenon of spin*, when the classical rotational symmetry group $\mathrm{SO}(3)$ is replaced in quantum theory by its double cover $\mathrm{SU}(2)$ (cf. Remark 10.3). Other such examples include the *Virasoro central charge* in 2-dimensional conformal field theory (Section 12.7) and the central charge for the $N = 2$ supersymmetry in four dimensions discussed in Section 14.16.

The quantum Noether charges corresponding to these symmetries simply define the corresponding projective Lie algebra representation

$$\pi_* : \mathfrak{g} \to \mathrm{End}(\mathcal{S}),$$

where $\mathcal{S}$ is a certain dense subspace of $\mathcal{H}$ (of smooth vectors) on which all the operators $\pi_*(y)$ are defined. For instance, in quantum mechanics, as in classical one, the time translation corresponds to the Hamiltonian $\widehat{H}$, the spatial translations to the momentum operators $\widehat{p}_j := -i\hbar\partial_{x^j}$, and rotations around $x^k = x^j = 0$ to the angular momentum operators

$$\widehat{M}_{kj} := -i\hbar(x^k\partial_j - x^j\partial_k).$$

Finally, in quantum field theory, by analogy with classical one, a quantum Noether charge is an operator of the form

$$Q = \int_{\mathbb{R}^{d-1}} J(\mathrm{x})d\mathrm{x},$$

where $J(\mathrm{x})$ is a quantum local operator called the *quantum Noether current*. For example, in the case of a free massive boson, the currents $J(\mathrm{x})$ and $J_k(\mathrm{x})$ for time and space translations are given by the same formulas (11.3) and (11.4), but now with $\phi(t, \mathrm{x})$ being the quantum field corresponding to the massive boson (say, in the setting of Wightman axioms) rather than the classical field, and with normal ordered product instead of the usual product,

$$J = \frac{1}{2}(:\phi_t^2: + \sum_{j=1}^{d-1} :\phi_{x^j}^2: + m^2:\phi^2:),$$

$$J_k = :\phi_t\phi_{x^k}:,$$

and the corresponding charges, as in the classical case, are given by integration of the current over the space. For example, the free boson

$$\widehat{H} = \int_{\mathbb{R}^{d-1}} J(\mathrm{x})d\mathrm{x} \tag{11.11}$$

is the quantum Hamiltonian, and

$$\widehat{P}_k := \int_{\mathbb{R}^{d-1}} J_k(\mathrm{x})d\mathrm{x}$$

are the quantum momentum operators.

It is also possible to construct quantum Noether currents and charges of symmetries of a QFT in Lagrangian formalism. This is analogous to the case of classical field theory described in Section 11.3, except that as usual classical observables are replaced by operators. In particular, the output is an operator-valued Noether current $\mathbb{J}$ (a $d-1$-form on spacetime), which is conserved in the sense that $d\mathbb{J} = 0$, and the corresponding charge $Q = \int_C \mathbb{J}$, where C is a space cycle. In a free field theory, the formula for the current is analogous to the classical case, with appropriate normal ordering. Finally, as in classical field theory, for a pure spacetime symmetry given by a vector field $v = v^j\partial_{x^j}$ on the spacetime X, the quantum Noether current $\mathbb{J}_v$ can be expressed in terms of the quantum energy-momentum tensor T by the formula $\mathbb{J}_v = *Tv$.

11.5. Types of symmetry

One can distinguish the following types of symmetry that arise in field theory.

11.5.1. Spacetime (Poincaré) symmetry. The spacetime symmetry is the action of the Poincaré group $\widetilde{\mathbf{P}}$ present by definition in any relativistic field theory on $\mathbb{R}^d$.

11.5.2. Global (or internal) symmetry. A global symmetry is an action of a (possibly disconnected) Lie group G which commutes with the Poincaré group. The term "global" refers to the property of such symmetry to act "in the same way" at all points of spacetime, as opposed to *local symmetries* (Poincaré symmetry, conformal symmetry, gauge symmetry) which act differently at different points. For instance, in the theory of a massive scalar boson valued in a Euclidean space E with Lagrangian

$$\mathcal{L} = \frac{1}{2}(|\nabla\phi|^2 - m^2|\phi|^2), \tag{11.12}$$

there is a global $O(E)$-symmetry rotating ϕ.

The *Coleman-Mandula theorem* states that in a conventional quantum field theory in $d \geq 3$ dimensions *with a mass gap and nontrivial scattering matrix*, every even symmetry is a composition of a Poincaré symmetry and a global symmetry; i.e., the group of symmetries is a direct product of the Poincaré group

with the group of global symmetries (a compact Lie group). This is a rigorous theorem which can be proved in the context of Wightman axioms, whose precise formulation can be found in [**QFS**].[36]

11.5.3. Conformal symmetry. Conformal symmetry is the symmetry of a field theory under conformal transformations of spacetime. Recall that a *conformal map* between (pseudo-)Riemannian manifolds X_1, X_2 is a diffeomorphism $f : X_1 \to X_2$ whose differential df_x at every point $x \in X_1$ is the composition of an isometry and a dilation by some factor $\lambda_f(x)$; i.e., it rescales the metric by multiplying it by a function of x.[37] Thus conformal symmetry includes Poincaré symmetry, and on $\mathbb{R}^d$, $d \geq 3$, in the Minkowski (respectively, Euclidean) signature it is the symmetry under the finite-dimensional conformal group $\mathrm{Conf}(1, d-1) = O(2, d)$ (respectively, $\mathrm{Conf}(0, d) = O(1, d+1)$). Note that $\mathrm{Conf}(0, d)$ is generated by Poincaré symmetries (rotations and translations) as well as dilations and the inversion $x \mapsto \frac{x}{|x|^2}$.

In $d = 2$ dimensions, conformal symmetry is much richer, since any orientable Riemannian surface has a canonical complex structure, and any univalent holomorphic function of one variable defines a conformal map. Thus in 2-dimensional conformal field theory we have the action of an infinite-dimensional Lie algebra of conformal symmetries—the Lie algebra polynomial vector fields $f(z)\partial_z$ on $\mathbb{C}^\times$. This is the *Witt algebra* W (also known as *centerless Virasoro algebra*), see Section 12.7. Even though there is no infinite-dimensional complex Lie group whose Lie algebra is W (there is only a semigroup defined by G. Segal, see Section 12.12), the infinitesimal symmetries defined by this Lie algebra greatly facilitate the study of 2-dimensional conformal field theories. Moreover, since W acts on the theory only $\mathbb{R}$-linearly, we obtain an action of the complexification $W_{\mathbb{C}} = W \oplus \overline{W}$. 2-Dimensional conformal field theory

[36] The mass gap condition in the Coleman-Mandula theorem cannot be dropped; e.g., scale-invariant (thus necessarily gapless) theories have dilation symmetries that do not commute with Poincaré symmetries (see Remark 11.14).

Similarly, the condition of a nontrivial scattering matrix cannot be removed; e.g., it rules out free theories for which the Coleman-Mandula theorem fails. For example, consider the free theory of a complex scalar ϕ of mass m in $\mathbb{R}^d$ for $d > 2$. Let $\widehat{\phi}$ be the Fourier transform of ϕ with respect to the space coordinates. Then the Hamiltonian is

$$H = \int_{\mathbb{R}^{d-1}} H(\mathrm{p}) d\mathrm{p}, \text{ where } H(\mathrm{p}) := \frac{1}{2}(: |\frac{d\widehat{\phi}}{dt}|^2 : + (\mathrm{p}^2 + m^2) : |\widehat{\phi}|^2 :)$$

(a superposition of independent harmonic oscillators labeled by $\mathrm{p} \in \mathbb{R}^{d-1}$). For any smooth function $\theta : \mathbb{R}^{d-1} \to \mathbb{R}$, $\widehat{\phi}(\mathrm{p}) \mapsto e^{i\theta(\mathrm{p})}\widehat{\phi}(\mathrm{p})$ is a symmetry, which is not of the form claimed by the Coleman-Mandula theorem unless θ is constant. Moreover, the Noether charge of this symmetry is

$$Q = \int_{\mathbb{R}^{d-1}} \theta(\mathrm{p}) \, \mathrm{Im}(: \frac{d\widehat{\phi}}{dt}\overline{\widehat{\phi}} :) d\mathrm{p},$$

which is local in ϕ if θ is a polynomial. Taking θ of arbitrary degree, we see that such symmetries can have arbitrarily high spin.

[37] In physics literature, the symmetry of the theory under rescaling of the spacetime metric by a function is often called *Weyl invariance*.

is discussed in Chapter 12; there are also important conformal field theories in higher dimensions $d \geq 3$ (thus, with a finite-dimensional group of conformal symmetries), but we will not discuss them.

More generally, one may talk about conformal invariance of a *family* of field theories parametrized by the Riemannian (or Lorentzian) metrics on the spacetime. In this case conformal invariance by definition means that the action does not change under multiplying the metric by a positive scalar function. There is an obvious criterion for such conformal invariance in terms of the energy-momentum tensor (Section 11.2):

Proposition 11.9. *A classical field theory depending on a metric is conformal if and only if the trace of the energy-momentum tensor vanishes:* $\operatorname{Tr} T = T_i^i = 0$.

Then, in particular, the theory on the flat space will be conformal in the above sense, and the Noether charges for its conformal symmetries can be written in terms of the energy-momentum tensor as explained in Section 11.3. We will see an example of this in Chapter 12.

Remark 11.10. Note that the notion of conformal invariance of a family is strictly stronger than the notion of conformal invariance of an individual theory on a flat spacetime. In particular, the latter does not imply tracelessness of the energy-momentum tensor.

For instance, in Example 11.1 we have

$$\operatorname{Tr} T = (1 - \tfrac{d}{2})|\nabla\phi|^2$$

(and this is also true on a general Riemannian manifold), so we conclude that the classical free scalar field theory with its usual energy-momentum tensor T is conformally invariant (in the above family sense) if and only if $d = 2$. This is of course also easy to check directly: the Lagrangian density is $\frac{1}{2} \int d\phi \wedge *d\phi$, and the Hodge $*$-operator on 1-forms is conformally invariant if and only if $d = 2$ (since in general $*$ on p-forms rescales by the factor λ^{d-2p} when the metric transforms as $g_{ij} \mapsto \lambda^2 g_{ij}$).

So, what happens for classical free scalar field theory in $d \geq 3$ dimensions? It turns out that conformal invariance is still present provided that the field ϕ is rescaled under conformal transformations $x \mapsto x' = f(x)$:

$$\phi'(x') = \lambda_f(x)^{-\frac{d-2}{2}} \phi(x)$$

(i.e., ϕ has *conformal weight* $\frac{d-2}{2}$). This is reflected in the fact that even though in this case the standard energy-momentum tensor T is not traceless, it can be *improved* to achieve tracelessness while keeping it conserved. Namely, define the *improved energy-momentum tensor*

$$\widetilde{T}_{ij} = T_{ij} + \frac{d-2}{4(d-1)}(\delta_{ij}\Delta - \partial_i\partial_j)\phi^2,$$

where the additional summand accounts for the rescaling of ϕ. Then, using the equation of motion $\Delta\phi = 0$, we obtain

$$\operatorname{Tr} \widetilde{T} = 0,$$

exhibiting conformal invariance of the theory in flat space with simultaneous rescaling of ϕ with conformal weight $\frac{d-2}{2}$. Moreover, this can be extended to any Riemannian or Lorentzian manifold.

Exercise 11.11. Show by explicit computation that $\widetilde{T}$ is conserved, that is, $\sum_j \partial_j \widetilde{T}^{jk} = 0$.

Similar statements apply in quantum theory, where T is a local operator.

Example 11.12. Consider the quantum theory of a free scalar field in $d \geq 3$ dimensions. Then the 2-point function is

$$\langle \phi(x)\phi(y)\rangle = |x-y|^{2-d}.$$

This is obviously invariant under dilations with conformal weight $\frac{d-2}{2}$ for ϕ. Moreover, it is invariant under the inversion $x \mapsto \frac{x}{|x|^2}$ with the same conformal weight. Indeed,

$$\left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right|^2 = \frac{|x-y|^2}{|x|^2|y|^2},$$

so

$$\left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right|^{2-d} = |x-y|^{2-d}\,|x|^{d-2}|y|^{d-2}.$$

Thus, this 2-point function is conformally invariant.

For $d = 2$ the situation is similar, except the operator ϕ is not quite well defined, and one should work with its derivatives, see Chapter 12. For example, if x denotes the complex coordinate of $\mathbb{R}^2$, then $\partial\phi$ has scaling dimension 1, and

$$\langle \partial\phi(x)\partial\phi(y)\rangle = \frac{1}{(x-y)^2},$$

which is conformally invariant.

Similarly, the theory of a free massless spinor (both classical and quantum) is conformally invariant in all dimensions.

The situation with gauge theory is similar but a bit more nuanced, since a gauge field A is not itself a physical observable, and only its curvature F_A (more precisely, gauge invariant expressions such as traces of its powers) define observables. To analyze conformal invariance of classical gauge theory, recall that the action of (Euclidean) gauge theory is

$$S = \int_X \operatorname{Tr} F_A \wedge *F_A,$$

and note that the Hodge $*$-operator on 2-forms is invariant under rescalings of the metric if and only if $d = 4$. Thus, for classical gauge theory the action is invariant under rescalings of the metric if and only if $d = 4$. In particular, on flat spacetime, classical gauge theory is conformally invariant precisely in four dimensions. In this case the theory is in fact conformally invariant on any 4-dimensional Riemannian or Lorentzian manifold, even in the nonabelian case, see Section 13.3.

11.5.4. Scaling symmetry. A part of conformal symmetry is the *scaling* (*dilation*) *symmetry* of the spacetime. In a scale-invariant theory, the algebra of observables is graded by scaling dimension, and correlation functions are homogeneous of degree determined by the scaling dimensions of the fields inside the correlator. The scaling dimension Δ of a field in the Lagrangian density is determined from the condition that its kinetic term (including the volume element) has dimension 0, where the coordinates x^i have dimension -1 (see also Section 13.3). For example, the scaling dimension for a scalar field is $\frac{d-2}{2}$, and for a spinor field it is $\frac{d-1}{2}$. The action of the scaling group on fields is then $\phi(x) \mapsto a^\Delta \phi(ax)$, which makes the correlation functions invariant.

One may also discuss scale invariance of a family of theories depending on a metric. Such scale invariance by definition means that the action does not change under rescaling the metric by a positive number. Similar to Proposition 11.9, we have an obvious criterion of scale invariance in this sense in terms of the energy-momentum tensor:

Proposition 11.13. *A classical field theory on a spacetime X is scale-invariant if and only if the trace of the energy-momentum tensor is a total derivative.*

For example, it is easy to see that the theory of a free scalar field defined by Lagrangian (11.12) is scale-invariant if and only if $m = 0$ (in which case it is conformally invariant as we have seen above). Namely, the Noether current on $\mathbb{R}^d$ for the scaling symmetry is

$$\mathbb{J} = \sum_{j,k} x_k \widetilde{T}_{jk} * dx^j,$$

where $\widetilde{T}$ is the improved energy-momentum tensor.

Another example of a scale-invariant theory is the free (abelian) gauge theory in any spacetime dimension d. As explained above, it is conformally invariant (in particular, scale-invariant) for $d = 4$. For $d \neq 4$, it is not conformally invariant (we cannot add an improvement term to T since the gauge field cannot be rescaled), but it is still scale-invariant: the correlation functions of F_A are homogeneous.

As before, the same story applies to quantum theories with T being an operator.

Remark 11.14. Note that scaling symmetry can only exist in quantum field theories without mass gap (as mass introduces a scale of energy, so the spectrum of the Hamiltonian cannot be invariant under dilations), so it does not fall under the Coleman-Mandula theorem and it is no contradiction that scaling does not commute with translations (which are part of Poincaré symmetry).

11.5.5. Topological and area-preserving symmetry. As we have already seen and will see below, in some field theories, there is a symmetry under the group $\mathrm{Diff}_0(X)$ of diffeomorphisms of the spacetime X homotopic to the identity: the quantum mechanics of a massless fermion (Section 9.2), 2-dimensional topological theories (Section 11.11), e.g., 2-dimensional gauge theory for a finite group (Section 11.10), Chern-Simons theory (Section 11.12), etc. There are also theories with slightly less symmetry, namely the subgroup $\mathrm{Diff}_0^a(X) \subset \mathrm{Diff}_0(X)$, of area-preserving diffeomorphisms, e.g., 2-dimensional gauge theory for a compact Lie group (Section 11.10). Such theories are discussed in more detail in Section 11.11.

11.5.6. Gauge symmetry. This is the action of the group $\mathcal{G}$ of automorphisms of a principal G-bundle P on the spacetime (the group of *gauge transformations*), when one of the fields is a connection A on this bundle (a *gauge field*), see Section 11.1. This symmetry is important in defining gauge theory, but it does not actually act on classical states or observables, because the space of classical states is defined as a quotient of the space of connections (and other fields, if present) by the action of this group. Similarly, it does not act on the Hilbert space of quantum states, which is defined as the space of $\mathcal{G}$-invariants in a certain representation $\widetilde{\mathcal{H}}$ of $\mathcal{G}$.

11.5.7. Supersymmetry. This is a special kind of odd symmetry exchanging bosons and fermions, which squares to the translation symmetry and transforms in a spin representation of the Lorentz group (being odd, it does not fall under the Coleman-Mandula theorem, so it is no contradiction that it does not commute with the Lorentz group). This is the subject of Chapter 14.

11.5.8. R-symmetry. This is a special kind of global symmetry that acts on the supercharges discussed in Section 14.13.

11.6. Anomalies and breaking of classical symmetries under quantization

It is important to note that a symmetry of a classical field theory may fail to survive quantization. If this happens, one says that the symmetry becomes

anomalous in quantum theory, or suffers a *quantum anomaly*. This is prohibited for Poincaré symmetries by the axioms of QFT. Other examples of anomalies that make the theory inconsistent are the *gravitational anomaly* in conformal field theory, when the left-moving and right-moving copies of the Virasoro algebra act with different central charges (see Chapter 12), the chiral anomaly (see Section 14.5) and anomalies of gauge symmetry ([**QFS**, p. 1166]). But this can happen for other types of symmetry without making the theory inconsistent: global symmetry, scaling (or conformal) symmetry, or supersymmetry. For example, the classical scaling symmetry of the theory of a scalar field with a critical interaction becomes anomalous quantum mechanically, as in perturbation theory, see Section 13.8.

Another possibility is that a certain symmetry acts on the algebra of observables but there is no invariant ground state; i.e., the symmetry group acts nontrivially on the moduli space of ground states (vacua) of the theory. In this case one speaks of *spontaneous symmetry breaking*, discussed in more detail in Section 11.7. Spontaneous symmetry breaking may already happen in classical theory and can be cured or not quantum mechanically. Or the symmetry may be present in classical theory and break when the theory is quantized (as it happens in the Gross-Neveu model, see Example 13.26).

11.7. Spontaneous symmetry breaking

Spontaneous symmetry breaking is an important and ubiquitous phenomenon in field theory, which arises when a system whose laws of motion are invariant under some group G settles down into a regime which is not G-invariant, but is only invariant under a subgroup $K \subset G$ (possibly the trivial one). In this case one says that the G-symmetry of the system *breaks down to* K. Let us briefly discuss this phenomenon, following [**QFS**, pp. 1121–1142].

Spontaneous symmetry breaking can be vividly observed already in classical mechanics. Let us start with *discrete symmetry breaking*, i.e., the situation when G is a finite group. Consider a particle on the real line with potential

$$U(q) = (q^2 - 1)^2;$$

we may imagine a ball rolling on the graph of this function, as plotted in Figure 11.1.

This system has a symmetry under $G = \mathbb{Z}/2$ acting by $q \mapsto -q$. On the other hand, if the ball has energy < 1, then it must oscillate around one of the ground states $q = 1$ or $q = -1$, since it does not have enough energy to go over the bump at $q = 0$. Thus the symmetry under $G = \mathbb{Z}/2$ is spontaneously broken (to the trivial group). The same applies to classical field theory.

For the *quantum* particle with the same potential this phenomenon disappears, however, and we have a single ground state ψ_0—the unique positive

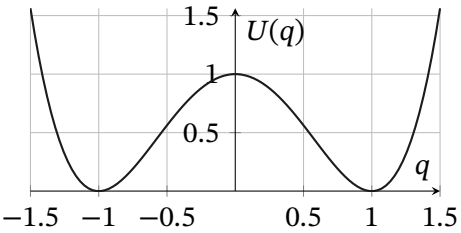


Figure 11.1

eigenfunction of the Hamiltonian of norm 1 with smallest eigenvalue (Section 7.5), which is therefore G-invariant. This happens because of *tunneling*—the quantum particle, unlike the classical one, can jump over the bump from one ground state to the other with positive probability even if its energy is less than 1.

It turns out that the same happens in field theory of a real scalar with Minkowskian Lagrangian

$$\mathcal{L} = \frac{1}{2}(|\nabla\phi|^2 - (\phi^2 - 1)^2)$$

in finite volume (e.g., compact space, such as a torus $(S^1)^{d-1}$), for the same reason. However, in infinite volume (on $\mathbb{R}^{d-1}$) we have a particle at every point $\mathrm{x} \in \mathbb{R}^{d-1}$ and for all of them to jump over the bump requires an infinite amount of energy, so is impossible even in quantum theory. As a result, the corresponding quantum field theory has two realizations (versions) localized around the two classical ground states $q = \pm 1$, with Hilbert spaces $\mathcal{H}_\pm$, but the group $\mathbb{Z}/2$ does not act in either of them; rather, it swaps $\mathcal{H}_+$ and $\mathcal{H}_-$. Thus in quantum field theory in infinite volume the $\mathbb{Z}/2$-symmetry is spontaneously broken in any dimension $d \geq 2$.

A similar thing happens in the Gross-Neveu model discussed in Example 13.26, but here the situation is a bit different: the $\mathbb{Z}/2$ symmetry is present classically (there is only one ground state), but in quantum theory it is broken and this ground state bifurcates into two, permuted by the symmetry.

The situation with *continuous symmetry breaking* (i.e., when G is a connected Lie group) is similar but with some crucial differences. As before, such symmetry breaking easily occurs in classical mechanics. Indeed, consider a particle moving on $\mathbb{R}^N$ with the potential

$$U(\mathbf{q}) = (|\mathbf{q}|^2 - 1)^2.$$

Then classical ground states form the sphere S^{N-1} defined by $|\mathbf{q}| = 1$. So the symmetry under $G = \mathrm{SO}(N)$ breaks down to $\mathrm{SO}(N-1)$ (stabilizer of a point in S^{N-1}). The same happens in classical field theory.

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Note that continuous global symmetry breaking in a relativistic classical field theory gives rise to massless modes. For example, consider the field theory with Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}|\nabla\phi|^2 + (|\phi|^2 - 1)^2,$$

where ϕ takes values in $\mathbb{R}^N$. Using spherical coordinates, we write $\phi = ru$, where $r = |\phi|$ and $u \in S^{N-1}$ is a unit vector. In terms of the variables r, u, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}|\nabla r|^2 + (r^2 - 1)^2 + \frac{1}{2}r^2|\nabla u|^2.$$

The potential depends only on r, so the vacuum manifold is the unit sphere S^{N-1} defined by the equation $r = 1$, and fluctuations in the u-directions are massless. Thus we obtain a massless scalar field u valued in S^{N-1}. The coordinates of u are the *Goldstone modes*.

In general, if a global symmetry breaks from a connected compact group G to a subgroup K, the Goldstone modes can be described as scalar fields taking values in the homogeneous space G/K; there is one such mode per broken generator, i.e. $\dim(G/K)$ of them.

Note that Goldstone modes need not be free. In the above example, after freezing the radial mode $r = 1$, we obtain a nonlinear sigma model with target S^{N-1} (Section 11.1). For $N = 2$ this target space is flat, so the Goldstone mode is equivalent to a free (circle-valued) massless scalar field. For $N > 2$, however, S^{N-1} is curved, and this curvature leads to interactions between Goldstone modes (i.e., cubic and higher terms in the Lagrangian when written in coordinates on S^{N-1}).

In quantum theory, under the usual assumptions of relativistic local QFT, spontaneous breaking of a continuous global symmetry again gives rise to Goldstone modes. This is the content of *Goldstone's theorem*, discussed, for example, in [**QFS**, p. 1139]: for each broken generator there corresponds a massless scalar quantum particle (a subrepresentation of $\mathbf{P}$ in $\mathcal{H}$ isomorphic to $L^2(X_0^+)$) which is created from the vacuum by the current $J(x)$ of the broken symmetry (i.e., $J(x)\Omega$ is not orthogonal to this subrepresentation). This particle is called a *Goldstone boson*. Thus, as in the classical case, we have $\dim G/K$ independent Goldstone bosons.

We note that, similar to the case of a discrete symmetry, a continuous symmetry can be spontaneously broken in the quantization of a classical field theory even if it is not broken in the classical theory itself. In this case, the symmetry breaking is a genuinely nonperturbative effect, since the perturbation series around a G-stable vacuum (defined in Chapter 13) is manifestly G-invariant. However, if the symmetry is already broken classically, then Goldstone bosons

can be seen in perturbation theory as quantizations of the classical Goldstone modes.

On the other hand, since massless bosons in spacetime dimension ≤ 2 violate the clustering property, as their Green's function does not decay at infinity (Section 10.2), we see that a continuous global symmetry cannot be spontaneously broken in quantum mechanics (just as in the discrete case), nor in 2-dimensional quantum field theory, and more generally whenever there is at most one noncompact spatial dimension (in contrast to the discrete case!). An informal explanation for this is that in the continuous case it is even easier for the system to move between ground states: it does not need to go over the bump any more, since the set of ground states is connected.

Thus, spontaneous breaking of a continuous global symmetry can only occur in quantum field theory starting with spacetime dimension $d = 3$, i.e. when there are at least two noncompact spatial dimensions. Detailed understanding of the mechanism and implications of this phenomenon is a central topic in quantum field theory, but it is beyond the scope of this book, and we refer the reader to [**QFS**] for more details.

11.8. Higgs mechanism

When spontaneous symmetry breaking occurs in gauge theory with a charged scalar, the initially massless gauge field acquires a nonzero mass. This important phenomenon is called the *Higgs mechanism*. Let us first explain how it happens classically.

Consider a 4-dimensional $U(1)$-gauge theory with gauge field A and a complex scalar ϕ in the tautological $U(1)$-representation with potential $(|\phi|^2 - 1)^2$. So the Euclidean Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(e^{-2} dA \wedge *dA + |\nabla_A \phi|^2) + (|\phi|^2 - 1)^2,$$

where e is the coupling constant (electric charge). It is convenient to rescale A, replacing it by eA; then we have

$$\mathcal{L} = \tfrac{1}{2}(dA \wedge *dA + |\nabla\phi + ieA\phi|^2) + (|\phi|^2 - 1)^2.$$

For fixed A there is a global $U(1)$-symmetry breaking, since there is a circle of vacuum configurations for ϕ, as discussed in the previous section. So, as before, let us write ϕ in polar coordinates,

$$\phi = r \exp(i\theta).$$

Then we have

$$\mathcal{L} = \tfrac{1}{2}(dA \wedge *dA + |\nabla r|^2 + r^2 |\nabla\theta + eA|^2) + (r^2 - 1)^2.$$

The difference with the previous section is that now we can gauge away θ: pick the (unique) gauge in which $\theta = 0$. Then the Lagrangian takes the form

$$\mathcal{L} = \tfrac{1}{2}(dA \wedge *dA + |\nabla h|^2 + e^2(1+h)^2|A|^2) + h^2(h+2)^2,$$

where $h := r - 1$ is the radial fluctuation of r around the vacuum value $r = 1$. The quadratic part of this Lagrangian is thus

$$\mathcal{L}_{\text{quadratic}} = \frac{1}{2}(dA \wedge *dA + e^2|A|^2 + |\nabla h|^2 + 8h^2),$$

i.e., we see that we have a massive vector A (of mass e) and massive real scalar h (instead of the initial massless vector and two real scalars). Thus, the Higgs mechanism allows the massless gauge field A to "eat" a massless scalar (the Goldstone mode of the broken symmetry) and become massive. The second (massive) scalar h remains intact.

In quantum theory, the Higgs mechanism results in the corresponding transformation for quantum particles. For $e = 0$, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}dA \wedge *dA + \tfrac{1}{2}|\nabla\phi|^2 + (|\phi|^2 - 1)^2,$$

i.e., A and ϕ are decoupled, and A is the gauge field of a free $U(1)$ gauge theory. So for the particle content we should have a massless vector and two real scalars—the massive radial mode h and the massless angular mode θ (the Goldstone boson). However, when e is turned on, the massless vector representation of $\mathbf{P}$, which alone could not deform to a massive one, combines with a massless scalar representation, and the direct sum of these representations deforms into a massive vector representation (see Exercise 10.24). This is the Higgs mechanism at the quantum level.

11.9. Dimensional reduction

Dimensional reduction is a procedure of reducing a field theory initially defined in d spacetime dimensions down to a space of smaller dimension $d - r$. In classical field theory this is done by restricting to field configurations independent of the last r coordinates (in either Lagrangian or Hamiltonian formulation). This means that the spacetime is taken to be $X \times \mathbb{R}^r$ where X is a $(d-r)$-dimensional manifold and all the fields are pulled back from X. This can be extended to quantum theory in the path integral approach by computing the path integral over such field configurations.

An important feature of dimensional reduction is that the reduced theory may have more fields than the original one. Namely, while reduction does nothing to the scalar fields, other types of fields fall apart into several components. For example, a vector in dimension d becomes a vector in dimension $d-r$ together with r scalars coming from the last r coordinates. Likewise, since the restriction of a spin representation of $\mathrm{Spin}(d)$ to $\mathrm{Spin}(d-r)$ is a direct sum of

several spin representations, a spinor in d dimensions falls into several spinors in dimension $d-r$.

Note that by construction, upon reduction from d to $d-r$ dimensions, the theory acquires a new residual global symmetry under the group $\mathrm{Spin}(r)$ acting on the variables on which the fields are now independent, commuting with all the symmetries coming from d dimensions.

As an example consider a 4-dimensional $U(1)$-gauge theory with a massless complex spinor field ψ valued in the tautological representation ρ of $U(1)$ (more precisely, in $\rho \otimes S_+$) together with the complex conjugate $\psi^\dagger$ valued in $\rho^* \otimes S_-$ (recall that in four dimensions, a kinetic term must include both a chiral and anti-chiral spinor). The Lagrangian of this theory is

$$\mathcal{L}_{4\text{-dim}}(A, \psi, \psi^\dagger) = |F_A|^2 + (\psi^\dagger, \mathbf{D}_A \psi),$$

where A is the gauge field (photon), F_A is the curvature of the connection A, and $\mathbf{D}_A$ is the covariant Dirac operator (cf. Section 10.7). What happens if this theory is reduced to three dimensions?

For simplicity we assume that A is a connection on the trivial bundle. Reduction to three dimensions means that fields that initially depended on t, x^1, x^2, x^3 will now depend only on t, x^1, x^2. Thus $A = \mathbb{A} + \phi dx^3$, where ϕ is a scalar field and $\mathbb{A}$ is a 1-form in the variables t, x^1, x^2. So

$$F_A = dA = d\mathbb{A} + d\phi \wedge dx^3 = F_{\mathbb{A}} + d\phi \wedge dx^3,$$

hence $|F_A|^2 = |F_{\mathbb{A}}|^2 + |\nabla \phi|^2$.

Now let us see what happens to the spinors $\psi, \psi^\dagger$. The 2-dimensional spin representation S_+ of $\mathrm{Spin}(4)$ remains irreducible when restricted to $\mathrm{Spin}(3)$, so we will still have a single spinor field ψ together with conjugate $\psi^\dagger$. However, the kinetic term $(\psi^\dagger, \mathbf{D}_A \psi) = (\psi^\dagger, (\mathbf{D} + A)\psi)$ will take the form $(\psi^\dagger, (\mathbf{D} + \mathbb{A})\psi) + \phi(\psi^\dagger, \psi)$ (the Clifford algebra element corresponding to x^3 switches S_+ and S_-). Note that the last term only makes sense in three dimensions, as there is no $\mathrm{Spin}(4)$-invariant pairing between S_+ and S_-; however, over $\mathrm{Spin}(3) = \mathrm{SU}(2)$, S_+ and S_- become isomorphic to S, which is self-dual, so an invariant pairing appears. Thus the Lagrangian of the reduced 3-dimensional theory is

$$\mathcal{L}_{3\text{-dim}}(\mathbb{A}, \phi, \psi, \psi^\dagger) = |F_{\mathbb{A}}|^2 + |\nabla \phi|^2 + (\psi^\dagger, \mathbf{D}_{\mathbb{A}} \psi) + \phi(\psi^\dagger, \psi).$$

What happens to this theory if we further reduce it to two dimensions? In this case we get $\mathbb{A} = \mathcal{A} + \eta dx^2$ where $\mathcal{A}$ is a 1-form and η is a new scalar field in the variables t, x^1. Also, the spin representation S falls into two 1-dimensional representations, so we get $\psi = (\psi_+, \psi_-)$, with the element of the Clifford algebra corresponding to x^2 acting on ψ_+ by $+1$ and on ψ_- by -1. Thus the reduced

Lagrangian looks like

$$\begin{aligned}\mathcal{L}_{2\text{-dim}}(\mathcal{A},\phi,\eta,\psi_+,\psi_-,\psi_+^\dagger,\psi_-^\dagger)\\ = |F_{\mathcal{A}}|^2 + |\nabla\phi|^2 + |\nabla\eta|^2 + (\psi_+^\dagger, \mathbf{D}_{\mathcal{A}}\psi_+) + (\psi_-^\dagger, \mathbf{D}_{\mathcal{A}}\psi_-)\\ + (\phi + i\eta)(\psi_-^\dagger,\psi_+) + (\phi - i\eta)(\psi_+^\dagger,\psi_-)\end{aligned}$$

(the factor i for η appears from the action of the Lie algebra of $U(1)$ on the tautological representation). In terms of the complex scalar $\chi := \phi + i\eta$, this can be written as

$$\begin{aligned}\mathcal{L}_{2\text{-dim}}(\mathcal{A},\chi,\psi_+,\psi_-,\psi_+^\dagger,\psi_-^\dagger)\\ = |F_{\mathcal{A}}|^2 + |\nabla\chi|^2 + (\psi_+^\dagger, \mathbf{D}_{\mathcal{A}}\psi_+) + (\psi_-^\dagger, \mathbf{D}_{\mathcal{A}}\psi_-)\\ + \chi(\psi_-^\dagger,\psi_+) + \overline{\chi}(\psi_+^\dagger,\psi_-).\end{aligned}$$

As expected, in addition to the $U(1)$-symmetry rotating ψ (and $\psi^\dagger$ in the opposite direction) which exists in the original 4-dimensional theory, the reduced 2-dimensional theory has another $U(1)$ symmetry (commuting with the previous one), which only rotates ψ_- (and $\psi_-^\dagger$ in the opposite direction) and χ. This symmetry appears from the $\mathrm{Spin}(2)$-action on the variables x^2, x^3.

Finally, let us reduce this theory to one dimension (quantum mechanics). In this reduction, the gauge field disappears, producing a third real boson ξ. We thus obtain the Lagrangian

$$\begin{aligned}\mathcal{L}_{1\text{-dim}}(\xi,\chi,\psi_+,\psi_-,\psi_+^\dagger,\psi_-^\dagger)\\ = |\dot\xi|^2 + |\dot\chi|^2 + (\psi_+^\dagger,\dot\psi_+) + (\psi_-^\dagger,\dot\psi_-)\\ + \xi(\psi_+^\dagger,\psi_+) - \xi(\psi_-^\dagger,\psi_-) + \chi(\psi_-^\dagger,\psi_+) + \overline{\chi}(\psi_+^\dagger,\psi_-).\end{aligned}$$

Introducing the scalar $\beta = \begin{pmatrix}\xi & \chi\\ \overline{\chi} & -\xi\end{pmatrix}$ valued in traceless hermitian 2-by-2 matrices, we can write this Lagrangian as

$$\mathcal{L}_{1\text{-dim}}(\beta,\psi,\psi^\dagger) = |\dot\beta|^2 + (\psi^\dagger,\dot\psi) + (\psi^\dagger,\beta\psi),$$

which makes manifest the expected $\mathrm{SU}(2) = \mathrm{Spin}(3)$-symmetry acting on ψ, dually on $\psi^\dagger$, and by conjugation on β. This symmetry comes from the action of this group on x^1, x^2, x^3.

11.10. 2-Dimensional gauge theory

While nonabelian quantum gauge theory in ≥ 3 dimensions is very complicated, 2-dimensional pure nonabelian quantum gauge theory is simple enough to be defined rigorously and completely solved. In this section we discuss (somewhat informally) its solution by E. Witten ([**Wi1**]).

Let G be a compact Lie group with a fixed invariant inner product on its Lie algebra $\mathfrak{g}$. Recall from Section 11.1 that the action of the pure gauge theory for G on a surface X is a functional of a connection A on a principal G-bundle on X given by

$$S(A) := \int_X \mathcal{L}(A), \quad \mathcal{L}(A) := \frac{1}{2}|F_A|^2 d\sigma,$$

where F_A is the curvature of A and $\mathcal{L}(A)$ the Lagrangian density. Note that $\mathcal{L}$ depends on the metric on X only through the area form $d\sigma$; indeed, given such a form, we can view the 2-form F_A with values in the adjoint bundle as a 0-form (i.e., a section), so the expression $|F_A|^2$ makes sense. This property will be preserved in quantum theory, so the Euclidean partition function

$$Z(X,\hbar) := \int e^{-\frac{S(A)}{\hbar}} DA$$

for a closed oriented surface X only depends on the topology and the area of X. Moreover, $S(A)$ scales reciprocally to $d\sigma$ and hence to the area of X, thus

$$Z(X,\hbar) = Z_{\mathrm{g}}(\hbar \cdot \mathrm{area}(X)),$$

where Z_{g} depends only on the genus g of X. Finally, when $\hbar \to 0$, the path integral localizes to minima of the action, which are flat connections ($F_A = 0$). So we should expect the asymptotics of $Z_{\mathrm{g}}(\hbar)$ as $\hbar \to 0$ to be expressed in terms of flat G-connections or, equivalently, G-local systems on the surface X.

Let us compute Z_{g}. To this end, let us first put this theory on the cylinder $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ is the time axis. As noted in Section 11.9, the phase space of the classical theory is $T^*\mathcal{M}$, where $\mathcal{M}$ is the space of gauge classes of G-connections on the circle $\mathbb{R}/\mathbb{Z}$ defined by (11.2). But this space is very easy to describe—it is just the space $G/\operatorname{Ad}(G)$ of conjugacy classes in G, since a connection A on the circle is completely determined by its holonomy $\mathrm{Hol}_{\mathbb{R}/\mathbb{Z}}(A)$. Thus in quantum theory, the Hilbert space should be

$$\mathcal{H} = L^2(\mathcal{M}) = L^2(G)^G.$$

Moreover, classically the Hamiltonian of the theory is $H = \frac{p^2}{2}$, where $p \in \mathfrak{g}^*$ is the momentum on $T^*G = G \times \mathfrak{g}^*$, so in quantum theory it is $H = -\frac{\hbar^2}{2}\Delta$, where Δ is the Laplace operator on G. It thus follows that the partition function on the finite cylinder $C_L := [0, L] \times \mathbb{R}/\mathbb{Z}$ of area L with boundary conditions[38] $\mathrm{Hol}(A) = x_0$ at 0 and $\mathrm{Hol}(A) = x_1$ at L ($x_0, x_1 \in G$) is[39]

$$Z(C_L, \hbar, x_0, x_1) = \langle \delta_{x_1}, e^{-\frac{LH}{\hbar}} \delta_{x_0} \rangle = \int_G K_{\hbar L}(y x_0 y^{-1} x_1^{-1}) dy,$$

[38] In this section we will use letters x, y for elements of G, not for points of the spacetime X as elsewhere in the book.

[39] We normalize the Haar measure on G to have unit volume.

where δ_x is the delta function of the conjugacy class of x and $K_t(y)$ is the heat kernel of G, i.e., the fundamental solution of the heat equation,

$$\partial_t u = \frac{1}{2}\Delta u.$$

Thus for the torus $T = T_L := \mathbb{R}/L\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ the partition function is

$$Z(T_L, \hbar) = \mathrm{Tr}_{\mathcal{H}}(e^{-\frac{LH}{\hbar}}) = \int_{G^2} K_{\hbar L}(yxy^{-1}x^{-1})dxdy.$$

In other words, the genus 1 partition function is given by

$$Z_1(\hbar) = \int_{G^2} K_{\hbar}(yxy^{-1}x^{-1})dxdy = \sum_{V \in \mathrm{Irrep}(G)} e^{-\frac{1}{2}\hbar C_V}, \tag{11.13}$$

where C_V is the eigenvalue of the quadratic Casimir C of G on its irreducible representation V.

Exercise 11.15. Show that

$$K_{\hbar}(y) = \sum_{V \in \mathrm{Irrep}(G)} e^{-\frac{1}{2}\hbar C_V} \dim(V)\chi_V(y),$$

where χ_V is the character of V.

Example 11.16. Let $G = U(1)$. Then we get

$$Z_1(\hbar) = \sum_{N \in \mathbb{Z}} e^{-\frac{\hbar N^2}{2}} = \vartheta(0, \tfrac{\hbar}{2\pi}),$$

where ϑ is the theta function defined by (6.11).

It is instructive to obtain the same answer from the path integral formulation (they must coincide by the Feynman-Kac formula). For this we must recall that topological classes of $U(1)$-bundles on a closed oriented surface X are labeled by integers: $E \mapsto c_1(E)$. The smallest value of the action on the bundle E_N with $c_1(E_N) = N$ is attained on a connection A_N with constant curvature, $F_{A_N} = 2\pi i N d\sigma$, and for other connections on the same bundle, $A = A_N + \omega$ for a 1-form ω on X, we have $F_A = F_{A_N} + d\omega = (2\pi i N + f_\omega)d\sigma$, where $d\omega = f_\omega d\sigma$, so $\int_X f_\omega d\sigma = 0$. Thus

$$S(A) = \frac{1}{2}\int_X |2\pi i N + f_\omega|^2 d\sigma = 2\pi^2 N^2 + \frac{1}{2}\int_X |f_\omega|^2 d\sigma.$$

It follows that the path integral over connections on E_N equals

$$I_N(\hbar) = e^{-\frac{2\pi^2 N^2}{\hbar}} I_0(\hbar),$$

where $I_0(\hbar)$ is a Gaussian integral independent of N which under correct normalization ends up being equal to $\sqrt{\frac{2\pi}{\hbar}}$. Thus we get

$$Z_1(\hbar) = \sum_{N\in\mathbb{Z}} I_N = \sqrt{\tfrac{2\pi}{\hbar}}\vartheta(0, \tfrac{2\pi}{\hbar}).$$

This agrees with the Hamiltonian answer by the modular invariance of the theta function.

One can make a generalization of this theory by adding to the Lagrangian a *topological term*

$$\mathcal{L}_\theta(A) := \mathcal{L}(A) + \frac{i\theta}{2\pi}F_A,$$

where θ is a real parameter called the *θ-angle*; the answer will depend on this parameter only modulo 2π. This term is called topological because the corresponding term in the action, $\frac{i\theta}{2\pi}\int_X F_A = i\theta c_1(E)$ only depends on the topology of the bundle. The introduction of this term results in weighting the contribution of bundles with $c_1 = N$ by the phase factor $e^{iN\theta}$, so we get the genus 1 Lagrangian partition function

$$Z_1(\hbar,\theta) = \sqrt{\tfrac{2\pi}{\hbar}}\sum_{N\in\mathbb{Z}} e^{iN\theta - \frac{2\pi^2N^2}{\hbar}} = \sqrt{\tfrac{2\pi}{\hbar}}\vartheta(\tfrac{\theta}{2\pi}, \tfrac{2\pi}{\hbar}).$$

Now let us pass to more general surfaces. First consider a genus 0 surface $X_{0,n+1}$ with $n+1$ boundary circles C_i, $i = 0,\dots,n$ (of unit area). So we need to specify boundary conditions on C_i-elements $g_0,\dots,g_n \in G$, and then we can consider the partition function

$$Z(X_{0,n+1},\hbar,g_0,\dots,g_n) = \int_{A:\mathrm{Hol}_{C_i}(A)\sim g_i} e^{-\frac{S(A)}{\hbar}}DA.$$

This function depends only on the conjugacy classes of $g_0,\dots,g_n$.

To compute this function, note that if we attach to a surface $X_{0,n+1}(t)$ of area t a cylinder $C_\hbar$ of area $\hbar$ at the 0th hole, we obtain a surface $X_{0,n+1}(t+\hbar)$ of area $t+\hbar$. Thus we get

$$Z(X_{0,n+1},t+\hbar,g_0,\dots,g_n) = \int_G K_\hbar(g_0u^{-1})Z(X_{0,n+1},t,u,g_1,\dots,g_n)du. \tag{11.14}$$

Also when $t\to 0$, the surface $X_{0,n+1}(t)$ becomes very thin and degenerates into a bouquet of n circles, so we get

$$Z(X_{0,n+1},0,u,g_1,\dots,g_n) = \int_{G^n}\delta(u\prod_{j=1}^n x_jg_jx_j^{-1})dx_1\dots dx_n.$$

Thus by (11.14)

(11.15)

$$Z(X_{0,n+1},\hbar,g_0,\dots,g_n)=\int_{G^{n+1}}K_\hbar(g_0u^{-1})\delta(u\prod_{j=1}^{n}x_jg_jx_j^{-1})dudx_1\dots dx_n$$
$$=\int_{G^n}K_\hbar(g_0\prod_{j=1}^{n}x_jg_jx_j^{-1})dx_1\dots dx_n.$$

This formula can be used to compute partition functions of surfaces of higher genus, with or without boundary. Namely, let $X_{\mathrm{g},n}$ be a surface of genus g with n holes, of unit area. Then $X_{\mathrm{g},n}$ is obtained by gluing $X_{0,n+\mathrm{g}+1}$ to $X_{0,\mathrm{g}+1}$ at $\mathrm{g}+1$ holes, so

$$Z(X_{\mathrm{g},n},\hbar,g_0,\dots,g_{n-1})$$
$$=\int_{G^{2\mathrm{g}+1}}Z(X_{0,n+\mathrm{g}+1},\hbar,g_0,\dots,g_{n-1},x_0^{-1},\dots,x_{\mathrm{g}}^{-1})\delta(x_0\prod_{j=1}^{\mathrm{g}}y_jx_jy_j^{-1})d\mathbf{x}d\mathbf{y}$$
$$=\int_{G^{2\mathrm{g}}}Z(X_{0,n+\mathrm{g}+1},\hbar,g_0,\dots,g_{n-1},\prod_{j=1}^{\mathrm{g}}y_jx_jy_j^{-1},x_1^{-1},\dots,x_{\mathrm{g}}^{-1})d\mathbf{x}d\mathbf{y}$$
$$=\int_{G^{3\mathrm{g}+n}}K_\hbar(\prod_{k=0}^{n-1}v_kg_kv_k^{-1}\prod_{j=1}^{\mathrm{g}}y_jx_jy_j^{-1}\prod_{j=1}^{\mathrm{g}}u_jx_j^{-1}u_j^{-1})d\mathbf{x}d\mathbf{y}d\mathbf{u}d\mathbf{v}$$
$$=\int_{G^{2\mathrm{g}+n}}K_\hbar(\prod_{k=0}^{n-1}v_kg_kv_k^{-1}\prod_{j=1}^{\mathrm{g}}y_jx_jy_j^{-1}\prod_{j=1}^{\mathrm{g}}x_j^{-1})d\mathbf{x}d\mathbf{y}d\mathbf{v}$$

(note that if $n\geq 1$, then we can set $v_0=1$ and integration over v_0 can be dropped). In particular, we get

$$Z_{\mathrm{g}}(\hbar)=Z(X_{\mathrm{g},0},\hbar)=\int_{G^{2\mathrm{g}}}K_\hbar(\prod_{j=1}^{\mathrm{g}}y_jx_jy_j^{-1}\prod_{j=1}^{\mathrm{g}}x_j^{-1})d\mathbf{x}d\mathbf{y}. \tag{11.16}$$

Note that the product $\prod_{j=1}^{\mathrm{g}}y_jx_jy_j^{-1}\prod_{j=1}^{\mathrm{g}}x_j^{-1}$ in (11.16) is one of the forms of the defining relation of the fundamental group of $\pi_1(X_{\mathrm{g},0})$ in some generators x_i,y_i. Another form of this relation (for another choice of generators) is $\prod_{j=1}^{\mathrm{g}}a_jb_ja_j^{-1}b_j^{-1}$. Thus by changing variables, we get

Proposition 11.17. *The partition function Z_{g} is given by the formula*

$$Z_{\mathrm{g}}(\hbar)=\int_{G^{2\mathrm{g}}}K_\hbar(\prod_{j=1}^{\mathrm{g}}a_jb_ja_j^{-1}b_j^{-1})d\mathbf{a}d\mathbf{b}. \tag{11.17}$$

For example, in the case $\mathrm{g} = 1$, this recovers formula (11.13), while for $\mathrm{g} = 0$ we get

$$Z_0(\hbar) = K_\hbar(1) = \sum_{V \in \mathrm{Irrep}(G)} e^{-\frac{1}{2}\hbar C_V} \dim(V)^2.$$

In particular, when G is finite,[40] then the Hamiltonian is zero (a topological quantum field theory), so $K_\hbar = K_0 = |G|\delta$ and we get

$$Z_\mathrm{g} = |G|^{1-2\mathrm{g}} \cdot |\{(a_1, \dots, a_\mathrm{g}, b_1, \dots, b_\mathrm{g}) \in G^{2\mathrm{g}} : \prod_{j=1}^{\mathrm{g}} a_j b_j a_j^{-1} b_j^{-1} = 1\}|. \tag{11.18}$$

Thus $|G|^{2\mathrm{g}-2} Z_\mathrm{g}$ provides the orbifold count of conjugacy classes of homomorphisms $\pi_1(X_{\mathrm{g},0}) \to G$ (i.e., G-bundles on $X_{\mathrm{g},0}$): every class P of bundles counts with (by now our favorite!) weight $\frac{1}{|\mathrm{Aut}(P)|}$.

Exercise 11.18. Show that the number of homomorphisms $\pi_1(X_{\mathrm{g},0}) \to G$ equals

$$N_\mathrm{g}(G) = |G|^{2\mathrm{g}-1} \sum_{V \in \mathrm{Irrep}(G)} \dim(V)^{2-2\mathrm{g}}.$$

Hint. Consider the central element $z \in \mathbb{C}G$ given by the formula

$$z := \sum_{x,y \in G} yxy^{-1}x^{-1}.$$

Split this sum into conjugacy classes of x, and deduce that for every irreducible representation V of G, $z|_V = \frac{|G|^2}{\dim(V)^2}$. Then compute the trace of z^g on the regular representation of G using Maschke's theorem.

Thus, we get that

$$Z_\mathrm{g} = \frac{N_\mathrm{g}(G)}{|G|^{2\mathrm{g}-1}} = \sum_{V \in \mathrm{Irrep}(G)} \dim(V)^{2-2\mathrm{g}}.$$

It turns out that, as was shown by E. Witten, this formula generalizes to compact simple Lie groups G. Namely, by analogy with Exercise 11.18, for every irreducible representation V of G we may consider the operator

$$z_V := \int_{G^2} \rho_V(yxy^{-1}x^{-1}) dx dy : V \to V.$$

As in Exercise 11.18, this operator is G-invariant, so it is a scalar, namely $\frac{1}{\dim(V)^2}$. Define the *Witten zeta function*

$$\zeta_G(s) := \sum_{V \in \mathrm{Irrep}(G)} \frac{1}{\dim(V)^s};$$

[40] In gauge theory one may assume that the gauge group is compact, but not necessarily connected.

for example $\zeta_{SU(2)}(s) = \zeta(s)$, the usual Riemann zeta function. It is easy to show that the series $\zeta_G(s)$ converges if $\mathrm{Re}(s) > 1$ and defines an analytic function in this half-plane. It follows that the operator z^{s+1} on $L^2(G)$ is of trace class if $\mathrm{Re}(s) > \frac{1}{2}$, and by the Peter-Weyl theorem

$$\mathrm{Tr}(z^{s+1}) = \zeta_G(2s).$$

In particular, if $\mathrm{g} \geq 2$ is an integer, then

$$Z_{\mathrm{g}}(0) = \int_{G^{2\mathrm{g}}} \delta(\prod_{j=1}^{\mathrm{g}} a_j b_j a_j^{-1} b_j^{-1}) d\mathbf{a} d\mathbf{b} = \mathrm{Tr}(z^{\mathrm{g}}) = \zeta_G(2\mathrm{g}-2).$$

Note that the integral in this equality is nothing but $\frac{1}{|Z(G)|}$ of the volume of the moduli space $\mathrm{Loc}_{\mathrm{g}}(G)$ of G-local systems on $X_{\mathrm{g},0}$ (i.e., representations of $\pi_1(X_{\mathrm{g}}, 0)$) in its natural symplectic structure, where $Z(G)$ is the center of G; indeed, the support of the delta function is a principal $G/Z(G)$-bundle over the generic locus of $\mathrm{Loc}_{\mathrm{g}}(G)$. Thus we obtain

Theorem 11.19 (Witten's volume formula)**.** *If the Haar measure of G is normalized to have unit volume, then*

$$\mathrm{Vol}(\mathrm{Loc}_{\mathrm{g}}(G)) = |Z(G)|\zeta_G(2\mathrm{g}-2).$$

Exercise 11.20. Show that for any G we have

$$Z_{\mathrm{g}}(\hbar) = \sum_{V \in \mathrm{Irrep}(G)} e^{-\frac{1}{2}\hbar C_V} \dim(V)^{2-2\mathrm{g}}.$$

Finally, let us compute expectation values of some observables in our theory. We assume that G is connected. One of the most important observables in gauge theory is the *Wilson line operator*. Namely, given a closed oriented curve γ in X and a finite-dimensional representation R of G, the Wilson line $W_{\gamma,R}$ is

$$W_{\gamma,R}(A) = \mathrm{Tr}(\mathrm{Hol}_\gamma(A), R),$$

the trace of the holonomy of A along γ in R.

Let us compute the expectation value of $W_{\gamma,R}(A)$ on the sphere S_t^2 of area t. Assume that γ is a simple curve that splits the sphere into a disk of area s and a disk of area $t-s$. Then, since the sphere is glued out of these two disks, using formula (11.15) for $n = 0$, we obtain

$$\langle W_{\gamma,R}(A)\rangle_{S_t^2} = \frac{1}{K_{\hbar t}(1)} \int_G K_{\hbar s}(\mathrm{g}) \chi_R(\mathrm{g}) K_{\hbar(t-s)}(\mathrm{g}^{-1}) d\mathrm{g}.$$

In particular, assuming R is irreducible, we obtain that for fixed s (using that $K_t(y) \to 1$ as $t \to \infty$ and $C_R = C_{R^*}$),

$$\langle W_{\gamma,R}(A)\rangle_{\mathbb{R}^2} = \lim_{t\to\infty} \langle W_{\gamma,R}(A)\rangle = \int_G K_{\hbar s}(\mathrm{g}) \chi_R(\mathrm{g}) d\mathrm{g} = e^{-\frac{1}{2}\hbar s C_R} \dim(R).$$

Note that for nontrivial R we have $C_R > 0$. Thus, the expectation of the Wilson line $W_{\gamma,R}$ (for nontrivial R) decays exponentially fast with the area s encircled by γ. This is called the *area law* in gauge theory, which is a mathematical formulation of *confinement*. In 4-dimensional nonabelian gauge theory the validity of the area law is one of the most central conjectures.

Exercise 11.21. Let X be a closed oriented surface of genus $\mathrm{g} \geq 2$ with area form $d\sigma$, let G be a simply connected compact simple Lie group, and let $\mathrm{Loc}_{\mathrm{g}}(G) = \mathrm{Hom}(\pi_1(X), G)/G$ be the moduli space of flat G-bundles on X. Let $L \in \mathrm{Loc}_{\mathrm{g}}(G)$ be a smooth point, then $T_L \mathrm{Loc}_{\mathrm{g}}(G) = H^1(X, \mathrm{ad}(L))$. Now given $v_1, v_2 \in T_L \mathrm{Loc}_{\mathrm{g}}(G)$, define the *Atiyah-Bott 2-form* [**AB**] by

$$\omega_{AB}(v_1, v_2) = (v_1 \wedge v_2, [X]),$$

where $v_1 \wedge v_2 \in H^2(X, \mathbb{R})$ is the cup product of v_1, v_2 composed with the inner product on $\mathfrak{g}$.

(i) Show that ω_{AB} is a symplectic form on the smooth part of $\mathrm{Loc}_{\mathrm{g}}(G)$, i.e., it is closed and nondegenerate.

Now consider the classical pure gauge theory on X with gauge group G and Lagrangian density $|F_A|^2 d\sigma$. Then the equation of motion is $d_A * F_A = 0$, i.e., the section $*F_A$ of the adjoint bundle is covariantly constant. Moreover, if the centralizer in G of the holonomy group of A is $Z(G)$ (which happens generically), then $*F_A = 0$, so $F_A = 0$ and A is flat. The space of such connections is a smooth open subset $\mathrm{Loc}^{\circ}_{\mathrm{g}}(G)$ of $\mathrm{Loc}_{\mathrm{g}}(G)$. According to the yoga of classical field theory, this manifold has a natural symplectic structure.

(ii) Show that this symplectic structure has the form

$$\omega = C\omega_{AB},\ C > 0.$$

(Note that this implies (i)). How does C depend on the area S of X?

(iii) Let $\Pi(X)$ be the set of conjugacy classes in $\pi_1(X)$, i.e., homotopy classes of closed oriented curves on X without basepoint. For two smooth closed oriented curves α, β on X in general position (i.e., only simple transversal intersections) and an intersection point $P \in \alpha \cap \beta$, define the **join** $\alpha *_P \beta$ of α and β at P to be the closed oriented curve obtained by cutting α, β at P and then reconnecting them into a single curve preserving orientations. Also define the sign of P to be $+$ if α approaches β from the left and $-$ if α approaches β from the right. Now following W. Goldman define the *Goldman bracket* on $\mathbb{Z}\Pi(X)$ by the formula

$$[\alpha, \beta] := \sum_{P \in \alpha \cap \beta} \mathrm{sign}(P) \alpha *_P \beta,$$

where on the right-hand side we choose representatives of α and β in general position. Show that this bracket is well defined (i.e., does not depend on the representatives) and is a Lie bracket on $\mathbb{Z}\Pi(X)$.

(iv) Let $G = U(N)$ and $R = \mathbb{C}^N$ be the vector representation of G. Show that

$$\{W_{\alpha,R}, W_{\beta,R}\} = W_{[\alpha,\beta],R},$$

where $W_{\gamma,R}(A) = \mathrm{Tr}(\mathrm{Hol}_\gamma(A), R)$ is the Wilson line functional and $\{,\}$ is the Poisson bracket defined by the symplectic form ω_{AB}.

Remark 11.22 (see [**Wi1**], [**AB**], [**CMR**])**.** Consider two-dimensional gauge theory for $G = \mathrm{SU}(2)$. As noted in Exercise 11.21, the equation of motion is $d_A * F_A = 0$, and its solutions with centralizer of holonomy group $Z(G) = \{\pm 1\}$ are flat connections. However, it also has nonflat solutions associated to direct sums $L \oplus L^*$, where L is a line bundle with $c_1(L) = N > 0$ and a unitary connection of constant curvature, forming a 2g-dimensional torus $\mathbb{T}^{2\mathrm{g}}(N)$.

The partition function of quantum theory has the form

$$Z_{\mathrm{g}}(\hbar) = \sum_{n\geq 1} n^{2-2\mathrm{g}} e^{-\hbar\frac{n^2-1}{4}} = \sum_{n\in\mathbb{Z}\setminus 0} f(n),$$

where $f(x) := \frac{1}{2}x^{2-2\mathrm{g}}e^{-\hbar\frac{x^2-1}{4}}$. Applying the Poisson summation formula to this equality, one obtains another series for $Z_{\mathrm{g}}(\hbar)$ which gives the exact expansion of $Z_{\mathrm{g}}(\hbar)$ at $\hbar \to 0$, including instanton corrections; for example, if $\mathrm{g} = 1$, this is just the modular invariance of the theta-function, which has already shown up many times. The terms of this series express via the confluent hypergeometric function ${}_1F_1$. (If $\mathrm{g} \geq 2$, the function f is not locally L^1 near 0, so one needs to extend f to a distribution before taking the Fourier transform).

This series can be interpreted as the semiclassical expansion around the space of solutions of the equation of motion. Namely, the perturbative part of the expansion comes from the moduli space of flat connections, while the other, exponentially small (instanton) terms come from expanding around $\mathbb{T}^{2\mathrm{g}}(N)$ for $N = 0, 1, 2, \ldots$. The fact that the expansion converges to the exact partition function follows from an infinite-dimensional version of the *Duistermaat-Heckman formula*.

Similar results hold for other compact Lie groups.

11.11. The cobordism picture of QFT

11.11.1. The general setup. A conceptual, albeit somewhat informal, way of thinking about Hamiltonian QFT on manifolds, which is quite useful for motivational purposes, is the following *cobordism picture*, due to G. Segal, which describes a (Euclidean) QFT as a kind of functor from a suitable cobordism category to the category of separable (super-)Hilbert spaces.

To explain this picture, as a running example we will use the theory of a scalar boson and restrict ourselves to compact manifolds, although the same approach extends to more general QFT and to the noncompact case (with fields

satisfying suitable decay conditions at infinity). Consider a d-dimensional compact oriented Riemannian manifold X with boundary ∂X, which is split into a disjoint union of incoming boundary $\partial_- X$ and outgoing boundary $\partial_+ X$ (i.e., $\partial_- X$ is equipped with the "incoming" orientation, while $\partial_+ X$ is equipped with the "outgoing" orientation). The most basic example is $X = [0, L] \times Y$, where Y is a $(d-1)$-dimensional space manifold and $[0, L]$ is a time interval; then $\partial_- X = 0 \times Y$ and $\partial_+ X = L \times Y$. Recall that the classical theory of a scalar $\phi : X \to \mathbb{R}$ has the action

$$S(\phi) = \int_X (\tfrac{1}{2}(\nabla\phi)^2 + U(\phi))dx,$$

where $U : \mathbb{R} \to \mathbb{R}$ is a bounded below potential function (for convenience with $\min U = 0$), and the Euler-Lagrange equation is the Laplace equation

$$\Delta\phi - U'(\phi) = 0.$$

Typically this equation has many solutions. To specify a particular solution, we need to impose boundary conditions $\phi|_{\partial_\pm X} = \phi_\pm$ (the Dirichlet problem); under suitable assumptions (e.g., for $U = 0$) this determines the solution uniquely.

In quantum theory, the output should therefore be the partition function $Z_X(\phi_-, \phi_+)$ given by the path integral

$$Z_X(\phi_+, \phi_-) = \int_{\phi: X \to \mathbb{R}: \phi|_{\partial_\pm X} = \phi_\pm} e^{-\frac{S(\phi)}{\hbar}} D\phi,$$

with integration over field configurations satisfying the boundary conditions (assuming that we have made sense of integration, for instance in perturbation theory using Feynman calculus). We may view this partition function as the Schwartz kernel of an integral operator

$$\mathcal{H}(X) : L^2(C^\infty(\partial_- X)) \to L^2(C^\infty(\partial_+ X))$$

given by

$$(\mathcal{H}(X)\Psi)(\phi_+) = \int_{C^\infty(\partial_- X)} Z_X(\phi_+, \phi_-)\Psi(\phi_-)D\phi_-$$

(again, assuming that we have made sense of integration over $C^\infty(\partial_\pm X)$).

Thus we obtain the following picture. To every compact oriented Riemannian $d-1$-manifold Y corresponds a Hilbert space $\mathcal{H}(Y) = L^2(C^\infty(Y))$, and to every cobordism X between such manifolds Y_-, Y_+ (i.e., a compact oriented Riemannian d-manifold with boundary and $\partial_- X = Y_-, \partial_+ X = Y_+$) an operator $\mathcal{H}(X) : \mathcal{H}(Y_-) \to \mathcal{H}(Y_+)$ (the "amplitude" of X). Note that $\mathcal{H}(\emptyset) = \mathbb{C}$, thus if X is a closed manifold, then $\mathcal{H}(X) = Z_X \in \mathbb{C}$ (the partition function of our QFT on X). Moreover, this assignment is compatible with disjoint union:

$$\mathcal{H}(Y_1 \sqcup Y_2) = \mathcal{H}(Y_1) \otimes \mathcal{H}(Y_2),\ \mathcal{H}(X_1 \sqcup X_2) = \mathcal{H}(X_1) \otimes \mathcal{H}(X_2).$$

Also note that $\mathcal{H}(Y)$ is the Hilbert space of the theory on $\mathbb{R}\times Y$ in the Hamiltonian formulation, and the Feynman-Kac formula implies that

$$\mathcal{H}([0,L]\times Y) = e^{-LH_Y},$$

where H_Y is the Hamiltonian, while the vacuum vector Ω_Y is defined by

$$\Omega_Y(\xi) = Z_{\mathbb{R}\times Y}^{-1/2} \int_{\phi:(-\infty,0]\times Y\to\mathbb{R};\phi(0,y)=\xi(y)} e^{-\frac{S(\phi)}{\hbar}} D\phi,\ \xi\in C^\infty(Y),$$

where

$$Z_{\mathbb{R}\times Y} := \int_{\phi:\mathbb{R}\times Y\to\mathbb{R}} e^{-\frac{S(\phi)}{\hbar}} D\phi$$

(the integrals are taken over finite energy configurations).

The main property of the assignment $X\mapsto\mathcal{H}(X)$ is compatibility with gluing. Namely, suppose X_1, X_2 are two Riemannian d-manifolds endowed with an isometry $\partial_+X_2\cong\partial_-X_1$, and with cylindrical metrics near the boundary. Then we can glue them along the boundary to obtain the Riemannian d-manifold $X_1\#X_2$ with $\partial_-(X_1\#X_2)=\partial_-X_2$ and $\partial_+(X_1\#X_2)=\partial_+X_1$, and compatibility of $\mathcal{H}(X)$ with gluing is expressed by the equality

$$\mathcal{H}(X_1\#X_2) = \mathcal{H}(X_1)\mathcal{H}(X_2);$$

this follows simply by integrating over $\phi|_{\partial_+X_2=\partial_-X_1}$. Similarly, if two isometric components of ∂X are glued to each other to produce a new manifold X', then

$$\mathcal{H}(X') = \operatorname{Tr}\mathcal{H}(X),$$

where Tr stands for the partial trace along the Hilbert space corresponding to the glued component. For example,

$$\mathcal{H}(\mathbb{R}/L\mathbb{Z}\times Y) = \operatorname{Tr}(e^{-LH_Y}).$$

Moreover, let $\overline{Y}$ be Y with opposite orientation, and denote by Y_L^-, Y_L^+ the Riemannian d-manifolds obtained by orienting both boundaries of $[0,L]\times Y$ as incoming (respectively, outgoing). Then we get

$$\mathcal{H}(Y_L^-) = \mathrm{ev}_L : \mathcal{H}(Y)\otimes\mathcal{H}(\overline{Y})\to\mathbb{C}$$

(the evaluation map) and

$$\mathcal{H}(Y_L^+) = \mathrm{coev}_L : \mathbb{C}\to\mathcal{H}(\overline{Y})\otimes\mathcal{H}(Y)$$

(the coevaluation map). Moreover, since $Y_L^+\#_{\overline{Y}}Y_K^- = [0,K+L]\times Y$, the gluing property implies that for any $v\in\mathcal{H}(Y)$

$$(\mathrm{ev}_K\otimes\mathrm{Id})\circ(v\otimes\mathrm{coev}_L) = e^{-(K+L)H_Y}v,$$

in particular

$$(\mathrm{ev}_0\otimes\mathrm{Id})\circ(v\otimes\mathrm{coev}_0) = v.$$

Similarly, for $w \in \mathcal{H}(\overline{Y})$,

$$(\mathrm{Id} \otimes \mathrm{ev}_0) \circ (\mathrm{coev}_0 \otimes w) = w.$$

This means that $\mathrm{ev}_0, \mathrm{coev}_0$ define an isomorphism $\mathcal{H}(\overline{Y}) \cong \mathcal{H}(Y)^* = \overline{\mathcal{H}(Y)}$. With these identifications, it follows that $\mathcal{H}(\overline{X}) = \mathcal{H}(X)^* = \mathcal{H}(X)^\dagger$ (Hermitian adjoint).

To summarize, the cobordism picture of QFT defines a symmetric monoidal functor from the category $\mathbf{RiemBord}_d$ of $(d-1)$-dimensional compact oriented Riemannian manifolds where morphisms are Riemannian cobordisms cylindrical near the boundary (including the height zero cylinders giving the identity morphisms) to the category of (separable) Hilbert spaces,

$$\mathcal{H} : \mathbf{RiemBord}_d \to \mathbf{Hilb},$$

which is compatible with conjugation and taking traces (in particular, we want the amplitudes $\mathcal{H}(X)$ for nonidentity cobordisms to be compact operators, so H_Y must have discrete spectrum). For a general Lagrangian this is, of course, only an aspirational picture (as the ingredients are not defined rigorously), but in some special cases (such as some topological quantum field theories, 2-dimensional rational conformal field theories, etc.) one can make sense of it in a rigorous manner.

As usual, local operators on $\mathcal{H}(Y)$ arising in the Hamiltonian picture are realized by insertions of fields in the path integral. For example, if $X = [0, L] \times Y$ and $0 < t_1 < \cdots < t_n < L$, $y_1, \ldots, y_n \in Y$, then the path integral

$$\int_{\phi: X \to \mathbb{R}: \phi(0,-)=\phi_-, \phi(L,-)=\phi_+} \phi(t_1, y_1) \ldots \phi(t_n, y_n) e^{-\frac{S(\phi)}{\hbar}} D\phi$$

gives the integral kernel for the operator

$$e^{-(L-t_n)H_Y} \widehat{\phi}(y_n) \ldots e^{-(t_2-t_1)H_Y} \widehat{\phi}(y_1) e^{-t_1 H_Y} : \mathcal{H}(Y) \to \mathcal{H}(Y).$$

Thus the entire theory (i.e., its correlation functions) can be recovered by considering amplitudes with insertions for d-dimensional manifolds without boundary.

11.11.2. Topological QFT. If the underlying theory has nontrivial spacetime symmetries, then the functor $\mathcal{H}$ may factor through a category of oriented manifolds with less structure than a Riemannian metric. For example, if the theory is topological (completely independent of the spacetime metric), then the functor $\mathcal{H}$ factors through the category $\mathbf{Bord}_d$ of smooth oriented compact $d-1$-manifolds, with morphisms being cobordisms up to diffeomorphism. Note that this category is rigid (has duality), with $Y^* = \overline{Y}$ (the evaluation and coevaluation maps are the cylinder cobordisms). In this case the Hamiltonian H_Y must be zero (since the isomorphism class of $[0, L] \times Y$ no longer depends on L, so $\mathcal{H}([0, L] \times Y) = \mathrm{Id}$ for all L). Hence $\dim \mathcal{H}(Y) = \mathcal{H}(\mathbb{R}/L\mathbb{Z} \times Y) < \infty$. Thus we

recover *Atiyah's definition of a d-dimensional topological quantum field theory*, which makes sense over any ground field:

Definition 11.23. A d-dimensional TQFT over a field k is a symmetric monoidal functor $\mathcal{H} : \mathbf{Bord}_d \to \mathrm{Vect}_k$ from the category $\mathbf{Bord}_d$ to the category of finite-dimensional vector spaces over k.[41]

More precisely, what we get from the above physical setup is a *unitary* TQFT, i.e., a symmetric monoidal functor $\mathcal{H}$ from $\mathbf{Bord}_d$ to the category of finite-dimensional Hilbert spaces compatible with complex conjugation.

This is a rigorous notion with especially interesting examples for $d = 3$ (e.g., Chern-Simons theory, see Section 11.12). Note that a d-dimensional TQFT over k gives rise to a k-valued diffeomorphism invariant of closed oriented d-manifolds, which is why such theories are important for differential topology. Note also that to a TQFT one may add insertions—operators concentrated on submanifolds of the spacetime manifold, e.g., Wilson lines concentrated on knots inside a 3-manifold in Chern-Simons theory.

Example 11.24. Here is a simple example of a d-dimensional TQFT called *finite gauge theory*. Let G be a finite group. For a closed oriented $d-1$-manifold Y, let

$$\mathcal{H}(Y) := \mathrm{Fun}(\mathrm{Conn}_G(Y), \mathbb{C}),$$

i.e., the space of functions on the set of isomorphism classes of G-connections (i.e., G-local systems) on Y; for example, if Y is connected, this is the space of G-invariant functions on $\mathrm{Hom}(\pi_1(Y), G)$. This space has a natural Hermitian inner product

$$(f_1, f_2) = \sum_{\rho \in \mathrm{Conn}_G(Y)} \frac{f_1(\rho)\overline{f_2(\rho)}}{|\mathrm{Aut}(\rho)|}.$$

Now, if X is a closed oriented d-manifold with $\partial_- X = Y_-$, $\partial_+ X = Y_+$, we define the operator $A(X) : \mathcal{H}(Y_-) \to \mathcal{H}(Y_+)$ by

$$(A(X)f)(\phi_+) = \sum_{\phi_- \in \mathrm{Conn}_G(Y_-)} N_X(\phi_+, \phi_-) f(\phi_-),$$

where

$$N_X(\phi_+, \phi_-) := \sum_{\eta \in \mathrm{Conn}_G(X)} \sum_{\alpha_- : \eta|_{Y_-} \cong \phi_-, \alpha_+ : \eta|_{Y_+} \cong \phi_+} \frac{1}{|\mathrm{Aut}(\eta, \alpha_+, \alpha_-)|},$$

where $\mathrm{Aut}(\eta, \alpha_+, \alpha_-)$ is the group of automorphisms of $(\eta, \alpha_+, \alpha_-)$. It is easy to check that this is a TQFT. The invariant of closed d-manifolds it defines is

[41] Note that any symmetric monoidal functor out of a rigid category is automatically compatible with duality.

not terribly interesting, however: we have

$$A(X) = \sum_{\eta \in \mathrm{Conn}_G(X)} \frac{1}{|\operatorname{Aut}(\eta)|},$$

so for connected X, we get

$$A(X) = \frac{|\operatorname{Hom}(\pi_1(X), G)|}{|G|}.$$

For $d = 2$, TQFT are easy to classify. Namely, recall that a *Frobenius algebra* is a finite-dimensional unital algebra A over a field k with a linear functional $T : A \to k$ such that the bilinear form $(a, b) = T(ab)$ on A is nondegenerate.

Theorem 11.25. *The data of a 2-dimensional TQFT over a field k is equivalent to the data of a commutative Frobenius algebra over k.*

Proof (Sketch)**.** Let A be a commutative Frobenius algebra over k. We define a functor $\mathcal{H} : \mathbf{Bord}_2 \to \mathrm{Vect}_k$ on objects by $\mathcal{H}(S^1) = A$. To the pair-of-pants surface with two incoming and one outgoing boundary, we attach the multiplication $\mu : A \otimes A \to A$; to the disk with outgoing boundary (cup), we attach $1 \in A = \operatorname{Hom}(k, A)$; and to the disk with incoming boundary (cap), we attach $T : A \to k$. The Frobenius algebra axioms then imply that this data defines a 2-dimensional TQFT. Conversely, given a 2-dimensional TQFT $\mathcal{H}$, let $A := \mathcal{H}(S^1)$. The pair of pants defines a commutative associative product on A, the cup defines a unit $1 \in A$, and the cap defines $T : A \to k$. The axioms of a Frobenius algebra for A follow in a straightforward manner. □

Let us compute the invariant $\chi_A(\mathrm{g}) := \mathcal{H}_A(\Sigma_\mathrm{g})$ of the closed surface Σ_g of genus g produced by the 2-dimensional TQFT defined by a Frobenius algebra A. By definition, $\chi_A(0) = T(1)$, $\chi_A(1) = \dim A$. More generally, let $\Delta : A \to A \otimes A$ be the amplitude of the pair of pants with one incoming and two outgoing boundaries, i.e., the dual μ^* of the multiplication operator under the inner product $(a, b) = T(ab)$ on A. The composition $B := \mu \circ \Delta$ is a linear operator $A \to A$ (the amplitude of the torus with two holes—one incoming and one outgoing). The surface Σ_g is obtained by concatenating g such tori and adding a cup and a cap, so we get

$$\chi_A(\mathrm{g}) = T(B^\mathrm{g} 1).$$

Note that for $\mathrm{g} \geq 1$ we can also obtain Σ_g by concatenating $\mathrm{g} - 1$ tori and then gluing the remaining two boundaries; thus we also have

$$\chi_A(\mathrm{g}) = \operatorname{Tr}(B^{\mathrm{g}-1}).$$

So we obtain the identity

$$T(B^\mathrm{g} 1) = \operatorname{Tr}(B^{\mathrm{g}-1}), \tag{11.19}$$

which holds in every commutative Frobenius algebra.

Exercise 11.26. Prove (11.19) algebraically.

Example 11.27 (Semisimple Frobenius algebras)**.** Let $A = \mathbb{C}\times\cdots\times\mathbb{C}$ (n times). Then $T(x_1,\dots,x_n) = \sum_i b_i x_i$ for some $b_i \neq 0$, and $1 = (1,\dots,1)$. Also $B = \operatorname{diag}(b_1^{-1},\dots,b_n^{-1})$. Thus both formulas give $\chi_A(\mathrm{g}) = \sum_{i=1}^n b_i^{1-\mathrm{g}}$.

In particular, the data of unitary TQFT over $\mathbb{C}$ is a commutative Frobenius *-algebra. This means that all such theories are given by Example 11.27 with $b_i > 0$.

Note that, as mentioned above, the whole theory can often be recovered just from the invariants of closed surfaces. Namely, given a 2-dimensional unitary TQFT $\mathcal{H}$, consider the vectors $v_{\mathrm{g}} \in \mathcal{H}(S^1)$ which correspond to the surface of genus g with one outgoing boundary. Then we have

$$(v_{\mathrm{g}_1}, v_{\mathrm{g}_2}) = \mathcal{H}(\Sigma_{\mathrm{g}_1+\mathrm{g}_2}) = \chi(\mathrm{g}_1+\mathrm{g}_2).$$

Thus, assuming that v_{g} span $\mathcal{H}(S^1)$, we may recover $\mathcal{H}(S^1)$ as the quotient $\overline{V}$ of the infinite-dimensional space V with basis e_{g} and nonnegative Hermitian inner product given by

$$(e_{\mathrm{g}_1}, e_{\mathrm{g}_2}) = \chi(\mathrm{g}_1+\mathrm{g}_2)$$

by the radical of this inner product. And indeed, in the setting of Example 11.27 we have $\chi(\mathrm{g}) = \sum_{i=1}^n b_i^{1-g}$, so if b_i are distinct, then v_{g} span $\mathcal{H}(S^1)$ and $\mathcal{H}(S^1) = \overline{V} = A$. Moreover, even if b_i are not all distinct, the sequence $\mathrm{g} \mapsto \sum_{i=1}^n b_i^{1-\mathrm{g}}$ completely determines the theory, as it determines both n and b_i up to permutation.

A similar principle applies in higher dimensions (notably $d = 3$).

Exercise 11.28. Consider finite gauge theory in $d = 2$ dimensions with gauge group G having r conjugacy classes (Example 11.24). By Theorem 11.25 it must correspond to a commutative Frobenius algebra A_G. Using Exercise 11.18, show that A_G is the semisimple Frobenius algebra $\mathbb{C}^r$ with the numbers $b_i = \frac{(\dim V_i)^2}{|G|^2}$, where V_i are the irreducible representations of G (Exercise 11.27).

11.11.3. QFT with area-preserving and conformal symmetries. Another possibility is that the theory depends only on the volume element on X; in this case for a $(d-1)$-dimensional manifold Y, $\mathcal{H}(Y)$ depends only on the smooth structure on Y, while the operator $\mathcal{H}(X)$ depends only on the smooth structure and volume of X (for connected X). Thus the functor $\mathcal{H}$ factors through the category $\widetilde{\mathbf{Bord}}_d$ with the same objects as $\mathbf{Bord}_d$ but with morphisms being pairs (X, v), where X is a smooth cobordism and v is a positive function on $\pi_0(X)$ (giving the volumes of components of X).

An example of this is 2-dimensional gauge theory considered in Section 11.10. In this case there is only one possibility for connected Y, namely $Y = S^1$,

and $\mathcal{H}(S^1) = L^2(G)^G$. The Hamiltonian $H_{S^1} = H$ is $-\frac{1}{2}\Delta$, where Δ is the Laplace operator on G (normalizing $\hbar$ to be 1). Moreover, for a connected oriented compact surface X with n_- incoming boundaries and n_+ outgoing ones, the operator $\mathcal{H}(X) : L^2(G)^{\otimes n_-} \to L^2(G)^{\otimes n_+}$ whose integral kernel is computed in the previous section depends only on the area of X.

Finally, if the theory is conformally invariant, then the functor $\mathcal{H}$ factors through the category $\mathbf{ConfBord}_d$ of oriented manifolds with conformal structure. In the 2-dimensional case, this is the category where objects are unions of circles (i.e., labeled by nonnegative integers) and morphisms are Riemann surfaces with boundary, and we obtain the *Segal picture* of 2-dimensional conformal field theory, to be discussed in Chapter 12.

11.12. Chern-Simons theory

One of the most interesting TQFTs is *Chern-Simons theory* attached to a connected compact Lie group G. The goal of this section is to briefly discuss this theory; for more detail see the seminal paper of E. Witten ([**Wi2**]).

Chern-Simons theory is a 3-dimensional gauge theory with gauge group G, but its kinetic term is different from the one in Yang-Mills theory, producing first order equations of motion. First consider this theory on $\mathbb{R}^3$.

The only field in Chern-Simons theory is a connection A on a (necessarily trivial) principal G-bundle on $\mathbb{R}^3$. Thus without loss of generality we may simply take A to be a 1-form on $\mathbb{R}^3$ with values in $\mathfrak{g} = \operatorname{Lie} G$, which gives rise to the connection $d + A$ on the trivial G-bundle. To define the action $S(A)$ of Chern-Simons theory, we need to produce a 3-form ω_A on $\mathbb{R}^3$ from A so that $S(A) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \omega_A$ is gauge invariant.

Fix an invariant inner product $(,)$ on $\mathfrak{g}$. The obvious candidate form is then $\omega_A = \operatorname{Tr}(A \wedge dA)$, where the symbol Tr means that we apply the inner product; i.e., if $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$, then

$$\begin{aligned}&\operatorname{Tr}(A \wedge dA)\\ &= ((A_1, \partial_2 A_3 - \partial_3 A_2) + (A_2, \partial_3 A_1 - \partial_1 A_3) + (A_3, \partial_1 A_2 - \partial_2 A_1))\, dx_1 \wedge dx_2 \wedge dx_3.\end{aligned}$$

If G is abelian, then an infinitesimal compactly supported gauge transformation $b \in C^\infty(\mathbb{R}^3, \mathfrak{g})$ acts on ω_A by the formula

$$\omega_A \circ b = \operatorname{Tr}(db \wedge dA).$$

So by Stokes' formula

$$\int_{\mathbb{R}^3} \omega_A \circ b = 0,$$

hence the action

$$S(A) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \omega_A$$

is gauge invariant.

When G is not necessarily abelian, this is no longer the case. But then we can attach to A another 3-form, $\mathrm{Tr}(A \wedge A \wedge A)$, where we apply the 3-form $([x, y], z)$ on $\mathfrak{g}$:

$$\mathrm{Tr}(A \wedge A \wedge A) = 3([A_1, A_2], A_3)dx_1 \wedge dx_2 \wedge dx_3.$$

So we may define ω_A by the formula

$$\omega_A = \mathrm{Tr}(A \wedge dA + \beta A \wedge A \wedge A)$$

for $\beta \in \mathbb{R}$ and look for the value of β making the integral $\int_{\mathbb{R}^3} \omega_A$ gauge-invariant. Applying a compactly supported infinitesimal gauge transformation $b \in C^\infty(\mathbb{R}^3, \mathfrak{g})$, we obtain

$$\omega_A \circ b = \mathrm{Tr}((db + [A, b]) \wedge dA + A \wedge d[A, b] + 3\beta db \wedge A \wedge A).$$

Using Stokes' formula, after a short calculation this yields

$$\int_{\mathbb{R}^3} \omega_A \circ b = (2 - 3\beta)\int_{\mathbb{R}^3} \mathrm{Tr}([A, b] \wedge dA).$$

Thus the suitable value of β is $\beta = \frac{2}{3}$. So for

$$\omega_A = \mathrm{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A),$$

we have $\int_{\mathbb{R}^3} \omega_A \circ b = 0$, thus the functional

$$CS(A) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \omega_A$$

is invariant under infinitesimal gauge transformations. The functional $CS(A)$ is called the *Chern-Simons functional* (or action) of A.

The Chern-Simons functional defines a classical field theory called the *classical Chern-Simons theory*. The Euler-Lagrange equation $\delta CS(A) = 0$ for this theory is

$$dA + \frac{1}{2}[A, A] = 0,$$

i.e., $F_A = 0$, where F_A is the curvature of A (this is another way to find the right value of β—for a general β one gets the equation $dA + \frac{3\beta}{4}[A, A] = 0$, which is only gauge invariant for $\beta = \frac{2}{3}$). Thus classical solutions for Chern-Simons theory are *flat connections*.

Note that classical Chern-Simons theory is topological: to define it, we didn't need any structure on $\mathbb{R}^3$ besides that of a smooth oriented manifold.

Thus it makes sense to put this theory on any smooth oriented 3-manifold X, just replacing $\mathbb{R}^3$ by X in all formulas.

There is, however, a slight subtlety arising when X is compact and connected and G is nonabelian (say, simple and simply-connected). In this case the group $\mathcal{G} = C^\infty(X, G)$ of gauge transformations is not connected. Namely, we have $\pi_0(\mathcal{G}) = \mathbb{Z}$; specifically, the degree of $g \in \mathcal{G}$ is the element $g^*(c) \in H^3(X, \mathbb{Z}) = \mathbb{Z}$, where $c \in H^3(G, \mathbb{Z})$ is the generator. Thus the above infinitesimal computation only proves the gauge invariance of $CS(A)$ under the identity component $\mathcal{G}^\circ$.

To understand how $CS(A)$ transforms under homotopically nontrivial gauge transformations, fix a compact connected oriented 4-manifold M with boundary such that $\partial_+ M = X$ (i.e., X is an outgoing boundary); it is well known that M exists, as the 3-dimensional cobordism group vanishes. Moreover, A can be extended to a connection on the trivial G-bundle on M; we will abuse notation and denote this extension also by A. It is then easy to compute that

$$d\omega_A = \mathrm{Tr}(F_A \wedge F_A),$$

so by Stokes' formula

$$CS(A) = \frac{1}{4\pi} \int_M \mathrm{Tr}(F_A \wedge F_A).$$

This gives yet another proof that $CS(A)$ is invariant under $\mathcal{G}^\circ$: any element $g \in \mathcal{G}^\circ$ extends to a gauge transformation over M, so the statement follows from the fact that the curvature F_A is a $\mathfrak{g}$-valued 2-form on M that transforms simply by conjugation by g.

However, if g is homotopically nontrivial, then it does not extend to a gauge transformation over M, since the map $H^3(M) \to H^3(X)$ vanishes (as $X = \partial_+ M$). To compute the effect of g on $CS(A)$, pick another compact connected 4-manifold N with $\partial_- N = X$, and extend A over N. Then the connections A on N and $g \circ A$ on M glue into a connection $\mathcal{A}$ on the G-bundle E_g on the closed 4-manifold $M \cup_X N$ obtained by gluing the trivial bundles on M and N along X using g. This bundle is in general nontrivial, and has a characteristic class $c_g \in H^4(M \cup_X N, \mathbb{Z}) \cong \mathbb{Z}$, which equals (up to a sign, depending on conventions) the degree of g. Note that for a suitable normalization of the inner product on $\mathfrak{g}$,

$$\int_{M \cup_X N} \mathrm{Tr}(F_{\mathcal{A}} \wedge F_{\mathcal{A}}) = 8\pi^2 c_g.$$

Thus we get

$$4\pi(CS(g \circ A) - CS(A)) = \int_M \mathrm{Tr}(F_{g\circ A} \wedge F_{g\circ A}) + \int_N \mathrm{Tr}(F_A \wedge F_A) = 8\pi^2 c_g.$$

It follows that

$$CS(g \circ A) = CS(A) + 2\pi c_g.$$

Thus we see that $CS(A)$ is not completely gauge invariant but rather is gauge invariant up to adding multiples of 2π. This is, however, enough to define a classical field theory, since for this we only need gauge-invariance of the differential $\delta CS(A)$, which does hold for all $g \in \mathcal{G}$.

Let us now consider quantization of Chern-Simons theory. This quantization is defined by the path integral

$$\int_{\mathrm{Conn}_G(X)/\mathcal{G}} e^{ikCS(A)} W(A) DA,$$

where W is a gauge-invariant observable and $k = \frac{1}{\hbar}$ is a coupling constant, provided, of course, that we can make sense of (i.e., regularize) this integral (a nontrivial procedure explained in [**Wi2**]). In particular, one may consider the partition function

$$\mathcal{Z}_X(G,k) = \int_{\mathrm{Conn}_G(X)/\mathcal{G}} e^{ikCS(A)} DA.$$

This defines the *quantum Chern-Simons theory at level k*.

Note that on $X = \mathbb{R}^3$, the expression $e^{ikCS(A)}$ is well defined (gauge invariant) for any $k \in \mathbb{C}$, but for compact X, since $CS(A)$ is only well defined up to $2\pi\mathbb{Z}$, it is only gauge invariant if $k \in \mathbb{Z}$. Thus the above path integral and the corresponding QFT potentially makes sense only for integer level k. We will assume, without loss of generality, that $k \geq 0$.

Since regularization of the Chern-Simons path integral involves choices that break diffeomorphism invariance (e.g., a Riemannian metric on X), it is not obvious that Chern-Simons theory remains topological at the quantum level. Luckily, it turns out to be the case, except for a slight anomaly called the *framing anomaly*. Namely, as was shown by Witten, there is a regularization of the quantum Chern-Simons theory depending only on a *framing* of X, i.e., a trivialization of its tangent bundle. It is well known that a framing exists on any compact oriented 3-manifold, and two different framings are related by a map $\xi : X \to \mathrm{SO}(3)$ up to homotopy. To every such ξ we can attach its degree $s(\xi) \in \mathbb{Z}$, and it turns out that the partition function changes under the change of framing as follows: if f_1, f_2 are two framings differing by ξ of degree $s = s(\xi)$, then

$$Z_X(G,k,f_2) = Z_X(G,k,f_1) e^{2\pi i s \frac{c(k,G)}{24}},$$

with

$$c(k,G) = \frac{k \dim G}{k + h^\vee},$$

where $h^\vee$ denotes the dual Coxeter number of G.

Thus, the Chern-Simons path integral gives rise to a *framed* 3-dimensional TQFT. In fact, it can be defined rigorously as a 3-dimensional TQFT in Atiyah's sense, using 3-dimensional topology and representation theory of affine Lie algebras or quantum groups ([**Tu**]). In particular, $Z_X(G,k,f)$ is a topological invariant of the framed 3-manifold (X,f). This very interesting invariant is called the *Reshetikhin-Turaev invariant* of X attached to G,k,f. For example, Witten showed that for $X = S^3$ with standard framing f and $G = \mathrm{SU}(2)$, one has

$$Z = Z_{S^3}(G,k,f) = \sqrt{\frac{2}{k+2}} \sin\frac{\pi}{k+2}.$$

Now take $X = S^3$ with standard framing and consider correlation functions of operators in Chern-Simons theory. Recall that a basic example of a gauge-invariant observable is the *Wilson line functional* corresponding to a map $\gamma : S^1 \to S^3$ and a finite-dimensional representation R of G, namely,

$$W_{\gamma,R}(A) = \mathrm{Tr}(\mathrm{Hol}_\gamma(A), R).$$

Thus, given smooth maps $\gamma_1, \dots, \gamma_n : S^1 \to S^3$, one can consider the correlation function

$$\langle W_{\gamma_1,R_1}(A) \dots W_{\gamma_n,R_n}(A)\rangle = \frac{1}{Z}\int_{\mathrm{Conn}_G(S^3)/\mathcal{G}} e^{ikCS(A)} W_{\gamma_1,R_1}(A) \dots W_{\gamma_n,R_n}(A) DA.$$

Since our theory is topological, this depends only on the isotopy class of $(\gamma_1,\dots,\gamma_n)$ and the framing of each γ_i (because of the framing anomaly). Thus if γ_i are simple nonintersecting curves, it depends only on the corresponding framed link in S^3. Moreover, the dependence on the framing can be removed by a suitable rescaling. This leads to many interesting invariants of links called *quantum link invariants*. For example, when $G = \mathrm{SU}(2)$ and $R_j = \mathbb{C}^2$ is the tautological representation, then one obtains the (suitably normalized) *Jones polynomial* of the link with parameter $q = e^{\frac{\pi i}{k+2}}$. More generally, for $G = \mathrm{SU}(2)$ and arbitrary R_j, one obtains the *colored Jones polynomial*, and for $G = \mathrm{SU}(N)$ and $R_j = \mathbb{C}^N$, the *HOMFLY polynomial*. For more details on these invariants see [**Tu**].

Chapter 12

Two-dimensional conformal field theory

Two-dimensional conformal field theory (CFT) is an especially important class of quantum field theories. There are several reasons for this:

1. Due to an infinite-dimensional symmetry algebra (the Virasoro algebra), conformal field theories are among the most tractable quantum field theories, and many of them are exactly solvable and can be constructed rigorously.

2. Conformal field theories often arise as *scaling limits of* 2*-dimensional models of statistical mechanics*, such as the *Ising model*.

3. Conformal field theories serve as a foundation for *string theory*, similarly to how (relativistic) quantum mechanics serves as a foundation of quantum field theory. Namely, in string theory particles are represented not by points but rather by small loops (strings), and consequently the role of Feynman graphs representing particle interaction scenarios is played by worldsheet Riemann surfaces which are traced out by such strings.

4. Conformal field theories have numerous deep connections to representation theory and complex geometry.

In this chapter we give a brief introduction into 2-dimensional conformal field theory. To keep the discussion within bounds, we will mostly discuss free CFT (except in the last two sections). For a much more in-depth treatment, see [**DMS**].

12.1. Classical free massless scalar in two dimensions

Consider a free massless scalar boson ϕ on $\mathbb{R}^2$ with Lagrangian $\mathcal{L} = \frac{1}{2}|\nabla\phi|^2$. In this case the local functional $\phi(t, \mathrm{x})$ satisfies the 2-dimensional wave (=string) equation

$$\phi_{tt} - \phi_{\mathrm{xx}} = 0,$$

so it splits into a sum of two functionals

$$\phi = \tfrac{1}{\sqrt{2}}\phi_L + \tfrac{1}{\sqrt{2}}\phi_R,$$

where

$$\phi_L(t, \mathrm{x}) = \eta_L(\mathrm{x} + t),\ \phi_R(t, \mathrm{x}) = \eta_R(\mathrm{x} - t),$$

which for obvious reasons are called the *left-mover and right-mover*. In other words, we have

$$(\partial_t - \partial_{\mathrm{x}})\phi_L = 0,\ (\partial_t + \partial_{\mathrm{x}})\phi_R = 0.$$

So we get

$$\phi_{\mathrm{x}} + \phi_t = \sqrt{2}\eta_L'(\mathrm{x} + t),\ \phi_{\mathrm{x}} - \phi_t = \sqrt{2}\eta_R'(\mathrm{x} - t).$$

Substituting this into (10.3), we get that the Poisson bracket of η_L', η_R' is given by

$$\{\eta_L'(\mathrm{x}), \eta_L'(\mathrm{y})\} = \delta'(\mathrm{x} - \mathrm{y}),\ \{\eta_R'(\mathrm{x}), \eta_R'(\mathrm{y})\} = -\delta'(\mathrm{x} - \mathrm{y}),$$
$$\{\eta_L'(\mathrm{x}), \eta_R'(\mathrm{y})\} = 0.$$

Let us now perform Wick rotation which replaces t with it. This makes ϕ_L, ϕ_R complex valued, so we will denote them by $\eta, \overline{\eta}$. Thus, setting $u := x + it$, we have

$$\overline{\partial}_u \eta = 0,\ \partial_u \overline{\eta} = 0,$$

i.e., $\eta(u)$ is holomorphic and $\overline{\eta}(\overline{u})$ is antiholomorphic.

Now consider the case when x runs over the circle $\mathbb{R}/2\pi\mathbb{Z}$, with Lebesgue measure normalized to have volume 1. Then, if we still want to have a decomposition of ϕ into a periodic left-mover and right-mover, we should "kill the zero mode" by requiring that $\int_0^{2\pi} \phi(t, \mathrm{x})d\mathrm{x} = 0$ (otherwise we have a solution $\phi(t, \mathrm{x}) = t$ of the string equation which cannot be written as a sum of a left-moving and a right-moving periodic wave). Then we may introduce the coordinate $z = e^{iu}$, which takes values in $\mathbb{C}^\times$, and $\eta, \overline{\eta}$ become holomorphic (respectively, antiholomorphic) fields on $\mathbb{C}^\times$, which we will denote by $\varphi, \overline{\varphi}$. So we have Laurent expansions

$$\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n z^{-n},\ \overline{\varphi}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{\varphi}_n \overline{z}^{-n},$$

with $\varphi_0 = \overline{\varphi_0} = 0$. When z is on the unit circle, these are just the Fourier expansions of $\phi_L(0, x)$, $\phi_R(0, x)$, and for

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} := i\partial_z\varphi(z),\ \overline{a}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{a}_n \overline{z}^{-n-1} := -i\overline{\partial}_z\overline{\varphi}(\overline{z}),$$

where $a_0 = \overline{a}_0 = 0$, we have

$$za(z) = \eta'(u),\ \overline{za}(\overline{z}) = \overline{\eta}'(\overline{u}).$$

Thus for $z = e^{iu}, w = e^{iv}$ we get

$$\{za(z), wa(w)\} = \delta'(u - v). \tag{12.1}$$

Note that

$$\delta'(u - v) = i \sum_{n \in \mathbb{Z}} n z^n w^{-n}.$$

So setting

$$\delta(w - z) := \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$$

(Fourier expansion of the distribution $\delta(w - z)$ on $(S^1)^2$, where $|z| = |w| = 1$), we can write (12.1) as

$$\{a(z), a(w)\} = -i\delta'(w - z). \tag{12.2}$$

In components, this takes the form

$$\{\sum_{m \in \mathbb{Z}} a_m z^{-m}, \sum_{n \in \mathbb{Z}} a_{-n} w^n\} = -i \sum_{n \in \mathbb{Z}} n z^{-n} w^n.$$

Thus we get

$$\{a_n, a_m\} = -in\delta_{n,-m}. \tag{12.3}$$

Similarly,

$$\{\overline{a}_n, \overline{a}_m\} = in\delta_{n,-m} \tag{12.4}$$

and

$$\{a_n, \overline{a}_m\} = 0, \tag{12.5}$$

which in terms of generating functions can be written as

$$\{\overline{a}(z), \overline{a}(w)\} = i\delta'(w - z),\ \{a(z), \overline{a}(w)\} = 0. \tag{12.6}$$

Finally, let us write down the Hamiltonian of the theory in terms of the Fourier (=Laurent) modes a_n. Recall that in the original notation it has the form

$$H = \frac{1}{2} \int_{\mathbb{R}/2\pi\mathbb{Z}} (\phi_t^2 + \phi_x^2) dx.$$

Thus we have

$$H = \frac{1}{4} \int_{\mathbb{R}/2\pi\mathbb{Z}} ((\overline{za} - za)^2 + (\overline{za} + za)^2) dx = \frac{1}{2} \int_{\mathbb{R}/2\pi\mathbb{Z}} (\overline{z}^2 \overline{a}^2 + z^2 a^2) dx,$$

i.e.,

$$H = \sum_{n>0} (a_{-n} a_n + \overline{a}_{-n} \overline{a}_n). \tag{12.7}$$

It satisfies the relations

$$\{a_m, H\} = -ima_m,\ \{\overline{a}_m, H\} = im\overline{a}_m.$$

12.2. Free quantum massless scalar on $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$ with killed zero mode

Consider now the free QFT of a massless scalar boson ϕ on the cylinder $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$ with Minkowskian metric $dt^2 - dx^2$, with killed zero mode, i.e., a quantization of the classical field theory (described in Section 12.1). Since this is not a theory on a vector space, it will not satisfy Wightman axioms. However, we can naturally quantize the commutation relations (12.3), (12.4), and (12.5) of the corresponding Euclidean theory (with $\hbar = 1$), by replacing them with

$$[a_n, a_m] = n\delta_{n,-m},\ [\overline{a}_n, \overline{a}_m] = n\delta_{n,-m},\ [a_n, \overline{a}_m] = 0.$$

In other words, for $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, $\overline{a}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{a}_n \overline{z}^{-n-1}$ we have

$$[a(z), a(w)] = \delta'(w - z),\ [\overline{a}(z), \overline{a}(w)] = \delta'(\overline{w} - \overline{z}),\ [a(z), \overline{a}(w)] = 0,$$

which quantize equations (12.2) and (12.6) (this is a field-theoretic generalization of the analysis of Section 7.7, with an infinite sequence of harmonic oscillators labeled by positive integers). Thus we see that the Euclidean space-locality property is satisfied.

This shows that we have an infinite system of independent harmonic oscillators. To restate this algebraically, consider the infinite-dimensional Heisenberg Lie algebra $\mathfrak{H}$ with basis $a_n, n \neq 0$ and K (central) with commutation relations

$$[a_n, a_m] = n\delta_{n,-m}K.$$

Then we see that some dense subspace of the Hilbert space $\mathcal{H}$ of our theory should carry a pair of commuting actions of $\mathfrak{H}$ (by left-movers and right-movers), with K acting by 1 (we will denote the second copy of $\mathfrak{H}$ by $\overline{\mathfrak{H}}$).

Let us now describe the Hilbert space $\mathcal{H}$. Note that the Lie algebra $\mathfrak{H}$ has an irreducible *Fock representation* $\mathcal{F}$ generated by Ω with defining relations

$$a_n\Omega = 0,\ n > 0, \quad K\Omega = \Omega.$$

As a vector space, $\mathcal{F}$ is the *Fock space*

$$\mathcal{F} = \mathbb{C}[X_1, X_2, \dots]$$

(with $\Omega = 1$), on which the operators a_{-n} for $n > 0$ act by multiplication by X_n and a_n act by $n\frac{\partial}{\partial X_n}$.

Now, the Hamiltonian of the system (which we rescale for convenience by a factor of 2) should satisfy the commutation relations

$$[a_n, H] = na_n,\ [\overline{a}_n, H] = n\overline{a}_n.$$

Thus we see that if we want the spectrum of H to be bounded below and if $\Omega \in \mathcal{H}$ is the lowest eigenvector of H, then we must have

$$a_n\Omega = 0,\ \overline{a}_n\Omega = 0$$

for $n > 0$. But in this case the space $\mathcal{D}$ generated from Ω by the action of $a_n, \overline{a}_n$ has to be the irreducible representation $\mathcal{F} \otimes \overline{\mathcal{F}}$ of the Lie algebra $\mathfrak{H} \oplus \overline{\mathfrak{H}}$, where $\overline{\mathcal{F}} := \mathbb{C}[\overline{X}_1, \overline{X}_2, \dots]$ with $\overline{a}_{-n}$ acting by multiplication by $\overline{X}_n$ and $\overline{a}_n \mapsto n\frac{\partial}{\partial \overline{X}_n}$ for $n > 0$.[42]

Thus the space $\mathcal{D}$ is the tensor product of the polynomial algebras $\mathbb{C}[X_j]$ and $\mathbb{C}[\overline{X}_j]$ over $j \geq 1$. Each of the algebras $\mathbb{C}[X_j]$ carries a positive inner product with X_j^n being an orthogonal basis and $\|X_j^n\|^2 = j^n n!$, and similarly for $\mathbb{C}[\overline{X}_j]$. This yields a positive inner product $\langle\ ,\ \rangle$ on $\mathcal{F}, \overline{\mathcal{F}}$ and $\mathcal{D}$, with respect to which $a_i^\dagger = a_{-i}$ and $\overline{a}_i^\dagger = \overline{a}_{-i}$. The Hilbert space $\mathcal{H}$ is the completion of $\mathcal{D}$ with respect to $\langle\ ,\ \rangle$.

This implies that the quantum Hamiltonian has to be given by the formula

$$H = \sum_{n>0}(a_{-n}a_n + \overline{a}_{-n}\overline{a}_n) + C$$

obtained by quantizing the classical Hamiltonian (12.7) (note that the annihilation operators are written on the right to make sure the infinite sum makes sense). We may write H as the sum of left-moving and right-moving parts:

$$H = H_L + H_R,$$

where

$$H_L := \sum_{n>0} a_{-n}a_n + \frac{C}{2},\ H_R := \sum_{n>0} \overline{a}_{-n}\overline{a}_n + \frac{C}{2}.$$

12.3. ζ-function regularization

At the moment it is not clear what the right value of C should be. To answer this question, recall that the Hamiltonian of a single harmonic oscillator is $z\partial_z + \frac{1}{2}$ acting on $\mathbb{C}[z]$. This suggests that the formula for H should be

$$H = \frac{1}{2}\sum_{n\neq 0}(a_{-n}a_n + \overline{a}_{-n}\overline{a}_n),$$

which is a more symmetric and natural formula for the quantization of H. Moving the annihilation operators to the right to make sense of the infinite sum, we thus get

$$H = \sum_{n>0}(a_{-n}a_n + \overline{a}_{-n}\overline{a}_n + n).$$

[42] Note that $\overline{X}_n$ are independent variables of X_n, not their complex conjugates.

This formula still makes no sense, however, since the series

$$\sum_{n>0} n = 1 + 2 + 3 + \dots$$

is divergent. We may, nevertheless, regularize it using a method called *ζ-function regularization*.

Namely, recall that the Riemann ζ-function is defined by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

It is well known that this function extends meromorphically to the entire complex plane with a unique (simple) pole at $s = 1$, and it satisfies the functional equation, which says that the function $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is symmetric under the change $s \mapsto 1 - s$:

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\zeta(s).$$

Now, it is natural to define

$$C = 1 + 2 + 3 + \dots := \zeta(-1).$$

But the functional equation for $s = 2$ implies that

$$\zeta(-1) = \frac{\pi^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})}\zeta(2) = -\frac{\pi^{-\frac{3}{2}}}{2\pi^{\frac{1}{2}}}\frac{\pi^2}{6} = -\frac{1}{12}.$$

So from this point of view it is natural to set

$$C := -\frac{1}{12}.$$

Remark 12.1. Recall that for integer $g \geq 1$

$$\zeta(2g) = (-1)^{g+1}2^{2g-1}\frac{B_{2g}}{(2g)!}\pi^{2g},$$

where B_m is the mth Bernoulli number. So the functional equation for ζ implies that

$$\zeta(1-2g) = \pi^{\frac{1}{2}-2g}\frac{\Gamma(g)}{\Gamma(\frac{1}{2}-g)} \cdot (-1)^{g+1}2^{2g-1}\frac{B_{2g}}{(2g)!}\pi^{2g} = -\frac{B_{2g}}{2g}.$$

Thus the Harer-Zagier theorem (Theorem 5.4) can be interpreted as the statement that the Euler characteristic of the moduli space of curves of genus g is $\zeta(1-2g)$. In particular, for $g = 1$, we get the Euler characteristic of $\mathrm{SL}_2(\mathbb{Z})$, which is $-\frac{1}{12}$.

12.4. Modularity of the partition function

The value $-1/12$ turns out indeed to be the most natural value of C. To see this, let us return to the Euclidean signature $|du|^2 = dt^2 + dx^2$ and put our theory on the complex torus $E = E_\tau = \mathbb{C}^\times/q^{\mathbb{Z}} \cong \mathbb{R}/2\pi T\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$, where

$$T > 0,\ \tau = iT,\ q = e^{2\pi i\tau} = e^{-2\pi T} \in (0,1).$$

In this case, as we know from quantum mechanics, we should consider the partition function

$$Z(\tau) := \mathrm{Tr}(e^{2\pi i\tau H}).$$

Note that

$$H(P \otimes Q) = (\deg P + \deg Q + C)P \otimes Q,$$

where $P \in \mathcal{F}$ and $Q \in \overline{\mathcal{F}}$, and the degree is given by

$$\deg(X_n) = \deg(\overline{X}_n) = n.$$

Thus we have

$$Z(\tau) = \frac{e^{2\pi i\tau(C+\frac{1}{12})}}{\eta(\tau)^2},$$

where

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

is the Dedekind η-function. Now recall that $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$ with respect to a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ containing the element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, thus

$$\eta(-\tfrac{1}{\tau}) = \sqrt{-i\tau} \cdot \eta(\tau).$$

So the partition function Z has a nice modular property for a unique value of C, which is exactly $-\frac{1}{12}$.

Let us now explain the physical reason why we should expect $Z(\tau)$ to have a modular property.

For this, recall that the Lagrangian of the theory

$$\mathcal{L}(\phi) = \frac{1}{4\pi}\int_E |\nabla\phi|^2 dx = \frac{1}{4\pi}\int_E d\phi \wedge *d\phi$$

is *conformally invariant* (up to exact terms) as it is written purely in terms of the Hodge $*$-operator which depends only on the conformal structure on E. The same is true for the equation of motion, which is the Laplace equation $\Delta\phi = 0$. In other words, our classical field theory is *conformal* (see Section 11.5). Thus we could hope that the corresponding quantum theory is conformal as well. This should mean that $Z(-\frac{1}{\tau}) = Z(\tau)$, since the complex tori $E_{-\frac{1}{\tau}}$ and E_τ are conformally equivalent.

This said, we note that this modular property is only satisfied up to a linear factor in τ; in fact, we have

$$Z(-\tfrac{1}{\tau}) = -\frac{1}{i\tau}Z(\tau).$$

This is because we have killed the zero mode, which we should, in fact, have included (after all, the space cycle in the torus E_τ is not $\mathrm{SL}_2(\mathbb{Z})$-invariant, hence neither is the condition that the integral of ϕ over this cycle vanishes). This is done in the next section.

12.5. Including the zero mode

The zero mode corresponds to the x-periodic solutions $\phi(t, \mathrm{x}) = \alpha + \mu t$ of the string equation ($\alpha, \mu \in \mathbb{R}$), which for nonzero μ cannot be split into a left-moving and right-moving periodic wave. So putting back the zero mode corresponds to replacing the Hilbert space $\mathcal{H}$ with $\mathcal{H}_{\text{full}} := \mathcal{H} \otimes L^2(\mathbb{R})$, where $L^2(\mathbb{R})$ is the Hilbert space of a quantum-mechanical free massless particle, and the Hamiltonian H by

$$H_{\text{full}} := H + \widehat{\mu}^2,$$

where $\widehat{\mu}$ is the quantum momentum operator for this quantum mechanical particle, acting on $L^2(\mathbb{R})$ by multiplication by the momentum μ. Thus we may write

$$\mathcal{H}_{\text{full}} = \int_{\mathbb{R}} \mathcal{H}_\mu d\mu,$$

with $\mathcal{H}_\mu = \mathcal{F}_\mu \otimes \overline{\mathcal{F}}_\mu$ where $\mathcal{F}_\mu = \mathcal{F}$ but with $a_0 = \mu$ instead of $a_0 = 0$, and similarly $\overline{\mathcal{F}}_\mu = \overline{\mathcal{F}}$ but with $\overline{a}_0 = \mu$ instead of $\overline{a}_0 = 0$. Then we still have

$$H = H_L + H_R,$$

where

$$H_L = \tfrac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n - \tfrac{1}{24},\ H_R = \tfrac{1}{2}\overline{a}_0^2 + \sum_{n>0} \overline{a}_{-n}\overline{a}_n - \tfrac{1}{24}.$$

According to Remark 7.31, the partition function of such a particle when time runs over $\mathbb{R}/L\mathbb{Z}$ is, up to scaling, $L^{-\frac{1}{2}}$. Thus the full partition function should be

$$\mathcal{Z}(\tau) = (-i\tau)^{-\frac{1}{2}}Z(\tau).$$

And then we have the genuine modular property,

$$\mathcal{Z}(-\tfrac{1}{\tau}) = \mathcal{Z}(\tau).$$

We note that the function $\mathcal{Z}(\tau)$ has a natural extension to arbitrary $\tau \in \mathbb{C}_+$ (not necessarily purely imaginary), which is just the path integral over a "non-rectangular" complex torus E_τ. To explain this, note that we have a natural

action of the translation group $\mathbb{R}/2\pi\mathbb{Z}$ on our spacetime, hence we should expect its action on the Hilbert space $\mathcal{H}$ preserving the unique vacuum vector Ω. The infinitesimal generator D of this group should satisfy the commutation relations

$$[a_n, D] = na_n,\ [\overline{a}_n, D] = -n\overline{a}_n$$

(which differs from the corresponding relations for H by the sign in the second relation). As $D\Omega = 0$, it follows that

$$D(P \otimes Q) = (\deg P - \deg Q)P \otimes Q,$$

i.e.,

$$D = H_L - H_R.$$

Let $s \in \mathbb{R}$ and $\tau := iT + s$. Then a twisted version of the Feynman-Kac formula implies that given $s \in \mathbb{R}$, we have

$$Z(\tau) = \operatorname{Tr}(e^{-2\pi TH} e^{2\pi isD}) = |q|^{-\frac{1}{12}} \operatorname{Tr}(q^{H_L} \overline{q}^{H_R}),$$

where $q = e^{-2\pi(T-is)} = e^{2\pi i\tau}$. Thus we still have

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2}.$$

Hence

$$\mathcal{Z}(\tau) = \frac{1}{\sqrt{T}|\eta(\tau)|^2} = \frac{1}{\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2},$$

which is a (real analytic) modular function for $\mathrm{PSL}(2,\mathbb{Z})$, i.e., invariant under $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ for $a, b, c, d \in \mathbb{Z}$, $ad-bc = 1$. Thus, here we have a genuine quantum conformal symmetry (as the moduli of complex tori E_τ is exactly $\mathbb{C}_+/\mathrm{PSL}_2(\mathbb{Z})$). Indeed, this function is obviously symmetric under $\tau \mapsto \tau + 1$, and we've seen that it is invariant under $\tau \mapsto -1/\tau$, but these two transformations generate $\mathrm{PSL}_2(\mathbb{Z})$.

12.6. Correlation functions on the cylinder and torus

We may also consider correlation functions of the quantum fields a and $\overline{a}$. They are computed separately in $\mathcal{F}$ and $\overline{\mathcal{F}}$ and can be easily found using representation theory. For example, we have $a_n a_{-n}\Omega = n\Omega$ for $n > 0$, so the 2-point function is given by

$$\langle\, \Omega, a(z)a(w)\Omega \,\rangle = \sum_{n=1}^{\infty} nz^{-n-1}w^{n-1} = \frac{1}{(z-w)^2}.$$

More precisely, the series converges only for $|w| < |z|$, but the function analytically continues to all $z \neq w$. Since our theory is free, the higher correlation functions are given by Wick's formula:

Proposition 12.2. *We have*

$$\langle \Omega, a(z_1)\dots a(z_{2k})\Omega \rangle = \sum_{\sigma\in\Pi_{2k}} \frac{1}{\prod_{j\in[1,2k]/\sigma}(z_j - z_{\sigma(j)})^2},$$

and the $2k+1$*-point correlation functions are zero.*

We note that since $\mathcal{F}$ is generated by Ω as an $\mathfrak{H}$-module, these functions determine $a(z)$ as a local operator (=quantum field). More generally, they determine the operators $a(z_1)\dots a(z_r)$ when $z_i \neq z_j$, which are symmetric in $z_1,\dots,z_r$ due to space locality. However, these operators are not well defined (have poles) on the diagonals $z_i = z_j$.

Exercise 12.3. Give a direct algebraic proof of Proposition 12.2.

Exercise 12.4. Compute the normalized 2-point correlation function of the quantum field $\tilde{a}(z) := za(z)$ on the torus $E := \mathbb{R}/2\pi T\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ in terms of theta functions.

Hint. This correlation function is given by

$$\frac{\langle \tilde{a}(z)\tilde{a}(w)\rangle_E}{\langle \emptyset \rangle_E} = \mathrm{Tr}_{\mathcal{F}}(\tilde{a}(z)\tilde{a}(w)e^{-2\pi T H_L}).$$

12.7. Infinitesimal conformal symmetry: the Virasoro algebra

We have already pointed out that the theory of a free massless scalar in two dimensions is classically conformally invariant and saw some manifestations of the fact that this invariance survives at the quantum level (modular invariance of the partition function on the torus). However, to study conformal symmetry systematically, we need to consider *infinitesimal conformal symmetry*, given by "infinitesimal conformal mappings", i.e., holomorphic vector fields on $\mathbb{C}^\times$ (cf. Section 11.5).

For simplicity we consider polynomial vector fields $P(z)\partial_z$, where P is a Laurent polynomial (this is sufficient since polynomial fields are dense in all holomorphic vector fields in an appropriate topology). Such vector fields form a Lie algebra called the *Witt algebra* (or *centerless Virasoro algebra* in the physics literature), and we will denote it by W. A convenient basis of W is $\{L_n = -z^{n+1}\partial_z, n \in \mathbb{Z}\}$ which satisfies the commutation relations

$$[L_n, L_m] = (n-m)L_{m+n},\ m,n \in \mathbb{Z}.$$

The Lie algebra W acts by symmetries of classical field theory of a free massless scalar, since its Lagrangian is conformally invariant. In fact, importantly, this action is only $\mathbb{R}$-linear and not $\mathbb{C}$-linear, which is a good thing. This means that we have an action of the complexification $W_{\mathbb{C}} = W \oplus \overline{W}$, where $\overline{W}$ is

the Lie algebra of antiholomorphic vector fields; in other words, we have two commuting actions of W.

If our theory is quantum-mechanically conformally invariant, then the Lie algebra $W \oplus \overline{W}$ should act on the space $\mathcal{D}$ in a way compatible with the action of $\mathfrak{H} \oplus \overline{\mathfrak{H}}$, i.e., $a = i\partial_z \varphi$ should transform as a (0,1)-form under (infinitesimal) conformal transformations. Thus we should have

$$[L_n, a(z)] = \mathrm{Lie}_{-z^{n+1}\partial_z} a(z) = z^{n+1}a'(z) + (n+1)z^n a(z),$$

and likewise

$$[\overline{L}_n, \overline{a}(\overline{z})] = \overline{z}^{n+1}\overline{a}'(\overline{z}) + (n+1)\overline{z}^n \overline{a}(\overline{z}),$$
$$[\overline{L}_n, a(z)] = [L_n, \overline{a}(\overline{z})] = 0,$$

or in components

$$[L_n, a_m] = -ma_{m+n},\ [\overline{L}_n, \overline{a}_m] = -m\overline{a}_{m+n},\ [L_n, \overline{a}_m] = [\overline{L}_n, a_m] = 0.$$

Is there such an action? To figure this out, first note that the operators $L_0, \overline{L}_0$ satisfy the same commutation relations with $a, \overline{a}$ as H_L, H_R, respectively. Since $\mathcal{D}$ is generated by Ω as an $\mathfrak{H} \oplus \overline{\mathfrak{H}}$-module and these operators must preserve $\mathbb{C}\Omega$ (as it is the common kernel of the subalgebra $(\mathfrak{H} \oplus \overline{\mathfrak{H}})_{>0} \subset \mathfrak{H} \oplus \overline{\mathfrak{H}}$ of annihilation operators, which is normalized by $L_0, \overline{L}_0$), this means that we must have

$$L_0 = H_L + C_L,\ \overline{L}_0 = H_R + C_R$$

for some constants C_L, C_R. Since $[L_n, L_0] = nL_n$, this shows that L_n has to shift the grading in $\mathcal{F}$ by n, and similarly for $\overline{L}_n$ and $\overline{\mathcal{F}}$.

Now by analogy with the formula

$$L_0 = \sum_{k \geq 1} a_{-k}a_k + \text{const},$$

define for $n \neq 0$

$$L_n := \frac{1}{2}\sum_{k \in \mathbb{Z}} a_{-k}a_{k+n}. \tag{12.8}$$

It is easy to check that this operator on $\mathcal{F}$ (and hence on $\mathcal{D} = \mathcal{F} \otimes \overline{\mathcal{F}}$) is well defined (as a_{-k} commutes with a_{k+n}) and satisfies the desired commutation relations

$$[L_n, a(z)] = z^{n+1}a'(z) + (n+1)z^n a(z),\ [L_n, \overline{a}(z)] = 0.$$

Thus we see that if the desired action of W exists at all, then L_n **must** be given by formula (12.8). Indeed, if $M_n := L_n - \frac{1}{2}\sum_{k \in \mathbb{Z}} a_{-k}a_{k+n}$, then $[M_n, a_m] = 0$ for all m, so $[M_n, L_0] = 0$. But we also have $[M_n, L_0] = nM_n$, hence $M_n = 0$.

So it remains to check if the constructed operators satisfy the commutation relations of W. Let $M_{nm} := [L_n, L_m] - (n-m)L_{m+n}$. Using the Jacobi

identity, we see that M_{nm} commutes with $a, \overline{a}$. First assume $n \neq -m$. Since $[M_{nm}, a_k] = [M_{nm}, \overline{a}_k] = 0$, it follows that M_{nm} commutes with L_0. But we also have $[M_{nm}, L_0] = (m+n)M_{nm}$, so $M_{nm} = 0$ and thus

$$[L_n, L_m] - (n-m)L_{m+n} = 0.$$

Now consider the case $n = -m > 0$. In this case, since $\mathcal{D}$ is generated by Ω as an $\mathfrak{H} \oplus \overline{\mathfrak{H}}$-module and $M_{n,-n}$ preserves $\mathbb{C}\Omega$, we have

$$[L_n, L_{-n}] - 2nL_0 = \gamma(n),$$

where $\gamma(n) \in \mathbb{C}$, and we have an action of W if $\gamma(n) = 0$ for all n. So let us compute $\gamma(n)$. To this end, note that the eigenvalue by which $[L_n, L_{-n}]$ acts on Ω is $2n(C_L - \frac{1}{24}) + \gamma(n)$. So it suffices to compute this eigenvalue, i.e., the vector $L_n L_{-n}\Omega$.

In terms of the polynomial realization, we have

$$L_{-n}\Omega = \tfrac{1}{2} \sum_{0<j<n} X_j X_{n-j}.$$

Thus

$$L_n L_{-n}\Omega = \tfrac{1}{4} \sum_{0<j<n} j(n-j) \tfrac{\partial^2}{\partial X_j \partial X_{n-j}} \sum_{0<j<n} X_j X_{n-j} = \tfrac{1}{2} \sum_{0<j<n} j(n-j) = \frac{n^3 - n}{12}.$$

So

$$\gamma(n) = \frac{n^3 - n}{12}.$$

Thus we see that we almost have an action of W, but not quite—no matter how we choose C_L, the cubic term in n will be present (this effect is called a *conformal anomaly*). Instead, we have a *projective* representation of W, which is, in fact, a representation of a *central extension* of W, as often happens in quantization, see Remark 11.8.

This motivates the following definition.

Definition 12.5. The *Virasoro algebra* is the 1-dimensional central extension of the Witt algebra W with basis $L_n, n \in \mathbb{Z}$, and C (a central element) with commutation relations

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{n^3 - n}{12}\delta_{n,-m}C.$$

Thus we have a 1-dimensional central ideal $\mathbb{C}C \subset \mathrm{Vir}$ spanned by C, and $\mathrm{Vir}/\mathbb{C}C \cong W$.

So we obtain

Theorem 12.6. *The formulas*

$$L_0 = \sum_{k \geq 1} a_{-k} a_k,\ L_n = \tfrac{1}{2} \sum_{k \in \mathbb{Z}} a_{-k} a_{k+n}, n \neq 0$$

define an action of Vir *on* $\mathcal{F}$ *with* C *acting by* 1.

It is easy to check that the same theorem holds more generally on the space $\mathcal{F}_\mu$ where $a_0 = \mu$. The only change is that L_0 acquires an additional summand $\frac{1}{2}\mu^2$,

$$L_0 = \tfrac{1}{2}\mu^2 + \sum_{k\geq 1} a_{-k}a_k.$$

If C acts on a representation $\mathbb{V}$ of Vir by a scalar c (as it will, for instance, on every irreducible representation), then one says that $\mathbb{V}$ has *central charge* c. Thus $\mathcal{F}_\mu$ is a representation of Vir of central charge $c = 1$.

Similarly, the formulas

$$\overline{L}_0 = \tfrac{1}{2}\mu^2 + \sum_{k\geq 1} \overline{a}_{-k}\overline{a}_k,\ \overline{L}_n = \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \overline{a}_{-k}\overline{a}_{k+n},\ n \neq 0$$

define an action of Vir on $\overline{\mathcal{F}}_\mu$ with the central element $\overline{C}$ acting by 1 (i.e., also with central charge $c^* = 1$). This copy of Vir is a central extension of $\overline{W}$, so it is denoted by $\overline{\mathrm{Vir}}$.

Thus we obtain two commuting projective actions of W on the space $\mathcal{D} = \mathcal{F} \otimes \overline{\mathcal{F}}$ which define usual linear actions only for the central extension Vir of W. Still, the corresponding actions of W on quantum observables are genuine linear actions, so this quantum field theory is *conformal*.

We also observe that since $c = c^*$, the anti-diagonal copy of W spanned by the vector fields

$$A_n = -z^{n+1}\partial_z + \overline{z}^{-n+1}\partial_{\overline{z}} = L_n - \overline{L}_{-n},$$

which on the unit circle $z = e^{i\theta}$ specialize to $ie^{in\theta}\partial_\theta$ (a basis of $\mathrm{Lie}(\mathrm{Diff}(S^1))_{\mathbb{C}}$, i.e., the Lie algebra of vector fields tangent to the unit circle) act linearly, i.e., with central charge 0. This means that this theory has no *gravitational anomaly* (see Section 11.5).

We note that the Virasoro action preserves the positive Hermitian form on $\mathcal{F}_\mu$ in the sense that

$$L_n^\dagger = L_{-n}.$$

Thus $\mathcal{F}_\mu$ is a *positive energy unitary representation of* Vir (positive energy means that L_0 is diagonalizable with spectrum bounded below).

More generally, we may consider the theory of ℓ massless scalars $\phi_1, \dots, \phi_\ell$. In this case $\mathcal{D} = \mathcal{F}^{\otimes\ell} \otimes \overline{\mathcal{F}}^{\otimes\ell}$, and $\mathcal{F}^{\otimes\ell}$ is a positive energy unitary representation of Vir with central charge $c = \ell$ (the tensor product of ℓ copies of the representation $\mathcal{F}$).

Exercise 12.7.

1. Show that Vir is a nontrivial central extension of W (i.e., not isomorphic to $W \oplus \mathbb{C}$ as a Lie algebra).

2. Show that Vir is a universal central extension of W (i.e., every nontrivial central extension of W by $\mathbb{C}$ is isomorphic to Vir).

3. Show that for a generic μ, $\mathcal{F}_\mu$ is an irreducible representation of Vir.

Hint. Consider the asymptotics of the operators $\frac{1}{\mu}L_n$, $n \neq 0$ and $\frac{2}{\mu^2}L_0$ as $\mu \to \infty$.

Exercise 12.8. Let $\mathrm{Vir}_{<0}$ be the Lie subalgebras of Vir generated by L_i with $i < 0$, and let $\mathrm{Vir}_{\geq 0}$ be the Lie subalgebra of Vir generated by L_i with $i \geq 0$ and C. For $h, c \in \mathbb{C}$ let $M(h,c)$ be the Vir-module generated by a vector v with defining relations $L_i v = 0$ for $i > 0$, $L_0 v = hv$, $Cv = cv$, i.e., the induced module

$$M(h,c) = U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}_{\geq 0})} \mathbb{C}_{h,c},$$

where $\mathbb{C}_{c,h}$ is the 1-dimensional representation of $\mathrm{Vir}_{\geq 0}$ spanned by a vector v with $L_0 v = hv$, $Cv = cv$, $L_i v = 0$ for $i > 0$ and $U(\mathfrak{g})$ denotes the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. $M(h,c)$ is called the *Verma module* over Vir with highest weight (h,c). Prove the following:

(i) The map $U(\mathrm{Vir}_{<0}) \to M(h,c)$ sending $b \in U(\mathrm{Vir}_{<0})$ to bv is an isomorphism of $\mathrm{Vir}_{<0}$-modules. Moreover, $M(h,c)$ has a unique irreducible quotient $L(h,c) = M(h,c)/J(h,c)$, where $J(h,c)$ is the maximal proper submodule of $M(h,c)$, i.e., the sum of all proper submodules of $M(h,c)$. Finally,

$$\mathrm{Tr}_{M(h,c)}(q^{L_0}) = q^h \prod_{n \geq 1} (1 - q^n)^{-1}.$$

Hence the eigenvalues of L_0 on $M(h,c)$ are in $h + \mathbb{Z}_{>0}$, and the $h+n$-eigenspace of L_0 on $M(h,c)$ has dimension $p(n)$, where $p(n)$ is the number of partitions of n. Thus the corresponding dimension for $L(h,c)$ is $\leq p(n)$.

(ii) If $h, c \in \mathbb{R}$, then the module $M(h,c)$ admits a unique invariant Hermitian form $(,)$ for Vir with $(v,v) = 1$ (i.e., such that $(u, L_n w) = (L_{-n}u, w)$), called the *Shapovalov form*, and the radical of $(,)$ is $J(h,c)$. Thus $(,)$ descends to a nondegenerate Hermitian form on $L(h,c)$.

(iii) Let V be an irreducible positive energy unitary representation of Vir with central charge $c \in \mathbb{R}$. Recall that L_0 is diagonalizable on V and has real eigenvalues bounded below. Let h be an eigenvalue of L_0 on V such that L_0 has no eigenvalues $\leq h - 1$ (it clearly exists). Let $\Omega \in V$ be an eigenvector of L_0 with eigenvalue h. Then $L_i \Omega = 0$ for $i > 0$.

(iv) V is generated by Ω as a $\mathrm{Vir}_{<0}$-representation, i.e., it is spanned by the vectors $\prod_{j \geq 1} L_{-j}^{n_j} \Omega$, where n_j are nonnegative integers which are almost all zero. Moreover, Ω is the unique up-to-scaling eigenvector of L_0 with eigenvalue

h, and the eigenvalues of L_0 on V belong to $h + \mathbb{Z}_{\geq 0}$ (in particular, h is the smallest eigenvalue).

(v) There is a unique homomorphism $\phi : M(h,c) \to V$ sending v to Ω, and ϕ is surjective. Moreover, $\operatorname{Ker}\phi = J(h,c)$. Thus $V \cong L(h,c)$ and the non-degenerate Hermitian form $(,)$ on $L(h,c)$ coincides with the unitary structure on V.

(vi) $(L_{-n}\Omega, L_{-n}\Omega) = 2nh + \frac{n^3-n}{12}c$, thus unitarity implies $c \geq 0$ (taking n large) and $h \geq 0$ (taking $n = 1$).

(vii) For $c = 0$ we have

$$\det\begin{pmatrix}(L_{-n}^2\Omega, L_{-n}^2\Omega) & (L_{-n}^2\Omega, L_{-2n}\Omega)\\ (L_{-2n}\Omega, L_{-n}^2\Omega) & (L_{-2n}\Omega, L_{-2n}\Omega)\end{pmatrix} = 4n^3h^2(8h-5n).$$

In a unitary representation, this must be nonnegative, so if $c = 0$, then $h = 0$, hence $V = \mathbb{C}$, the trivial representation.

Remark 12.9. Note that part (vii) of Exercise 12.8 implies that any interesting 2-dimensional quantum CFT (one carrying a nontrivial action of Vir, i.e., not topological) must have a conformal anomaly ($c > 0$).

12.8. Normal ordering, composite operators, and operator product expansion in conformal field theory

Let us now summarize the theory of normal ordering, composite operators, and operator product expansion from Section 10.16 in the case of conformal field theory, for the running example of a quantum massless scalar boson. We have seen that the operator product $a(z)a(w)$ is well defined only if $w \neq z$ and has a pole when $w = z$, leading to the local operator $a(z)^2$ not being well defined. So let us expand this operator product in a Laurent series near $w = z$ and identify the singular part involving negative powers of $w - z$. For this purpose consider the difference

$$: a(z)a(w) := a(z)a(w) - \frac{1}{(z-w)^2}.$$

The formula for the correlation functions for $a(z)$ implies that

$$\langle\, \Omega, a(z_1)\dots a(z_{i-1}) : a(z_i)a(z_{i+1}) : a(z_{i+2})\dots a(z_n)\Omega \,\rangle$$
$$= \sum_{\sigma\in\Pi_{2k}\,:\,\sigma(i)\neq i+1} \frac{1}{\prod_{j\in\Pi_{2k}/\sigma}(z_j - z_{\sigma(j)})^2}.$$

Note that this function is regular at $z_i = z_{i+1}$, hence the operator $: a(z)a(w) :$ is regular at $z = w$, i.e., defined for all $z, w \in \mathbb{C}^\times$. This operator is called the *normally ordered product* of $a(z)$ and $a(w)$. In particular, although the square $a(z)^2$ is not defined, we have a well-defined normally ordered square $: a(z)^2 :$.

In terms of Laurent coefficients,

$$: a(z)a(w) := \sum_{m,n\in\mathbb{Z}} : a_n a_m : z^{-n-1}w^{-m-1},$$

where $: a_n a_m := a_n a_m$ if $m \geq n$ and $: a_n a_m := a_m a_n$ if $m < n$ (normal ordering of modes). Of course, this ordering only matters if $m + n = 0$. In particular, we see that

$$\tfrac{1}{2} : a(z)^2 := T(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

the generating function of the Virasoro modes L_n. This operator is called the (holomorphic part of the) *energy-momentum tensor*. In a similar way one defines the antiholomorphic part of the energy-momentum tensor,

$$\overline{T}(\overline{z}) := \sum_{n\in\mathbb{Z}} \overline{L}_n \overline{z}^{-n-2}.$$

More precisely, as explained in Section 11.2, in this case the energy-momentum tensor classically has the form

$$T_+(z)(dz)^2 + \overline{T}_-(\overline{z})(d\overline{z})^2,$$

where $T_+ = (\partial\phi)^2, T_- = (\bar{\partial}\phi)^2$, and $T_+(z)(dz)^2$ is a holomorphic quadratic differential, while $T_-(\overline{z})(d\overline{z})^2$ is an antiholomorphic one. In quantum theory, the situation is similar (for convenience we denote T_+, T_- by $T, \overline{T}$). The only essential difference is that because of the conformal anomaly (central charge), T is no longer a quadratic differential but rather a deformation of that notion—a geometric object called a *projective connection* (see [**FZ**, Section 8.2]).

Thus we see that the Virasoro modes L_n may be viewed as Noether charges for the corresponding infinitesimal conformal symmetries, in the holomorphic sector of the theory (cf. Section 11.4). The corresponding Noether currents are $z^{n+1}T(z)$, as

$$L_n = \frac{1}{2\pi i}\oint z^{n+1}T(z)dz.$$

The Noether charges for the full theory are then $L_n + \overline{L}_n$ with currents $z^{n+1}T(z)+\overline{z^{n+1}T(z)}$. In particular, the Hamiltonian H, up to adding a constant, is $L_0 + \overline{L_0}$, which agrees with formula (11.11).

Similarly, we may define the normal ordered products of more than two factors, $: a(z_1)\dots a(z_n) :$. This can be done by induction in n. Namely, we have

(12.9)

$$: a(z_0)a(z_1)\dots a(z_n) := a(z_0) : a(z_1)\dots a(z_n) : - \sum_{k\in[1,n]} \frac{: \prod_{j\neq k} a(z_j) :}{(z_0 - z_k)^2}.$$

It is easy to see that the operator $: a(z_1)\dots a(z_n) :$ has no singularities and is well defined for all values $z_1, \dots, z_n \in \mathbb{C}^\times$. Thus for every $r_1, \dots, r_n$ we have the operator

$$: a^{(r_1)}(z_1)\dots a^{(r_n)}(z_n) := \partial_{z_1}^{r_1}\dots\partial_{z_n}^{r_n} : a(z_1)\dots a(z_n) :$$

Setting $z_1 = \dots = z_n$, we can then define the holomorphic local operator $: P(a)(z) :$ for any differential polynomial P in $a(z)$. This local operator, called a *composite operator*, is a quantization of the corresponding local functional $P(a)(z)$ in classical field theory. We denote the space of (polynomial) holomorphic local operators by $\mathcal{V}$.

Exercise 12.10 (The state-field correspondence)**.** Show that the map $P \mapsto P(a)(z)\Omega|_{z=0}$ is well defined and gives an isomorphism between $\mathcal{V}$ and the Fock space $\mathcal{F}$.

More generally, repeatedly using (12.9), we have

$$: a(z_1)\dots a(z_n) : \cdot : a(w_1)\dots a(w_m) :$$
$$:= \sum_{I\subset[1,n],J\subset[1,m],s:I\cong J} \frac{: \prod_{i\notin I} a(z_i) \prod_{j\notin J} a(w_j) :}{\prod_{i\in I}(z_i - w_{s(i)})^2}.$$

So setting $z_i = z, w_j = w$, we obtain

$$: a(z)^n : \cdot : a(w)^m := \sum_{k=0}^{\min(m,n)} k! \binom{n}{k}\binom{m}{k} \frac{: a(z)^{n-k}a(w)^{m-k} :}{(z-w)^{2k}}.$$

For example, for $n = m = 1$ we get the familiar identity

$$a(z)a(w) = \frac{1}{(z-w)^2} + : a(z)a(w) := \frac{1}{(z-w)^2} + \text{regular terms.}$$

More generally, for $n = 1$ and any m we get

$$a(z) : a(w)^m := \frac{m : a^{m-1}(w) :}{(z-w)^2} + : a(z)a(w)^m :$$

$$= \frac{m : a^{m-1}(w) :}{(z-w)^2} + \text{regular terms.}$$

For $m = 2$ this can be written as

$$a(z)T(w) = \frac{a(w)}{(z-w)^2} + \text{regular terms,}$$

which encodes the commutation relations between a_i and L_j.

For $n = 2$, $m = 2$, we get

$$: a(z)^2 :: a(w)^2 := \frac{2}{(z-w)^4} + \frac{4 : a(z)a(w) :}{(z-w)^2} + : a(z)^2 a(w)^2$$
$$:= \frac{2}{(z-w)^4} + \frac{4 : a(w)^2 :}{(z-w)^2} + \frac{4 : a(w)a'(w)}{z-w} + \text{regular terms.}$$

This can also be written as

$$T(z)T(w) = \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \text{regular terms,}$$

which encodes the commutation relations between L_i. More generally, at central charge c this relation would look like

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \text{regular terms.}$$

It can be interpreted as the above statement that $T(z)$ transforms under conformal changes of variable as a projective connection ([**FZ**, Section 8.2]).

These are the simplest examples of the *operator product expansion*. In fact, we have the following theorem, whose proof we will leave to the reader:

Theorem 12.11. *For any local operators $P, Q \in \mathcal{V}$, there exist a unique finite sequence of local operators $R_1, \dots, R_N \in \mathcal{V}$ such that*

$$P(a)(z)Q(a)(w) = \sum_{j=1}^{N} R_j(a)(w)(z-w)^{-j} + \text{regular terms,}$$

where $(z-w)^{-j} := \sum_{k \geq 0} \binom{k+j-1}{j-1} z^{-j-k} w^k$.

Note that the space-locality property implies that $Q(a)(w)P(a)(z)$ is given by the same formula, but with $(z-w)^{-j}$ expanded in the opposite direction, i.e., $(z-w)^{-j} := -\sum_{k<0} \binom{k+j-1}{j-1} z^{-j-k} w^k$. Thus we have

$$[P(a)(z), Q(a)(w)] = \sum_{j=1}^{N} \frac{1}{(j-1)!} R_j(a)(w)\delta^{(j-1)}(w-z).$$

So Theorem 12.11 gives us information about commutators between the modes of P and Q. For example, as we have seen above,

$$[a(z), a(w)] = \delta'(w-z),$$

and also

$$[a(z), T(w)] = a(w)\delta'(w-z),$$
$$[T(z), T(w)] = \frac{c}{12}\delta'''(w-z) + 2T(w)\delta'(w-z) + T'(w)\delta(w-z),$$

where in our example $c = 1$.

Moreover, it is clear that one can uniquely continue the expansion of Theorem 12.11 to also include terms of nonnegative degree; namely, we simply need

to expand the regular terms into a Taylor series with respect to $z - w$ for fixed w. For example, we have an asymptotic expansion

$$a(z)a(w) \sim \frac{1}{(z-w)^2} + \sum_{k=0}^{\infty} : a^{(k)}(w)a(w) : \frac{(z-w)^k}{k!}.$$

So in general we have

$$P(a)(z)Q(a)(w) \sim \sum_{j=-\infty}^{N} R_j(a)(w)(z-w)^{-j}.$$

This formula is called the *operator product expansion* of the product of P and Q. The operator product expansion satisfies certain axioms, which means that it defines on the space $\mathcal{V} \cong \mathcal{F}$, an algebraic structure called a *conformal vertex algebra*:

Definition 12.12 ([**FZ**])**.** A *conformal vertex algebra* is a quintuple $(V, \mathbb{Y}, v, T, \omega)$ where V is a complex vector space, $v \in V$ is a distinguished vector called the *vacuum*, $T \in \mathrm{End}(V)$ is a linear operator called the *translation operator*, $\omega \in V$ is the *conformal vector*, and $\mathbb{Y}$ is a linear map (the *state–field correspondence*)

$$\mathbb{Y}(\cdot, z) : V \otimes V \to V((z)), \qquad a \mapsto \mathbb{Y}(z)(a \otimes -) = \mathbb{Y}(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

where $a_{(n)} \in \mathrm{End}(V)$, such that the following axioms hold:

(1) *Vacuum axioms.*

  (1) $\mathbb{Y}(v, z) = 1$.

  (2) For all $a \in V$, one has

$$\mathbb{Y}(a, z)v \in a + zV[[z]].$$

(2) *Translation covariance.*

  (1) $Tv = 0$.

  (2) For all $a \in V$,

$$[T, \mathbb{Y}(a, z)] = \partial_z \mathbb{Y}(a, z).$$

(3) *Locality.* For all $a, b \in V$ there exists an integer $N \geq 0$ such that

$$(z-w)^N [\mathbb{Y}(a, z), \mathbb{Y}(b, w)] = 0$$

as a formal series in $\mathrm{End}(V)[[z^{\pm 1}, w^{\pm 1}]]$.

(4) *Conformal structure.* The field corresponding to ω has the form

$$\mathbb{Y}(\omega, z) = T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

where the operators L_n satisfy the Virasoro commutation relations with central charge $c \in \mathbb{C}$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}.$$

Moreover,

$$L_{-1} = T,$$

and L_0 gives a $\mathbb{Z}$-grading

$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \qquad V_n = \{v \in V \mid L_0 v = nv\},$$

with $V_n = 0$ for $n \ll 0$ and $\dim V_n < \infty$ for all n.

Exercise 12.13. Check that the state-field correspondence defined above endows $\mathcal{V} = \mathcal{F}$ with the structure of a conformal vertex algebra of central charge $c = 1$.

12.9. Vertex operators

Vertex operators are obtained by quantizing the local functional $e^{i\lambda\varphi(z)}$, where

$$\varphi(z) = -i\int a(z)dz = -i(a_0 \log z + \sum_{n\neq 0} \tfrac{a_{-n}}{n} z^n + a_0^\vee)$$

and $a_0^\vee$ is a constant of integration (dual variable to a_0). In other words, we have

$$e^{i\lambda\varphi(z)} = e^{\lambda\int a(z)dz} = e^{\lambda(a_0\log z + \sum_{n\neq 0}\frac{a_{-n}}{n}z^n)} e^{\lambda a_0^\vee}.$$

A natural quantization of this functional is the operator

$$\begin{aligned} X(\lambda, z) &:= T^\lambda \,:\, e^{\lambda(a_0\log z + \sum_{n\neq 0}\frac{a_{-n}}{n}z^n)} \\ &:= e^{\lambda\sum_{n>0}\frac{a_{-n}}{n}z^n} e^{-\lambda\sum_{n>0}\frac{a_n}{n}z^{-n}} T^\lambda z^{\lambda a_0}, \end{aligned}$$

where T^λ maps $\mathcal{F}_\mu$ to $\mathcal{F}_{\mu+\lambda}$ inducing the identity map on the polynomial algebras. Thus $X(\lambda, z)$ acts from $\mathcal{F}_\mu$ to $\mathcal{F}_{\mu+\lambda}$ by $X_0(\lambda, z)z^{\lambda\mu}$, where

$$X_0(\lambda, z) := e^{\lambda\sum_{n>0}\frac{a_{-n}}{n}z^n} e^{-\lambda\sum_{n>0}\frac{a_n}{n}z^{-n}}.$$

(Here we work over the group algebra of the group $\mathbb{C}$ with basis $z^\alpha, \alpha \in \mathbb{C}$.)

Now note that if $[A, B]$ commutes with A, B, then by the Campbell-Hausdorff formula

$$e^A e^B = e^B e^A e^{[A,B]},$$

and

$$[\sum_{n>0}\frac{a_n}{n}z^{-n}, \sum_{n>0}\frac{a_{-n}}{n}w^n] = \sum_{n>0}\frac{z^{-n}w^n}{n} = -\log(1-\tfrac{w}{z}).$$

Thus

$$X_0(\lambda, z)X_0(\nu, w) = (1-\tfrac{w}{z})^{\lambda\nu} \,:\, X_0(\lambda, z)X_0(\nu, w) \,:$$

for $|w| < |z|$. So we get

$$X(\lambda, z)X(\nu, w) = (z-w)^{\lambda\nu} : X(\lambda, z)X(\nu, w) :$$

for $|w| < |z|$, where the normal ordering puts T to the left of a_0. More generally, we see that

$$X(\lambda_1, z_1)\dots X(\lambda_n, z_n) = \prod_{1\le j<k\le n} (z_j - z_k)^{\lambda_j\lambda_k} : X(\lambda_1, z_1)\dots X(\lambda_n, z_n) :$$

for $|z_1| > \dots > |z_n|$. In particular, denoting the highest weight vector of $\mathcal{F}_\mu$ by Ω_μ and setting $\lambda := \sum_{j=1}^n \lambda_j$, we have

$$\langle\, \Omega_{\mu+\lambda}, X(\lambda_1, z_1)\dots X(\lambda_n, z_n)\Omega_\mu \,\rangle = \prod_{j=1}^n z_j^{\lambda_j\mu} \prod_{1\le j<k\le n} (z_j - z_k)^{\lambda_j\lambda_k}.$$

for $|z_1| > \dots > |z_n|$.

We see that this correlation function admits analytic continuation to the complement of the diagonals $z_i \ne z_j$, but this continuation is not, in general, single valued. In other words, the fields $X(\lambda, z)$ in general do not satisfy space locality. Instead, we have

$$X(\lambda, z)X(\nu, w) = e^{\pi i\lambda\nu} X(\nu, w)X(\lambda, z), \tag{12.10}$$

which is understood in the sense of analytic continuation along a path where $v := w/z$ passes from the region $|v| < 1$ to the region $|v| > 1$ along positive reals, avoiding the point $v = 1$ from above. In particular,

$$X(\lambda, z)X(\lambda, w) = e^{\pi i\lambda^2} X(\lambda, w)X(\lambda, z),$$

i.e., $X(\lambda, z)$ has "statistics $\lambda^2/2$" (where statistics $\alpha \in \mathbb{R}/\mathbb{Z}$ means that switching the order produces a phase factor $e^{2\pi i\alpha}$; e.g., statistics 0 corresponds to bosons and statistics 1/2 to fermions).

Note that if we apply commutation relation (12.10) twice, we obtain a multiplier $e^{2\pi i\lambda\nu}$, which corresponds to the fact that the operator product $X(\lambda, z)X(\nu, w)$ is multivalued in general. This is an example of appearance of a *braiding* in conformal field theory. Namely, relation (12.10) is called *braided space-locality* (or *braided commutativity*), since it can be viewed as commutativity in a suitable braided monoidal category.

Note also that

$$X'(\lambda, z) = \lambda : a(z)X(\lambda, z) :$$

where $X' := \partial_z X$, and

$$[a_n, X(\lambda, z)] = \lambda z^n X(\lambda, z).$$

Hence

$$[L_n, X(\lambda, z)] = z^{n+1}X'(\lambda, z) + \frac{\lambda^2}{2}(n+1)z^n X(\lambda, z),$$

which implies that $X(\lambda, z)$ has spin $\frac{\lambda^2}{2}$ (i.e., it transforms as a section of $(T\mathbb{C}^\times)^{\otimes\lambda^2/2}$). Thus we have the spin-statistics property for $X(\lambda, z)$, which generalizes the usual one: spin modulo $\mathbb{Z}$ equals statistics.

As noted in Remark 10.3, quantum fields $X(\lambda, z)$ are called "anyons" (as they can have any spin and statistics) and can exist only in two dimensions. The most general spin-statistics property for these anyons says that

$$X(z)Y(w) = e^{\pi i(s_{XY}-s_X-s_Y)}Y(w)X(z),$$

where s_X, s_Y, s_{XY} are the spins of X, Y and XY (note that in general $s_{XY} \neq s_X + s_Y$). In particular, we see that if $\lambda^2 \in \mathbb{Z}$ is odd, then the operators $X(n\lambda, z)$ behave like fermions for odd n and like bosons for even n with respect to each other (i.e., the corresponding operators $X(n_1\lambda, z)$ and $X(n_2\lambda, z)$ commute if $n_1 n_2$ is even and anticommute if $n_1 n_2$ is odd), while for even λ^2 they all behave like bosons (i.e., the operators commute).

12.10. The circle-valued theory

Now consider the theory of a massless scalar on $\mathbb{C}^\times$ with values in the circle $\mathbb{R}/2\pi r\mathbb{Z}$. This theory is the same as the line-valued one, except for the zero mode, which entails the following circle-valued solutions of the string equation:

$$\phi(t, \mathrm{x}) = \alpha + \mu t + Nr\mathrm{x},$$

where $\alpha \in \mathbb{R}/2\pi r\mathbb{Z}$, $\mu \in \mathbb{R}$, and N is an integer (the winding number). Upon Wick rotation, we get

$$\phi = \alpha + \mu i t + Nr\mathrm{x} = \alpha + \tfrac{1}{2}\mu(u-\overline{u}) + \tfrac{1}{2}Nr(u+\overline{u}) = \alpha + \tfrac{1}{2}(\mu+Nr)u + \tfrac{1}{2}(-\mu+Nr)\overline{u}.$$

Thus the space of such solutions is a disjoint union of cylinders T^*S^1 with canonical coordinates α, μ, labeled by values of $N \in \mathbb{Z}$. So in quantum theory μ should be replaced by the operator $\widehat{\mu} = -i\partial_\alpha$ on $L^2(\mathbb{R}/2\pi r\mathbb{Z})$, which has eigenfunctions $e^{i\ell r^{-1}\alpha}$ with eigenvalues ℓr^{-1}, $\ell \in \mathbb{Z}$. Thus altogether we get the Hilbert space

$$\mathcal{H}_r^\circ = \bigoplus_{N,\ell\in\mathbb{Z}} \mathcal{H}_r^\circ(N, \ell),$$

where $\mathcal{H}_r^\circ(N, \ell)$ is the completion of $\mathcal{F}_{\frac{1}{\sqrt{2}}(\ell r^{-1}+Nr)} \otimes \overline{\mathcal{F}}_{\frac{1}{\sqrt{2}}(-\ell r^{-1}+Nr)}$. Thus we obtain the following formula for the partition function on the torus E_τ:

$$\mathcal{Z}_r^\circ(\tau) = |\eta(\tau)|^{-2}\vartheta_r(\tau, \overline{\tau}),$$

where

$$\vartheta_r(\tau,\overline{\tau}) := \sum_{\ell,N\in\mathbb{Z}} e^{\frac{1}{2}\pi i\tau(\ell r^{-1}+Nr)^2-\frac{1}{2}\pi i\overline{\tau}(\ell r^{-1}-Nr)^2}$$
$$= \sum_{\ell,N\in\mathbb{Z}} e^{-\pi(\ell^2r^{-2}+N^2r^2)\operatorname{Im}\tau+2\pi i\ell N\operatorname{Re}\tau}.$$

This shows an interesting duality $\mathcal{Z}_r^\circ(\tau)=\mathcal{Z}_{r^{-1}}^\circ(\tau)$; in fact, we see that the whole theory with parameter r is equivalent to the one with parameter r^{-1}. This duality is called *T-duality*, and it plays an important role in string theory.

Also we note that ϑ_r is a real modular form of weight 1:

$$\vartheta_r\left(-\frac{1}{\tau},-\frac{1}{\overline{\tau}}\right)=|\tau|\vartheta_r(\tau,\overline{\tau}),$$

which leads to modular invariance of the function $\mathcal{Z}_r^\circ(\tau)$, as expected in conformal field theory. To see this, it is enough to note that in the exponential we have a quadratic form on $\mathbb{Z}^2$ with matrix

$$Q(\tau)=\begin{pmatrix} r^2\operatorname{Im}\tau & -i\operatorname{Re}\tau \\ -i\operatorname{Re}\tau & r^{-2}\operatorname{Im}\tau\end{pmatrix}.$$

So

$$Q(\tau)^{-1}=|\tau|^{-2}\begin{pmatrix} r^{-2}\operatorname{Im}\tau & i\operatorname{Re}\tau \\ i\operatorname{Re}\tau & r^2\operatorname{Im}\tau\end{pmatrix}=\begin{pmatrix} r^{-2}\operatorname{Im}\tau' & -i\operatorname{Re}\tau' \\ -i\operatorname{Re}\tau' & r^2\operatorname{Im}\tau'\end{pmatrix}=\mathbf{S}^TQ(\tau')\mathbf{S},$$

where $\tau' := -\frac{1}{\tau}$ and $\mathbf{S}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Thus the result follows from the Poisson summation formula.

We see that if $r^2=\frac{p}{q}\in\mathbb{Q}$ (in lowest terms), this conformal field theory has a special property called *rationality* (see Section 12.13): the Hilbert space $\mathcal{H}_r^\circ$ is the completion of a finite sum of "sectors" $\bigoplus_{i=1}^n \mathcal{V}_i\otimes\overline{\mathcal{V}}_i'$, where the left-moving fields act on $\mathcal{V}_i$ and right-moving ones in $\overline{\mathcal{V}}_i'$, so that $\vartheta_r(\tau,\overline{\tau})$ and hence $\mathcal{Z}_r^\circ(\tau)$ are finite sums of products of a holomorphic and an antiholomorphic function (in fact, it is easy to see that $n=2pq$). For example, the vacuum vector Ω is contained in the tensor product $\mathcal{V}(pq)\otimes\overline{\mathcal{V}}(pq)$, where for $s\in\mathbb{Z}_{>0}$ we defined $\mathcal{V}(s):=\bigoplus_{m\in\mathbb{Z}}\mathcal{F}_{m\sqrt{2s}}$. The space $\mathcal{V}(s)$ is a vertex algebra called the *lattice vertex algebra* attached to the even lattice $\sqrt{2s}\mathbb{Z}$ ([**FZ**]). This algebra is generated by the vertex operators $X(m\sqrt{2s},z)$ (which, as we know, satisfy the bosonic version of space locality).

Example 12.14. Consider the case $r=1$. In this case we have two sectors, the vacuum sector $\mathcal{V}(1)\otimes\overline{\mathcal{V}}(1)$ and another one, $\mathcal{W}\otimes\overline{\mathcal{W}}$, where $\mathcal{W}=\bigoplus_{n\in2\mathbb{Z}+1}\mathcal{F}_{\frac{n}{\sqrt{2}}}$. The particles corresponding to $\mathcal{F}_{\frac{n}{\sqrt{2}}}$ for odd n are anyons with statistics $\frac{1}{4}$, so

they satisfy the braided commutativity relation of the form

$$X(z)Y(w) = iY(w)X(z).$$

It is not difficult to show that the Fourier modes of the vertex operators $X(\sqrt{2}, z)$ and $X(-\sqrt{2}, z)$ generate a projective action of the Lie algebra $\mathfrak{sl}_2[t,t^{-1}]$ on $\mathcal{V}(1) = \mathbb{L}_0$ and on $\mathcal{W} = \mathbb{L}_1$. More specifically, define *the affine Kac-Moody Lie algebra*

$$\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2[t, t^{-1}] \oplus \mathbb{C}K,$$

with K central and commutator,[43]

$$[b_1(t), b_2(t)] = [b_1, b_2](t) + \mathrm{Residue}_{t=0}\,\mathrm{Tr}(b_1'(t)b_2(t))dt \cdot K,\ b_1, b_2 \in \mathfrak{sl}_2[t, t^{-1}].$$

Then one can show that $\mathbb{L}_0$ and $\mathbb{L}_1$ are exactly the irreducible integrable highest weight representations of $\widehat{\mathfrak{sl}}_2$ at level $k = 1$ (i.e., K acts by 1). Namely $X(\sqrt{2}, z)$, $X(-\sqrt{2}, z)$, $\sqrt{2}a(z)$ give the currents $e(z)$, $f(z)$, and $h(z)$, where for $b \in \mathfrak{sl}_2$

$$b(z) := \sum_n (b \otimes t^n) z^{-n-1}.$$

This is the so-called *Frenkel-Kac vertex operator construction* of level 1 irreducible integrable modules (defined for any finite-dimensional simply laced simple Lie algebra) in the simplest special case $\mathfrak{g} = \mathfrak{sl}_2$ (see [**KRR**]). Thus the circle-valued theory of a free boson for $r = 1$ is the so-called *Wess-Zumino-Witten* (*WZW*) *model* in the simplest example of the Lie algebra $\mathfrak{sl}_2$ and level 1.

Example 12.15. Let $r = \sqrt{2}$. In this case we have four sectors: $\mathcal{V}_j \otimes \overline{\mathcal{V}}_{-j}$, $j = 0, 1, 2, 3$, where $\mathcal{V}_j = \oplus_{n \in 4\mathbb{Z}+j} \mathcal{F}_{\frac{n}{2}}$. In particular, $\mathcal{V}_0 = \mathcal{V}(2)$ and the particles in $\mathcal{V}_2 = \oplus_{m \in \mathbb{Z}} \mathcal{F}_{2m+1}$ are the fermions arising in the boson-fermion correspondence ([**KRR**, Lecture 5]).

12.11. Free massless fermion (Ising model CFT)

In a similar way to free massless bosons, one can describe the theory of a free massless fermion. As explained in Section 10.4, on $\mathbb{R}^2$ it makes sense to consider a standalone chiral Majorana-Weyl spinor ψ taking values in the tautological representation of $\mathrm{Spin}(2) = U(1)$, with Euclidean kinetic term

$$\frac{1}{2}(\psi, \mathbf{D}\psi) = \frac{1}{2}(\psi, \overline{\partial}\psi).$$

Note that ψ is a $(\frac{1}{2}, 0)$-form, so $(\psi, \overline{\partial}\psi)$ is a $(1, 1)$-form (i.e., a density), which can be canonically integrated over the spacetime surface. The corresponding action $S = \frac{1}{2}\int_{\mathbb{R}^2}(\psi, \overline{\partial}\psi)$ is therefore classically conformally invariant: it depends only on the complex structure on the spacetime. However, the corresponding quantum field theory exhibits a *gravitational anomaly*: the central charges of

[43] One can show that this is the universal central extension of $\mathfrak{sl}_2[t, t^{-1}]$.

the left and right copies of the Virasoro algebra do not coincide (one is equal to $\frac{1}{2}$ and the other to 0), so the theory is not consistent when put on an arbitrary surface (see Section 11.5).

To fix this, we should add an anti-chiral Majorana-Weyl spinor $\overline{\psi}$ with kinetic term $\frac{1}{2}(\overline{\psi}, \partial\overline{\psi})$. So the total Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\psi, \bar{\partial}\psi) + \frac{1}{2}(\overline{\psi}, \partial\overline{\psi}).$$

Thus the equations of motion are $\bar{\partial}\psi = 0,\ \partial\overline{\psi} = 0$, i.e., ψ is holomorphic and $\overline{\psi}$ is antiholomorphic.

Let us consider this theory on the cylinder, i.e., spacetime $\mathbb{R} \times S^1 = \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. We can put two kinds of periodicity conditions on the spinors around the space circle—periodic and antiperiodic (these are the only periodicity conditions making the kinetic term periodic). In other words, ψ can be a section of two different real line bundles on S^1—the trivial bundle $\mathbb{S}_+$ whose total space is a cylinder and the nontrivial bundle $\mathbb{S}_-$ whose total space is a Möbius strip. Sections of $\mathbb{S}_+$ are of the form $f(z)(dz/z)^{1/2}$, where $f(z)$ is a periodic function decomposing into a Fourier series $\sum_{n\in\mathbb{Z}} c_n z^{-n}$, while sections of $\mathbb{S}_-$ are of the form $f(z)(dz/z)^{1/2}$, where f is an antiperiodic function that multiplies by -1 when continued around the circle, hence decomposing as $\sum_{n\in\mathbb{Z}+\frac{1}{2}} c_n z^{-n}$. Therefore, our theory should have two sectors—the *Ramond sector* corresponding to $\mathbb{S}_+$ and the *Neveu-Schwarz sector* corresponding to $\mathbb{S}_-$. This means that the Hilbert space $\mathcal{H}$ of the theory should decompose as $\mathcal{H}_R \oplus \mathcal{H}_{NS}$, the orthogonal direct sum of the subspaces corresponding to the Ramond and the Neveu-Schwarz sectors.

The basic holomorphic quantum field of the theory is thus

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-\frac{1}{2}}$$

in the Ramond sector and

$$\psi(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi_n z^{-n-\frac{1}{2}}$$

in the Neveu-Schwarz sector, and it comes together with the conjugate antiholomorphic quantum field $\overline{\psi}(\overline{z})$ which anticommutes with $\psi(z)$. (Here for brevity we drop the factor $(dz)^{1/2}$.)

The field $\psi(z)$ in both cases satisfies the relation

$$[\psi(z), \psi(w)]_+ = \delta(z - w),$$

where $[,]_+$ is the supercommutator and

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1} = \sum_{n\in\mathbb{Z}+\frac{1}{2}} z^n w^{-n-1}.$$

(Note that the delta-function has scaling dimension $[\delta] = 1$, so ψ has scaling dimension $[\psi] = \frac{1}{2}$, as a fermionic field should.) This yields for both sectors the Clifford algebra relations

$$\psi_n\psi_m + \psi_m\psi_n = \delta_{m,-n},$$

but with ψ_n labeled by $n \in \mathbb{Z}$ in the Ramond sector and $n \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz sector. Denote by Cl_R, Cl_{NS} the infinite-dimensional Clifford algebras defined by these relations in the Ramond and Neveu-Schwarz cases, respectively.

The algebra Cl_{NS} has a unique irreducible positive energy representation $\Lambda_{NS} = \wedge(\psi_{-1/2}, \psi_{-3/2}, \dots)$ on which ψ_j act by multiplications for $j < 0$ and by differentiations for $j > 0$. There is an invariant positive Hermitian inner product on Λ_{NS} in which the wedge monomials in ψ_j, $j < 0$ form an orthonormal basis (invariance means that $\psi_j^\dagger = \psi_{-j}$).[44] Thus the Hilbert space $\mathcal{H}_{NS}$ of the Neveu-Schwarz sector is the completion of $\mathcal{D}_{NS} := \Lambda_{NS} \otimes \overline{\Lambda}_{NS}$, where $\overline{\Lambda}_{NS}$ is the conjugate of Λ_{NS} corresponding to the antiholomorphic field $\overline{\psi}(\overline{z})$.

The Hamiltonian H is supposed to satisfy the commutation relations

$$[H, \psi_n] = -n\psi_n,\ [H, \overline{\psi}_n] = -n\overline{\psi}_n,$$

which are quantizations of the corresponding Poisson bracket relations. So on $\mathcal{H}_{NS}$ we have

$$H = H_L + H_R,$$

where

$$H_L = \sum_{n\in\frac{1}{2}+\mathbb{Z}_{\geq 0}} n\psi_{-n}\psi_n + \frac{C}{2}$$

for some $C \in \mathbb{R}$, and similarly for H_R. The Virasoro algebra action is defined by

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}+\frac{1}{2}} n : \psi_n\psi_{-n+m} :,$$

where $: \psi_r\psi_s := \psi_r\psi_s$ if $r \leq s$ and $: \psi_r\psi_s := -\psi_s\psi_r$ if $r \geq s$. Namely, this formula can be proved similarly to the bosonic case (Section 12.7) or obtained from the expression of the energy-momentum tensor for this theory (similar to Section 12.8). Thus $H_L = L_0 + \frac{C}{2}$, $H_R = \overline{L}_0 + \frac{C}{2}$.

[44] To lighten the notation, we normalize the inner product on the odd subspace $\Lambda_{NS,1} \subset \Lambda_{NS}$ to make it positive in the usual sense and remove the i factors.

Now, we have

$$\mathrm{sTr}_{\Lambda_{NS}}(q^{H_L}) = e^{\pi i C\tau}\prod_{j=0}^{\infty}(1-q^{j+\frac{1}{2}}) = e^{\pi i\tau(C+\frac{1}{24})}\frac{\eta(\frac{\tau}{2})}{\eta(\tau)},$$

where $q = e^{2\pi i\tau}$. Recall that $\eta(\tau)$ is a modular form of weight $1/2$ for a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ which changes under $\mathrm{PSL}_2(\mathbb{Z})$ by a character of order 24. This implies that $\eta(\frac{\tau}{2})$ is also a modular form of weight $1/2$, changing by a character under the action of the congruence subgroup $\Gamma_0(2) \subset \mathrm{PSL}_2(\mathbb{Z})$ of matrices which are lower triangular modulo 2. Indeed, we have

$$\eta(\tfrac{\tau}{2\tau+1}) = e^{-\frac{2\pi i}{12}}(2\tau+1)^{1/2}\eta(\tau),$$

thus

$$\eta(\tfrac{1}{2}\tfrac{\tau}{\tau+1}) = \eta\left(\frac{\frac{\tau}{2}}{2\frac{\tau}{2}+1}\right) = e^{-\frac{2\pi i}{12}}(\tau+1)^{1/2}\eta(\tfrac{\tau}{2}),\ \eta(\tfrac{\tau+2}{2}) = e^{\frac{2\pi i}{24}}\eta(\tfrac{\tau}{2}),$$

but the matrices $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ generate $\Gamma_0(2)$. It follows that the ratio $\frac{\eta(\frac{\tau}{2})}{\eta(\tau)}$ is a modular function (of weight 0) with respect to a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. This suggests that to have a chance for modular invariance, we must put

$$C = -\frac{1}{24}$$

to cancel the exponential prefactor. Thus

$$H_L = L_0 - \frac{1}{48},\ H_R = \overline{L}_0 - \frac{1}{48}.$$

Let us now compute the partition function for the Neveu-Schwarz sector on the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau) = \mathbb{C}^\times/q^{\mathbb{Z}}$. We obtain

$$\mathcal{Z}^+_{NS}(\tau) = \mathrm{sTr}_{\mathcal{H}_{NS}}(q^{H_L}\overline{q}^{H_R}) = |q|^{-\frac{1}{24}}|\prod_{j=0}^{\infty}(1-q^{j+\frac{1}{2}})|^2 = \left|\frac{\eta(\frac{\tau}{2})}{\eta(\tau)}\right|^2.$$

(If you are wondering what the "plus" superscript is for, you will soon find out!)

This is indeed a real analytic modular function, but only with respect to the congruence subgroup $\Gamma_0(2)$ of $\mathrm{PSL}_2(\mathbb{Z})$, of index 3. At first sight, this sounds like a disappointment if we want to check that the theory is conformal. But on reflection, this is to be expected: since this theory involves spinors, its partition function depends not just on the elliptic curve E but also on a *spin structure* on it. A spin structure on E is the square root of the (trivial) canonical bundle, so there are four of them, which we denote by $\mathbb{S}_{\pm\pm}$; namely, $\mathbb{S}_{\varepsilon_1\varepsilon_2}$ has holonomy ε_1 around the space circle and holonomy ε_2 around the time circle. The

Neveu-Schwarz sector is then responsible for the bundles $\mathbb{S}_{-\pm}$, while the Ramond sector is responsible for the bundles $\mathbb{S}_{+\pm}$. Note that $\mathrm{SL}_2(\mathbb{Z})$ acts on these spin structures, factoring through

$$\mathrm{PSL}_2(\mathbb{Z})/\Gamma(2) = \mathrm{PSL}_2(\mathbb{Z}/2) = S_3,$$

where $\Gamma(2) \subset \mathrm{PSL}_2(\mathbb{Z})$ is the normal congruence subgroup of matrices equal to I mod 2, and the S_3 permutes $\mathbb{S}_{+-}, \mathbb{S}_{-+}$, and $\mathbb{S}_{--}$ and preserves $\mathbb{S}_{++}$. Thus we should compute three more partition functions – $\mathcal{Z}^-_{NS}$ (for $\mathbb{S}_{+-}$) and $\mathcal{Z}^\pm_R$ (for $\mathbb{S}_{+\pm}$), and the combinations invariant under the full group $\mathrm{PSL}_2(\mathbb{Z})$ should be $\mathcal{Z}^+_R$ and $\mathcal{Z}^+_{NS} + \mathcal{Z}^-_{NS} + \mathcal{Z}^-_R$, with S_3 permuting the summands in this sum.[45].

So let us proceed with computing $\mathcal{Z}^-_{NS}$. A twisted version of the Feynman-Kac formula implies that it differs from $\mathcal{Z}^+_{NS}$ by replacing the supertrace by the ordinary trace (as ψ changes sign when we go around the time circle). That is, we should replace $q^{1/2}$ by $-q^{1/2}$. Thus we get

$$\mathrm{Tr}_{\Lambda_{NS}}(q^{H_L}) = e^{\pi i C\tau}\prod_{j=0}^{\infty}(1+q^{j+\frac{1}{2}}) = e^{\pi i\tau(C+\frac{1}{24})}\frac{e^{-\frac{\pi i}{24}}\eta(\frac{\tau+1}{2})}{\eta(\tau)}$$

and

$$\mathcal{Z}^-_{NS}(\tau) = \mathrm{Tr}_{\mathcal{H}_{NS}}(q^{H_L}\overline{q}^{H_R}) = |q|^{-\frac{1}{24}}|\prod_{j=0}^{\infty}(1+q^{j+\frac{1}{2}})|^2 = \left|\frac{\eta(\frac{\tau+1}{2})}{\eta(\tau)}\right|^2.$$

It remains to compute the Ramond sector. First we need to describe the space $\mathcal{H}_R[0]$ of ground states. This space should be the irreducible 2-dimensional representation of the Clifford algebra generated by $\psi_0, \overline{\psi}_0$ with relations

$$\psi_0^2 = \overline{\psi}_0^2 = \frac{1}{2},\ \psi_0\overline{\psi}_0 + \overline{\psi}_0\psi_0 = 0.$$

Let

$$\xi = e^{\pi i/4}\psi_0 + e^{-\pi i/4}\overline{\psi}_0,\ \eta = e^{-\pi i/4}\psi_0 + e^{\pi i/4}\overline{\psi}_0,$$

then

$$\xi^2 = \eta^2 = 0,\ \xi\eta + \eta\xi = 2.$$

So $\mathcal{H}_R[0] = \wedge\xi$ with η acting as $2\partial_\xi$.

Now consider the Clifford algebra Cl^0_R generated by ψ_n with $n \neq 0$; so $\mathrm{Cl}_R = \mathrm{Cl}^0_R \otimes \wedge\psi_0$. The algebra Cl^0_R has a unique irreducible positive energy representation $\Lambda_R = \wedge(\psi_{-1}, \psi_{-2}\dots)$ on which ψ_j act by multiplication for $j < 0$ and by differentiation for $j > 0$. There is an invariant positive Hermitian inner

[45] Of course, if we know modular invariance, we can just compute $\mathcal{Z}^-_{NS}, \mathcal{Z}^-_R$ using the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{Z}^+_{NS}$, but we want to compute independently from the Hamiltonian picture and then compare as a check for modular invariance.

product on Λ_R in which the wedge monomials in ψ_j, $j < 0$, form an orthonormal basis. Thus the Hilbert space $\mathcal{H}_R$ of the Ramond sector is the completion of

$$\mathcal{D}_R := \mathcal{H}_R[0] \otimes \Lambda_R \otimes \overline{\Lambda}_R,$$

where $\overline{\Lambda}_R$ is the conjugate of Λ_R corresponding to the antiholomorphic field $\overline{\psi}(\overline{z})$.

This shows that $\mathcal{Z}_R^+ = 0$: the even and odd subspaces of $\mathcal{H}_R$ are isomorphic together with the grading, so the supertrace of $q^{H_L}\overline{q}^{H_R}$ on $\mathcal{H}_R$ vanishes. So it remains to compute $\mathcal{Z}_R^-$, which is obtained by taking the ordinary trace of $q^{H_L}\overline{q}^{H_R}$ instead of a supertrace.

One can show that the Virasoro algebra action on Λ_R is defined by

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} n : \psi_n\psi_{-n+m} : + \frac{\delta_{m,0}}{16}$$

([**KRR**, Lecture 3]). As before, we have $H = H_L + H_R$ and $H_L = L_0 + \frac{C}{2}$, $H_R = \overline{L}_0 + \frac{C}{2}$.

Exercise 12.16. Show that the operators L_m in both the Neveu-Schwarz and Ramond sectors satisfy the Virasoro commutation relations with central charge $c = \frac{1}{2}$.

Now,

$$\mathrm{Tr}_{\Lambda_R}(q^{H_L}) = e^{\pi i(C+\frac{1}{8})\tau}\prod_{j=1}^{\infty}(1+q^j) = e^{\pi i(C+\frac{1}{8})}e^{-\frac{\pi i\tau}{12}}\frac{\eta(2\tau)}{\eta(\tau)}.$$

The function $\eta(2\tau)$ is a modular form of weight 1/2 for a congruence subgroup. Thus if we want to have a chance for modular invariance, we must again put $C = -\frac{1}{24}$ to cancel the exponential prefactor, and

$$\mathcal{Z}_R^-(\tau) = 2\left|\frac{\eta(2\tau)}{\eta(\tau)}\right|^2$$

(here the factor 2 is the dimension of $\mathcal{H}_R[0]$). So for the total partition function we get

$$\mathcal{Z}(\tau) = \frac{|\eta(\frac{\tau}{2})|^2 + |\eta(\frac{\tau+1}{2})|^2 + 2|\eta(2\tau)|^2}{|\eta(\tau)|^2}.$$

We can write this answer slightly differently. To this end, note that $\Lambda_{NS} = \Lambda_{NS,0} \oplus \Lambda_{NS,1}$, the decomposition into even and odd states. It follows from the above computations that setting

$$\chi_j(\tau) := \mathrm{Tr}\,|_{\Lambda_{NS,j}}(q^{H_L}),\ j = 0, 1,$$

we have

$$\chi_0(\tau) = \frac{e^{-\frac{\pi i}{24}}\eta(\frac{\tau+1}{2}) + \eta(\frac{\tau}{2})}{2\eta(\tau)},\ \chi_1(\tau) = \frac{e^{-\frac{\pi i}{24}}\eta(\frac{\tau+1}{2}) - \eta(\frac{\tau}{2})}{2\eta(\tau)}.$$

So setting

$$\chi_2(\tau) := \mathrm{Tr}\,|_{\Lambda_R}(q^{H_L}) = \frac{\eta(2\tau)}{\eta(\tau)},$$

we get

$$\mathcal{Z}(\tau) = 2(|\chi_0(\tau)|^2 + |\chi_1(\tau)|^2 + |\chi_2(\tau)|^2).$$

The functions χ_0, χ_1, χ_2 have a representation-theoretic meaning. Namely, they are the characters of the Virasoro algebra representations $\Lambda_{NS,0}$, $\Lambda_{NS,1}$, Λ_R, i.e., the traces of $q^{L_0 - \frac{1}{48}}$ in these representations. It turns out that these representations are irreducible. Namely, recall from Exercise 12.8 that given two numbers $h, c \in \mathbb{C}$, we can define the *Verma module* $M(h,c)$ over Vir with highest weight (h,c) which has a unique irreducible quotient $L(h,c)$, and that if $h, c \in \mathbb{R}$, then $L(h,c)$ has a unique nondegenerate Hermitian form in which $L_n^\dagger = L_{-n}$, called the *Shapovalov form*. If this form is positive definite, the representation $L(h,c)$ is unitary.

Proposition 12.17 ([**KRR**])**.**

(i) *One has*

$$\Lambda_{NS,0} \cong L(0, \tfrac{1}{2}),\ \Lambda_{NS,1} \cong L(\tfrac{1}{2}, \tfrac{1}{2}),\ \Lambda_R \cong L(\tfrac{1}{16}, \tfrac{1}{2}).$$

Thus these irreducible representations are unitary.

(ii) $L(h, \frac{1}{2})$ *is unitary if and only if* $h = 0, \frac{1}{2}$, *or* $\frac{1}{16}$.

(iii) *The characters of these representations are given by*

$$\mathrm{ch}\, L(0, \tfrac{1}{2}) = \chi_0,\ \mathrm{ch}\, L(\tfrac{1}{2}, \tfrac{1}{2}) = \chi_1,\ \mathrm{ch}\, L(\tfrac{1}{16}, \tfrac{1}{2}) = \chi_2.$$

Thus the even part $\mathcal{H}_0$ of the Hilbert space of our theory is the completion of

$$\mathcal{D}_0 := L(0, \tfrac{1}{2}) \otimes \overline{L(0, \tfrac{1}{2})} \oplus L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes \overline{L(\tfrac{1}{2}, \tfrac{1}{2})} \oplus L(\tfrac{1}{16}, \tfrac{1}{2}) \otimes \overline{L(\tfrac{1}{16}, \tfrac{1}{2})},$$

as a Vir $\oplus$ Vir-module, and correspondingly

$$\mathrm{Tr}\,|_{\mathcal{H}_0}(q^{H_L}\overline{q}^{H_R}) = \frac{1}{2}\mathcal{Z}(\tau) = |\chi_0(\tau)|^2 + |\chi_1(\tau)|^2 + |\chi_2(\tau)|^2.$$

We can now check that this function is really modular invariant. Indeed, it is clear that $\mathcal{Z}$ is invariant under $\tau \to \tau + 1$. Furthermore, using the modular transformation formula for η,

$$\eta(-\tfrac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau),$$

we have

$$\chi_2(-\tfrac{1}{\tau}) = \frac{\eta(-\frac{2}{\tau})}{\eta(-\frac{1}{\tau})} = \frac{1}{\sqrt{2}}\frac{\eta(\frac{\tau}{2})}{\eta(\tau)} = \frac{1}{\sqrt{2}}(\chi_0(\tau) - \chi_1(\tau)).$$

On the other hand, it follows from the Jacobi triple product identity that

$$\vartheta(0, -2i\tau) = \frac{e^{-\frac{\pi i}{24}}\eta^2(\frac{\tau+1}{2})}{\eta(\tau)},$$

where ϑ is the theta function defined by (6.11). Thus the modular transformation formulas for ϑ (equation (7.10)) and η imply that

$$\eta\left(\frac{-\frac{1}{\tau}+1}{2}\right) = \sqrt{-i\tau}\,\eta(\tfrac{\tau+1}{2}).$$

Hence

$$\chi_0(-\tfrac{1}{\tau}) + \chi_1(-\tfrac{1}{\tau}) = \chi_0(\tau) + \chi_1(\tau).$$

It follows that the matrix S of the transformation $\tau \mapsto -\frac{1}{\tau}$ in the basis χ_0, χ_1, χ_2 is

$$S = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}. \tag{12.11}$$

This matrix is unitary, which implies that $\mathcal{Z}$ is modular invariant, as claimed.

Thus we see that the holomorphic functions χ_0, χ_1, χ_2 span a 3-dimensional projective unitary representation of $\mathrm{PSL}_2(\mathbb{Z})$ in which they form an orthonormal basis. Namely, the transformation $\tau \mapsto -\frac{1}{\tau}$ acts in this basis by the matrix S given by (12.11), while $\tau \mapsto \tau + 1$ acts by the matrix

$$T = \begin{pmatrix} e^{-\frac{\pi i}{24}} & 0 & 0 \\ 0 & -e^{-\frac{\pi i}{24}} & 0 \\ 0 & 0 & e^{-\frac{\pi i}{12}} \end{pmatrix}.$$

The purely bosonic conformal field theory with Hilbert space $\mathcal{H}_0$ is the *Ising model CFT*, called this because it is the scaling limit of the 2-dimensional Ising model of statistical mechanics. It is the simplest of *minimal models*, which are CFT of central charges $c = 1 - \frac{6}{m(m+1)}$, $m \geq 3$ (the Ising model is the case $m = 3$).[46] It turns out that these are the only values of $0 < c < 1$ for which there exist unitary representations $L(h, c)$ of the Virasoro algebra ([**KRR**]), and therefore the only values of c in this range for which there can exist (unitary) conformal field theories.[47] Moreover, by a result of Friedan, Qiu, and Shenker,

[46] We note that for $m \geq 4$ minimal models do not have a Lagrangian description.

[47] Recall that by Exercise 12.8(vii), if $L(h, c)$ is unitary, then $c \geq 0$, and the only unitary representation $L(h, 0)$ is the trivial representation $L(0, 0) = \mathbb{C}$, so there are no unitary conformal field theories with $c < 0$ and the ones for $c = 0$ are trivial.

all unitary conformal field theories in this range reduce to minimal models. For a discussion of minimal models, see [**DMS**, Chapters 7 and 8].

12.12. The cobordism picture of 2-dimensional CFT

Let us now discuss Segal's cobordism picture (Section 11.11) for 2-dimensional (quantum) CFT; for more details see [**Se**]. According to the general setup, such a theory would first of all entail a Hilbert space $\mathcal{H}$ attached to the circle $S^1 = \mathbb{R}/\mathbb{Z}$. More precisely, there should be a Hilbert space $\mathcal{H}_a$ attached to every Riemannian metric $a(s)ds^2$ on S^1. However, our theory should be *conformal*, i.e., invariant under rescaling of the metric, so we should have $\mathcal{H}_a = \mathcal{H}$ for all a. Thus every orientation-preserving diffeomorphism $g : S^1 \to S^1$ should give rise to a unitary operator $\rho(g) : \mathcal{H} \to \mathcal{H}$ such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$; in other words, ρ is a unitary representation of $\mathrm{Diff}_+(S^1)$ on $\mathcal{H}$. Also, to a disjoint union of n copies of S^1 corresponds the space $\mathcal{H}^{\otimes n}$.

Next, according to the cobordism picture of QFT, to every compact oriented Riemannian 2-manifold X with boundary, we are supposed to have its amplitude—a compact operator

$$A(X) : \mathcal{H}(\partial_- X) \to \mathcal{H}(\partial_+ X).$$

Now, $\partial_- X \cong (S^1)^n$ and $\partial_+(X) \cong (S^1)^m$ for some m, n, so $\mathcal{H}(\partial_- X) \cong \mathcal{H}^{\otimes n}$ and $\mathcal{H}(\partial_+ X) \cong \mathcal{H}^{\otimes m}$. Thus $A(X)$ should be an operator $\mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes m}$. However, to represent it in this way, we must fix orientation-preserving diffeomorphisms $(S^1)^n \cong \partial_- X, (S^1)^m \cong \partial_+ X$, i.e., parametrizations of the boundary components. Thus to any X equipped with such parametrizations, we are supposed to attach a compact operator

$$A(X) : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes m}$$

(the amplitude of X), such that change of parametrization of a component of ∂X by a diffeomorphism $g \in \mathrm{Diff}_+(S^1)$ multiplies $A(X)$ by the corresponding operator in the appropriate $\mathcal{H}$-factor (by $\rho(g)$ on the left for $\partial_+ X$ and by $\rho(g)^{-1}$ on the right for $\partial_- X$). Moreover, we would like $A(X)$ to be invariant under rescaling of the metric of X by a positive function, i.e., to depend only on the underlying complex structure on X.

Unfortunately, the last condition turns out to be a bit too strong to allow any nontrivial examples. This corresponds to the fact that all interesting 2-dimensional quantum CFT have a conformal anomaly—a nonzero Virasoro central charge (Remark 12.9). To accommodate for that, we should relax the requirement that $A(X)$ is completely invariant under rescaling of the metric, and ask for such invariance only up to a positive scalar. Thus to every Riemann surface X with boundary and parametrizations g_i^-, g_j^+ of its boundary components, there corresponds an operator $A(X, g_i^-, g_j^+)$ defined up to scaling by a positive number. We will fix this scaling somehow.

Now, recall that the main property of amplitudes is the gluing property: $A(X_1 \# X_2) = A(X_1)A(X_2)$. Thus for Riemann surfaces X_1, X_2 with parametrizations $g^-_{1i}, g^+_{1j}, g^-_{2j}, g^+_{2k}$, we have

$$A(X_1, g^-_{1i}, g^+_{1j})A(X_2, g^-_{2j}, g^+_{2k}) \sim A(X_1 \#_{g^-_{1j} \circ (g^+_{2j})^{-1}} X_2, g^-_{1i}, g^+_{2k}),$$

where we glue X_1, X_2 using the maps $g^-_{1j} \circ (g^+_{2j})^{-1}$, and $\sim$ means that the two sides differ by a positive scaling factor. Similarly, if we attach a component of $\partial_- X$ to a component of $\partial_+ X$ according to their parametrizations, then the new amplitude equals the trace of the old one in the corresponding copy of $\mathcal{H}$, up to a positive factor. Finally, $A(\overline{X}) = A(X)^\dagger$. Thus we recover the *Segal axiomatics* of a 2-dimensional quantum CFT.

Now let us try to describe the amplitudes $A(X)$ more explicitly.

First consider the case when X is the unit disk D with standard parametrization of the boundary, $g_0(s) = e^{2\pi i s}$. Then $A(X) = A(D, g_0)$ is a nonzero vector $\Omega \in \mathcal{H}$, which we can normalize to be a unit vector; it is the *vacuum vector* of the theory. Amplitudes of D with other parametrizations of the boundary then equal $\rho(g)\Omega$, where $g \in \mathrm{Diff}_+(S^1)$.

Consider now the case when X is a cylinder (i.e., annulus). In this case there is a unique $\beta > 0$ and an orientation-preserving conformal map $f : \mathbb{R}/\mathbb{Z} \times [0, \beta] \to X$. Thus $A(X) = A(g^-, \beta, g^+)$, where $g^\pm : S^1 \cong S^1$ are the parametrizations of the boundaries of the annulus $\mathbb{R}/\mathbb{Z} \times [0, \beta]$. Triples (g^-, β, g^+) (i.e., conformal classes of annuli with parametrized boundaries) form a semigroup under gluing according to parametrizations; it is called the *Segal semigroup* or *semigroup of annuli*, and we will denote it by $\mathcal{A}^\circ$.[48] The union $\mathcal{A} := \mathrm{Diff}_+(S^1) \sqcup \mathcal{A}^\circ$ is in a natural way a monoid (elements $g \in \mathrm{Diff}_+(S^1)$ may be viewed as infinitely thin annuli $(g^-, 0, g^+)$ where $(g^+)^{-1} g^- = g$).

Let $C(\beta) := \mathbb{R}/\mathbb{Z} \times [0, \beta]$ with standard parametrizations (so $C(0) = 1$). Then a general element of $\mathcal{A}$ is of the form $(g^+)^{-1} \circ C(\beta) \circ g^-$, where $g^+, g^- \in \mathrm{Diff}_+(S^1)$; the subgroup $\mathrm{Diff}_+(S^1) \subset \mathcal{A}$ corresponds to $\beta = 0$. Moreover, for $\beta > 0$

$$(g^+_1)^{-1} \circ C(\beta) \circ g^-_1 = (g^+_2)^{-1} \circ C(\beta) \circ g^-_2$$

iff $g^\pm_2 = T_s g^\pm_1$ for some $s \in [0, 1)$, where $T_s(x) := x + s$ on $\mathbb{R}/\mathbb{Z}$ is a translation. Thus the factorization $a = (g^+)^{-1} \circ C(\beta) \circ g^-$ of $a \in \mathcal{A}^\circ$ is uniquely determined if we require that $g^-(0) = 0$. Using this normalization, denote $\exp(2\pi i g^+(0))$ by $\zeta(a)$. Then we may define a metric on $\mathcal{A}^\circ$ by

$$\mathrm{dist}(a_1, a_2) = |\zeta(a_1) - \zeta(a_2)| + \sup |(g^+_1)' - (g^+_2)'| + |\beta_1 - \beta_2| + \sup |(g^-_1)' - (g^-_2)'|.$$

It is easy to see that the multiplication on $\mathcal{A}^\circ$ is continuous in this metric.

[48] Note that if $(g^-, \beta, g_+) = (g^-, \beta_1, h^+) \circ (h^-, \beta_2, g^+) = (\tilde{g}^-, \beta, \tilde{g}^+)$, then $\beta \neq \beta_1 + \beta_2$ in general.

Now, we have a projective representation A of $\mathcal{A}$ by bounded operators on $\mathcal{H}$ which extends ρ, such that $\mathcal{A}^\circ$ acts by compact operators. It is defined by

$$A((g^+)^{-1} \circ C(\beta) \circ g^-) = \rho(g^+)^{-1} A(C(\beta))\rho(g^-).$$

We may normalize A so that $A(C(\beta))A(C(\gamma)) = A(C(\beta+\gamma))$, so $A(C(\beta))$ is a 1-parameter semigroup of compact self-adjoint operators (they are self-adjoint because $C(\beta) \cong \overline{C(\beta)}$). By the Hilbert-Schmidt theorem, such a semigroup admits a spectral decomposition

$$\mathcal{H} = \widehat{\bigoplus}_{n\geq 0} \mathcal{H}(E_n)$$

(Hilbert-completed orthogonal direct sum), where $A(C(\beta))$ acts on $\mathcal{H}(E_n)$ by $\Lambda_n = e^{-\beta E_n}$ and $E_0 \leq E_1 \leq \cdots \in \mathbb{R}$, $E_n \to \infty$. Indeed, all Λ_n are strictly positive (so E_n are defined) and the eigenspaces $\mathcal{H}(E_n)$ are finite dimensional since $A(C(\beta)) \to 1$ in the weak topology as $\beta \to 0$. Thus we can define the (unbounded) self-adjoint operator

$$H := -\frac{d}{d\beta}|_{\beta=0} A(C(\beta))$$

acting on $\mathcal{H}(E_n)$ by E_n, which is the Hamiltonian of the theory. We see that this operator is bounded below: $H \geq E_0$. It is convenient to define the dense subspace of H-finite vectors $\mathcal{H}_{\mathrm{fin}} = \oplus_{n\geq 0}\mathcal{H}(E_n) \subset \mathcal{H}$ (algebraic direct sum), then H is essentially self-adjoint on $\mathcal{H}_{\mathrm{fin}}$.

The partition function of the theory therefore is

$$Z(\beta) = A(\mathbb{C}/(\mathbb{Z} \oplus i\beta\mathbb{Z})) = \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}) = \sum_{n\geq 0} \dim \mathcal{H}(E_n) e^{-\beta E_n},$$

and this sum must be finite for all $\beta > 0$.

As mentioned above, the complex Lie algebra $\mathrm{Vect}(\mathbb{C}^\times)$ of holomorphic vector fields on $\mathbb{C}^\times$ (unlike its real Lie subalgebra of vector fields tangent to the unit circle), is **not** the Lie algebra of any infinite-dimensional complex Lie group. In some sense, the semigroup $\mathcal{A}$ is a replacement for this nonexistent group. Namely, recall that if G is an ordinary real or complex Lie group with Lie algebra $\mathfrak{g}$, then for every $v \in \mathfrak{g}$ we have a 1-parameter subsemigroup $\exp(\bullet v) : \mathbb{R}_{\geq 0} \to G$ such that for every $n \geq 1$, $v, w \in \mathfrak{g}$,

$$\mathrm{dist}(\exp(tv)\exp(tw), \exp(CH_n(tv, tw))) = O(t^{n+1}),\ t \to 0, \tag{12.12}$$

where CH_n is the degree $\leq n$ part of the Campbell-Hausdorff series, e.g.,

$$\mathrm{dist}\Big(\exp(tv)\exp(tw), \exp(t(v+w) + \tfrac{t^2}{2}[v,w])\Big) = O(t^3),\ t \to 0, \tag{12.13}$$

and also

$$\exp(tv)\exp(tw) = \exp(tw)\exp(tv) = \exp(t(v+w)) \tag{12.14}$$

if $[v, w] = 0$. Moreover, the linear map $L : \mathfrak{g} \to T_1 G$ defined by the formula $L(v) := \frac{d}{dt}|_{t=0} \exp(tv)$ is an isomorphism. We will now construct a 1-parameter subsemigroup $\exp(\bullet v) : \mathbb{R}_{\geq 0} \to \mathcal{A}$ defined for vector fields $v \in \mathrm{Vect}(\mathbb{C}^\times)$ directed inside the unit circle (i.e., $\mathrm{Re}(v(z)/z) < 0$ for $|z| = 1$) with similar properties. To this end, consider the differential equation

$$\dot{z} = v(z),$$

and let $z(t, s)$ be the solution of this equation with initial condition

$$z(0, s) = e^{2\pi i s},\ 0 \leq s < 1$$

(it exists for small enough t by Picard's theorem). There exists $\varepsilon_v > 0$ such that for $0 \leq t < \varepsilon_v$ the curve $s \mapsto z(t, s)$ is simple and $\partial_s z(t, s)$ is nonvanishing. Now consider the annulus $X(v, t) \subset \mathbb{C}$ whose incoming boundary is the circle $|z| = 1$ with standard parametrization $s \mapsto e^{2\pi i s}$ and outgoing boundary is the parametrized curve $s \mapsto z(t, s)$. Then $X(v, t) \in \mathcal{A}$ (with $X(v, 0) = 1$), and we set

$$\exp(tv) := X(v, t),\ 0 \leq t < \varepsilon_v.$$

More generally, taking the incoming boundary to be $s \mapsto e^{\beta + 2\pi i s}$, we can similarly define the annulus $X_\beta(v, t)$ for small enough $t \geq 0$, and if $\beta > 0$, then this definition makes sense for any $v \in \mathrm{Vect}(\mathbb{C}^\times)$. Moreover, if v is directed inside the unit circle, then

$$X_\beta(v, t) = \exp(vt) C(\beta).$$

Exercise 12.18.

(i) Show that

$$\exp(t_1 v) \exp(t_2 v) = \exp((t_1 + t_2)v)$$

for $t_1, t_2 \geq 0$, $t_1 + t_2 < \varepsilon_v$, and use this identity to extend the function $t \mapsto \exp(tv)$ to all $t \geq 0$ so that it holds for all $t_1, t_2 \geq 0$.

(ii) Prove that (12.12), (12.13), and (12.14) hold for $v, w \in \mathrm{Vect}(\mathbb{C}^\times)$ directed inside the unit circle.

(iii) Realize the tangent space T_β to $\mathcal{A}^\circ$ at $C(\beta)$ as

$$T_\beta = \mathrm{Vect}_0(S^1) \oplus \mathbb{R} \oplus \mathrm{Vect}(S^1),$$

where $\mathrm{Vect}_0(S^1)$ is the Lie algebra of vector fields on S^1 vanishing at 0 (deformations of g^-, β, g^+), and compute the $\mathbb{R}$-linear map

$$L : \mathrm{Vect}(\mathbb{C}^\times) \to T_\beta$$

given by

$$L(v) := \frac{d}{dt}|_{t=0} X_\beta(v, t).$$

Show that L is injective and has dense image (in C^∞-topology).

In particular, we have unbounded densely defined operators $L_n, \overline{L}_n$ on $\mathcal{H}$ such that

$$L(-z^{n+1}\partial_z)|_{\mathcal{H}} = L_n + \overline{L_n},\ L(-iz^{n+1}\partial_z)|_{\mathcal{H}} = i(L_n - \overline{L_n}).$$

It is not difficult to show that $L_{-n}, \overline{L}_{-n} : H(E) \to H(E+n)$, so they preserve $\mathcal{H}_{\text{fin}}$ and satisfy on $\mathcal{H}_{\text{fin}}$ the commutation relations of two commuting copies of the Virasoro algebra, $\text{Vir} \oplus \overline{\text{Vir}}$. Moreover, the complexified Lie algebra of $\text{Diff}_+(S^1)$ in this realization is the anti-diagonal copy of Vir generated by $L_n - \overline{L}_{-n}$, which must act with zero central charge (no gravitational anomaly!), so the central charges of the two Virasoro algebras must be the same; we denote their common value by c.[49] Also, the Hamiltonian is

$$H = L_0 + \overline{L}_0$$

up to adding a constant, since $C(\beta) = X(-z\partial_z, \beta)$.

Note that the semigroup $\mathcal{A}$ contains the subsemigroup $\mathcal{A}_+$ of univalent holomorphic maps $f : D \to D$ (smooth on the boundary) under composition; namely, $f \in \mathcal{A}_+$ maps to the annulus $C(f)$ bounded by $\partial_- C(f) = U(1)$ (the unit circle) and $\partial_+ C(f) := f(U(1))$, with the corresponding parametrizations $s \mapsto e^{2\pi i s}$, $s \mapsto f(e^{2\pi i s})$. The Lie algebra of this semigroup in the above sense is the Lie algebra $\text{Vect}(\mathbb{C})$ of holomorphic vector fields on $\mathbb{C}$, whose (topological) basis consists of $-z^n\partial_z$ for $n \geq 0$.

If we glue $C(f)$ to the standard disk D respecting parametrizations, we again obtain D. Thus we should have $\mathcal{A}_+\Omega = \Omega$ and hence, passing to vector fields, $L_n\Omega = \overline{L}_n\Omega = 0$ for $n \geq -1$. More precisely, a priori we have $L_n\Omega = \overline{L}_n\Omega \sim \Omega$, but the Lie algebra $W_+ = \text{Vect}_{\text{pol}}(\mathbb{C})$ of polynomial vector fields on $\mathbb{C}$ spanned by $L_n, n \geq -1$ is simple and has no nontrivial 1-dimensional representations, so it follows that $L_n\Omega = \overline{L}_n\Omega = 0$, $n \geq -1$.

The remaining content of the theory is thus contained in the amplitudes $A(X)$, where X is a sphere with two incoming boundaries and one outgoing boundary (pair of pants), since every Riemann surface can be glued out of pairs of pants and disks. The moduli space of such Riemann surfaces is 3-dimensional (over $\mathbb{R}$), since gluing such a surface to itself yields a generic closed Riemann surface of genus 2 with a real structure and three real ovals. To realize such X, we may consider the disk $|z| \leq r_\infty$ for some $r_\infty > 1$ with two deleted disks inside, $|z| \leq r_0$ and $|z-1| \leq r_1$ for some $r_0 + r_1 < 1 < r_\infty - r_1$; see Figure 12.1.

Denote the corresponding surface with standard boundary parametrizations (uniform clockwise starting at 3 o'clock) by $X(r_0, r_1, r_\infty)$. Then it remains

[49] This also implies that the eigenvalues of $L_0 - \overline{L}_0$ on $\mathcal{H}_{\text{fin}}$ are integers, as this is the infinitesimal operator of the rotation group of S^1.

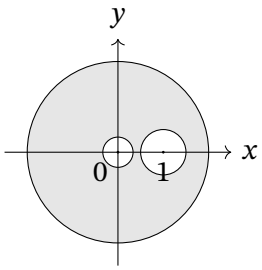

Figure 12.1

to describe the operator $A(X(r_0, r_1, r_\infty)) : \mathcal{H}^{\otimes 2} \to \mathcal{H}$, in fact just the corresponding operator $\mathcal{H}_{\mathrm{fin}}^{\otimes 2} \to \widehat{\mathcal{H}}_{\mathrm{fin}}$, where $\widehat{\mathcal{H}}_{\mathrm{fin}} := \overline{\mathcal{H}}_{\mathrm{fin}}^*$ is the completion of $\mathcal{H}_{\mathrm{fin}}$ with respect to its grading by eigenvalues of H. We denote this operator by $Y(r_0, r_1, r_\infty)$. It is clear that for $\rho_0, \rho_1, \rho_\infty > 0$,

$$\rho_\infty^{-H} Y(r_0, r_1, r_\infty)(\rho_0^H \otimes \rho_1^H) = Y(r_0\rho_0, r_1\rho_1, r_\infty\rho_\infty)$$

whenever both sides make sense. Thus

$$Y(1,1,1) := r_\infty^H Y(r_0, r_1, r_\infty)(r_0^{-H} \otimes r_1^{-H})$$

for $r_0 + r_1 < 1 < r_\infty - r_1$ does not depend on r_0, r_1, r_∞ and makes sense as an operator $\mathcal{H}_{\mathrm{fin}}^{\otimes 2} \to \widehat{\mathcal{H}}_{\mathrm{fin}}$ (albeit it does not extend to $\mathcal{H}^{\otimes 2}$). Also

$$Y(r_0, r_1, r_\infty) = r_\infty^{-H} Y(1,1,1)(r_0^H \otimes r_1^H).$$

So really it remains to describe $Y(1,1,1)$.

In fact, it is more convenient to consider the operator

$$Y(z, \overline{z}) := Y(1,1,1) z^{L_0} \overline{z}^{\overline{L}_0} = Y(1,1,1) r^{L_0 + \overline{L}_0} e^{i\theta(L_0 - \overline{L}_0)},\ z = re^{i\theta} \in \mathbb{C}^\times,$$

as it has an expansion

$$Y(z, \overline{z}) = \sum_{h \in \mathbb{R}, n \in \mathbb{Z}} Y_{h,n} z^{\frac{h+n}{2}} \overline{z}^{\frac{h-n}{2}},$$

where $Y_{h,n} : \mathcal{H}_{\mathrm{fin}}^{\otimes 2} \to \mathcal{H}_{\mathrm{fin}}$ is an operator of weight h with respect to $H = L_0 + \overline{L}_0$ and n with respect to $L_0 - \overline{L}_0$. The operator $Y(z, \overline{z})$ is called the *state-field correspondence operator*: given a state $\psi \in \mathcal{H}_{\mathrm{fin}}$, we have a quantum field

$$Y(\psi, z, \overline{z}) := Y(z, \overline{z})(\psi \otimes \bullet) : \mathcal{H}_{\mathrm{fin}} \to \widehat{\mathcal{H}}_{\mathrm{fin}}.$$

In the next section, we will describe this operator more explicitly in several examples.

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12.13. Rational conformal field theories

One of the simplest yet very interesting classes of 2-dimensional quantum CFT are ones whose genus 1 partition function is a *finite* sum of holomorphic functions of τ multiplied by antiholomorphic ones:

$$\mathcal{Z}(\tau) = \sum_{j=0}^{n} \chi_j(\tau)\overline{\chi_j^\vee(\tau)}.$$

Such theories are called *finitely factorized.* We have already seen several examples of such theories—the circle-valued boson for circle radius $\sqrt{p/q}$, $p, q \in \mathbb{Z}_{\geq 1}$, and the Ising model CFT.

It turns out that the known examples of finitely factorized theories actually have a lot more structure, namely that of a *rational conformal field theory*. This entails the following structures and properties.

1. Vacuum sector. There is a positive energy unitary representation V of Vir with a unit vector v annihilated by L_j, $j \geq -1$ and an isometric embedding $V \otimes \overline{V} \hookrightarrow \mathcal{H}_{\text{fin}}$ of $\text{Vir} \oplus \overline{\text{Vir}}$-modules such that $\Omega = v \otimes \overline{v}$.[50] Moreover, the components of the state-field correspondence map send the vector space $(V \otimes \overline{V}) \otimes (V \otimes \overline{V})$ to $V \otimes \overline{V}$, namely

$$Y(z, \overline{z})|_{(V\otimes\overline{V})\otimes(V\otimes\overline{V})} = \mathbb{Y}(z) \otimes \overline{\mathbb{Y}(z)},$$

where $\mathbb{Y} : V \otimes V \to V((z))$. The Segal axioms then imply that the map $\mathbb{Y}$ equips V with the structure of a *conformal vertex algebra* (Definition 12.12).

Moreover, the state-field correspondence map makes the space $\mathcal{H}_{\text{fin}}$ into a bimodule over the vertex algebra V—we have commuting actions of V and $\overline{V}$ on $\mathcal{H}_{\text{fin}}$ in the sense of the following definition.

Definition 12.19. Let $(V, \mathbb{Y}, v, T, \omega)$ be a conformal vertex algebra as in Definition 12.12. A (*conformal*) V*-module* is a triple $(M, \mathbb{Y}^M, T_M)$ where M is a complex vector space, $T_M \in \text{End}(M)$ is a linear operator, and

$$\mathbb{Y}^M(\,\cdot\,, z) : V \otimes M \to M((z)), \qquad a \otimes \mathbf{m} \mapsto \mathbb{Y}^M(a, z)\mathbf{m} = \sum_{n\in\mathbb{Z}} a^M_{(n)}\mathbf{m}\, z^{-n-1},$$

$a^M_{(n)} \in \text{End}(M)$, such that the following axioms hold:

(1) *Vacuum axiom.*

$$\mathbb{Y}^M(v, z) = 1.$$

(2) *Translation covariance.*

$$[T_M, \mathbb{Y}^M(a, z)] = \partial_z \mathbb{Y}^M(a, z).$$

[50] Here $\overline{V}$ is V viewed as a representation of $\overline{\text{Vir}}$, and $\overline{v} \in \overline{V}$ is the vector corresponding to v.

(3) *Locality.* For all $a, b \in V$ there exists an integer $N \geq 0$ such that

$$(z-w)^N\big[\mathbb{Y}^M(a,z), \mathbb{Y}^M(b,w)\big] = 0$$

as a formal series in $\mathrm{End}(M)[[z^{\pm 1}, w^{\pm 1}]]$.

(4) *Conformal structure.* The field corresponding to ω on M has the form

$$\mathbb{Y}^M(\omega, z) = T^M(z) = \sum_{n\in\mathbb{Z}} L_n^M z^{-n-2},$$

and the operators L_n^M satisfy the Virasoro commutation relations with central charge c (the same c as for V):

$$[L_m^M, L_n^M] = (m-n)L_{m+n}^M + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}.$$

Moreover, $T_M = L_{-1}^M$ and

$$M = \bigoplus_{h\in\mathbb{C}} M_h, \qquad M_h = \{m \in M \mid L_0^M m = hm\},$$

with $M_h = 0$ for $\mathrm{Re}(h) \ll 0$.

2. Finite factorization. In the literature one can find several slightly different definitions of a rational CFT. One of the possible definitions is as follows.

Definition 12.20. A CFT with vacuum sector as in axiom (1) is said to be *rational* if there is a finite collection of irreducible V-modules $V_0 = V, V_1, \ldots, V_n$, such that we have an orthogonal decomposition of V-bimodules

$$\mathcal{H}_{\mathrm{fin}} = \oplus_{i,j=0}^n m_{ij} V_i \otimes \overline{V}_j,\ m_{ij} \in \mathbb{Z}_+,$$

and the state-field correspondence map $Y(z, \overline{z})$ is a finite sum of tensor products of operators

$$\mathbb{Y}_{ij}^k(z) = \sum_s \mathbb{Y}_{ij}^{k,s} z^s,\ \mathbb{Y}_{ij}^{k,s} : V_i \otimes V_j \to V_k,$$

with their complex conjugates. The summands $V_i \otimes \overline{V}_j$ are called the *sectors*. The operators $\mathbb{Y}_{ij}^k(z)$ are called the *chiral vertex operators*.

Thus for a rational CFT the genus 1 partition function is

$$\mathcal{Z}(\tau) = \sum_{i,j=0}^n m_{ij}\,\mathrm{ch}\,V_i(e^{2\pi i\tau})\overline{\mathrm{ch}\,V_j(e^{2\pi i\tau})},$$

i.e., the theory is finitely factorized. Moreover, since this function must be real for $\tau \in i\mathbb{R}$ (as the corresponding genus 1 Riemann surface X is isomorphic to $\overline{X}$), we have $m_{ij} = m_{ji}$.

Also, since $\mathcal{Z}$ must be modular invariant, the linear span of $\mathrm{ch}\,V_i$ is a finite-dimensional projective representation ρ of $\mathrm{PSL}(2,\mathbb{Z})$ with an invariant Hermitian form defined by the matrix m_{ij}. This representation ρ (in particular,

its matrices S and T in the basis of ch V_i) is an important invariant of rational conformal field theory.

Example 12.21. One of the most basic examples of a rational CFT (with central charge $c = 1$) is the theory of circle-valued boson of Section 12.10 with rational squared radius, $r^2 = \frac{p}{q}$. In this case, the conformal blocks comprising the state-field correspondence map are given by the suitable vertex operators $X(\alpha, z)$. For example, if $r = 1$, then we have

$$\mathcal{H}_{\text{fin}} = V_0 \otimes \overline{V}_0 \oplus V_1 \otimes \overline{V_1},$$

where V_0, V_1 are the integrable irreducible highest weight representations of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ at level 1. As explained in Section 12.10, this is the simplest nontrivial example of a Wess-Zumino-Witten model.

Another simple example of a rational CFT is the Ising model CFT (Section 12.11); it has Virasoro central charge 1/2. Further examples include minimal models, as well as Wess-Zumino-Witten models for compact simple Lie groups. In the latter case the corresponding topological QFT is the Chern-Simons theory (Section 11.12). However, a detailed discussion of these models is beyond the scope of this book; it can be found, for instance, in [**DMS**, Chapter 15].

In fact, the representation of the partition function as a finite sum of products of holomorphic and antiholomorphic functions is a key property of a rational CFT which actually holds in all genera. This property is discussed in detail in Exercises 12.22, 12.23, and 12.24.

Exercise 12.22. Let M be a connected complex manifold and let $F : M \to \mathbb{R}$ be a function which locally around every point $p \in M$ can be written as

$$F(z, \overline{z}) = \sum_{j=1}^{n} a_j(z)\overline{b_j(z)}, \tag{12.15}$$

where a_j, b_j are holomorphic functions. Show the following.

(i) The space $V(p)$ of germs of holomorphic functions on M near p spanned by a_j is independent of the presentation (12.15), and has dimension n if and only if this presentation is shortest possible; moreover, $\dim V(p)$ is the same for all $p \in M$.

(ii) The functions $b_j(z)$ also span $V(p)$, and $F \in V(p) \otimes \overline{V(p)}$ is a nondegenerate element defining a nondegenerate Hermitian inner product $h_p(,)$ on the space $E(p) := V(p)^*$. Moreover, $E(p)$ contains a unique vector $\sigma(p)$ such that $(\sigma(p), f) = f(p)$ for all $f \in V(p)$, and $F(p) = h_p(\sigma(p), \sigma(p))$.

(iii) The assignment $p \mapsto (E(p), \nabla, h_p, \sigma(p))$ defines a Hermitian holomorphic vector bundle E on M with a flat (pseudo-)unitary connection ∇ and a

holomorphic section σ (in general, not flat). In particular, we obtain a holonomy representation of $\pi_1(M, p)$ on $E(p)$.

(iv) Recall that every Hermitian holomorphic vector bundle on a complex manifold carries a canonical (Chern) connection. Show that for E, this connection is ∇.

(v) Consider the example $M = \mathbb{C}^\times$, $F(z, \overline{z}) = |z|^{2s}$, $s \in \mathbb{R}$. Write $F(z, \overline{z})$ locally as $a(z)\overline{a(z)}$, where $a(z)$ is a branch of z^s. Let $a_*(z)$ be the dual vector to a in $E(z)$ (also defined locally) and show that

$$\nabla a_* = 0,\ h_p(a_*, a_*) = 1,$$

$s(z) := z^s a_*(z)$ is a globally defined holomorphic (but nonflat) section of E, and $F(z, \overline{z}) = h_z(s(z), s(z)) = |z|^{2s}$. Compute the holonomy representation corresponding to F.

Exercise 12.23. Consider the following generalization of Exercise 12.22. Let M be a complex manifold, let $\mathcal{L}$ be a holomorphic line bundle on M, let $s \in \mathbb{R}$, and let F be a section of the real line bundle $|\mathcal{L}|^{2s}$ (with transition functions $|g_{ij}(z)|^{2s}$, where $g_{ij}(z)$ are transition functions of $\mathcal{L}$). Locally near each $p \in M$, let ξ_p be a nonvanishing holomorphic section of $\mathcal{L}$, and suppose that

$$|\xi_p(z)|^{-2s} F(z, \overline{z}) = \sum_{j=1}^{n} a_j(z)\overline{b_j(z)}, \tag{12.16}$$

where a_j, b_j are holomorphic functions (clearly, this property does not depend on the choice of ξ_p); in other words, suppose that F has a representation (12.15) in any holomorphic local trivialization of $\mathcal{L}$. Generalize the results of Exercise 12.22 to this setting. Namely, show the following.

(i) F still uniquely defines a holomorphic vector bundle E on M. However, the Hermitian structure on E is only defined up to positive rescaling in each fiber, and the (flat) connection ∇ is only uniquely defined for the corresponding projective bundle $\mathbb{P}E$; in particular, we only get a representation of $\pi_1(M, p)$ on $\mathbb{P}E(p)$, i.e., a *projective* representation defined up to tensoring with 1-dimensional representations.

(ii) If in addition $\mathcal{L}$ is equipped with a Hermitian structure, then so is E, so it carries the canonical Chern connection ∇. This connection is not flat in general, but its curvature is scalar, namely s times the curvature of the Chern connection of $\mathcal{L}$.

As in Chapter 5, let M_g denote the moduli space of closed Riemann surfaces of genus g. If we did not have conformal anomaly, the partition function of a CFT on $\Sigma \in M_g$ would have been a positive function on M_g. However, due to conformal anomaly, this function is defined only up to positive rescaling; more precisely, it is only well defined as a section of a certain $\mathbb{R}_{>0}$-bundle. It

turns out that this bundle is $|\operatorname{Det}|^c$, where Det is the *determinant bundle* for the $\overline{\partial}$-operator on Σ, i.e.,

$$\operatorname{Det}(\Sigma) = \det H^0(\Sigma, \mathcal{O}) \otimes \det H^1(\Sigma, \mathcal{O})^*.$$

Exercise 12.24. Show that in rational conformal field theory, in every local holomorphic trivialization of Det, the partition function $\mathcal{Z}(\Sigma)$ of the theory can be represented as a finite sum

$$\mathcal{Z}(\Sigma) = \sum_{i=1}^{N_g} \mathcal{Z}_i^L(\Sigma)\overline{\mathcal{Z}_i^R(\Sigma)},$$

where the functions $\mathcal{Z}_i^L(\Sigma)$, $\mathcal{Z}_i^R(\Sigma)$ are holomorphic. Then use Exercise 12.23(i) to construct a natural projective representation $E(\Sigma)$ of the mapping class group $\Gamma_g = \pi_1^{\mathrm{orb}}(M_g)$.[51]

Remark 12.25. The determinant bundle Det on M_g actually has a natural Hermitian metric called the *Quillen metric*, so Exercise 12.22(ii) also applies.

The assignment $\Sigma \mapsto E(\Sigma)$ is called the *modular functor*. It turns out that this functor is a part of data of a 3-dimensional topological quantum field theory, attached canonically to our rational CFT. A modular functor is attached to an algebraic structure called a *modular tensor category*. The theory of modular functors and modular tensor categories originated from the work of Segal and Moore-Seiberg in late 1980s (see [**Se**],[**MS**]),[52] and it is a purely mathematical theory discussed, for example, in [**BK**]. In particular, this theory allows one to prove that the Virasoro central charge of a rational CFT is a rational number (the Anderson-Moore-Vafa theorem, see [**EGNO**, Section 8.18]).

Any modular tensor category gives rise to a (projective) unitary representation of $\mathrm{SL}(2,\mathbb{Z})$ given by two matrices S, T with rows and columns labeled by the simple objects V_i, representing the action of the standard generators of this group. In our situation, V_i are the simple modules over the vertex algebra V_0, and $\mathrm{SL}(2,\mathbb{Z})$ acts by the modular transformation of their characters $\chi_j(\tau)$, as mentioned above. The matrix T is diagonal, with eigenvalues θ_j being roots of unity, which are called the *twists* of V_j; in our case, they are $\theta_j = e^{2\pi i E_j}$, where E_j is the lowest eigenvalue of L_0 on V_j. The matrix S is a unitary matrix called the *S-matrix* of the category. In Section 12.11 we computed the matrices S, T for the Ising model CFT.

It follows that in our setting the matrix $m_{ij} = \delta_{ij}$ is always preserved by S, T, so one can define a CFT with Hilbert $\mathcal{H} = \oplus_{j=0}^n V_j \otimes \overline{V_j}$. Such a theory is called *diagonal*. For example, the Ising CFT is diagonal, as are the

[51] Since M_g is not a manifold but an orbifold, this requires a slight generalization of Exercise 12.23.

[52] See also M. Kontsevich, *Rational Conformal Field Theory and Invariants of 3-Dimensional Manifolds*, Marseille preprint CPT-88/P.2189.

Wess-Zumino-Witten and the minimal models, whereas the theory of Example 12.15 is not. Other interesting examples of nondiagonal models are the Capelli-Itzykson-Zuber theories; see [**Ga**].

Chapter 13

Perturbative expansion for interacting QFT

13.1. General strategy of quantization

We now pass to nonfree field theories defined by the action $S(\phi) := \int \mathcal{L}(\phi)dx$ in Minkowski space $V \cong \mathbb{R}^d$, where $\mathcal{L}(\phi)dx$ is a local Poincaré-invariant Lagrangian density (up to exact differentials). The general strategy of quantization of such theories is as follows.

Step 1. Write down the Euclidean path integral correlators for the theory:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \int \phi(x_1) \dots \phi(x_n) e^{-\frac{S_E(\phi)}{\hbar}} D\phi.$$

Compute the corresponding formal expansion in $\hbar$ using the Feynman rules (as we have done in the case of quantum mechanics, $d = 1$; see Section 6.3).

Step 2. Perform Borel summation of this formal series to obtain actual functions defined for small enough $\hbar > 0$.

Step 3. Perform the Wick rotation of these functions to Minkowski space to obtain Wightman correlation functions W_n.

Step 4. Use the functions W_n to define a Wightman QFT; i.e., extract the Hilbert space $\mathcal{H}$, the representation π of the (double cover of the) Poincaré group on $\mathcal{H}$, the vacuum vector Ω, and the field map ϕ.

All these steps are nontrivial, and while Step 1 can be performed fully rigorously, starting from Step 2 a rigorous implementation is only known for a handful of theories treated in constructive field theory (for many Lagrangians

the ultimate Wightman QFT, in fact, does not exist). For most physically interesting theories, doing these steps rigorously is still an open problem. In this chapter, we will only discuss Step 1.

13.2. The ϕ^3-theory

As a running example, we will use the theory of a scalar boson ϕ with Euclidean Lagrangian

$$\mathcal{L}_E(\phi) := \frac{1}{2}(|\nabla\phi|^2 + m^2\phi^2) - \frac{g}{6}\phi^3,$$

which we will call the *ϕ^3-theory*. This theory is a deformation of the theory of a free scalar boson obtained by adding a single interaction term $-\frac{g}{6}\phi^3$, which in Feynman calculus corresponds to a 3-valent vertex. Physically this vertex corresponds to an interaction in which two particles collide and transform into a third one.

We will set $\hbar = 1$ (which can be done by rescaling ϕ and g) and consider the formal expansion of correlation functions in powers of g (which is equivalent to Step 1).

Let us compute the 1-loop correction to the 1-particle irreducible 2-point correlation function of the free theory

$$\widehat{G}_0(p) = \frac{1}{p^2 + m^2}$$

in the momentum space presentation. It is easy to see that this correction is given by a single Feynman diagram shown in Figure 13.1.

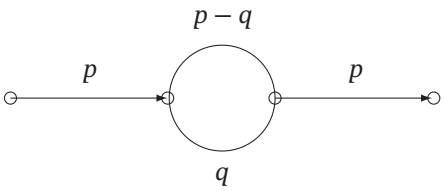

Figure 13.1

According to the Feynman rules (see Section 6.3), the amplitude of this Feynman diagram is

$$A(p) = \frac{g^2}{2(p^2+m^2)^2}\Sigma_1(p^2),\ \Sigma_1(p^2) := (2\pi)^{-d}\int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)}$$

(thanks to rotational symmetry, this integral depends only on p^2). The expression $\frac{g^2}{2}\Sigma_1(p^2)$ is the 1-loop approximation to the *self-energy* $\Sigma(p^2)$, which is the sum of amplitudes of all amputated 1-particle irreducible Feynman diagrams with two external edges. The meaning of self-energy is that the (connected)

momentum-space 2-point function (the exact propagator) equals $\frac{1}{p^2+m^2-\Sigma(p^2)}$. Thus the exact squared mass m^2_{ex} of the quantum particle would be computed as the pole of the exact propagator, i.e., from the equation

$$-m^2_{\text{ex}} + m^2 - \Sigma(-m^2_{\text{ex}}) = 0.$$

In particular, the 1-loop approximation m_1^2 to the exact squared mass is given by

$$m_1^2 = m^2 - \frac{g^2}{2}\Sigma_1(-m^2) + O(g^4)$$

(as $m_1^2 = m^2 + O(g^2)$).

If $d < 4$, the integral defining $\Sigma_1(p^2)$ is convergent and can be computed explicitly. To this end, we may use the following simple lemma from multivariable calculus, which is known in physics literature as the *Feynman's famous formula*:

Lemma 13.1. *Let Δ_n be the $(n-1)$-dimensional simplex defined in $\mathbb{R}^n$ by the equation*

$$y_1 + \cdots + y_n = 1,$$

and let dy be the Lebesgue measure on Δ_n of volume 1. Then for positive numbers $a_1, \dots, a_n$ we have

$$\int_{\Delta_n} \frac{dy}{(a_1y_1 + \cdots + a_ny_n)^n} = \frac{1}{a_1 \dots a_n}.$$

Proof. We have

$$\frac{1}{(a_1y_1 + \cdots + a_ny_n)^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-(a_1y_1+\cdots+a_ny_n)t}dt.$$

So we get

$$\begin{aligned}
\int_{\Delta_n} \frac{dy}{(a_1y_1 + \cdots + a_ny_n)^n} &= \frac{1}{(n-1)!}\int_{\Delta_n}\int_0^\infty t^{n-1}e^{-(a_1y_1+\cdots+a_ny_n)t}dtdy \\
&= \int_0^\infty \frac{1}{(n-1)!}\int_{\Delta_n} t^{n-1}e^{-(a_1y_1+\cdots+a_ny_n)t}dydt \\
&= \frac{1}{(n-1)!}\int_0^\infty \int_{t\Delta_n} e^{-a_1z_1-\cdots-a_nz_n}dzdt \\
&= \int_{z_1,\dots,z_n\geq 0} e^{-a_1z_1-\cdots-a_nz_n}dz = \prod_{j=1}^n \int_0^\infty e^{-a_jz_j}dz_j = \frac{1}{a_1 \dots a_n}. \qquad \square
\end{aligned}$$

Applying Feynman's famous formula to our integral and making a change of variable $q \mapsto q + (1-y)p$, we have

$$\int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} = \int_0^1 \int_V \frac{dq}{((1-y)q^2+y(p-q)^2+m^2)^2} dy$$
$$= \int_0^1 \int_V \frac{dq}{(q^2+M^2(y,p))^2} dy,$$

where

$$M^2(y,p) := y(1-y)p^2 + m^2.$$

Now, using spherical coordinates

$$\int_V \frac{dq}{(q^2+M^2)^2} = C_d \int_0^\infty \frac{r^{d-1}dr}{(r^2+M^2)^2},$$

where C_d is the area of the unit sphere in $\mathbb{R}^d$. Thus for $d = 2$,

$$\int_V \frac{dq}{(q^2+M^2)^2} = 2\pi \int_0^\infty \frac{rdr}{(r^2+M^2)^2} = \pi \int_0^\infty \frac{ds}{(s+M^2)^2} = \frac{\pi}{M^2}.$$

It follows that

$$\Sigma_1(p^2) = \frac{1}{(2\pi)^2} \int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} = \frac{1}{4\pi} \int_0^1 \frac{dy}{y(1-y)p^2+m^2}$$
$$= \frac{1}{\pi p^2 \sqrt{\frac{4m^2}{p^2}+1}} \operatorname{arccotanh} \sqrt{\frac{4m^2}{p^2}+1}.$$

Thus the 1-loop approximation m_1^2 to the squared mass of the quantum particle is computed from the equation

$$m_1^2 = m^2 - \frac{g^2}{2} \cdot \frac{1}{\pi m^2 \sqrt{3}} \operatorname{arccotan} \sqrt{3} = m^2(1 - \tfrac{1}{12\sqrt{3}}(\tfrac{g}{m^2})^2).$$

The case $d = 3$ is computed similarly. In this case

$$\int_V \frac{dq}{(q^2+M^2)^2} = 4\pi \int_0^\infty \frac{r^2 dr}{(r^2+M^2)^2} = \frac{\pi^2}{M}.$$

It follows that

$$\Sigma_1(p^2) = \frac{1}{(2\pi)^3} \int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)}$$
$$= \frac{1}{8\pi} \int_0^1 \frac{dy}{\sqrt{y(1-y)p^2+m^2}} = \frac{1}{4\pi|p|} \operatorname{arccotan}(\tfrac{2m}{|p|}).$$

Thus the 1-loop approximation m_1^2 to the squared mass of the quantum particle is computed from the equation

$$m_1^2 = m^2 - \frac{g^2}{2} \cdot \frac{1}{4\pi m} \operatorname{arccotanh}(2) = m^2(1 - \frac{\log 3}{16\pi}(\frac{g}{m^{3/2}})^2) + O(g^4).$$

However, for $d = 4$ we encounter our first difficulty: the integral logarithmically diverges (as the integrand behaves at infinity as $|q|^{-4}$). More specifically, for a *momentum cutoff* Λ, define

$$A_\Lambda(p) := \frac{g^2}{2(p^2+m^2)^2}(2\pi)^{-4} \int_{|q|\le\Lambda} \frac{dq}{(q^2+m^2)((p-q)^2+m^2)},$$

the integral over the ball in V of radius Λ. Then

$$A_\Lambda(p) \sim \frac{g^2}{16\pi^2(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}),\ \Lambda \to \infty,$$

for $d = 4$ and

$$A_\Lambda(p) \sim \frac{C_d}{(2\pi)^d} \frac{g^2}{2(d-4)(p^2+m^2)^2} \Lambda^{d-4},\ \Lambda \to \infty,$$

if $d > 4$. A way to remedy this difficulty is to add a Λ-dependent term in the Lagrangian, called a *counterterm*, which blows up as $\Lambda \to \infty$, and thus cancels this divergence in the sense that when integration is performed over the ball $|q| \le \Lambda$, then the integral has a finite limit as $\Lambda \to \infty$.

For example, consider $d = 4$. In this case modulo g^3 the momentum space 2-point function computed with cutoff Λ looks like

$$\widehat{G}_{\Lambda,m^2}(p) = \frac{1}{p^2+m^2} + \frac{g^2}{16\pi^2(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}) + \dots$$

(here we explicitly indicate dependence of $\widehat{G}$ on m^2 since we are about to vary it). Let us try to fix the divergence by replacing the parameter m^2 by $m^2 + Kg^2 \log(\frac{\Lambda}{\Lambda_0})$ for a constant K, where Λ_0 is a fixed scale of momenta introduced to have a dimensionless quantity under the logarithm.

So we have

$$\widehat{G}_{\Lambda,m^2+Kg^2\log(\frac{\Lambda}{\Lambda_0})}(p) = \frac{1}{p^2+m^2+Kg^2\log(\frac{\Lambda}{\Lambda_0})} + \frac{g^2}{16\pi^2(p^2+m^2)^2}\log(\tfrac{\Lambda}{m}) + \dots$$

$$= \frac{1}{p^2+m^2} + (\tfrac{1}{16\pi^2} - K)\frac{g^2}{(p^2+m^2)^2}\log(\tfrac{\Lambda}{\Lambda_0}) + \frac{g^2}{16\pi^2(p^2+m^2)^2}\log(\tfrac{\Lambda_0}{m}) + \dots,$$

where we ignore terms of order higher than g^2. Thus to cancel the divergence, we should take $K = \frac{1}{16\pi^2}$, i.e., replace the Lagrangian with

$$\mathcal{L}_{E,\Lambda} := \frac{1}{2}(|\nabla\phi|^2 + (m^2 + \tfrac{g^2}{16\pi^2}\log(\tfrac{\Lambda}{\Lambda_0}))\phi^2) - \frac{g}{6}\phi^3.$$

For this Lagrangian, if integration is performed with cutoff Λ, then the 2-point function modulo g^3 will have a finite limit as $\Lambda \to \infty$, given by

$$\widehat{G}(p) = \frac{1}{p^2+m^2} + \frac{g^2}{2(p^2+m^2)^2}\Sigma_1(p),$$

where

$$\Sigma_1(p) = \frac{1}{(2\pi)^4} \lim_{\Lambda\to\infty} \left(\int_{|q|\le\Lambda} \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} - 2\pi^2 \log(\tfrac{\Lambda}{\Lambda_0}) \right).$$

This limit is easy to compute using Feynman's famous formula. Namely, computing similarly to the $d<4$ case, we get

$$\Sigma_1(p) = \int_0^1 I(p,y)dy,\ I(p,y) := \frac{1}{8\pi^2} \lim_{\Lambda\to\infty} \left(\int_0^\Lambda \frac{r^3 dr}{(r^2+M^2(y,p))^2} - \log(\tfrac{\Lambda}{\Lambda_0}) \right).$$

So

$$I(p,y) = \frac{1}{8\pi^2} \left(\log(\tfrac{\Lambda_0}{m}) - \tfrac{1}{2}(1+\log(y(1-y)\tfrac{p^2}{m^2}+1)) \right).$$

Thus

$$\Sigma_1(p) = \frac{1}{8\pi^2} \left(\log(\tfrac{\Lambda_0}{m}) + \tfrac{1}{2} - \sqrt{\tfrac{4m^2}{p^2}+1} \cdot \operatorname{arccotanh} \sqrt{\tfrac{4m^2}{p^2}+1} \right).$$

For $d>4$ the calculation becomes a bit more elaborate. Namely, while for $d=5$, we have

$$A_\Lambda(p) \sim C_5 \frac{g^2}{2(p^2+m^2)^2}\Lambda + O(1),\ \lambda\to\infty,$$

so the procedure is the same, with mass modification $m^2 \mapsto m^2 + K\Lambda$, already for $d=6$ we will have to take a deeper expansion of the divergent integral,

$$A_\Lambda(p) \sim \frac{g^2}{4(p^2+m^2)^2}(C_6\Lambda^2 + Cp^2\log(\tfrac{\Lambda}{m}) + O(1)),\ \Lambda\to\infty.$$

We can cancel the most singular term $C_6\Lambda^2$ by means of the mass modification $m^2 \mapsto m^2 + K\Lambda^2$, but after that we will still have logarithmic divergence, $Cp^2\log(\frac{\Lambda}{m})$, which depends on p. To kill this divergence, we must modify the coefficient of $\frac{1}{2}|\nabla\phi|^2$ in the Lagrangian by a counterterm, changing it from 1 to

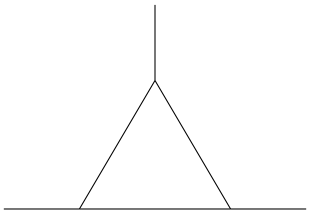

Figure 13.2

$1 + C'g^2 \log(\frac{\Lambda}{\Lambda_0})$ for an appropriate constant C'. Also we find that the 1-loop correction to the 3-point function coming from the Feynman diagram (in Figure 13.2) is logarithmically divergent: the corresponding contribution in the momentum presentation is, up to scaling,

$$\frac{g^3}{\prod_{j=1}^3 (p_j^2 + m^2)} J(p_1, p_2, p_3)\delta(p_1 + p_2 + p_3),$$

where

$$J(p_1, p_2, p_3) = (2\pi)^{-d} \int_V \frac{dq}{((q-p_1)^2 + m^2)((q-p_2)^2 + m^2)((q-p_3)^2 + m^2)}$$

for $p_1 + p_2 + p_3 = 0$, which is divergent and asymptotically proportional to $\log \Lambda$ when computed over the ball of radius Λ. So to kill this divergence, we must change the coefficient of ϕ^3 in the Lagrangian by a counterterm, changing it from $-\frac{g}{6}$ to $-\frac{g}{6} + C''g^3 \log(\frac{\Lambda}{\Lambda_0})$.

We are starting to see the main idea of *renormalization theory*, which allows us to regularize divergent integrals coming from Feynman diagrams in all orders of perturbation series. The idea is that the coefficients of the Lagrangian are actually **not** meaningful physical quantities—they are just mathematical parameters depending on the scale (cutoff) Λ at which we are doing the computation, and they may blow up when $\Lambda \to \infty$ (called the *ultraviolet limit*, as Λ has the meaning of frequency of oscillation). Rather, the meaningful quantity is the answer, the correlation functions $\langle \phi(x_1) \dots \phi(x_n) \rangle$ (or their Fourier transforms, if we work in the momentum realization), and related quantities, e.g., free energies and exact masses of particles. This answer depends on some parameters, which are the actual parameters of the theory. So the coefficients in the Lagrangian must be adjusted in such a way that the answer has a finite limit as $\Lambda \to \infty$. The specific answer we will get will depend on the adjustment procedure, but in good cases (called *renormalizable*) will lie in a nice universal family (often, but not always depending on finitely many parameters).

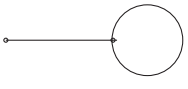

Figure 13.3

Remark 13.2. Note that the 1-particle irreducible 1-point function of the ϕ^3-theory requires renormalization for any $d \geq 2$. For example, for $d = 2$ the amplitude of the 1-loop tadpole diagram in Figure 13.3 (which contributes to the 1-point function) is given by the integral

$$\frac{g}{8\pi^2} \int_{p \in \mathbb{R}^2 : |p| \leq \Lambda} \frac{dp}{p^2 + m^2} = \frac{g}{4\pi} \int_0^\Lambda \frac{rdr}{r^2 + m^2} = \frac{g}{8\pi} \log(1 + \frac{m^2}{\Lambda^2}) + \frac{g}{4\pi} \log(\frac{\Lambda}{m}).$$

Thus to renormalize this divergence, we should add a counterterm and replace the Lagrangian $\mathcal{L} = \frac{1}{2}(|\nabla\phi|^2 + m^2\phi^2) - \frac{g}{6}\phi^3$ with

$$\mathcal{L}(\Lambda) = \frac{1}{2}(|\nabla\phi|^2 + m^2\phi^2) - \frac{g}{6}\phi^3 + \frac{g}{4\pi}\log\Big(\frac{\Lambda}{m}\Big)\phi.$$

Taking the limit $\Lambda \to \infty$, we see that this renormalization would set the 1-point function of ϕ to zero modulo g^2. In fact, this renormalization allows us to ignore all Feynman diagrams with a self-loop, since it defines the contribution of the self-loop to be zero.

The situation is similar in higher dimensions. For example, for $d = 3$ the amplitude of the tadpole diagram is

$$\frac{g}{16\pi^3}\int_{p\in\mathbb{R}^3:|p|\le\Lambda}\frac{dp}{p^2+m^2} = \frac{g}{4\pi^2}\int_0^\Lambda \frac{r^2dr}{r^2+m^2} = \tfrac{gm}{4\pi^2}(\tfrac{\Lambda}{m} - \arctan(\tfrac{\Lambda}{m})),$$

so in order to set the self-loop to zero, for the counterterm we should take $\frac{g}{4\pi^2}(\Lambda - \frac{\pi}{2}m)\phi$.

Exercise 13.3. Generalize the analysis of this section to the ϕ^4*-theory* with Euclidean Lagrangian

$$\mathcal{L}_E(\phi) := \frac{1}{2}(|\nabla\phi|^2 + m^2\phi^2) + \frac{g}{24}\phi^4.$$

(Note that in this theory, because of the symmetry $\phi \to -\phi$, n-point functions vanish for odd n.) Namely:

(i) Show that the 1-loop 2-point function requires renormalization in dimensions $d \ge 2$. For $d = 2$, compute the 1-loop correction to m^2 that sets the self-loop to zero (i.e., the exact mass is $m_1^2 = m^2 + O(g^2)$ in 1-loop approximation). With this correction, compute the self-energy in 2-loop approximation, and the corresponding correction to the exact mass, $m_2^2 = m^2 - cg^2 + O(g^3)$.

(ii) For $d = 3$, compute the 1-loop correction to m^2 that sets the self-loop to zero. Show that even after this correction, the 2-point function has a logarithmic divergence in two loops, owing to the Feynman diagram in Figure 13.4 and compute the 2-loop correction to m^2 of the form $cg^2\log(\Lambda/m)$ making the 2-point function finite in two loops.

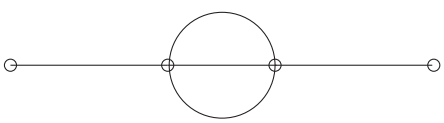

Figure 13.4

(iii) For $d = 4$, show that the 4-point function has a logarithmic divergence in one loop, and compute the 1-loop correction to g of the form $cg^2\log(\Lambda/m)$ that makes the 4-point function finite in one loop.

13.3. Super-renormalizable, renormalizable, and nonrenormalizable theories

Let us now discuss this more systematically. Consider the theory of a scalar boson with a general Lagrangian. Given a Feynman diagram Γ, we have the corresponding Feynman integral I_Γ in the momentum space realization, which is an integral of a rational volume form over a real vector space. We can define the *superficial degree of divergence* $D(\Gamma)$ to be the degree of the numerator of this form (where the differentials of coordinates have degree 1) minus the degree of its denominator. We will say that Γ is *superficially divergent* if $D(\Gamma) \geq 0$ and *superficially convergent* if $D(\Gamma) < 0$.

It is clear that if $D(\Gamma) \geq 0$ (i.e., Γ is superficially divergent), then the integral I_Γ diverges. However, the converse is false (which explains the adjective "superficial"): if $D(\Gamma) < 0$, the integral may still diverge. Indeed, suppose Γ_1, Γ_2 are Feynman diagrams such that Γ_2 is connected with n external edges, and let v be an n-valent internal vertex of Γ_1. Let σ be a bijection between the edges of v and external edges of Γ_2. Then we can form the diagram

$$\Gamma := \Gamma_1 \#_{v,\sigma} \Gamma_2$$

obtained by inserting Γ_2 into Γ_1 instead of v using σ. In this case Γ_2 is said to be a (connected) *subdiagram* of Γ. The Tonelli-Fubini theorem then implies that the Feynman integral I_Γ equals the Feynman integral over Γ_1 with insertion of I_{Γ_2} at v. Thus if Γ contains a subdiagram Γ_2 with $D(\Gamma_2) \geq 0$, then the Feynman integral over Γ diverges even if $D(\Gamma) < 0$ (as I_{Γ_2} is infinite). For example, in the ϕ^3-theory in $d \geq 2$ dimensions, every Feynman diagram containing a self-loop produces a divergent integral, even though many such diagrams have $D(\Gamma) < 0$.

This turns out to be the only obstruction to convergence, however. Namely, one has the following theorem, which plays a central role in the theory of perturbative renormalization.

Theorem 13.4 (Weinberg's theorem)**.** *If every subdiagram of a Feynman diagram Γ is superficially convergent, then the Feynman integral I_Γ corresponding to Γ absolutely converges.*

A detailed proof of this theorem (originally stated by Dyson in 1949 and proved rigorously by Weinberg in 1960) can be found in [**HZi**]; see also [**Col**, Section 5.8].

Let us now compute $D(\Gamma)$. The degree of the denominator is easy to compute. It is just $2e(\Gamma)$ where $e(\Gamma)$ is the number of internal edges of Γ (indeed, every edge contributes a propagator, which is the inverse of a quadratic function). On the other hand, the number of integrations over V is the number of loops, i.e, $d(e(\Gamma) - v(\Gamma) + 1)$, where $v(\Gamma)$ is the number of internal vertices. Finally, the terms in the Lagrangian containing derivatives of ϕ contribute the

number of such derivatives to the degree of the numerator. It follows that

$$D(\Gamma) = (d-2)e(\Gamma) - dv(\Gamma) + d + N,$$

where N is the total number of derivatives in vertex monomials. In particular, when there are no derivatives, we have

$$D(\Gamma) = (d-2)e(\Gamma) - dv(\Gamma) + d.$$

This shows that we may compute $D(\Gamma)$ as a sum of contributions over vertices, defining the degree $D(\Phi)$ of a differential monomial Φ standing at a fully internal vertex (one in which all edges are internal) as the contribution of this vertex to $D(\Gamma)$. Indeed, every Φ contributes

$$D(\Phi) = \frac{d-2}{2}e(\Phi) - d + N_\Phi,$$

where $e(\Phi)$ is the number of edges of Φ (i.e., its degree with respect to ϕ) and N_Φ is the number of derivatives in Φ.

We see that a more natural invariant is

$$[\Phi] := D(\Phi) + d.$$

Definition 13.5. The number $[\Phi]$ is called the *classical scaling dimension* of the differential monomial Φ.

In fact, we have already encountered this definition in Section 11.5 when discussing scaling symmetry. Namely, if we make the change of variable $x \mapsto sx$ in the path integral, the term $g\Phi$ in the Lagrangian is replaced by $gs^{[\Phi]}\Phi$ (see Remark 1.13).

Note that the classical scaling dimension is additive:

$$[\Phi_1\Phi_2] = [\Phi_1] + [\Phi_2].$$

This is not surprising since Φ comes with a volume factor dx, so $D(\Phi)$ is actually the scaling dimension of Φdx; thus to get the scaling dimension of Φ, we need to add d (as $d(a^{-1}x) = a^{-d}dx$, so the scaling dimension of dx is $-d$).

Thus for a Feynman diagram Γ we have

$$D(\Gamma) = d - \frac{k(d-2)}{2} + \sum_\Phi D(\Phi), \tag{13.1}$$

where k is the number of external vertices of Γ.

For example, for $\Phi = \phi^n$ we get

$$D(\phi^n) = \tfrac{n}{2}(d-2) - d = (\tfrac{n}{2} - 1)d - n.$$

Each derivative adds a 1 to the degree, so for instance

$$D(\phi^{n-2}|\nabla\phi|^2) = (\tfrac{n}{2} - 1)(d-2).$$

So for the 1-loop Feynman diagram Γ for the k-point function (a cycle with k legs), we have

$$D(\Gamma) = d - \frac{k}{2}(d-2) + kD(\phi^3) = d - \frac{k}{2}(d-2) + \frac{k}{2}(d-6) = d - 2k.$$

Definition 13.6. Let Φ be an interaction term, i.e., a differential monomial in ϕ. We will say that Φ is *super-renormalizable* if $D(\Phi) < 0$, *renormalizable* (or critical) if $D(\Phi) = 0$, and *nonrenormalizable* if $D(\Phi) > 0$.

Thus super-renormalizable terms improve convergence, renormalizable ones do not affect it, and nonrenormalizable ones worsen it.

Example 13.7.

1. The kinetic term $|\nabla\phi|^2$ has $D = 0$, so it is renormalizable; in fact, this is so by definition in any QFT. Note that this can be used to easily compute the classical scaling dimensions of monomials. Namely, we have $[|\nabla\phi|^2] = D(|\nabla\phi|^2) + d = d$, so $2[\phi] + 2 = d$, i.e., $[\phi] = \frac{d-2}{2}$. Using multiplicativity, we now immediately compute $[\Phi]$ for any Φ.

2. The mass term ϕ^2 has $D = -2$, so it is super-renormalizable. The term ϕ^3 has $D = \frac{1}{2}d - 3$, so it is super-renormalizable for $d < 6$, renormalizable for $d = 6$, and nonrenormalizable for $d > 6$.

Definition 13.8. A Lagrangian is called

• *super-renormalizable* if all its terms except the kinetic term are super-renormalizable;

• *renormalizable* (or critical) if all its terms are at worst renormalizable and there is at least one renormalizable nonkinetic (i.e., interacting) term;

• *nonrenormalizable* if it contains nonrenormalizable terms.

Clearly, every Lagrangian is of exactly one of these three types.

Proposition 13.9.

(i) *If a Lagrangian is super-renormalizable, then the superficial degree of divergence of the corresponding Feynman diagrams*[53] *is bounded above, and there are finitely many superficially divergent diagrams with any given number of external edges. Moreover, if $d > 2$, then there are finitely many superficially divergent diagrams altogether.*

(ii) *If a Lagrangian is renormalizable, then there are infinitely many superficially divergent diagrams with a fixed number of external edges, but the superficial degree of divergence of these diagrams is still bounded above.*

(iii) *If a Lagrangian is nonrenormalizable, then the degree of superficial divergence of diagrams with a fixed number of external edges is unbounded above.*

[53] Here we consider only connected diagrams.

Proof. This is clear from formula (13.1). □

This means that for a nonrenormalizable Lagrangian, regularization of divergent integrals will definitely get out of control. Namely, if we want to regularize diagrams with unbounded above superficial degree of divergence, then we will have to introduce counterterms with an unlimited number of derivatives, and our renormalized Lagrangian will no longer depend on a finite number of derivatives of ϕ.

On the other hand, if the Lagrangian is renormalizable, then for $d > 2$ there are only finitely many terms that we will need to modify in the renormalization procedure; namely, these are the possible super-renormalizable and renormalizable terms in the Lagrangian. The fact that this procedure works to all orders of perturbation theory is a rather nontrivial fact which we will not prove here, but the result is a finite-parametric family of perturbative QFT.

In two dimensions, there is an additional feature—there can be infinitely many (super-)renormalizable terms in the Lagrangian; but they all have at most two derivatives.

Finally, in the super-renormalizable case the renormalization procedure is completed in finitely many steps.

Example 13.10. Consider the ϕ^3-theory in $d = 2$ dimensions. Formula (13.1) yields

$$D(\Gamma) = 2 - 2i,$$

where i is the number of internal vertices of Γ. It follows that the only connected superficially divergent diagram (with $D = 0$) is the tadpole diagram with one internal 3-valent vertex and a self-loop (Remark 13.2). Thus by Weinberg's theorem, the Feynman integral I_Γ for any connected diagram Γ without self-loops is convergent. So once we set the tadpole diagram to zero by adding a linear counterterm in ϕ as done above, i.e., discard all diagrams with self-loops, the corresponding QFT is well defined to all orders of perturbation theory. The same conclusion applies to the ϕ^4 theory in two dimensions considered in Exercise 13.3: after setting the self-loop to zero, the theory is well defined to all orders.

Similarly, in ϕ^3-theory in $d = 3$ dimensions, formula (13.1) yields

$$D(\Gamma) = 3 - \frac{k + 3i}{2},$$

where k is the number of external vertices of Γ. Thus, the only superficially divergent diagrams are those with $k + 3i \leq 6$. We must have $k, i \geq 1$, so the only possibilities are $i = 1$, $k = 1, 3$, but for $k = 3$ there is no integration. So the only superficially divergent diagram is again the tadpole diagram, and once it is set to zero, the theory is well defined to all orders.

On the other hand, in ϕ^4-theory in $d = 3$ dimensions, formula (13.1) yields

$$D(\Gamma) = 3 - \frac{k+2i}{2},$$

so since k has to be even and positive, the only possibilities for superficial divergence are $(k,i) = (2,1),(2,2),(4,1)$, but in the last case there is no integration. Thus the only divergent diagrams are two 2-valent ones with one loop (a self-loop) and two loops (see Exercise 13.3(ii)). After adding the corresponding corrections to m^2 (linear and logarithmic terms in Λ), we may sum over the diagrams which do not contain these two as subdiagrams, and the theory is again well defined in all orders.

For renormalizable theories, the story is a bit more complicated, since there are divergent diagrams with any number of loops, so a superficially divergent diagram Γ may contain superficially divergent subdiagrams. In this case, before rendering the amplitude of Γ finite by adding a counterterm, one must first have this done to all the divergent subdiagrams of Γ. This is somewhat complicated by the fact that a diagram may contain two superficially divergent subdiagrams that intersect each other (*overlapping divergences*). For example, in the ϕ^3 theory in $d = 6$ dimensions, the two 1-loop subdiagrams of the 2-loop diagram in Figure 13.5 are logarithmically divergent and have an edge in common. Nonetheless, the systematic recursive process of making all diagrams finite by adding counterterms still works: once all subdiagrams of Γ have been rendered finite and a suitable counterterm has been added to compensate for the divergence of Γ itself, the amplitude of Γ becomes finite, so we obtain a perturbative QFT well defined to all orders. This stronger version of Weinberg's theorem is discussed in [**Col**].

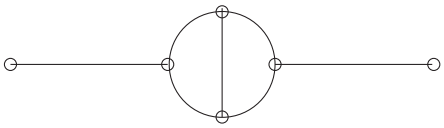

Figure 13.5

13.4. Dimensional regularization

We have seen that giving sense to a perturbation expansion of a QFT involves two steps—*regularization* of Feynman diagram amplitudes (introducing a momentum cutoff Λ) and *renormalization* (adding to the Lagrangian Λ-dependent counterterms which blow up at large Λ and cancel the ultraviolet divergences of the amplitude integrals, and then computing the limit $\Lambda \to \infty$ to extract the perturbative expansion of correlation functions). However, the particular

regularization and renormalization procedure of Section 13.2 is rather cumbersome, as we have to deal with multiple types of counterterms (logarithmic, linear, quadratic in Λ, etc.). It turns out that there exists a more convenient scale-independent procedure, which also introduces a momentum scale μ but produces only μ-independent counterterm coefficients, with μ-dependence showing up only in the renormalized answer. It also has the added benefit of preserving various symmetries, such as gauge symmetry. This is the procedure of *dimensional regularization*, which is based on defining amplitudes of Feynman diagrams in a fictitious "spacetime of complex dimension D" as meromorphic functions of D, and then setting D to equal the actual spacetime dimension d and evaluating the corresponding function, or its regular part if there is a pole (renormalization by *minimal subtraction*). This corresponds to adding a counterterm to cancel the singular part.

Thus, even though, unlike momentum cutoff, dimensional regularization has no direct physical meaning (nor any meaning whatsoever outside perturbation theory), it is a powerful mathematical method to regularize and renormalize perturbative QFT.

Let us now briefly describe how dimensional regularization works, following [**QFS**, pp. 560–562] (for more details see [**QFS**, p. 597]). Suppose we have a Feynman graph Γ with M external edges and momenta $p_1, \dots, p_M$ at $(p_1 + \cdots + p_M = 0)$. Let $q_1, \dots, q_{N-M}$ be the loop variables over which we are integrating. To uniformize the notation, we set $q_{N-M+i} := p_i$. Then the amplitude of the graph is (up to a factor) given by the formula

$$F_\Gamma(\mathbf{p}) = \frac{1}{(2\pi)^{D(N-M)}} \int_{V^{N-M}} \frac{Q(\mathbf{q},\mathbf{p})d\mathbf{q}}{\prod_{j=1}^{n}(\ell_j(\mathbf{q},\mathbf{p})^2 + m_j^2)}, \tag{13.2}$$

where $\mathbf{q} = (q_1, \dots, q_{N-M}), \mathbf{p} = (p_1, ..., p_M)$, $\ell_j : V^N \to V$ are given by linear forms on $\mathbb{R}^N$, $m_j > 0$, and Q is a polynomial in the inner products $q_i \cdot q_k$, $q_i \cdot p_k$ and $p_i \cdot p_k$.

Unfortunately, the integral (13.2) is divergent, so let us explain how to regularize it using dimensional regularization. As does any regularization scheme, dimensional regularization works inductively in the number of loops (or internal vertices) of the graph. So we assume that the coefficients in the Lagrangian have been adjusted at the previous step of the process in such a way that all proper subgraphs of Γ are convergent.

In order to regularize the integral corresponding to Γ, we pretend that it exists and, using the formula

$$\int_0^\infty e^{-at}dt = a^{-1},$$

transform it to the form

$$F_\Gamma(\mathbf{p}) = (2\pi)^{D(M-N)} \int_{V^{N-M}} Q(\mathbf{q},\mathbf{p}) \left(\int_{\mathbf{t}\in[0,\infty)^n} e^{-\sum_{j=1}^n (t_j \ell_j(\mathbf{q},\mathbf{p})^2 + t_j m_j^2)} d\mathbf{t} \right) d\mathbf{q}$$
$$= (2\pi)^{D(M-N)} \int_{\mathbf{t}\in[0,\infty)^n} e^{-\sum_{j=1}^n t_j m_j^2} \left(\int_{V^{N-M}} Q(\mathbf{q},\mathbf{p}) e^{-\sum_{j=1}^n t_j \ell_j(\mathbf{q},\mathbf{p})^2} d\mathbf{q} \right) d\mathbf{t},$$

where $\mathbf{t} := (t_1, ..., t_n)$.

The $\mathbf{q}$-integral in the brackets in the last expression is a Gaussian integral, so it can be computed using Wick's formula. Namely, let

$$\sum_{s=1}^n t_s \ell_s^2(\mathbf{q},\mathbf{p}) = (\mathbf{q},\mathbf{p}) \begin{pmatrix} B(\mathbf{t}) & A(\mathbf{t}) \\ A(\mathbf{t})^T & C(\mathbf{t}) \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix},$$

where B, C are symmetric matrices and $B > 0$. Letting $\mathbf{q}_* := \mathbf{q} + B(\mathbf{t})^{-1} A(\mathbf{t})\mathbf{p}$, we get

$$\sum_{s=1}^n t_s \ell_s^2(\mathbf{q},\mathbf{p}) = \mathbf{q}_*^T B(\mathbf{t}) \mathbf{q}_* + \mathbf{p}^T C_*(\mathbf{t})\mathbf{p},$$

where $C_* := C - A^T B^{-1} A$ is a nonnegative definite matrix (as the LHS is ≥ 0). Now changing variables from $\mathbf{q}$ to $\mathbf{q}_*$ (completion of the square) and computing the Gaussian integral using Wick's formula, we obtain

$$\int Q(\mathbf{q},\mathbf{p}) e^{-\sum_{j=1}^n t_j \ell_j(\mathbf{q},\mathbf{p})^2} d\mathbf{q} = \psi(\mathbf{t}, P, d) e^{-\mathbf{p}^T C_*(\mathbf{t})\mathbf{p}} \det(B(\mathbf{t}))^{-\deg(Q)-\frac{d}{2}},$$

where P is the matrix with entries $p_i \cdot p_j$, $1 \leq i, j \leq M$, ψ is a polynomial in $\mathbf{t}, P, d$, and $\deg(Q)$ is the degree of Q with respect to the variables $q_i \cdot q_j$ (the factor $\det(B(\mathbf{t}))^{-\deg(Q)}$ appears from the $\deg(Q)$ factors of the form $B(\mathbf{t})^{-1}(\cdot,\cdot)$ in the Wick formula, computing the inverse matrix by Cramer's rule).

Note that $\mathbf{p}^T C_*(\mathbf{t})\mathbf{p} = \mathrm{Tr}(PC_*(\mathbf{t}))$. Thus we get

$$F_\Gamma(\mathbf{p}) = (2\pi)^{D(M-N)} \int_{\mathbf{t}\in[0,\infty)^n} e^{-\sum_{j=1}^n t_j m_j^2} \psi(\mathbf{t}, P, d) e^{-\mathrm{Tr}(PC_*(\mathbf{t}))} (\det B(\mathbf{t}))^{-\deg(Q)-\frac{d}{2}} d\mathbf{t}. \quad (13.3)$$

Of course, this integral is also divergent. However, since it no longer contains integration with respect to $\mathbf{q}$ and depends on $\mathbf{p}$ only through P, we can replace d in it with a generic complex number D. It is easy to see that the obtained integral

$$F_\Gamma(P, D) = (2\pi)^{D(M-N)} \int_{\mathbf{t}\in[0,\infty)^n} e^{-\sum_{j=1}^n t_j m_j^2} \psi(\mathbf{t}, P, D) e^{-\mathrm{Tr}(PC_*(\mathbf{t}))} (\det B(\mathbf{t}))^{-\deg(Q)-\frac{D}{2}} d\mathbf{t}$$

is convergent for $\mathrm{Re}\, D \ll 0$ absolutely and uniformly on compact sets, hence defines a holomorphic function of D in this region.

Theorem 13.11. *The function $F_\Gamma(P, D)$ extends to a meromorphic function of $D \in \mathbb{C}$.*

Proof. This follows from J. Bernstein's theorem on the existence of the Bernstein-Sato polynomial, see [**QFS**], v. 1, p.605 and [**C**], p.94. □

We can now define the following renormalization procedure that goes with dimensional regularization, called *minimal subtraction*.

Definition 13.12. $F_\Gamma(\mathbf{p})$ is the value of the regular part of the meromorphic function $F_\Gamma(P, D)$ at $D = d$.

Thus, if the analytic continuation of integral (13.3) is regular at $D = d$, no counterterm is needed, while if it is singular at $D = d$, then we should adjust the Lagrangian by adding a counterterm (infinite at $D = d$) in such a way that the singular part cancels and the regular part remains unchanged. After this we can go to the next step of the process.

Example 13.13. Consider dimensional regularization of the divergent integral discussed in Remark 13.2. Note that when the dimension $d = 2$ of spacetime is deformed into the complex domain, the dimension of the coupling g going with ϕ^3 should also change accordingly: since $|\nabla\phi|^2$ has dimension D, we get that ϕ has dimension $\frac{D-2}{2}$, so for $g\phi^3$ to have dimension D we must have $[g] = D - 3\frac{D-2}{2} = 3 - \frac{D}{2}$. But initially the dimension of g is $3 - \frac{d}{2} = 2$, so to make things physically meaningful, we must replace g with $g\mu^{\frac{2-D}{2}}$, where μ is a reference mass scale. Also, in the linear term in ϕ that we are going to introduce, the dimension of the coefficient is $D - \frac{D-2}{2} = 1 + \frac{D}{2}$, so the counterterm should have the form $cg\mu^{D-2}\phi$, where c is a dimensionless quantity.

Now, following the above algorithm, we have

$$\int_{\mathbb{R}^D} \frac{dp}{p^2 + m^2} = \int_{\mathbb{R}^D} \int_0^\infty e^{-t(p^2+m^2)} dt dp = \int_0^\infty \int_{\mathbb{R}^D} e^{-t(p^2+m^2)} dp dt$$
$$= \pi^{D/2} \int_0^\infty t^{-D/2} e^{-tm^2} dt = m^{D-2}\pi^{D/2}\Gamma(1 - \tfrac{D}{2}).$$

We see that this function is indeed meromorphic in D and has poles at nonnegative even integers. So if d is odd, then the Feynman amplitude of the tadpole diagram in Remark 13.2 requires no renormalization. On the other hand, if $d = 2r$ is even, then using the identity

$$m^{D-2}\pi^{D/2}\Gamma(1 - \tfrac{D}{2}) = (-1)^r m^{D-2}\pi^{D/2} \frac{\Gamma(r + 1 - \frac{D}{2})}{(\frac{D}{2} - 1)\dots(\frac{D}{2} - r)},$$

we see that this function has a simple pole at $D = 2r$ with residue $\frac{2(-1)^r m^{2r-2}\pi^r}{(r-1)!}$. So the counterterm to be introduced is $\frac{(-1)^r \mu^{D-2r} m^{2r-2}\pi^r}{(r-1)!(\frac{D}{2}-r)} g\phi$, which is indeed independent of μ up to the necessary dimension factor μ^{D-2r}. The "integral" giving the amplitude of this diagram computed in dimension $D \in \mathbb{C}$ with this counterterm then has a finite limit as $D \to 2r$, obtained by applying minimal subtraction to the above integral (multiplied by $(2\pi)^{-D}$):

$$(-1)^r m^{2r-2} g \lim_{D\to 2r} \left((\tfrac{m}{\mu})^{D-2r} \frac{2^{-D}\pi^{-D/2}\Gamma(r+1-\frac{D}{2})}{(\frac{D}{2}-1)\dots(\frac{D}{2}-r)} - \frac{2^{-2r}\pi^{-r}}{(r-1)!\,(\frac{D}{2}-r)} \right)$$
$$= (-1)^r m^{2r-2} g \frac{\mathrm{d}}{\mathrm{d}D}\Big|_{D=2r} (\tfrac{m}{\mu})^{D-2r} \frac{2^{-D}\pi^{-D/2}\Gamma(r+1-\frac{D}{2})}{(\frac{D}{2}-1)\dots(\frac{D}{2}-r+1)}.$$

Obviously, this expression is logarithmic with respect to μ.

Remark 13.14. In higher loops the amplitude can have a pole of higher order at $D = d$, which may produce μ-dependent singular terms in the Laurent expansion, e.g.,

$$\frac{(\frac{\mu}{\mu_0})^{D-d}}{(D-d)^2} = \frac{1}{(D-d)^2} + \frac{\log(\frac{\mu}{\mu_0})}{D-d} + O(1).$$

But this dependence disappears when one uses the corresponding coupling $g = g(\mu)$ suitably renormalized at the previous stages of the process to keep correlation functions μ-independent (see Section 13.8).

Remark 13.15. In perturbation expansions of QFT containing classically massless fields, there can also be another kind of divergences in Feynman amplitude integrals, which arise for small, rather than large, values of the internal momenta q. For example, if ϕ is a massless scalar field, then the propagator for ϕ is $\frac{1}{p^2}$, so, e.g., the integral corresponding to the 1-loop diagram in the ϕ^3 theory with two external edges is

$$\int_{\mathbb{R}^d} \frac{dq}{q^2(p-q)^2},$$

and it diverges logarithmically near $q = 0$ and $q = p$ in $d = 2$ dimensions. Such divergences are called *infrared divergences*. These divergences usually cancel when one computes physically meaningful quantities; we will not discuss them here and instead refer the reader to [**PS**] and other textbooks.

13.5. Critical dimensions of some important QFT

For interacting QFT defined by Lagrangians, the theory is only (super-)renormalizable in small dimensions, and becomes nonrenormalizable when the dimension grows. If a theory is renormalizable in some dimension d and nonrenormalizable for bigger dimensions, we say that d is the *critical dimension* of the theory. Let us now discuss critical dimensions of the most important types of QFT.

13.5.1. Scalar bosons and nonlinear sigma models. Since for a scalar boson,

$$D(\phi^n) = (\tfrac{n}{2} - 1)d - n,$$

a term ϕ^n is (super-)renormalizable iff

$$d \leq \frac{2n}{n-2}.$$

Thus, in a (super-)renormalizable theory, the term ϕ^3 can be present only for $d \leq 6$, ϕ^4 only for $d \leq 4$, ϕ^5 and ϕ^6 only for $d \leq 3$. Also, since $D(\phi^{n-2}|\nabla\phi|^2) = (\frac{n}{2} - 1)(d-2)$, such terms with $n > 2$ cannot be present in a (super-)renormalizable theory unless $d = 2$. With more derivatives things get even worse. So we obtain

Proposition 13.16. *For a scalar ϕ, the most general (super-)renormalizable non-quadratic Poincaré-invariant Lagrangian is (up to scaling):*

- $d > 6$*: none,*
- $d = 5, 6$*:* $\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + P_3(\phi)$ *(the ϕ^3-theory),*
- $d = 4$*:* $\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + P_4(\phi)$ *(the ϕ^4-theory),*
- $d = 3$*:* $\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + P_6(\phi)$ *(the ϕ^6-theory),*
- $d = 2$*:* $\mathcal{L} = \frac{1}{2}g(\phi)(\nabla\phi)^2 + U(\phi)$*,*

where P_m is a polynomial of degree m, and U and g are arbitrary (real analytic) functions.

Note that without loss of generality, one may assume that P_m are missing the constant and linear terms. Thus the number of parameters for the theory with Lagrangian $\frac{1}{2}(\nabla\phi)^2 + P_m(\phi)$ is $m - 1$ (the coefficients of P_m).

Similarly, the above computations imply that the nonlinear sigma model (Section 11.1) for a nonflat metric on the target manifold M is renormalizable only in dimension $d = 2$, but in this case the metrics g_{ij} on X, G_{kl} on M, and the potential U can be arbitrary.

13.5.2. Fermions. Recall that for a fermionic field ψ the kinetic term looks like $(\psi, \mathbf{D}\psi)$. This implies that

$$2[\psi] + 1 = d,$$

i.e.,

$$[\psi] = \frac{d-1}{2},$$

which is always positive. So for mass terms $(\psi, M\psi)$ we have $D = -1$ and they are super-renormalizable. Beyond quadratic, we see that the only possible (super-)renormalizable terms in ψ for $d \geq 2$ are of the general shape ψ^{2k}, and

$$D(\psi^{2k}) = 2k[\psi] - d = k(d-1) - d = (k-1)(d-1) - 1.$$

The only case when this is (super-)renormalizable is $d = 2$ and $k = 2$, i.e., the term of schematic form ψ^4, in which case $D = 0$ (critical). Such terms indeed occur in the so-called *Gross-Neveu model*.

For $d > 2$, any fermionic term in a renormalizable Lagrangian must therefore be quadratic in the fermions. But it can contain other (bosonic) fields as factors. For example, $[\phi^n \psi^2] = n\frac{d-2}{2} + d - 1$, so

$$D(\phi^n \psi^2) = n\frac{d-2}{2} - 1.$$

This shows that in three dimensions we can have a term $\phi\psi^2$ (Yukawa interaction) and $\phi^2\psi^2$, while in four dimensions we can have only the Yukawa term $\phi\psi^2$, and for $d > 4$ there are no possible (super-)renormalizable terms.

13.5.3. Gauge theory. Recall from Section 10.7 that to define a gauge theory on $\mathbb{R}^d$, we need to fix a compact Lie group G (for example, $U(n)$) and the field is a 1-form A on $\mathbb{R}^d$ with values in $\mathfrak{g} = \operatorname{Lie} G$ which we view as a connection $\nabla_A = d + A$ on a trivial principal G-bundle P on V. The curvature of ∇_A is given by the formula

$$F_A = dA + \frac{1}{2}[A, A],$$

and the Lagrangian of the pure gauge theory is

$$\mathcal{L} = \frac{1}{2}|F_A|^2.$$

The field A is a vector under the action of $\mathbf{P}$, but for the purposes of dimension count we may treat it as a collection of scalars. Thus the degree of A is

$$[A] = \frac{d-2}{2}.$$

Also, if G is nonabelian, then the nonquadratic terms in the Lagrangian are of schematic form $A^2 dA$ and A^4, so their degrees are $\frac{1}{2}(d-4)$ and $d-4$, respectively. Thus we see that nonabelian gauge theory is critical in dimension 4 (the

physical case!), super-renormalizable in lower dimensions, but nonrenormalizable for $d > 4$.

Note also that for $d \leq 4$, we can also consider (super-)renormalizable Lagrangians with terms $(\nabla_A \phi)^2$ or $(\psi, \mathbf{D}_A \psi)$, where ϕ is a scalar and ψ is a spinor with values in the associated bundle $P \times_G \rho$, where ρ is a finite-dimensional representation of G (it is easy to check that all occurring monomials have $D \leq 0$). Such terms do occur in the *standard model*; the simplest case is $(\psi, \mathbf{D}_A \psi)$ where A is a $U(1)$-connection and ψ is a spinor valued in the tautological representation of $U(1)$, corresponding to an electron, which occurs in quantum electrodynamics.

13.5.4. Gravity. The theory of gravity (general relativity) is a theory of a bosonic field $h(x)$ taking values in symmetric tensors $S^2 V^*$; i.e., the Minkowskian metric on V is perturbed by setting $g = g_0 + h$, where g_0 is the standard Minkowskian metric. The Lagrangian of general relativity is

$$\mathcal{L} = R(g),$$

where R is the scalar curvature of the metric g. Since curvature is expressed in terms of second derivatives of the metric, up to scaling this can be schematically written in terms of h as

$$\mathcal{L} = (\nabla h)^2 + \dots,$$

where the dots stand for terms having at most two derivatives in h. Thus the general shape of this Lagrangian (for the purposes of computing classical scaling dimensions) is the same as for the sigma model; so this theory is only renormalizable in two dimensions. This is one of the main reasons why it has not yet been possible to incorporate gravity into the standard model, which lives in four spacetime dimensions.

Remark 13.17. We have seen in Section 10.16 that even in a free quantum field theory, the composite operators such as $\phi^2(x)$ are not automatically defined, and they require a normal ordering procedure to regularize them. This is all the more so in an interacting QFT.

It turns out that the normal ordering procedure, composite operators, and operator product expansion in a critical perturbative QFT can be defined analogously to the free case, using renormalization theory. We will not discuss it here and refer the reader to [**QFS**, vol. 1, p. 452].

13.6. Faddeev-Popov ghosts in gauge theory

Perturbative renormalization of gauge theories is complicated by the fact that the path integral is taken not over the space of connections Conn on a given topological G-bundle P over the spacetime X, but rather over its quotient $\mathrm{Conn}/\mathcal{G}$ by the gauge group $\mathcal{G}$, which is a nonlinear object. To compute the path

integral, we need to introduce coordinates on $\mathrm{Conn}/\mathcal{G}$, i.e., generically identify $\mathrm{Conn}/\mathcal{G}$ with an affine space. As we have seen above (Section 10.7, Remark 10.8), a convenient way to do so is to use a gauge fixing procedure—a surjective affine linear map $\nu : \mathrm{Conn} \to \mathrm{Lie}\,\mathcal{G}$ whose zero fiber $\nu^{-1}(0)$ is a transversal slice to the orbits of the $\mathcal{G}$-action, i.e., intersects a generic orbit transversally at exactly one point. Then we get a generic identification of $\mathrm{Conn}/\mathcal{G}$ with the affine space $\nu^{-1}(0)$.

To see how this helps compute the path integral, consider the following finite-dimensional model. Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$ and Haar measure dk of unit volume; its evaluation at $1 \in K$ yields a volume element du on $\mathfrak{k}$. Let Z be a manifold with a volume element dz and a free volume-preserving action of K. Let $Z^\circ \subset Z$ be an open subset, let $\nu : Z^\circ \to \mathfrak{k}$ be a smooth map such that 0 is its regular value, and assume that $S := \nu^{-1}(0)$ is a slice for the K-action, i.e., intersects each K-orbit in Z transversally at exactly one point. For $z \in Z$ let $D_z : \mathfrak{k} \to \mathfrak{k}$ be the invertible linear map defined by the formula

$$D_z b := \frac{d}{dt}|_{t=0} \nu(e^{tb} z);$$

we assume that $\det D_z > 0$. Since for $s \in S$ we have a surjective linear map $d\nu_s : T_s Z \to \mathfrak{k}$ whose kernel is $T_s S$, there is a unique volume element ds on S such that $dz = ds \cdot \nu^*(du)$.

Let $\pi : Z \to Z/K \cong S$ be the natural map. Another volume element on S is $\pi_* dz$. It is easy to check that $\pi_* dz = \det D_s ds$. Thus we obtain

Proposition 13.18. *Let f be a K-invariant compactly supported continuous function on Z. Then*

$$\int_Z f(z) dz = \int_{Z/K} f \pi_* dz = \int_S f(s) \det D_s ds.$$

We will need the following more general version of Proposition 13.18. Let $h \in C(\mathfrak{k})$ be a bump function supported in a small neighborhood of the origin, such that $\int_{\mathfrak{k}} h(u) du = 1$.

Proposition 13.19. *We have*

$$\int_Z f(z) dz = \int_Z f(z) h(\nu(z)) \det D_z dz.$$

Note that Proposition 13.18 is obtained from Proposition 13.19 by the limit $h(u) \to \delta(u)$.

Proof. Let

$$I := \int_Z f(z) h(\nu(z)) \det D_z dz.$$

Making a change of variable $z = ks \mapsto (k, s)$, $k \in K, s \in S$, and using that it transforms dz into $dk \cdot \pi_* dz$, we get

$$I = \int_{K \times S} f(s) h(\nu(ks)) \det \frac{\partial \nu(ks)}{\partial k} dk \cdot \pi_* dz.$$

Let us first do the K-integral (i.e., integrate over K-orbits). To this end, make a change of variable $(s, k) \mapsto (s, u = \nu(ks))$ (a diffeomorphism on the support of h). We obtain

$$I = \int_S f(s) \left(\int_{\mathfrak{k}} h(u) du \right) \pi_* dz. = \int_{Z/K} f(z) \pi_* dz = \int_Z f(z) dz.$$

□

Example 13.20. Let $Z = \mathbb{C}^\times$ with complex coordinate $z = x + iy$, $Z^\circ = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, $K = U(1)$ acting on Z by multiplication, $\nu(z) = \operatorname{Im} z = y$, so $\nu^{-1}(0) = \mathbb{R}_{>0}$, with coordinate x. Let $dxdy$ be the standard area element in the plane. Then we have $ds = 2\pi dx$ and $D_z = \operatorname{Re} z = x$. Thus if h is a bump function with $\int_{\mathbb{R}} h(u) du = 1$, then for a K-invariant function f on Z Proposition 13.19 yields

$$\int_{\mathbb{C}^\times} f(z) dxdy = 2\pi \int_{\mathbb{R}^2 \setminus 0} f(x + iy) h(y) x dxdy.$$

In particular, taking the limit $h(u) \to \delta(u)$, we obtain

$$\int_{\mathbb{C}^\times} f(z) dxdy = 2\pi \int_{\mathbb{R}_{>0}} f(x) x dx,$$

the standard formula for integration in polar coordinates.

Now let us apply Proposition 13.19 to gauge theory. Assume that P is the trivial bundle (it is the only one contributing to the perturbation expansion of the path integral in $\hbar$). For ν let us take the Lorentz gauge, i.e., $\nu(A) = *d*A$ (Remark 10.8). The left action of an infinitesimal gauge transformation b on A is given by the infinitesimal version of (10.9):

$$b \circ A = -A \circ b = -db - [A, b].$$

Hence we have

$$D_A b = d^*(db + [A, b]),$$

where $d^* := - * d *$. The operator D_A is called the *Faddeev-Popov operator*. Thus, taking $h(u) \sim e^{-\frac{|u|^2}{2\hbar\varepsilon}}$, for any gauge-invariant insertion $W(A)$ we get

$$\langle W(A) \rangle = \frac{1}{\mathcal{Z}} \int_{\mathrm{Conn}} e^{-\frac{S_\varepsilon(A)}{\hbar}} W(A) \det(d^*(d + \mathrm{ad}_A)) DA, \tag{13.4}$$

where $\mathcal{Z}$ is the partition function (the same integral without insertions), $S(A) = \frac{1}{2}\int_X |F_A|^2 dx$ is the Yang-Mills action and

$$S_\varepsilon(A) = S(A) + \frac{1}{2\varepsilon}\int_X |d^*A|^2 dx.$$

(Admittedly, h is not compactly supported, but this does not matter for the perturbation expansion in $\hbar$.) The determinant $\det(d^*(d + \mathrm{ad}_A))$ under the integral is called the *Faddeev-Popov determinant*.

By itself formula (13.4) is not yet very helpful, since $\det(d^*(d + \mathrm{ad}_A))$ is a nonlocal expression, thus is not convenient for perturbation theory. However, recall that by Proposition 8.16, if Y is an odd vector space with an operator $T : Y \to Y$, then

$$\int_{Y\oplus Y^*} \exp(Ty, y^*)(dydy^*)^{-1} = \det T.$$

Thus, the identity in Proposition 13.19 can be written as

$$\int_Z f(z)dz = \int_{Z\times(\Pi\mathfrak{k}\oplus\Pi\mathfrak{k}^*)} f(z)h(\nu(z))e^{(D_z c,c^*)}dz(dcdc^*)^{-1},$$

where $c \in \Pi\mathfrak{k}$, $c^* \in \Pi\mathfrak{k}^*$.

Similarly, in gauge theory, introducing auxiliary odd scalar fields

$$c : X \to \mathfrak{g},\ c^* : X \to \mathfrak{g}^*,$$

we may write the Faddeev-Popov determinant as

$$\det(d^*(d + \mathrm{ad}_A)) = \int_{\Pi C\oplus\Pi C^*} \exp\left(\frac{1}{\hbar}\int_X (d^*(d + \mathrm{ad}_A)c, c^*)dx\right)(DcDc^*)^{-1},$$

where C, C^* are the (even) spaces of c, c^*. Substituting this into the path integral, and identifying C^* with C using an invariant inner product on $\mathfrak{g}$, we see that for any $\varepsilon > 0$

$$\langle W(A)\rangle = \frac{1}{\mathcal{Z}}\int_{\mathrm{Conn}\times(\Pi C\oplus\Pi C)} e^{-\frac{S_\varepsilon(A,c,c^*)}{\hbar}} W(A)DA(DcDc^*)^{-1}, \tag{13.5}$$

where

$$S_\varepsilon(A, c, c^*) = \int_X \mathcal{L}_\varepsilon(A, c, c^*)dx$$

with

$$\mathcal{L}_\varepsilon(A, c, c^*) = \frac{1}{2}|F_A|^2 + \frac{1}{2\varepsilon}|d^*A|^2 + (dc^*, dc) + (dc^*, [A, c]).$$

This is already a local Lagrangian, so path integral (13.5) can be expanded in $\hbar$ by the usual methods of Feynman calculus.

When $\varepsilon \to 0$, equation (13.5) simplifies to

$$\langle W(A)\rangle = \frac{1}{\mathcal{Z}} \int_{\mathrm{Conn}^{\mathrm{L}} \times (\Pi C \oplus \Pi C)} e^{-\frac{S(A,c,c^*)}{\hbar}} W(A) DA (DcDc^*)^{-1}, \tag{13.6}$$

where $\mathrm{Conn}^{\mathrm{L}} \subset \mathrm{Conn}$ is the space of connections satisfying the Lorentz gauge condition $d^*A = 0$ and

$$S(A, c, c^*) = \int_X \mathcal{L}(A, c, c^*) dx$$

with

$$\mathcal{L}(A, c, c^*) = \frac{1}{2}|F_A|^2 + (dc^*, dc) + (dc^*, [A, c]).$$

However, formula (13.5) is a bit more convenient for the Feynman expansion than (13.6) since it does not involve a gauge fixing condition for A.

The auxiliary fields c, c^* are called the Faddeev-Popov *ghosts*; they were introduced by Faddeev and Popov in 1967 along with the FP operator and determinant. The term "ghosts" is motivated by the fact that these fields do not correspond to actual particles. Note that ghosts violate spin-statistics: they are fermions and yet transform as scalars under Poincaré symmetry and have kinetic term (dc^*, dc) typical for scalar fields.

13.7. The BRST complex

We may also introduce an auxiliary even scalar field $B : X \to \mathfrak{g}$ and write (13.5) in the form

$$\langle W(A)\rangle = \frac{1}{\mathcal{Z}} \int_{\mathrm{Conn} \times (\Pi C \oplus \Pi C) \times C} e^{-\frac{S_\varepsilon(A,c,c^*,B)}{\hbar}} W(A) DADB (DcDc^*)^{-1}, \tag{13.7}$$

where

$$S_\varepsilon(A, c, c^*, B) = \int_X \mathcal{L}_\varepsilon(A, c, c^*, B) dx$$

with

$$\mathcal{L}_\varepsilon(A, c, c^*, B) = \frac{1}{2}|F_A|^2 + \frac{\varepsilon}{2}|B|^2 + i(B, d^*A) + (dc^*, dc) + (dc^*, [A, c]).$$

Then (13.5) is recovered by integrating out B.

The auxiliary field B is called the *Nakanishi-Lautrup field*. One of its advantages is that in the presence of B, we may write the algebra of local functionals of our theory (or operators, in the quantum setting) as the cohomology of a certain complex, which places the problem of quantization of gauge theory in the context of homological algebra.

To explain this, for simplicity consider classical gauge theory. Let $\mathcal{C}^\bullet$ be the algebra of local functionals of A, c, c^*, B (without any equations of motion for now), with grading given by

$$\deg A = \deg B = 0, \deg c = 1, \deg c^* = -1;$$

the degree of a local functional with respect to this grading is called the *ghost number*. We introduce a degree 1 derivation $\delta : \mathcal{C}^\bullet \to \mathcal{C}^{\bullet+1}$ representing the universal infinitesimal gauge symmetry, i.e.,

$$\delta A = -dc - [A, c].$$

We would like δ to be a differential, i.e., $\delta^2 = 0$. We have

$$\delta^2 A = (d + \mathrm{ad}_A)(-\delta c + \frac{1}{2}[c, c]),$$

so for $\delta^2 = 0$ we must have

$$\delta c = \frac{1}{2}[c, c].$$

It is natural to set

$$\delta c^* = B,$$

hence

$$\delta B = 0.$$

Then δ can be uniquely extended to all local functionals using the Leibniz rule and the relation $[\delta, d] = 0$. Thus we obtain a complex $(\mathcal{C}^\bullet, \delta)$,

$$\ldots \to \mathcal{C}^{-1} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \ldots$$

(in fact, a differential graded algebra). It is easy to see that a local functional of A is gauge-invariant if and only if it is a 0-cocycle in $\mathcal{C}^\bullet$. The complex $\mathcal{C}^\bullet$ is called the *BRST complex*, after Becchi, Rouet, Stora, and Tyutin, who proposed this approach to gauge theory around 1975.

Note that we have a factorization of complexes, $\mathcal{C}^\bullet = \mathcal{C}^\bullet_{A,c} \otimes \mathcal{C}^\bullet_{c^*,B}$, where the factors comprise the local functionals of A, c and of c^*, B, respectively. Moreover, the complex $\mathcal{C}^\bullet_{c^*,B}$ is manifestly acyclic except for 1-dimensional cohomology in degree 0. On the other hand, it is easy to see that $\mathcal{C}^\bullet_{A,c}$ is the Chevalley-Eilenberg complex $CE^\bullet(\mathrm{Lie}\, \mathcal{G}, \mathcal{A})$ for the Lie algebra cohomology of Lie $\mathcal{G}$ with coefficients in the algebra $\mathcal{A}$ of (not necessarily gauge-invariant) local functionals of A. It thus follows from the Künneth formula that

$$H^i(\mathcal{C}^\bullet) = H^i(\mathrm{Lie}\, \mathcal{G}, \mathcal{A}).$$

Hence, $H^0(\mathcal{C}^\bullet)$ is the algebra $\mathcal{A}^\mathcal{G}$ of gauge invariant local functionals, i.e., ones on $\mathrm{Conn}/\mathcal{G}$, and the cohomology groups in negative degrees vanish. This cohomology is called the *BRST cohomology*.

Now, since the field equations (i.e., the Yang-Mills equations) are gauge-invariant, they are respected by δ, so we can descend to the quotient by the

differential ideal generated by these equations. Then H^0 of this reduced BRST complex will give the actual algebra $\overline{\mathcal{A}}$ of classical observables.

Note that the complex $(\mathcal{C}^\bullet, \delta)$ does not depend on a gauge fixing procedure. However, the field equations, before taking cohomology, do. Indeed, for example, for the Lorentz gauge we have $B = \frac{1}{i\varepsilon} d^* A$. Thus in this case, eliminating B, the formula for δ becomes

$$\delta A = -dc - [A, c],\ \delta c = \frac{1}{2}[c, c],\ \delta c^* = \frac{1}{i\varepsilon} d^* A.$$

But now $\delta^2 = 0$ holds only modulo the remaining field equations (i.e., on-shell), in contrast with the action of δ on $\mathcal{C}^\bullet$, where $\delta^2 = 0$ off-shell.

We now see the role of the auxiliary field B: it is needed as a partner of c^* (introduced by the Faddeev-Popov procedure) so that the part of the complex related to c^* is acyclic, and it also allows the universal gauge symmetry δ to become off-shell.

A similar BRST procedure can be carried out in the Hamiltonian setting, with $X = \mathbb{R} \times X_s$. Recall from Section 10.7 that in this case the configuration space of the classical theory is $\mathrm{Conn}^0/\mathcal{G}^0$, where Conn^0 and $\mathcal{G}^0$ are the space of connections and the group of gauge transformations on X_s. Hence the phase space is $T^*(\mathrm{Conn}^0/\mathcal{G}^0)$, the Hamiltonian reduction $(T^*\mathrm{Conn}^0)/\!\!/\mathcal{G}^0$ of $T^*\mathrm{Conn}^0$ by the action of $\mathcal{G}^0$.

Also recall from Exercise 11.7 that the Hamiltonian reduction of a symplectic manifold M by a free action of a connected Lie group K with moment map $\mu : M \to \mathfrak{k}^*$ is the symplectic manifold $M /\!\!/ K$ such that $C^\infty(M /\!\!/ K) = C^\infty(\mu^{-1}(0))^K$. This has a cohomological interpretation: since $\mu^{-1}(0)$ is cut out inside M by the equation $\mu(z) = 0$, one can compute $C^\infty(\mu^{-1}(0))$ as $H_0(\mathcal{K}_\bullet(C^\infty(M), \mu))$, where $\mathcal{K}_\bullet(C^\infty(M), \mu)$ is the corresponding Koszul complex (living in nonpositive degrees); moreover, since the action is free, 0 is a regular value of μ, so this complex is acyclic. Thus

$$C^\infty(M /\!\!/ K) = H^0(CE^\bullet(\mathfrak{k}, \mathcal{K}_\bullet(C^\infty(M), \mu))),$$

the 0th cohomology of the total complex of the double complex $CE^\bullet(\mathfrak{k}, \mathcal{K}_\bullet(C^\infty(M), \mu))$.

Similarly, in the Hamiltonian picture of gauge theory, the algebra $\overline{\mathcal{A}}$ of local observables is computed as H^0 of the BRST complex $CE^\bullet(\mathrm{Lie}\, \mathcal{G}^0, \mathcal{K}_\bullet(\widetilde{\mathcal{A}}, \mu))$, where $\widetilde{\mathcal{A}}$ is the algebra of (not necessarily gauge-invariant) local functionals on $T^*\mathrm{Conn}^0$, i.e., in A and $\dot{A}$ on X_s.

This discussion extends routinely to gauge theories with matter. Moreover, the whole story extends to quantum theory, in both the Lagrangian and Hamiltonian settings. In particular, in the Hamiltonian framework, the Hilbert space $\mathcal{H}$ of the theory is computed as $H^0(\mathcal{H}^\bullet)$, where $\mathcal{H}^\bullet$ is a certain complex with

BRST differential δ. The space $\mathcal{H}^0$ carries a nondegenerate Hermitian inner product, but it is not positive definite. In particular, its restriction to $\operatorname{Ker}\delta$ is degenerate and positive semidefinite, with radical $\operatorname{Im}\delta$. Thus the quotient $H^0 = \operatorname{Ker}\delta/\operatorname{Im}\delta = \mathcal{H}$ acquires a positive inner product, as it should.

This method of quantization of gauge theory is called *BRST quantization*. For details see [**QFS**, pp. 1159–1170].

13.8. Renormalization group

Recall that the aspirational procedure of Euclidean QFT is to compute the perturbative expansion of correlation functions using renormalization theory, and then perform Borel summation to obtain nonperturbative correlation functions for numerical values of parameters. The second step of this process (Borel summation) is highly nontrivial, however, and has been accomplished in very few cases beyond quantum mechanics. Yet it turns out that one can get a glimpse of nonperturbative behavior of quantum field theories remaining solely in the realm of perturbation theory. This is what is accomplished by the theory of renormalization group. In fact, this theory effectively performs a partial summation of the perturbative series, summing logarithmically enhanced terms associated with specific classes of Feynman graphs (namely, chain-shaped ones). Here we will briefly review this theory and refer the reader to the lectures of D. Gross [**QFS**, pp. 551–595] for more detail.

The goal of the theory of renormalization group is to compare the behaviors of a given QFT at different length (or momentum) scales; see Remark 1.13. To explain how this works, it is convenient to use the following informal *Wilsonian picture of renormalization*. We will give rigorous meaning to this picture in perturbation theory, but in principle it is not linked to perturbation expansion and can be used in nonperturbative contexts.

For simplicity consider the theory of a single massive scalar field ϕ on $\mathbb{R}^d$, $d \geq 3$. We consider perturbative QFT defined by Euclidean Lagrangians $\mathcal{L}(\phi)$, allowing all possible interactions, or all interactions even under the symmetry $\phi \mapsto -\phi$, including nonrenormalizable ones.[54] Thus $\mathcal{L}$ is a point in an infinite-dimensional space of Lagrangians $\mathcal{C}$. Because of divergences in Feynman diagrams, to define the perturbative QFT corresponding to $\mathcal{L} \in \mathcal{C}$, we need to fix a momentum cutoff Λ, and then cutoff correlation functions, obtained by computing integrals attached to Feynman diagrams over momenta of magnitude $\leq \Lambda$, are expressed as functions of Λ and $\mathcal{L}$, denoted collectively by $CF(\mathcal{L}, \Lambda)$. In other words, the cutoff correlation functions are computed using the *cutoff propagator* $\frac{\mathbf{1}_\Lambda(p)}{p^2+m^2}$, where $\mathbf{1}_\Lambda$ is the characteristic function of the ball $\mathbb{B}(0, \Lambda)$

[54] We eliminate the linear term in ϕ by applying the translation symmetry $\phi \mapsto \phi + \text{const}$.

of radius Λ in momentum space with center at the origin, instead of the usual propagator $\frac{1}{p^2+m^2}$.

Now, given cutoffs Λ_1, Λ_2, we can define the map $R(\Lambda_2, \Lambda_1) : \mathcal{C} \to \mathcal{C}$ by the equations

$$CF(R(\Lambda_2, \Lambda_1)(\mathcal{L}), \Lambda_2) = CF(\mathcal{L}, \Lambda_1).$$

It is not hard to show that this system of equations has a unique solution $R(\Lambda_2, \Lambda_1)(\mathcal{L})$. Informally speaking, $R(\Lambda_2, \Lambda_1)(\mathcal{L})$ is the Lagrangian that at scale Λ_2 gives the same correlation functions as $\mathcal{L}$ at scale Λ_1. It is clear that

$$R(\Lambda_3, \Lambda_2) \circ R(\Lambda_2, \Lambda_1) = R(\Lambda_3, \Lambda_1).$$

In perturbation theory, the map $R(\Lambda_2, \Lambda_1)$ is invertible and makes sense for any Λ_1, Λ_2, but beyond perturbation theory and physically it only makes sense if $\Lambda_2 \leq \Lambda_1$. Namely, in the language of the path integral, the momentum space correlation functions computed with cutoff Λ are given by

$$\langle \widehat{\phi}(p_1) \dots \widehat{\phi}(p_n) \rangle_\Lambda = \int_{\phi:\ \operatorname{supp} \widehat{\phi} \subset \mathbb{B}(0,\Lambda)} \exp\left(- \int_{\mathbb{R}^d} \mathcal{L}(\phi) dx \right) \widehat{\phi}(p_1) \dots \widehat{\phi}(p_n) D\phi,$$

where $\widehat{\phi}$ is the Fourier transform of ϕ. Thus the Lagrangian $\mathcal{L}_2 := R(\Lambda_2, \Lambda_1)\mathcal{L}_1$ is the *effective Lagrangian* at scale Λ_2 for $\mathcal{L}_1$ at scale Λ_1, obtained by integrating out (=averaging over) the degrees of freedom corresponding to values of $\widehat{\phi}(p)$ for $\Lambda_2 < |p| \leq \Lambda_1$: for ϕ with $\operatorname{supp} \widehat{\phi} \subset \mathbb{B}(0, \Lambda_2)$, we have

$$\exp\left(- \int_{\mathbb{R}^d} \mathcal{L}_2(\phi) dx \right) = \int_{\eta:\ \operatorname{supp} \widehat{\eta} \subset \mathbb{B}(0,\Lambda_1) \setminus \mathbb{B}(0,\Lambda_2)} \exp\left(- \int_{\mathbb{R}^d} \mathcal{L}_1(\phi + \eta) dx \right) D\eta.$$

Now fix Λ and consider the map $R(\mu, \Lambda)$ for varying $\mu < \Lambda$. The key property of this map, which follows just by dimensional analysis, is that if c is a coefficient (coupling) in the Lagrangian $\mathcal{L}$ of scaling dimension $D = D(c)$ (cf. Remark 1.13), then in all orders of perturbation theory

$$c(R(\mu, \Lambda)\mathcal{L}) \sim \mu^D,\ \mu \to 0,$$

up to logarithmic factors in μ; thus

$$c(R(\mu, \Lambda)\mathcal{L}) = O(\mu^{D-\varepsilon}),\ \mu \to 0,$$

for every $\varepsilon > 0$. It follows that all the nonrenormalizable couplings (i.e., those with $D > 0$) are suppressed in the limit $\mu \to 0$, and the family $R(\mu, \Lambda)\mathcal{L}$ asymptotically approaches the subspace $\mathcal{C}_{\mathrm{r}} \subset \mathcal{C}$ of renormalizable Lagrangians.

Now let $\mathcal{C}_{\mathrm{n}}$ be the space of purely nonrenormalizable Lagrangian corrections (i.e., containing only nonrenormalizable couplings). Thus we have a decomposition $\mathcal{C} = \mathcal{C}_{\mathrm{r}} \times \mathcal{C}_{\mathrm{n}}$.

Proposition 13.21.

(i) ([**QFS**, p. 558, Lemma 3]) *Given a renormalizable Lagrangian* $\mathcal{L} \in \mathcal{C}_{\mathrm{r}}$, *the intersection* $(\mathcal{L} + \mathcal{C}_{\mathrm{n}}) \cap R(\mu, \Lambda)\mathcal{C}_{\mathrm{r}}$ *consists of one point.*

(ii) ([**QFS**], *p. 558, Theorem 4*(*i*)) *Let the unique intersection point in* (*i*) *be denoted* $B(\mu, \Lambda)\mathcal{L}$. *Then there exists a limit*

$$B(\mu)\mathcal{L} := \lim_{\Lambda \to \infty} B(\mu, \Lambda)\mathcal{L} \in \mathcal{C}$$

(*i.e., all the coefficients converge*).

Let $Y(\mu) := B(\mu)\mathcal{C}_{\mathrm{r}} \subset \mathcal{C}$. Then it is easy to see that

$$R(\mu_2, \mu_1)Y(\mu_1) = Y(\mu_2).$$

Using these identifications, we can view the family of spaces $Y(\mu)$ as a single space $\mathcal{T}$ which is *the space of renormalizable theories*, with isomorphisms $\Phi_\mu : \mathcal{T} \cong Y(\mu)$, and given a theory $T \in \mathcal{T}$, we can compute its correlation functions by the formula

$$CF(T) = CF(\mathcal{L}(T, \mu), \mu), \ \mathcal{L}(T, \mu) := \Phi_\mu(T).$$

By design, this is independent of μ.

Thus, for a renormalizable Lagrangian $\mathcal{L}$, the Lagrangian $B(\mu)\mathcal{L}$ is obtained from $\mathcal{L}$ by adding nonrenormalizable terms (uniquely determined by $\mathcal{L}$), so that the correlation functions computed for $B(\mu)\mathcal{L}$ with cutoff μ equal the correlation functions of a renormalizable theory $T \in \mathcal{T}$ (also uniquely determined by $\mathcal{L}$, namely, $T = \Phi_\mu^{-1}(\mathcal{L})$), which can be alternatively defined using counterterms as we did in Section 13.2. Thus, while in the renormalization group approach we do not explicitly introduce counterterms into the Lagrangian, in the end we obtain equivalent results.

This gives rise to a flow on $\mathbb{R} \times \mathcal{C}_{\mathrm{r}}$ (or, equivalently, $\mathbb{R} \times \mathcal{T}$) with flowlines being the curves

$$\mu \mapsto (\mathcal{L}(T, \mu), \mu)$$

for various $\mathcal{T} \in T$:

$$\gamma^t(\mathcal{L}(T, \mu), \mu) = (\mathcal{L}(T, e^{-t}\mu), e^{-t}\mu).$$

This flow is called the *renormalization group* (*RG*) *flow*, even though in the physical context it is only a semigroup (as the map γ^t is defined only for $t \geq 0$). Note that if $\mathcal{L}$ is free (only in quadratic terms), then the renormalization procedure does nothing, so $\mu \mapsto (\mathcal{L}, \mu)$ is a flowline of the RG flow.

In perturbation theory the renormalization group flow can be encoded by a vector field $v(\mu)$ on $\mathcal{C}_{\mathrm{r}}$ such that

$$\mu \frac{d\mathcal{L}(T, \mu)}{d\mu} = v(T, \mu);$$

this vector field is simpler to work with than the flow γ^t. In terms of renormalizable couplings $\mathbf{g} = (g_i)$, this can be written as the differential equation

$$\mu \frac{d\mathbf{g}}{d\mu} = \beta(\mathbf{g}, \mu). \tag{13.8}$$

This equation is called the *renormalization group* (*RG*) *equation*. By definition, any physical quantity F is constant along the flowlines of this equation, i.e. $v(T, \mu)F = 0$. Explicitly this can be written as

$$\mu \frac{d}{d\mu} F(\mathbf{g}(\mu), \mu) = 0,$$

or, using the chain rule and (13.8),

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_j \beta_j(\mathbf{g}, \mu) \frac{\partial}{\partial g_j} \right) F(\mathbf{g}, \mu) = 0.$$

This equation, which is equivalent to the RG equation, is called the *Callan-Symanzik equation*. In particular, this equation says that renormalized correlation functions, being physically meaningful quantities, are scale-independent and stable under the RG flow.

Note that we have the group $\mathbb{R}_+^2$ acting on $\mathbb{R} \times \mathcal{C}_r$ by rescaling x and ϕ and rescaling μ accordingly, which commutes with the RG flow. Let $\overline{\mathcal{C}_r} := \mathcal{C}_r/\mathbb{R}_+^2$. So we get an identification $(\mathbb{R} \times \mathcal{C}_r)/\mathbb{R}_+^2 \cong \mathbb{R} \times \overline{\mathcal{C}_r}$ by fixing the quadratic part of the Lagrangian $\mathcal{L}$ to be $\frac{1}{2}(|\nabla\phi|^2 + m^2\phi^2)$ for a fixed mass m. This way, the RG flow becomes a flow on $\mathbb{R} \times \overline{\mathcal{C}_r}$.

We note that there is nothing special about the renormalization procedure we have used. For example, one can replace $\mathbf{1}_\Lambda$ with $h(|p|/\Lambda)$, where h is a smooth nonnegative function on $\mathbb{R}$ supported on $[0, 1]$ with $h(z) = 1$ for $z \leq \frac{1}{2}$, and use the *smeared cutoff propagator* $\frac{h(|p|/\Lambda)}{p^2+m^2}$. This is, in fact, better suited for mathematically rigorous formulation of the Wilsonian RG flow (see [**QFS**, pp. 555–557 and p. 557, Remark 4]). One may also use a collection of values of correlation functions at scale μ as coordinates on $\mathcal{T}$. Another convenient possibility is to use dimensional regularization and minimal subtraction (see Section 13.4). So it is natural to ask to what extent the RG flow depends on the choice of the renormalization procedure. It turns out that while the vector field $\beta(\mathbf{g}, \mu)$ does depend on this choice, the vector fields attached to two different procedures differ by a μ-dependent change of variable $\tilde{\mathbf{g}} = f_\mu(\mathbf{g})$ with μ-dependence disappearing in the limit $\mu \to \infty$. In other words, at large μ they are isomorphic as dynamical systems and differ only by the recipe of *finite renormalization*, i.e., choice of μ-dependent parametrization of $\mathcal{C}_r$ (or $\overline{\mathcal{C}_r}$).

Moreover, if the finite renormalization procedure is chosen to be *scale-independent* (such as dimensional regularization and minimal subtraction, see

Section 13.4 and [**QFS**, p. 568]), then dimensional analysis implies that the RG equation for dimensionless (critical) couplings $\mathbf{g}_c$ does not involve dimensionful (super-renormalizable) couplings, nor explicit dependence on μ ([**QFS**, p. 569]). In other words, this equation has the form

$$\mu \frac{d\mathbf{g}_c}{d\mu} = \beta(\mathbf{g}_c),$$

and since free Lagrangians are preserved by renormalization group, we have $\beta(0) = 0$, i.e. $\mathbf{g}_c = 0$ is a fixed point of the RG flow. The vector field β is called the *beta function*. In many interesting theories there is only one critical coupling g, in which case the RG equation has the form

$$\mu \frac{dg}{d\mu} = \beta(g),$$

where β is a function of one variable, which justifies this terminology.

The leading term of the Taylor expansion of β at 0 is determined by an ℓ-loop calculation for some $\ell > 0$ (the smallest number of loops for which the contribution to the β-function is nontrivial). This implies that in the case of one critical coupling we have $\beta'(0) = 0$, as a Feynman diagram contributing to renormalization of g must contain at least two internal vertices. Namely, if g is the coefficient of ϕ^s, then the corresponding vertex has s edges, and a connected ℓ-loop diagram with k such vertices has $k + \ell - 1$ internal edges. Thus it has $ks - 2(k + \ell - 1)$ external edges. So for such a diagram to contribute to renormalization of g, we need

$$ks - 2(k + \ell - 1) = s,$$

i.e.,

$$\ell = \frac{(k-1)(s-2)}{2},$$

and we should take the smallest $k > 1$ for which this is an integer. So if s is even, the smallest integer value of ℓ is obtained for $k = 2$ and is $\ell = \frac{s-2}{2}$, while for s odd we have $k = 3$ and $\ell = s-2$, and the leading term of $\beta(g)$ has the form Cg^k where C is the coefficient of $\log \Lambda$ in the asymptotics of the amplitudes of the corresponding Feynman diagram cut off at Λ as $\Lambda \to \infty$. Thus the RG equation has a *parabolic fixed point* at 0 (i.e., $\beta'(0) = 0$).

Example 13.22. For the ϕ^6 theory in three dimensions one has $k = 2, \ell = 2$ and $\beta(g) \sim C_3 g^2$. Likewise, for the ϕ^4 theory in four dimensions $k = 2, \ell = 1$, $\beta(g) \sim C_4 g^2$, and for the ϕ^3 theory in six dimensions $k = 3, \ell = 1, \beta(g) \sim C_6 g^3$.

Solving the approximate RG equation

$$\mu \frac{dg}{d\mu} = Cg^k$$

with initial condition $g(\mu_0) = g_0$ by separation of variables, we get

$$g_0^{1-k} - g^{1-k} = (k-1)C\log(\tfrac{\mu}{\mu_0}),$$

so

$$g(\mu) = (g_0^{1-k} - (k-1)C\log(\tfrac{\mu}{\mu_0}))^{\frac{1}{1-k}} = \frac{g_0}{(1-(k-1)Cg_0^{k-1}\log(\tfrac{\mu}{\mu_0}))^{\frac{1}{k-1}}}. \tag{13.9}$$

Thus we see that RG indeed performs partial summation of the perturbation series: e.g., for $k = 2$ (13.9) is the sum of the geometric series

$$g(\mu) = \sum_{n=0}^{\infty} (C\log(\tfrac{\mu}{\mu_0}))^n g_0^{n+1}.$$

Remark 13.23. We see that nonrenormalizable interactions (couplings) are suppressed by the RG flow in the infrared limit at a power rate; for this reason they are called *irrelevant*. On the contrary, critical interactions (also called *marginal*) and super-renormalizable ones (also called *relevant*) remain important in the infrared, and moreover relevant interactions blow up at a power rate in the infrared limit.

Exercise 13.24. Pick a normalization of g and compute the exact values of C_3, C_4, and C_6.

13.9. Asymptotic freedom

The constant C depends on the normalization of g, but its sign is well defined (assuming $g > 0$) and is of crucial importance. Indeed, if $C < 0$ (i.e., $\beta(g) < 0$ for small g), then 0 is an *attracting* fixed point of the RG flow, so the solution $g(\mu)$ exists for all $\mu > \mu_0$ and goes to 0 as $\mu \to \infty$. In other words, in the UV limit, the theory becomes free. Such theories (in which the Lagrangian becomes free as $\mu \to \infty$ when we run the RG flow backwards in perturbation theory, i.e., marginal interactions are suppressed at a logarithmic rate by the reverse RG flow) are called *asymptotically free*. On the other hand, if $C > 0$ (i.e., $\beta(g) > 0$ for small g), then 0 is a *repelling* fixed point of the RG flow, so the solution $g(\mu)$ cannot go to zero at $\mu \to \infty$ and in fact in the above approximation (13.9) develops a singularity at

$$\mu_s = \mu_0 \exp\left(\frac{1}{(k-1)Cg_0^{k-1}}\right) > \mu_0 \tag{13.10}$$

(the so-called *Landau pole*).

Physically, asymptotic freedom suggests that perturbation theory is valid at high energies μ, since the couplings are small for such μ. In fact, it can be shown that asymptotic freedom is a necessary condition for a renormalizable perturbative QFT to be Borel summable to a nonperturbative theory, or,

as physicists say, UV complete, i.e., fundamentally defined at high energies. For example, every super-renormalizable theory is automatically asymptotically free, since its couplings are suppressed at a power rate as $\mu \to \infty$.

On the other hand, asymptotic freedom for critical theories is a very strong restriction. For example, it is clear that the constants C_3, C_4 in scalar field theory are positive, since each of them is the logarithmic divergence coefficient of an integral of a positive function. So the ϕ^6 theory in 3 dimensions and ϕ^4 theory in 4 dimensions are **not** asymptotically free, and thus cannot be defined nonperturbatively in the UV limit.[55] In fact, it is known that in four dimensions, any renormalizable asymptotically free theory must contain nonabelian gauge fields. The leading coefficient C of the β-function of a pure nonabelian gauge theory turns out to be negative (due to negative definiteness of the Killing form of a compact simple Lie group), so such a theory is asymptotically free. Adding matter fields (scalars and spinors) that couple to this gauge field (i.e., are sections of associated bundles to the principal bundle on which the gauge field is a connection) makes C less negative, so the theory only remains asymptotically free if the number of scalars and spinors does not exceed a certain upper bound depending on the gauge group. This upper bound is actually satisfied in the *standard model*, which is the quantum field theory believed to describe particle interactions in nature.

On the other hand, in two dimensions there exist critical theories with scalar fields only which are asymptotically free. A notable example is *nonlinear sigma models* introduced in Section 11.1, with target a Riemannian manifold of positive Ricci curvature.

Example 13.25. Consider the nonlinear 2-dimensional sigma model with basic field $\phi : \mathbb{R}^2 \to M$, where M is a Riemannian manifold, and Euclidean Lagrangian

$$\mathcal{L} = \tfrac{1}{2}|d\phi|^2 = \tfrac{1}{2}g_{ij}(\phi)d\phi^i d\phi^j.$$

This theory has infinitely many dimensionless couplings, or rather it has a functional dimensionless coupling—the metric g_{ij}. It turns out that the RG flow for this theory in the 1-loop approximation is the *Ricci flow*,

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$

where R_{ij} is the Ricci tensor of g_{ij} and $t = \log(\mu^{-2})$ is the RG "time".[56]

[55] On the other hand, the β-function of the ϕ^3-theory in 6 dimensions is negative ($C_6 < 0$), since the positive Feynman diagram amplitude multiplies by $(-g)^3$. However, this theory is unstable since the function e^{-gx^3} is unbounded for any sign of g. To stabilize it, one may use the Yang-Lee convention, under which the real coupling g is replaced by ig. But then the β-function changes sign and becomes positive, so the theory is not asymptotically free any more.

[56] The Ricci flow was introduced by R. Hamilton in 1982 and studied deeply by Hamilton, S.-T. Yau, and other mathematicians in subsequent years. This eventually led to solutions, by Hamilton and others,

For example, if M is a 2-dimensional surface, then any metric on M can be written as a conformal metric $e^{-2\rho}|dz|^2$, where z is a complex coordinate, and the 1-loop β-function in terms of ρ has the form

$$\beta_{1\text{-loop}}(\rho) = -e^{2\rho}\Delta\rho = -K,$$

where K is the Gaussian curvature of M. So the RG equation in the 1-loop approximation is the 2-dimensional Ricci flow

$$\frac{\partial\rho}{\partial t} = -K = e^{2\rho}\Delta\rho. \tag{13.11}$$

For example, suppose g_0 is a metric of constant curvature K_0, then the Ricci flow will just rescale the metric, i.e., $g(t) = C(t)g_0$, with Gaussian curvature $K(t) = K_0C(t)^{-1}$. The RG equation for $C(t)$ is therefore

$$\frac{dC}{dt} = -2K_0,$$

or in terms of curvature

$$\frac{dK}{dt} = 2K^2.$$

Thus, given that $C(0) = 1$, we have

$$C(t) = 1 - 2K_0t,\ K(t) = \frac{K_0}{1 - 2K_0t}.$$

So we see that if $K_0 > 0$ (i.e., $\beta < 0$), then we have the asymptotically free situation in the UV limit $t \to -\infty$ (the sphere expands and in the limit becomes flat), while in the IR we have a singularity at $t = \frac{1}{2K_0}$ (the sphere shrinks and collapses to a point in finite time). On the other hand, if $K_0 < 0$ (i.e., $\beta > 0$), then we have the opposite situation: in the UV a singularity forms in finite time (a hyperbolic surface shrinks and collapses to a point), while in the IR limit the theory is free (a hyperbolic surface expands and in the limit becomes flat). The situation is similar for variable curvature of constant sign.

We see that the nonlinear 2-dimensional sigma model into a surface is asymptotically free iff this surface has nonnegative Gaussian curvature. In this case (at least for constant curvature, i.e., $M = S^2$ with round metric) it is expected to exist nonperturbatively.

This analysis extends to higher dimensions of the target manifold M: the theory is asymptotically free if M has positive Ricci curvature (i.e., the Ricci tensor is positive definite). For example, if M is a compact symmetric space (e.g., a higher dimensional sphere S^{N-1}) with an invariant metric, then the nonlinear 2-dimensional sigma model into M is asymptotically free and is believed to exist nonperturbatively. For spheres such models are considered in [**QFS**, p. 1171], and for Grassmannians in [**QFS**, p. 1183].

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of many important problems in differential geometry and topology, notably the proofs of the 3-dimensional Poincaré and Thurston geometrization conjectures by G. Perelman in 2003.

Another notable example of asymptotic freedom in two dimensions is the *Gross-Neveu model*, ([**QFS**, p. 585]).

Example 13.26. The Gross-Neveu model is a theory of a Dirac fermion ψ on $\mathbb{R}^2$ valued in a finite-dimensional space $\mathbb{C}^N$ with standard Hermitian inner product and Minkowskian Lagrangian

$$\mathcal{L} = i(\psi^\dagger, \mathbf{D}\psi) + \frac{g^2}{2}(\psi^\dagger, \psi)^2.$$

Classically the Gross-Neveu model has a global $U(N)$ symmetry of the space $\mathbb{C}^N$ and also a chiral symmetry $\psi_\pm \mapsto \pm\psi_\pm$, where $\psi = (\psi_+, \psi_-)$ is the decomposition of ψ into chiral and anti-chiral components. In particular, the chiral symmetry prohibits a mass term $(\psi^\dagger, \psi)$ (as it couples ψ_+ with ψ_- and hence changes sign under the chiral symmetry), so in perturbation theory ψ is massless and the $U(N)$-symmetry implies that there are no couplings other than g. This theory turns out to be asymptotically free ($\beta < 0$) if $g^2 > 0$, and one can show that it exists nonperturbatively (in fact, it is exactly solvable in the large N limit and partially solvable for finite N).

The Gross-Neveu model exhibits the following features:

1) The $U(N)$-symmetry is preserved in quantum theory (as there is no continuous symmetry breaking in dimension 2).

2) The chiral $\mathbb{Z}/2$-symmetry, which is present classically, is broken in quantum theory, producing two realizations (ground states) permuted by this symmetry (the phenomenon discussed in Section 11.7).

3) As the chiral symmetry is broken, the mass of ψ is no longer prohibited in quantum theory. In fact, the mass $m(g)$ of ψ (i.e., the pole of the 2-point function in Minkowskian momentum space) is *generated dynamically*, as a nonperturbative correction exponentially small in g as $g \to 0$. (Note that since the theory is classically scale-invariant, i.e., all terms in the Lagrangian are critical, the mass of ψ cannot appear in perturbation theory, say using dimensional regularization.)

Let us discuss the last effect (dynamical mass generation) in a bit more detail. Suppose that we have a theory with classically the only parameter being a dimensionless coupling g, and $\beta(g) \neq 0$. Then by dimensional analysis the mass $m(g)$ of a particle in this theory has to have the form

$$m(g) = \mu f(g),$$

where μ is a mass scale introduced in quantization (note that since $\beta(g) \neq 0$, the scaling symmetry is anomalous quantum-mechanically, so any method of quantization will introduce a scale).

Now recall that all physical quantities have to be constant along the RG flow. Thus we have the Callan-Symanzik equation

$$(\mu\partial_\mu + \beta(g)\partial_g)(\mu f(g)) = 0.$$

This yields

$$f(g) + \beta(g)f'(g) = 0,$$

i.e.,

$$f(g) = c\exp\left(\int_g^{g_0} \frac{du}{\beta(u)}\right).$$

Thus

$$m(g) = c\mu\exp\left(\int_g^{g_0} \frac{du}{\beta(u)}\right).$$

If $\beta < 0$ for small g (i.e., our theory is asymptotically free), then this function goes to 0 as $g \to 0$, so it is conceivable that $m(g) \neq 0$ (*dynamical mass generation*); however, since $\beta(g) = O(g^2)$ as $g \to 0$, the function $m(g)$ is exponentially small near $g = 0$ and its perturbative expansion is identically zero, as expected. Of course, the coefficient c may or may not be zero, and whether it is nonzero (i.e., whether mass generation actually occurs) depends on the details of the theory.

Now let us apply this argument to the Gross-Neveu model. Taking the large N limit, one can solve the model and show that the exact β-function (to all orders of perturbation theory and beyond) is

$$\beta(g) = -\frac{N}{2\pi}g^3.$$

It also turns out that $c \neq 0$ (this is nontrivial and shown in [**QFS**, pp. 588–593]). Thus, we may absorb c into μ, and in leading order in $1/N$, we obtain

$$m(g) = \mu\exp(-\tfrac{\pi}{g^2N}).$$

13.10. Quantum scale-invariance

Thus we see that the theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}|\nabla\phi|^2 + \frac{g}{r!}\phi^r,$$

where $r = 3, 4, 6$ for $d = 6, 4, 3$, which is classically scale-invariant (as it has only one critical interaction), ceases to be scale-invariant quantum-mechanically, even in perturbation theory (one or two loops): the coupling g changes with scale according to the RG flow. In other words, the scaling symmetry becomes *anomalous*. In order for the classical scaling symmetry to be preserved quantum-mechanically, at least in perturbation theory, the theory must have vanishing β-function to all orders of perturbation theory.

This is a very strong restriction! For instance, as we have seen in Example 13.25, in a nonlinear sigma model with target manifold M already in a 1-loop we get the condition that the Ricci tensor must vanish, i.e., M should be Ricci flat. So if $\dim M = 2$, we are reduced to the boring case of a flat target manifold, in which case the theory is free. In higher dimensions there are Ricci flat metrics which are not flat, but higher orders of perturbation theory impose further restrictions on the metric.

It is therefore amazing that nonfree theories retaining scale-invariance at the quantum level (at least in all orders of perturbation theory) do exist nonetheless! Namely, this is the case for the supersymmetric sigma model whose target M is a Ricci-flat compact Kähler manifold (also known as a *Calabi-Yau manifold*).[57] Specifically, as explained in Section 14.11, the sigma model with a Calabi-Yau target has an $\mathcal{N} = (2, 2)$ supersymmetric extension, and supersymmetry turns out to imply that the 1-loop β-function (which is zero in the Calabi-Yau case) is actually exact, i.e., the β-function is zero to all orders of perturbation theory. In fact, these supersymmetric sigma models are believed to exist nonperturbatively and be scale-invariant and moreover conformally invariant quantum-mechanically. For a more in-depth discussion see [**QFS**, pp. 1291–1341].

13.11. Low energy effective theories

Microscopic physical processes observed in nature happen at low energies. Thus the main goal of quantum field theory is to describe the low-energy (i.e., infrared) behavior of a given QFT (say, an asymptotically free theory defined perturbatively at high energies, or in some other way). In other words, we would like to describe the behavior of correlation functions of the theory in the infrared limit, i.e., at large distances (or small momenta). Since values of correlation functions at scale μ can be used as renormalized couplings of the theory, this is tantamount to determining what happens to the theory under the renormalization group flow in the infrared limit $\mu \to 0$. In the Hamiltonian approach, this means understanding the low energy spectrum of the Hamiltonian of the QFT in its Hilbert space of states.

As even asking this question requires going beyond perturbation theory, this is a very difficult problem, open for most physically interesting quantum field theories. For example, in the asymptotically free case $\beta < 0$, the function $g(\mu)$ given by (13.9) still develops a singularity at the point given by (13.10), but now with $\mu_s < \mu_0$, so we definitely cannot use perturbation theory near or beyond this point. It is therefore amazing that in some nontrivial cases (such

[57] Yau's theorem (which used to be Calabi's conjecture) states that if M is a compact complex manifold with $c_1(M) = 0$, then in every nonempty Kähler class of M (i.e., a class in $H^2(M, \mathbb{R})$ represented by a Kähler form) there is a unique Ricci-flat Kähler metric.

as supersymmetric field theories, but not only those) one can nevertheless say something interesting about this question. While this topic, called *QFT dynamics*, is, unfortunately, mostly beyond the scope of this book (as it requires a separate book), let us at least outline the language in which it is normally discussed.

The central notion used to discuss infrared behavior of QFT is that of a *low energy effective theory* for a given QFT. In general, given a quantum field theory $\mathcal{T}$, an effective theory is obtained when a part of degrees of freedom of $\mathcal{T}$ is integrated out (or, equivalently, averaged over). We have already seen an example of this in Section 3.13, when we defined the effective classical action for a QFT (in which case all the quantum degrees of freedom are integrated out). In a similar vein, a QFT $\mathcal{T}_{\text{eff}}$ is said to be a low energy effective theory for a QFT $\mathcal{T}$ if the correlation functions of $\mathcal{T}$ are well approximated by correlation functions of $\mathcal{T}_{\text{eff}}$ at long distances (or small momenta). Note that $\mathcal{T}_{\text{eff}}$ need not be fundamentally defined in the UV limit: its correlation functions only need to make sense at low energies. Thus such a theory need not be asymptotically free, not even renormalizable. In particular, while nonfree renormalizable QFT do not exist beyond dimension $d = 4$ (see Section 13.5), a nonfree effective theory can live in dimension $d > 4$, being the low energy approximation of a QFT defined in the UV using string theory. For example, this is expected to happen in dimension 6 for the so-called (2,0)-theory attached to a compact Lie group.

Thus, a low energy effective theory can be used to describe the infrared limit of a given QFT. Such an effective theory includes a number of massless particles, and one of the main problems of QFT dynamics is to determine this particle content. For example, if the theory has a mass gap (no massless particles), one says that its IR limit is *trivial* (i.e., below a sufficiently low energy threshold, the only state is the ground state). Note that this can happen even if the Lagrangian of the original QFT (defined fundamentally in the UV) contains massless fields, through the phenomenon of *dynamical mass generation*. For example, it is believed that the pure nonabelian gauge theory in $d \geq 3$ dimensions is infrared-trivial (i.e., has a mass gap), even though gauge fields are classically massless due to gauge symmetry. This is one of the most important conjectures in QFT (especially for dimension 4). A more elementary example of this, where dynamical mass generation can be proved rigorously, is the Gross-Neveu model (see Example 13.26).

More generally, a low energy effective theory may be used to describe low energy behavior of a QFT at a certain scale μ without going all the way to the infrared limit $\mu \to 0$. In this case such a theory may include not only massless particles, but also massive particles with masses of order $\leq \mu$. For example, the expected low energy particle content of a nonabelian pure gauge theory in four dimensions includes *glueballs*—massive scalar, vector, and tensor particles.

A nice example of a low energy effective theory arises for the theory of the scalar field $\phi : \mathbb{R}^d \to \mathbb{R}^N$ with Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}|\nabla\phi|^2 + \frac{g}{4!}(|\phi|^2 - a^2)^2,\ a > 0,\ N \geq 2.$$

This theory is super-renormalizable and is believed to exist nonperturbatively in $d \leq 3$ dimensions, and it is rigorously known to exist for $d = 2$ within the framework of constructive field theory. The low energy effective theory for this theory turns out to be the nonlinear sigma model of maps from $\mathbb{R}^d$ into the set of minima of the potential $(|\phi|^2 - a^2)^2$, which is the sphere S^{N-1} defined by the equation $|\phi| = a$ ([**QFS**, p. 1141]). Note that if $N \geq 3$, this theory is not free, since the sphere of dimension ≥ 2 has nonzero curvature. In fact, for $d = 3$ this theory is not renormalizable (but as we mentioned, this is OK for a low energy theory), and in the infrared limit $\mu \to 0$ it flows to the theory of $N - 1$ free massless scalars, i.e., a scalar valued in $\mathbb{R}^{N-1}$ (flattening of S^{N-1}). On the other hand, for $d = 2$ the nonlinear sigma model is renormalizable and even asymptotically free, but its infrared behavior for $N \geq 3$ is more subtle: one can show that at low energies it has N *massive* scalar particles, so the infrared limit is trivial ([**QFS**, p. 1171]).

13.12. Lattice models

As we have seen above, making the Lagrangian approach to quantum field theory rigorous is quite a challenge because usually the path integral is not rigorously defined. One answer to this challenge is perturbation theory, which produces power series in the Planck constant $\hbar$ that are conjecturally Borel summable to the actual correlation functions of the theory. However, there is another approach—putting the theory on a lattice. In this approach we can rigorously obtain numerical answers depending on a lattice spacing parameter ε, which are expected to converge to the actual correlation functions as $\varepsilon \to 0$. Unlike perturbation theory, this method is good for computer simulations of strong coupling behavior of quantum field theories, but its big drawback compared to perturbation theory is that geometric features and symmetries of the theory (such as Poincaré symmetries, conformal symmetries) are typically lost.

Let us explain how such lattice approximation works on the example of d-dimensional Euclidean theory of a real scalar ϕ with potential U tending to $+\infty$ at least quadratically at $\pm\infty$. In this approximation ϕ is a real-valued function on the lattice $\varepsilon\mathbb{Z}^d$. The action is defined by discretizing the action of the usual, continuous theory:

$$S_\varepsilon(\phi) = \varepsilon^d \sum_{\mathbf{x}\in\varepsilon\mathbb{Z}^d} \left(\frac{1}{2} \sum_{j=0}^{d-1} \left(\frac{\phi(\mathbf{x}+\varepsilon\mathbf{e}_j)-\phi(\mathbf{x})}{\varepsilon} \right)^2 + U(\phi(\mathbf{x})) \right),$$

where $\{\mathbf{e}_j,\ 0 \le j \le d-1\}$ is the standard basis of $\mathbb{R}^d$. Thus the Euler-Lagrange equation has the form

$$-\Delta_\varepsilon \phi + U'(\phi) = 0,$$

where Δ_ε is the discrete (lattice) Laplacian,

$$(\Delta_\varepsilon \phi)(\mathbf{x}) = \sum_{j=0}^{d-1} \frac{\phi(\mathbf{x}+\varepsilon\mathbf{e}_j) - 2\phi(\mathbf{x}) + \phi(\mathbf{x}-\varepsilon\mathbf{e}_j)}{\varepsilon^2}.$$

Thus the correlation functions of the corresponding quantum theory are given by the formula

$$\langle \phi(\mathbf{x}_1)\dots\phi(\mathbf{x}_n)\rangle = \int \phi(\mathbf{x}_1)\dots\phi(\mathbf{x}_n) e^{-\frac{S_\varepsilon(\phi)}{\hbar}} D\phi,$$

where the measure $D\phi$ is proportional to the product of $d\phi(x)$ over all nodes x and is normalized so that

$$\int e^{-\frac{S_\varepsilon(\phi)}{\hbar}} D\phi = 1.$$

Note that by putting the theory on a lattice, we have resolved the small distance (ultraviolet) issues (we no longer have any positive distances smaller than ε), and so this path integral is a perfectly rigorous object in probability theory for any d and U, with $e^{-\frac{S_\varepsilon(\phi)}{\hbar}} D\phi$ being the *Gibbs measure* ([**G**]).

Consider the special case when $U(\phi) = \frac{m^2\phi^2}{2}$ (free massive scalar). Since this theory is free, the 2-point function $\langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle$ is $G_\varepsilon(\mathbf{x}-\mathbf{y})$, where $G_\varepsilon(\mathbf{x}) = G_\varepsilon(\mathbf{x}, m)$ is the lattice Green's function—the fundamental solution of lattice Poisson equation

$$(-\Delta_\varepsilon + m^2) G_\varepsilon(\mathbf{x}) = \varepsilon^{-d}\delta_{\mathbf{x},0} \tag{13.12}$$

(i.e., one decaying at infinity); it is easy to see that for $\mathbf{x} \in \mathbb{Z}^d$

$$G_\varepsilon(\varepsilon\mathbf{x}, m) = \varepsilon^{2-d} G_1(\mathbf{x}, \varepsilon m).$$

Moreover, higher correlation functions are expressed in terms of G_ε via the Wick formula. The function G_ε can be found by considering its Fourier transform, i.e., the Fourier series

$$\widehat{G}_\varepsilon(\mathbf{p}) = \varepsilon^d \sum_{\mathbf{x}\in\varepsilon\mathbb{Z}^d} G_\varepsilon(\mathbf{x}) e^{i\mathbf{p}\mathbf{x}}$$

(the lattice propagator). This is a function on the torus $\mathbb{R}^d/(2\pi\varepsilon^{-1}\mathbb{Z})^d$, and equation (13.12) yields

$$\left(\sum_{j=0}^{d-1}\left(\frac{2\sin(\frac{\varepsilon p_j}{2})}{\varepsilon}\right)^2 + m^2\right)\widehat{G}_\varepsilon(\mathbf{p}) = 1,$$

where $\mathbf{p} = (p_1, \dots, p_d)$. Thus

$$\widehat{G}_\varepsilon(\mathbf{p}) = \frac{1}{\sum_{j=0}^{d-1} \left(\frac{2\sin(\frac{\varepsilon p_j}{2})}{\varepsilon} \right)^2 + m^2}.$$

Thus, as expected, when $\varepsilon \to 0$, the function $\widehat{G}_\varepsilon(\mathbf{p})$ converges to the usual momentum space propagator of a massive scalar, $\widehat{G}(p) = \frac{1}{p^2+m^2}$, and consequently, $G_\varepsilon(\mathbf{x})$ converges to its position space propagator $G(x)$.

This analysis also applies for $m = 0$, but only for $d \geq 3$, since for $d = 1, 2$ the theory of a free massless scalar (on a lattice or continuous space) is ill defined: its Green's function does not decay at infinity (see Section 6.11 for $d = 1$ and Section 10.2 for $d = 2$). In this case, the function $\frac{G_1(\mathbf{x})}{G_1(0)}$ has an additional probabilistic meaning: it is the probability for a simple random walk on $\mathbb{Z}^d$ emanating from $\mathbf{x}$ to ever visit the origin. Moreover, the probability of return for a random walk that has left the origin is given by *Pólya's formula*,

$$P = 1 - \frac{1}{2dG_1(0)}, \tag{13.13}$$

where

$$G_1(0) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{d\mathbf{p}}{\sum_{j=0}^{d-1} \left(2\sin(\frac{p_j}{2})\right)^2}. \tag{13.14}$$

Thus $P < 1$, i.e., the simple random walk in $d \geq 3$ dimensions is *transient* (never returns with positive probability). On the other hand, for $d = 1, 2$ integral (13.14) diverges, so $P = 1$ and the random walk is *recurrent* (returns with probability 1). This is a classical result of Pólya (1921).

Indeed, if $d \geq 3$, then it is easy to see that the probability $P(\mathbf{x})$ of ever coming from $\mathbf{x}$ to 0 satisfies the lattice Laplace equation $\Delta_1 P(\mathbf{x}) = 0$ outside the origin and decays at infinity, so it is a multiple of $G_1(\mathbf{x})$, namely $P(\mathbf{x}) = \frac{G_1(\mathbf{x})}{G_1(0)}$ since $P(0) = 1$. So $-\Delta_1 P(\mathbf{x}) = \frac{\delta_{\mathbf{x},0}}{G_1(0)}$. Hence, setting $\mathbf{x} = 0$, we get

$$G_1(\mathbf{e}_0) = G_1(0) - \frac{1}{2d}.$$

Thus the probability of returning to the origin is

$$P = \frac{G_1(\mathbf{e}_0)}{G_1(0)} = \frac{G_1(0) - \frac{1}{2d}}{G_1(0)},$$

which yields (13.13).

To make the model more finite (and therefore more amenable to computer simulations), one may further put the theory in finite volume, replacing $\mathbb{Z}^d$ by $(\mathbb{Z}/N\mathbb{Z})^d$. This makes the path integral finite dimensional.

Exercise 13.27. Compute the 2-point function of a free massive scalar on $(\mathbb{Z}/N\mathbb{Z})^d$ with lattice spacing $2\pi/N$, and show that as $N \to \infty$ it tends to the 2-point function of the quantum field theory of a free massive scalar on the d-dimensional torus.

In a similar way one can make lattice approximation of fermionic fields. We leave details to the reader.

As another example of lattice approximation consider pure *lattice gauge theory* on $\mathbb{Z}^d$, see Section 10.7 (it can then be rescaled to allow the lattice spacing to go to 0). Matter fields can then be added as described above.

Let G be a compact Lie group. Fix a faithful finite-dimensional representation ρ of G and define the distance from $g \in G$ to 1 to be Hilbert-Schmidt norm of $\rho(g) - 1$, i.e.,

$$\|\rho(g) - 1\|^2 = \operatorname{Tr}(\rho(g) - 1)(\rho(g^{-1}) - 1) = 2\dim\rho - \operatorname{Tr}\rho(g) - \operatorname{Tr}\rho(g^{-1}).$$

We will think of $\mathbb{Z}^d$ as a graph in which neighboring vertices are connected with an edge. A *lattice connection* (*gauge field*) on $\mathbb{Z}^d$ with gauge group G is a function $A : E(\mathbb{Z}^d) \to G$, where $E(\mathbb{Z}^d)$ is the set of ordered pairs of neighboring vertices in $\mathbb{Z}^d$ (i.e., oriented edges), such that $A(\mathbf{x}, \mathbf{y})A(\mathbf{y}, \mathbf{x}) = 1$. Thus a connection is determined by a collection of functions $A_j : \mathbb{Z}^d \to G$, where $A_j(\mathbf{x}) = A(\mathbf{x} + \mathbf{e}_j, \mathbf{x})$; we should think of this element as the holonomy of the connection from $\mathbf{x}$ to $\mathbf{x} + \mathbf{e}_j$ along the corresponding edge. Then the curvature of the connection A is the collection of functions $F_{ij} : \mathbb{Z}^d \to G$ given by

$$F_{A,ij}(\mathbf{x}) = A_i(\mathbf{x} + \mathbf{e}_i)^{-1}A_j(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j)^{-1}A_i(\mathbf{x} + \mathbf{e}_j)A_j(\mathbf{x}),$$

i.e., the holonomies of A at $\mathbf{x}$ around the squares spanned by $\mathbf{e}_j$ and $\mathbf{e}_i$. The action is then

$$S(A) = \sum_{\mathbf{x}\in\mathbb{Z}^d} \sum_{0\le i<j\le d-1} \|\rho(F_{A,ij}) - 1\|^2.$$

The group of gauge transformations is $\mathcal{G} = \operatorname{Fun}(\mathbb{Z}^d, G)$, and it acts on the space $\mathcal{A} = \operatorname{Fun}(E(\mathbb{Z}^d), G)$ by

$$(g \circ A)_j(x) = g(\mathbf{x} + \mathbf{e}_j)A_j(\mathbf{x})g(\mathbf{x})^{-1}.$$

It is easy to see that curvature F_A is equivariant under this action,

$$F_{g\circ A} = gF_Ag^{-1},$$

thus the action $S(A)$ is gauge-invariant.

In a similar way, replacing $\mathbb{Z}$ by $\mathbb{Z}/N\mathbb{Z}$, we can put gauge theory on $(\mathbb{Z}/N\mathbb{Z})^d$.

Example 13.28. Consider pure gauge theory with gauge group $G = U(1)$ on $(\mathbb{Z}/N\mathbb{Z})^2$. Thus a connection is a pair of functions

$$a, b : (\mathbb{Z}/N\mathbb{Z})^2 \to \mathbb{R}/2\pi\mathbb{Z} \cong U(1)$$

with curvature being

$$\theta(r,s) = -a(r+1,s) - b(r+1,s+1) + a(r,s+1) + b(r,s),\ r,s \in \mathbb{Z}/N\mathbb{Z}.$$

Let ρ be the tautological representation of $U(1)$. Then the action is given by

$$S(a,b) = \sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} (2 - 2\cos\theta(r,s)).$$

It follows that the Euler-Lagrange equation for extremizing S is

$$\sin\theta(r,s) = \text{const}.$$

Moreover, taking second derivatives, we see that the extra conditions for a local minimum of the action are that $\cos\theta(r,s) \geq 0$, so $\theta(r,s) = \theta$ is a constant and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. It is easy to show that connections with $\theta(r,s) = \theta$ exist if and only if $N^2\theta \in 2\pi\mathbb{Z}$, i.e., $\theta = \frac{2\pi k}{N^2}$, $k \in \mathbb{Z}$, $|k| \leq \frac{N^2}{4}$ (check it!). One may thus informally think that the corresponding bundle has first Chern class $c_1 = k$. In this way we can see how topology of G-bundles on the 2-torus emerges from the discrete picture as N gets large.

Quantum lattice gauge theory can now be described by the path integral

$$\int_{\mathcal{A}/\mathcal{G}} f_1(A)\dots f_n(A)e^{-\frac{S(A)}{\hbar}}DA,$$

where $f_i(A)$ are gauge-invariant quantities (physical observables). If we work on $(\mathbb{Z}/N\mathbb{Z})^d$, this is a finite-dimensional integral, so it is perfectly well defined and amenable to computer simulations. Even on $\mathbb{Z}^d$ it is rigorously defined in the realm of probability theory using the appropriate Gibbs measure. However, good limiting behavior for $G = \mathrm{SU}(m)$, $m \geq 2$, as the lattice spacing goes to 0 is only expected in dimensions $d \leq 4$, where the corresponding quantum field theory is renormalizable and asymptotically free, and establishing such behavior and the desired properties of the limit (such as mass gap) for $d = 4$ is literally a million dollar question—it is one of the Millennium Problems!

Nevertheless, there have been impressive numerical simulations on a lattice exhibiting the expected phenomena in 4-dimensional nonabelian gauge theory (such as mass gap and confinement) ([**Ro**]). Note that such results are highly nontrivial to obtain even on the most powerful computers. Indeed, if we put just ten nodes along each axis, the path integral for the pure $\mathrm{SU}(2)$ gauge theory is over a space of dimension about 10^5 (e.g., in the temporal gauge we have three spatial components A_1, A_2, A_3 of the gauge field valued in $\mathfrak{su}(2) \cong \mathbb{R}^3$ at each of the 10^4 nodes of the lattice). Computing such integrals with reasonable precision is an enormous numerical challenge, which is addressed using Monte Carlo methods.

For more details on lattice gauge theories see [**T4**, Section 4].

Chapter 14

Supersymmetry

In Sections 11.3 and 11.4 in theories that involve both bosons and fermions, we only considered even symmetries that preserve parity of fields. But it turns out to be very useful to also consider odd symmetries which change the parity; i.e., to allow the Lie algebra of symmetries to be a Lie superalgebra. Especially important is a special kind of such symmetries in terms of which the vector field of time translation can be expressed quadratically. These are called *supersymmetries*.

In this section we will provide a brief review of supersymmetry in field theory, in the Euclidean setting. For more details the reader can consult [**T1**] and for higher dimensions [**QFS**], [**T3**], [**B**].

14.1. $N = 1$ supersymmetric classical mechanics

Let us begin with mechanics ($d = 1$) and a single supersymmetry ($N = 1$). We first consider classical mechanics in the Lagrangian setting.

Recall the Lie superalgebra $\mathfrak{q} = \text{Lie } \mathcal{Q}$ of Example 8.9(3) spanned by an odd element Q and an even element H with commutation relations

$$[Q, H] = 0,\ [Q, Q] = 2H.$$

Definition 14.1.

(i) An $N = 1$ supersymmetry of a Lagrangian $\mathcal{L}$ is a left action of $\mathfrak{q}$ on the algebra $\mathcal{A}$ of (real) classical observables attached to $\mathcal{L}$ (i.e., local functionals of fields modulo equations of motion) preserving the Poisson bracket, such that the element H acts by $-\frac{d}{dt}$. In other words, it is an odd Poisson derivation Q of $\mathcal{A}$ such that $Q^2 = -\frac{d}{dt}$.

(ii) An $N = k$ supersymmetry[58] is a collection of k linearly independent (super-)commuting $N = 1$ supersymmetries.[59]

Example 14.2. Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\psi\dot{\psi}, \tag{14.1}$$

where ϕ is a real boson and ψ is a real fermion. So the equations of motion are

$$\ddot{\phi} = 0,\ \dot{\psi} = 0.$$

Define Q by the equalities

$$Q\phi = \psi,\ Q\psi = -\dot{\phi}.$$

Thus $Q^2 = -\frac{d}{dt}$ and

$$Q\dot{\phi} = \dot{\psi},\ Q\dot{\psi} = -\ddot{\phi},$$

so Q preserves the equations of motion. It is also easy to check that Q preserves the Poisson bracket, which is defined by the equalities

$$\{\phi, \dot{\phi}\} = \{\psi, \psi\} = 1,\ \{\psi, \phi\} = \{\psi, \dot{\phi}\} = 0.$$

Thus this theory is $N = 1$ supersymmetric.

The same holds if ϕ, ψ are valued in some Euclidean space E.

In the Hamiltonian picture, by Noether's theorem, the supersymmetry must arise from Poisson bracket with a uniquely determined odd observable called the *classical supercharge*, which we, abusing notation, will also denote by Q. For instance, in Example 14.2 we have

$$Q = -\dot{\phi}\psi.$$

Likewise, if in Example 14.2 we take ψ to be a fermion valued in some Euclidean space E of dimension k, then the theory has $N = k$ supersymmetry defined by the derivations Q_j such that in an orthonormal basis of E,

$$Q_j\phi = \psi_j,\ Q_j\psi_i = -\delta_{ij}\dot{\phi}. \tag{14.2}$$

Indeed, then $Q_j^2\phi = -\dot{\phi}$, $Q_j^2\psi_i = -\delta_{ij}\dot{\psi}_j = -\dot{\psi}_i$ since $\dot{\psi}_j = 0$ by the equations of motion, so $Q_j^2 = -\frac{d}{dt}$. Also it is easy to check that $Q_iQ_j + Q_jQ_i = 0$, as needed.

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[58] Some sources call this $N = \frac{k}{2}$ supersymmetry, see [**T1**, 1.4.2, 1.4.3].

[59] Note that if the theory is not topological, i.e., $\frac{d}{dt} \neq 0$ (the time evolution is nontrivial), then any k commuting supersymmetries are automatically linearly independent, as they satisfy the relations $Q_iQ_j + Q_jQ_i = -2\delta_{ij}\frac{d}{dt}$.

Example 14.3. Consider the theory of Example 14.2 but now with both ϕ and ψ valued in a Euclidean space E. We have shown that this theory has $N = 1$ supersymmetry, but maybe sometimes it has more? Let's explore.

We can modify Q to build other supercharges on the same theory: if $a : E \to E$ is an invertible operator, then we can define the supercharge Q_a by

$$Q_a\phi = a\psi,\ Q_a\psi = -a^{-1}\dot{\phi},$$

and then $Q_a^2 = -\frac{d}{dt}$. Recall also that the Poisson bracket in this theory is given in an orthonormal basis of E by the formulas

$$\{\phi_i, \dot{\phi}_j\} = \delta_{ij},\ \{\psi_i, \phi_j\} = \{\psi_i, \dot{\phi}_j\} = 0,\ \{\psi_i, \psi_j\} = \delta_{ij}.$$

So, since Q_a must preserve the Poisson bracket, we should have

$$0 = Q_a\{\phi_i, \psi_j\} = \{(a\psi)_i, \psi_j\} - \{\phi_i, (a^{-1}\dot{\phi})_j\} = a_{ij} - (a^{-1})_{ji},$$

so a must be an orthogonal operator.

Now we may try to define $Q_j, 1 \le j \le k$, by taking $Q_j = Q_{a_j}$ for $1 \le j \le k$ with $a_1 = 1$. Then we have

$$(Q_jQ_m + Q_mQ_j)\phi = -(a_j^{-1}a_m + a_m^{-1}a_j)\dot{\phi}.$$

Thus the supersymmetry algebra relations $Q_jQ_m + Q_mQ_j = 0$, $m \neq j$, are equivalent to the relations

$$(a_m^{-1}a_j)^2 = -1,\ m \neq j.$$

This is equivalent to saying that

$$a_j^2 = -1$$

for all $j \ge 2$ (taking $m = 1$) and

$$a_ja_m = -a_ma_j$$

for $j \neq m$, $j, m \ge 2$. It follows that the collection of elements $a_2, \dots, a_k$ defines an action of the real Clifford algebra $\mathrm{Cl}^-_{k-1}(\mathbb{R})$ of the quadratic form $-\sum_{i=1}^{k-1} x_i^2$ on E. The classification of real Clifford algebras implies that the dimension d_k of a real irreducible representation of the algebra $\mathrm{Cl}^-_{k-1}(\mathbb{R})$ satisfies the recursion

$$d_{k+8} = 16d_k,$$

and for $1 \le k \le 8$ the values of d_k are $1, 2, 4, 4, 8, 8, 8, 8$. Moreover, since the operators a_j generate a finite group, they act in any such representation orthogonally in a unique up-to-scaling invariant metric. So the amount of supersymmetry of our theory coming from this construction is the largest N such that d_N divides $\dim E$. For example, for $\dim E = 2$, we have $N = 2$, for $\dim E = 4$, we have $N = 4$, and so on.

14.2. $N = 1$ supersymmetric quantum mechanics

Now let us pass to quantum mechanics. The above discussion of the classical Hamiltonian picture suggests that in quantum theory the supersymmetry algebra should act unitarily on the Hilbert space $\mathcal{H}$ of the theory. Again we denote the corresponding operator by Q; so Q is an odd self-adjoint operator on $\mathcal{H}$ with $Q^2 = H$, where H is the Hamiltonian, called the *quantum supercharge* (note that this implies $H \geq 0$).

The benefit of supersymmetry in quantum theory is demonstrated by the following very simple but very important theorem. Suppose we are given an $N = 1$ supersymmetric quantum mechanical model with Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, Hamiltonian H, and quantum supercharge Q.

Theorem 14.4. *Let $\mathcal{I}(L)$ be the super-partition function of this model on the circle $\mathbb{R}/L\mathbb{Z}$, i.e.,*[60]

$$\mathcal{I}(L) = \mathrm{sTr}(e^{-LH}).$$

Then $\mathcal{I}(L)$ does not depend on L. Namely, $\mathcal{I}(L)$ equals the superdimension of the kernel of H.

Proof. Let $\mathcal{H}_i(\lambda)$ be the λ-eigenspace of H in $\mathcal{H}_i$. Then

$$\mathcal{I}(L) = \sum_{\lambda \in \mathbb{R}} (\dim \mathcal{H}_0(\lambda) - \dim \mathcal{H}_1(\lambda)) e^{-\lambda L}.$$

Since for $\lambda \neq 0$, the operator Q maps $\mathcal{H}_i(\lambda)$ to $\mathcal{H}_{1-i}(\lambda)$ and $Q^2 = \lambda \cdot I$ on these spaces, Q defines an isomorphism $\mathcal{H}_0(\lambda) \to \mathcal{H}_1(\lambda)$, which implies the statement. □

The same theorem holds if the theory is twisted by a unitary automorphism $g : \mathcal{H} \to \mathcal{H}$ which commutes with Q: we have

$$Z_{\mathrm{tw}}(L, g) := \mathrm{sTr}(ge^{-LH}) = \mathrm{sTr}(g, \mathrm{Ker}\, H).$$

Remark 14.5. The quantity $\mathrm{sTr}(e^{-LH})$, which by Theorem 14.4 is L-independent and equal to the superdimension of $\mathrm{Ker}\, H$ (i.e., the index of the operator Q) is called the *Witten index*. Similarly, $\mathrm{sTr}(ge^{-LH}) = \mathrm{sTr}(g, \mathrm{Ker}\, H)$ is called the *twisted Witten index*.[61]

14.3. $N = 2$ and $N = 4$ supersymmetric classical mechanics

The story becomes more interesting when we consider Lagrangian (14.1) with the fermion ψ taking values in a 2-dimensional Euclidean space, since in this case, as we have seen in Section 9.2, we can add a mass term for ψ. We have seen above that such a theory without a mass term has $N = 2$ supersymmetry.

[60] We assume that H has purely point spectrum with finite multiplicities and this trace exists.

[61] Note that in physics literature the length L of the circle is usually denoted by β.

It turns out that this supersymmetry can be preserved if we add a mass term for ψ, but then we must also add a mass term for ϕ in a coordinated manner. Let us consider this in detail.

Recall that $N = 2$ supersymmetry means that we have a pair of odd Poisson derivations Q_1, Q_2 of the algebra $\mathcal{A}$ of (real) classical observables such that

$$Q_1^2 = Q_2^2 = -\frac{d}{dt},\ Q_1Q_2 + Q_2Q_1 = 0.$$

Alternatively, these relations can be written in terms of the pair of conjugate complex derivations $Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2)$, $Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2)$ satisfying the $N = 2$ *supersymmetry algebra relations*

$$Q^2 = 0,\ (Q^\dagger)^2 = 0,\ QQ^\dagger + Q^\dagger Q = -2\frac{d}{dt}.$$

In this realization, it is convenient to introduce the complex-valued fermion $\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$, with complex conjugate $\psi^\dagger$, and write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - \omega^2\phi^2) + \mathrm{Re}(\psi^\dagger\dot{\psi}) + i\omega\psi^\dagger\psi, \tag{14.3}$$

where we have added mass terms for both ϕ and ψ. Actually, the imaginary part of the kinetic term is a total derivative and does not affect the action, so the real part sign can be dropped in the calculations below. The equations of motion for this Lagrangian are

$$\ddot{\phi} + \omega^2\phi = 0,\ \dot{\psi} + i\omega\psi = 0,\ \dot{\psi}^\dagger - i\omega\psi^\dagger = 0.$$

Define $Q, Q^\dagger$ by the equalities

$$Q\phi = \psi,\ Q^\dagger\phi = \psi^\dagger,\ Q\psi = Q^\dagger\psi^\dagger = 0,\ Q\psi^\dagger = -\dot{\phi} - i\omega\phi,\ Q^\dagger\psi = -\dot{\phi} + i\omega\phi.$$

Note that then $Q^2\psi^\dagger = -\dot{\psi} - i\omega\psi = 0$ by the equations of motion, so $Q^2 = 0$, and similarly $(Q^\dagger)^2 = 0$. Also

$$(QQ^\dagger + Q^\dagger Q)\psi = Q(-\dot{\phi} + i\omega\phi) = -\dot{\psi} + i\omega\psi = -2\dot{\psi}$$

and

$$(QQ^\dagger + Q^\dagger Q)\phi = Q\psi^\dagger + Q^\dagger\psi = -2\dot{\phi},$$

so the supersymmetry algebra relations are satisfied.

Now,

$$Q(\ddot{\phi} + \omega^2\phi) = \ddot{\psi} + \omega^2\psi = 0,\ Q(\dot{\psi}^\dagger - i\omega\psi^\dagger) = -\ddot{\phi} - \omega^2\phi = 0.$$

It is also straightforward to check that $Q, Q^\dagger$ preserve the Poisson bracket. Thus this theory is $N = 2$ supersymmetric.

Note that the masses (or, classically, frequencies) for ϕ and ψ in the Lagrangian are the same. This is, in fact, necessary to have supersymmetry: it is easy to show that the supersymmetry breaks when masses are different. Since Q maps ϕ to ψ, one may say that in our theory, supersymmetry preserves masses

of particles (fields); this is how this property (which is actually true much more generally) is formulated by physicists.

More generally, one may consider a nonfree theory in which the mass term ω for ψ depends on ϕ. In this case, it turns out that one can still save the supersymmetry, but at the cost of adding a potential for ϕ related in a certain way to ω. Namely, consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - U(\phi) + \mathrm{Re}(\psi^\dagger\dot{\psi}) + i\omega(\phi)\psi^\dagger\psi.$$

The equations of motion are

$$\ddot{\phi} + U'(\phi) - i\omega'(\phi)\psi^\dagger\psi = 0,\ \dot{\psi} + i\omega(\phi)\psi = 0,\ \dot{\psi}^\dagger - i\omega(\phi)\psi^\dagger = 0.$$

We want to keep the relations $Q\phi = \psi, Q^\dagger\phi = \psi^\dagger$, so $Q\psi = Q^\dagger\psi^\dagger = 0$. Let

$$Q\psi^\dagger = -\dot{\phi} - ih(\phi),\ Q^\dagger\psi = -\dot{\phi} + ih(\phi)$$

for some function h, then $Q^2\psi^\dagger = -\dot{\psi} - ih'(\phi)\psi$, so we need $h'(x) = \omega(x)$ for this to be zero (by the equations of motion). Assuming this, we have

$$\ddot{\psi} = \frac{d}{dt}(-ih'(\phi)\psi) = -ih''(\phi)\dot{\phi}\psi - h'(\phi)^2\psi,$$

so

$$\begin{aligned} Q(\dot{\psi}^\dagger - ih'(\phi)\psi^\dagger) &= -\ddot{\phi} - ih'(\phi)\dot{\phi} + ih''(\phi)\psi^\dagger\psi + ih'(\phi)(\dot{\phi} + ih(\phi)) \\ &= -\ddot{\phi} + ih''(\phi)\psi^\dagger\psi - h'(\phi)h(\phi) = U'(\phi) - h'(\phi)h(\phi). \end{aligned}$$

To have supersymmetry, this must vanish, so we need $U'(x) = h'(x)h(x)$, i.e.,

$$U(x) = \frac{1}{2}h(x)^2$$

(up to adding an inessential constant). If so, then

$$Q(\ddot{\phi} + U'(\phi) - ih''(\phi)\psi^\dagger\psi) = (U''(\phi) - h'(\phi)^2 - h(\phi)h''(\phi))\psi = 0$$

and

$$Q(\dot{\psi} + ih'(\phi)\psi) = 0,$$

and it is easy to check that Q preserves Poisson bracket, i.e., this theory is indeed $N = 2$ supersymmetric.

Summarizing, the supersymmetric Lagrangian looks like

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - h(\phi)^2) + \mathrm{Re}(\psi^\dagger\dot{\psi}) + ih'(\phi)\psi^\dagger\psi \tag{14.4}$$

(the free massive case corresponds to $h(x) = \omega x$).

As before, in the Hamiltonian picture supersymmetries must arise from Poisson brackets with uniquely determined odd observables (supercharges), which we, abusing notation, will also denote by $Q, Q^\dagger$. For instance, for Lagrangian (14.4) we have

$$Q = -(\dot{\phi} + i\omega(\phi))\psi,\ Q^\dagger = -(\dot{\phi} - i\omega(\phi))\psi^\dagger.$$

We may also consider a version of this theory with the boson ϕ valued in a Euclidean space E and the fermion ψ valued in $E_{\mathbb{C}}$, initially with Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}|\dot{\phi}|^2 + \mathrm{Re}(\psi^\dagger, \dot{\psi}). \tag{14.5}$$

As we know, it has $N = 2$ supersymmetry given by

$$Q\phi = \psi,\ Q^\dagger\phi = \psi^\dagger,\ Q\psi = Q^\dagger\psi^\dagger = 0,\ Q\psi^\dagger = Q^\dagger\psi = -\dot{\phi},$$

and it is natural to ask which potential terms can be added so that this supersymmetry is preserved.

The Lagrangian with the potential terms is

$$\mathcal{L} = \frac{1}{2}|\dot{\phi}|^2 - U(\phi) + \mathrm{Re}(\psi^\dagger, \dot{\psi}) + \mathrm{Re}(\psi^\dagger, i\omega(\phi)\psi),$$

where $U : E \to \mathbb{R}$ and $\omega : E \to \mathrm{End}(E_{\mathbb{C}})$. We may assume that $\omega(x)$ is Hermitian since the anti-Hermitian part of $\omega(x)$ does not contribute to the Lagrangian. As before, we deform the supercharge Q by setting

$$Q\psi^\dagger = -\dot{\phi} - ih(\phi),$$

where $h : E \to E$, and obtain $\omega = \nabla h$. Thus ω is real and ∇h is symmetric. It follows that $h = \nabla V$ for some function $V : E \to \mathbb{R}$, hence $\omega = \mathrm{Hess}(V)$ (where we identify E with E^* using the inner product). The rest of the calculation proceeds exactly as in the 1-dimensional case and leads to the equality

$$U(x) = \frac{1}{2}|h(x)|^2 = \frac{1}{2}|\nabla V(x)|^2.$$

Thus the $N = 2$ supersymmetric Lagrangian has the form

$$\mathcal{L} = \frac{1}{2}|\dot{\phi}|^2 - \frac{1}{2}|\nabla V(\phi)|^2 + \mathrm{Re}(\psi^\dagger, \dot{\psi}) + i(\psi^\dagger, \mathrm{Hess}(V)(\phi)\psi). \tag{14.6}$$

Example 14.6. Let us return to the free massless case with Lagrangian $\mathcal{L}_0$ given by (14.5). As in Example 14.3, given a collection of orthogonal operators $a_j : E \to E$ which define a representation of the real Clifford algebra $\mathrm{Cl}^-_{k-1}(\mathbb{R})$ on E (with $a_1 = 1$), we can define the supercharges $Q_j, Q_j^\dagger$, $1 \le j \le k$ by

$$Q_j\phi = a_j\psi,\ Q_j^\dagger\psi = a_j\psi^\dagger,\ Q_j\psi = Q_j^\dagger\psi^\dagger = 0,\ Q_j^\dagger\psi = Q_j\psi^\dagger = -a_j^{-1}\dot{\phi}.$$

So the amount of supersymmetry of our theory coming from this construction (i.e., the number of supercharges) is $N = 2k$, where k is the largest integer for which d_k divides $\dim E$.

For example, if $\dim E$ is even, then we have $k \ge 2$ (i.e., $N \ge 4$), so we have at least one nonscalar orthogonal operator $a_2 = I$ on E such that $I^2 = -1$. The operator I gives E a unitary structure—a complex structure compatible to the

Euclidean metric (i.e., E is a unitary module over $\mathrm{Cl}_1^-(\mathbb{R}) = \mathbb{C}$). Then in addition to the supercharges $Q, Q^\dagger$, we have another conjugate pair of supercharges $Q_*, Q_*^\dagger$ given by

$$Q_*\phi = I\psi,\ Q_*^\dagger\phi = I\psi^\dagger,\ Q_*\psi = Q_*^\dagger\psi^\dagger = 0,\ Q_*\psi^\dagger = Q_*^\dagger\psi = I\dot{\phi}.$$

Moreover, if $\dim E$ is divisible by 4, then we have $k \geq 4$ (i.e., $N \geq 8$), so there are at least three independent nonscalar orthogonal operators $a_2 = I$, $a_3 = J$, $a_4 = K$ on E such that

$$I^2 = J^2 = K^2 = -1,\ IJ = -JI,\ JK = -KJ,\ KI = -IK.$$

Then $T := JIK$ commutes with I, J, K and $T^2 = 1$, so replacing K with KT, we may assume without loss of generality that $T = 1$, i.e., I, J, K satisfy the quaternion relations

$$I^2 = J^2 = K^2 = -1,\ IJ = -JI = K,\ JK = -KJ = I,\ KI = -IK = J.$$

These operators give E a quaternionic unitary structure—a quaternionic structure compatible to the Euclidean metric (i.e., E is a unitary module over $\mathrm{Cl}_3^-(\mathbb{R}) = \mathbb{H} \oplus \mathbb{H}$ factoring through the first summand). Thus in addition to the supercharges $Q, Q^\dagger$ we have three other conjugate pairs of supercharges $Q_{a*}, Q_{a*}^\dagger$, $a = I, J, K$ given by

$$Q_{a*}\phi = a\psi,\ Q_{a*}^\dagger\phi = a\psi^\dagger,\ Q_{a*}\psi = Q_{a*}^\dagger\psi^\dagger = 0,\ Q_{a*}\psi^\dagger = Q_{a*}^\dagger\psi = a\dot{\phi}.$$

One may now ask under what condition on V the $N = 4$ supersymmetry of Example 14.6 for even dimensional E lifts to the theory with Lagrangian (14.6). Tracing through the calculations, we see that for this we need the operator $\mathrm{Hess}(V)(x) : E \to E$ to anticommute with I, i.e., to be complex antilinear. This condition is equivalent to the equations

$$\partial_{z_i}\overline{\partial}_{z_j}V = 0,$$

for some coordinates z_i on E. These equations are, in turn, equivalent to the condition that V is the sum of a holomorphic and antiholomorphic function, so, given that V is real, we have

$$V(z) = \mathrm{Re}\, W(z),$$

where W is a holomorphic function on E. Thus we get

$$\mathcal{L} = \frac{1}{2}|\dot{\phi}|^2 - \frac{1}{2}|\partial W(\phi)|^2 + \mathrm{Re}(\psi^\dagger, \dot{\psi}) + i(\psi^\dagger, \mathrm{Hess}(\mathrm{Re}\, W)(\phi)\psi). \tag{14.7}$$

The theory defined by $\mathcal{L}$ is called the (1-dimensional) *Landau-Ginzburg model*. The function W is called the *Landau-Ginzburg superpotential*.

14.4. $N = 2$ supersymmetric quantum mechanics

Now let us pass to quantum mechanics. The above discussion of the classical Hamiltonian picture suggests that in quantum theory the supersymmetry algebra should act unitarily on the Hilbert space $\mathcal{H}$ of the theory. Again we denote the corresponding operators (quantum supercharges) by $Q, Q^\dagger$; so $Q^\dagger$ is the adjoint of Q, and

$$Q^2 = 0,\ (Q^\dagger)^2 = 0,\ QQ^\dagger + Q^\dagger Q = 2H.$$

Moreover, the commutation relations between $Q, Q^\dagger$ and local operators should be given by quantizations of the classical supersymmetry relations.

Since $N = 2$ supersymmetry is just a combination of two commuting supersymmetries, Theorem 14.4 holds in any $N = 2$ supersymmetric theory (for the proof we may use either Q_1 or Q_2). In addition, we also have the following result.

Theorem 14.7. *In an $N = 2$ supersymmetric theory, the operators $Q, Q^\dagger$ act by 0 on* Ker H*, and the cohomology of Q and of $Q^\dagger$ on (the H-finite part of) $\mathcal{H}$ equals* Ker H.

Proof. If $Hv = 0$, then

$$0 = \langle v, (QQ^\dagger + Q^\dagger Q)v\rangle = \|Q^\dagger v\|^2 + \|Qv\|^2,$$

so $Qv = Q^\dagger v = 0$. On the other hand, if $Hv = \lambda v$ for $\lambda \neq 0$ and $Qv = 0$, then $v = \frac{1}{2\lambda}(QQ^\dagger + Q^\dagger Q)v = Q(\frac{1}{2\lambda}Q^\dagger v)$, so the cohomology of Q on $\mathcal{H}(\lambda)$ is zero, and similarly for $Q^\dagger$. □

Example 14.8. Consider the theory defined by Lagrangian (14.3). Since the boson and the fermion are decoupled, we have

$$\mathcal{I}(L) = Z_{\text{boson}}(L)\mathcal{I}_{\text{fermion}}(L),$$

where Z_{boson} is the partition function for bosons and $\mathcal{I}_{\text{fermion}}$ is the super-partition function for fermions. Recall that

$$Z_{\text{boson}}(L) = \frac{1}{e^{\frac{\omega L}{2}} - e^{-\frac{\omega L}{2}}}$$

(formula (6.5)) and

$$\mathcal{I}_{\text{fermion}}(L) = e^{\frac{\omega L}{2}} - e^{-\frac{\omega L}{2}}$$

(formula (9.1)). Thus $\mathcal{I}(L) = 1$. This agrees with Theorem 14.4, since $\mathcal{H}_0 = \mathcal{H}_1 = L^2(\mathbb{R})$ and $\operatorname{Ker} H \subset \mathcal{H}_0$ is 1-dimensional.

Exercise 14.9. Consider the quantum version of the theory of Example 14.6, with Hilbert space $L^2(V) \otimes \Lambda V$ (square integrable mixed differential forms on V) and Hamiltonian $H = -\frac{1}{2}\Delta$, where Δ is the Laplace operator on V (acting on the coefficients of differential forms). Show that the classical supersymmetry

lifts to this quantum theory with $Q = d$, $Q^\dagger = d^\dagger$, where d is the de Rham differential.

14.5. $N = 1$ supersymmetric quantum mechanics of a particle on a Riemannian manifold and index of the Dirac operator

The $N = 1$ supersymmetric classical mechanics (Section 14.1) has an important interacting (curved) generalization, when the flat space E is replaced by a Riemannian manifold M. Namely, we may consider supersymmetric quantum mechanics valued in M whose fields are pairs (ϕ, ψ), where $\phi : \mathbb{R} \to M$ is a C^∞-map and ψ is a section of the odd vector bundle $\phi^*\Pi TM$ on $\mathbb{R}$; i.e., $\psi(t)$ is an element of $\Pi T_{\phi(t)}M$ (a real fermion). For such fields, Lagrangian (14.1) makes sense (with $\dot\psi$ being the covariant time derivative of ψ with respect to the Levi-Civita connection), and one can check that the theory remains $N = 1$ supersymmetric with the same formulas for the supercharge Q as in the flat case.

Let us now consider the quantization of this theory. As in the flat case, a nice quantization requires $\dim M = 2n$ to be even, and moreover we need M to be a spin manifold (see Section 11.1). The Hilbert space of the quantum theory is then the space of L^2 sections of the total spin bundle

$$\mathcal{H} = L^2(M, SM) = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where $\mathcal{H}_\pm := L^2(M, S_\pm M)$ with $\mathcal{H}_+$ being even and $\mathcal{H}_-$ being odd (the curved analogue of the space $L^2(M) \otimes S$ arising in the flat case), and the Hamiltonian is the *spin Laplacian*

$$H = \mathbf{D}^2,$$

where $\mathbf{D} = \mathbf{D}_+ \oplus \mathbf{D}_-$ is the *Dirac operator*, with $\mathbf{D}_\pm : \mathcal{H}_\pm \to \mathcal{H}_\mp$ defined using the Levi-Civita connection (we remind the reader that $\mathbf{D}$ is an unbounded self-adjoint operator, so it is in fact defined only on a dense subspace of $\mathcal{H}_\pm$, namely the Sobolev space $W^{1,2}(M, S_\pm M)$). The supercharge Q is then the Dirac operator $\mathbf{D}$. Thus for compact M, when $\mathbf{D}$ has discrete spectrum, Theorem 14.4 implies that the partition function of our theory on the circle of length L is independent of L and equals the index of the Dirac operator $\mathbf{D}_+$ regarded as a Fredholm operator $W^{1,2}(M, S_+M) \to L^2(M, S_-M)$,

$$\mathrm{sTr}_{\mathcal{H}}(e^{-LH}) = \mathrm{ind}(\mathbf{D}_+) = \dim S_+^{\mathrm{har}} - \dim S_-^{\mathrm{har}},$$

where $S_\pm^{\mathrm{har}}$ are the spaces of *harmonic spinors*, i.e., solutions of the Dirac equation $\mathbf{D}\psi = 0$ or, equivalently, the spin Laplace equation $\mathbf{D}^2\psi = 0$ in $\mathcal{H}_\pm$ (they are canonically identified with the kernel and cokernel of the operator $\mathbf{D}_+ : W^{1,2}(M, S_+M) \to L^2(M, S_-M)$ since $\mathbf{D}_- = \mathbf{D}_+^\dagger$). As any index, $\mathrm{ind}(\mathbf{D}_+)$

is stable under Fredholm deformations, so it is independent of the Riemannian metric, i.e., is a topological invariant of M. A topological formula for this extremely important invariant is provided by the *Atiyah-Singer index theorem*.

Let us now explain how the Atiyah-Singer formula can be derived from the Feynman-Kac formula for this theory (we will follow E. Witten's lecture on this subject in [**QFS**, v. 1, pp. 477–485]). Namely, recall that the path integral expression for the partition function is

$$Z = \int_{\mathcal{M}_L} \exp\left(-\int_0^L \left(\frac{1}{2}|\dot{\phi}|^2 + \frac{1}{2}(\psi, \dot{\psi})\right) dt\right) D\phi(D\psi)^{-1}$$

where the integration is carried out over the set $\mathcal{M}_L$ of L-periodic pairs $(\phi(t), \psi(t))$. Replacing t by Lt and ψ by $\sqrt{L}\psi$, we get

$$Z = \int_{\mathcal{M}_1} \exp\left(-\frac{1}{2L}\int_0^1 \left(|\dot{\phi}|^2 + (\psi, \dot{\psi})\right) dt\right) D\phi(D\psi)^{-1} \tag{14.8}$$

(the infinite power of L which results from rescaling $D\psi$ is removed in the regularization procedure).

Since our theory is supersymmetric, this expression is, in fact, independent of L, so it can be computed by taking the limit $L \to 0$. In this limit, the integral localizes on constant loops $(\phi(t), \psi(t)) = (\phi_0, \psi_0) \in \Pi TM$, i.e.,

$$Z = \int_{\Pi TM} I(\phi_0, \psi_0) d\phi_0 (d\psi_0)^{-1},$$

where $I(\phi_0, \psi_0)$ is the Gaussian integral obtained by taking the quadratic part of the action near the constant loop (ϕ_0, ψ_0) and $d\phi_0(d\psi_0)^{-1}$ is the canonical density defined in Exercise 8.14. Recall from this exercise that a function on ΠTM is the same thing as a mixed degree differential form on M, so this integral can be written in more classical terms as

$$Z = \int_M I(\phi_0, \psi_0)$$

(the integral of the degree $\dim M$ component of I).

Let us now compute I. A direct computation in normal coordinates at ϕ_0 shows that the quadratic part of the Lagrangian near (ϕ_0, ψ_0) has the form

$$\mathcal{L}_{\text{quadratic}}(a, \psi) = \frac{1}{2}|\dot{a}|^2 + \frac{1}{2}(\psi, \dot{\psi}) - \frac{1}{2}(a, R(\phi_0)(\psi_0, \psi_0)\dot{a}),$$

where $a(t) \in T_{\phi_0}M$, $\psi(t) \in \Pi T_{\phi_0}M$ are perturbations of ϕ_0, ψ_0 such that $\int_0^1 a(t)dt = 0$, $\int_0^1 \psi(t)dt = 0$, and R is the Riemann curvature tensor.[62] Thus

[62] The extra factor 1/2 in the formula for I on p. 482 of [**QFS**] should be dropped. This error cancels out the error in the second displayed formula on p. 485 where $4\pi k$ should be $2\pi k$.

the Gaussian integral

$$I(\phi_0,\psi_0)=\int \exp\left(-\int_0^1 \mathcal{L}_{\text{quadratic}}(a,\psi)dt\right) Da(D\psi)^{-1}$$

equals the product of the fermionic integral and the bosonic integral, which yields

$$I(\phi_0,\psi_0)=\frac{\sqrt{\det_0(\partial_t)}}{\sqrt{\det_0(\partial_t^2+R(\phi_0)(\psi_0,\psi_0)\partial_t)}}=\frac{1}{\sqrt{\det_0(\partial_t+R(\phi_0)(\psi_0,\psi_0))}},$$

where $\det_0(A)$ is the regularized determinant of an operator A on the space of vector-valued functions on the circle with integral 0.

Recall that for $c\in\mathbb{C}$ the eigenvalues of the operator ∂_t+c on the circle $\mathbb{R}/\mathbb{Z}$ are $2\pi in+c$, $n\in\mathbb{Z}$ (with eigenfunctions $e^{2\pi int}$), so its regularized determinant on functions with integral zero is

$$\begin{aligned}\det_0(\partial_t+c)&=\left(\prod_{n\in\mathbb{Z}\setminus 0}(2\pi in+c)\right)_{\text{reg}}=\left(\prod_{n=1}^{\infty}(2\pi in+c)(-2\pi in+c)\right)_{\text{reg}}\\&=\left(\prod_{n=1}^{\infty}(4\pi^2n^2+c^2)\right)_{\text{reg}}=C\prod_{n=1}^{\infty}(1+\tfrac{c^2}{4\pi^2n^2})=C\frac{\sinh(c/2)}{c/2},\end{aligned}$$

where C is a universal constant coming from the regularization procedure. Thus if $\mathbf{M}$ is an m-by-m matrix, then

$$\det_0(\partial_t+\mathbf{M})=C^m\det\frac{\sinh(\mathbf{M}/2)}{\mathbf{M}/2}.$$

It follows that

$$Z=\operatorname{ind}(\mathbf{D}_+)=C^n\int_M\sqrt{\det\frac{R(x)/2}{\sinh(R(x)/2)}}.$$

This shows that the index vanishes unless n is even, i.e., $\dim M$ is divisible by 4. For example, since

$$\sqrt{\frac{z/2}{\sinh(z/2)}}=1-\frac{z^2}{48}+O(z^4),$$

for a 4-manifold we get

$$\operatorname{ind}(\mathbf{D}_+)=-\frac{1}{48}C^2\int_M\operatorname{Tr}(R(x)^2),$$

where $\operatorname{Tr}(R(x)^2)$ is a 4-form on M. This can be written as

$$\operatorname{ind}(\mathbf{D}_+)=-\frac{1}{24}(2\pi)^2C^2p_1(M),$$

where

$$p_1(M) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(R(x)^2) \tag{14.9}$$

is the first Pontryagin class of M, which is an element of $H^4(M,\mathbb{Z}) = \mathbb{Z}$.

The normalization constant C can now be determined by considering an example of a 4-manifold where the index is nonzero. The simplest such example is a K3-surface (for instance, a smooth quartic surface in $\mathbb{CP}^3$), which is known to have $p_1(M) = -48$, and $\mathrm{ind}(\mathbf{D}_+) = 2$. Thus we get that

$$C = \frac{1}{2\pi},$$

so we obtain

Theorem 14.10 (Atiyah-Singer index theorem for the Dirac operator)**.**

$$\mathrm{ind}(\mathbf{D}_+) = \frac{1}{(2\pi)^n}\int_M \sqrt{\det \frac{R(x)/2}{\sinh(R(x)/2)}}.$$

Note that the Bianchi identity implies that the mixed differential form

$$\frac{1}{(2\pi)^n}\sqrt{\det \frac{R(x)/2}{\sinh(R(x)/2)}}$$

is closed, so it represents a class $\widehat{A}(M) \in H^\bullet(M,\mathbb{R})$ (in fact, in $H^\bullet(M,\mathbb{Q})$) which is independent of the metric on M. This characteristic class is called the $\widehat{A}$-*genus* (*A-roof genus*) of M, and we have

$$\mathrm{ind}(\mathbf{D}_+) = \int_M \widehat{A}(M) = (\widehat{A}(M), [M]),$$

where $[M]$ is the fundamental class of M.

In particular, for 4-manifolds we get

$$\mathrm{ind}(\mathbf{D}_+) = -\frac{1}{24}p_1(M).$$

Thus we see that for any spin 4-manifold M, $p_1(M)$ must be divisible by 24 (in fact, it is even divisible by 48, i.e., the index of $\mathbf{D}_+$ is even, but this is not clear from the above considerations).

The possibility of nonzero index of the Dirac operator on spin 4-manifolds leads to an important phenomenon called *chiral anomaly* in 4-dimensional quantum field theories with spinor fields. Namely, let X be a compact spin 4-manifold and consider the theory of a chiral spinor ψ on X (a section of the chiral spin bundle S_+X) with Lagrangian $\mathcal{L} = (\psi^\dagger, \mathbf{D}\psi)$, where $\psi^\dagger$ is a section of S_-X. Classically this theory has a $U(1)$ symmetry acting by $\psi \mapsto e^{i\alpha}\psi$, hence $\psi^\dagger \mapsto e^{-i\alpha}\psi^\dagger$ to preserve the action. However, this symmetry becomes anomalous quantum-mechanically. To see this (somewhat informally), consider the

measure of integration $(D\psi D\psi^\dagger)^{-1}$ in the path integral. Decomposing this as a product over eigenvalues of the Dirac operator, we obtain that under the $U(1)$-symmetry, the measure multiplies by $e^{i\alpha\cdot \mathrm{ind}(\mathbf{D}_+)}$. So the fact that the index of $\mathbf{D}_+$ may be nonzero signals that the $U(1)$ symmetry suffers a quantum anomaly, which is called *chiral anomaly*. More formally, as explained above, the index of $\mathbf{D}_+$ is up to normalization the first Pontryagin class of X, which is obtained by integrating over X a certain 4-form—the trace of the exterior square of the curvature (formula (14.9)). It turns out that locally this 4-form is the divergence (differential) of the quantized current of the $U(1)$-symmetry (which is a 3-form on X). Thus the current is not conserved and the symmetry becomes anomalous (on any 4-manifold). For more details on chiral anomaly see [**T4**, Section 3].

14.6. $N = 2$ supersymmetric quantum mechanics of a particle on a Riemannian manifold and the Hodge theorem

The $N = 2$ supersymmetric theory of Exercise 14.9 also has an important curved generalization, when the flat space E is replaced by a Riemannian manifold M. Namely, as in the $N = 1$ case, we may consider supersymmetric quantum mechanics valued in M whose fields are pairs (ϕ, ψ), where $\phi : \mathbb{R} \to M$ is a C^∞-map and ψ is now a *complex* fermion, i.e., a section of the odd vector bundle $S \otimes \phi^* TM_{\mathbb{C}}$ on $\mathbb{R}$, so that $\psi(t)$ is an element of $\Pi T_{\phi(t)} M_{\mathbb{C}}$. For such fields, although the Lagrangian (14.5) makes sense, it is no longer supersymmetric. It turns out that in order to restore the $N = 2$ supersymmetry (with the same formulas for the supercharges $Q, Q^\dagger$), we have to add a curvature term quartic in ψ:

$$\mathcal{L} = \frac{1}{2}|\dot\phi|^2 + \mathrm{Re}(\psi^\dagger, \dot\psi) - \frac{1}{2}(\psi^\dagger, R(\psi, \psi^\dagger)\psi), \tag{14.10}$$

where R is the Riemann curvature tensor.[63]

Moreover, this supersymmetry extends to quantum theory, in which the Hilbert space $\mathcal{H}$ is the space $\Omega^\bullet_{L^2}(M)$ of mixed square integrable differential forms on M, with inner product

$$(\xi, \eta) = \int_M \overline{\xi} \wedge *\eta,$$

where $*$ is the *Hodge $*$-operator* $\Omega^i(M) \to \Omega^{\dim M - i}(M) \otimes \mathrm{Or}_M$, with Or_M the orientation bundle on M. The Hamiltonian of this quantum theory is $H = -\frac{1}{2}\Delta$, where Δ is the Laplace operator on differential forms (the *Hodge Laplacian*). The classical supersymmetry lifts to the quantum level by setting $Q =$

[63] The coefficient in front of R is uniquely determined but depends on notational conventions concerning the curvature tensor.

$d, Q^\dagger = d^\dagger$, where d is the de Rham differential; in particular,

$$dd^\dagger + d^\dagger d = -\Delta.$$

Thus if M is compact (so that H has pure point spectrum and e^{-LH} is trace class), then Theorem 14.7 says that the cohomology of d on forms of degree i (i.e., the ith de Rham cohomology $H^i(M, \mathbb{C})$) coincides with the space of $\eta \in \Omega^i(M)$ such that $\Delta\eta = 0$, i.e., the space of *harmonic i-forms*. Thus we obtain

Theorem 14.11 (Hodge theorem for Riemannian manifolds)**.** *Every harmonic differential form on M is closed, and every cohomology class of M has a unique harmonic representative. In other words, every closed form on M can be uniquely written as a sum of a harmonic form and an exact form.*

Theorem 14.4 for this supersymmetric model also has a nice geometric meaning. Namely, let $K(x, y, t)$ be the *heat kernel* for differential forms on M, i.e., the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u,\ u \in \Omega^\bullet(M).$$

In other words, $K(x, y, t)$ is the Schwartz kernel of the integral operator $e^{\frac{t\Delta}{2}}$ on $\Omega^\bullet_{L^2}(M)$. It is well known that K is a smooth function of $x, y \in M$ and $t > 0$, valued in $\oplus_i \operatorname{Hom}(\wedge^i T^*_x M, \wedge^i T^*_y M)$. Then Theorem 14.4 takes the following form.

Corollary 14.12. *For all $t > 0$*

$$\sum_i (-1)^i \int_M \operatorname{Tr}(K(y, y, t), \wedge^i T^*_y M) dy = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

Theorem 14.4 can also be used to derive the *Lefschetz fixed point formula* in the special case of diffeomorphisms of M of finite order or, more generally, ones belonging to a compact Lie group G acting faithfully on M. Namely, let M be a compact manifold, and let $g : M \to M$ be such a diffeomorphism with finitely many fixed points.

Theorem 14.13 (Lefschetz fixed point formula)**.** *We have*[64]

$$|M^g| = \sum_i (-1)^i \operatorname{Tr}(g, H^i(M, \mathbb{C})). \tag{14.11}$$

Proof. Pick a g-invariant Riemannian metric on M (it can be obtained by averaging any metric over G), and let Δ be the Hodge Laplacian for this metric. By Theorem 14.11, as $L \to \infty$, $\operatorname{sTr}(ge^{\frac{L\Delta}{2}})$ tends to the right-hand side of (14.11).

[64] Note that if A is an orthogonal matrix without eigenvalue 1, then $\det(\mathrm{I} - A) > 0$. Thus all fixed points of g have index $+1$.

On the other hand, it is easy to check that as $L \to 0$, the same supertrace tends to the left-hand side of (14.11). Thus Theorem 14.13 follows from Theorem 14.4. □

In fact, the assumption of finitely many fixed points in Theorem 14.13 can be dropped. To state this generalization, note that by Bochner's linearization theorem ([**Bo**]), if $g : M \to M$ is a diffeomorphism of a compact manifold M belonging to a compact Lie group G acting faithfully on M, then the fixed set M^g is a disjoint union of finitely many connected components Y which are compact manifolds, possibly of different dimensions.

Theorem 14.14 (Lefschetz formula for nonisolated fixed points)**.** *We have*

$$\chi(M^g) = \sum_i (-1)^i \operatorname{Tr}(g, H^i(M, \mathbb{C})). \tag{14.12}$$

Proof. The proof is the same as that of Theorem 14.13, using Corollary 14.12 for the $L \to 0$ limit. □

Remark 14.15. For cellular cohomology (which is isomorphic to de Rham cohomology by de Rham's theorem) and g of finite order, Theorem 14.14 is easy to prove by elementary algebraic topology, by lifting a cell decomposition of $M/\langle g\rangle$ compatible to the inclusion $M^g \hookrightarrow M/\langle g\rangle$ to M and computing the alternating sum of traces of g on cellular cochains in two different ways. Moreover, the general case easily follows from the finite order case. (Exercise: How?) However, derivation of Theorem 14.14 from a supersymmetric quantum mechanical model is still quite instructive. This use of supersymmetry based on comparing the limits $L \to 0$ and $L \to \infty$, which was also used in our derivation of the Atiyah-Singer formula, is prototypical for applications of supersymmetric quantum field theory in geometry and topology.

14.7. $N = 4$ supersymmetric quantum mechanics of a particle on a Kähler manifold

It is interesting that if M is a Kähler manifold, then the $N = 2$ supersymmetry of the theory of Section 14.6 extends to $N = 4$ supersymmetry. Indeed, when $M = E$ is a complex vector space with unitary structure, we have seen in Example 14.6 that this theory has $N = 4$ supersymmetry. Namely, the complexification $E_{\mathbb{C}}$ has the form $E_+ \oplus E_-$, where $E_+ = E$ and $E_- = \overline{E}$, so the fermion $\psi \in E$ splits as $\psi = \psi_+ + \psi_-$. Likewise, the supercharges $Q, Q^\dagger$ each split into two summands,

$$Q = Q_+ + Q_-,\ Q^\dagger = Q_+^\dagger + Q_-^\dagger$$

so that

$$Q_\pm \phi = \psi_\pm,$$
$$Q_+\psi_- = Q_+\psi_-^\dagger = Q_-\psi_+ = Q_-\psi_+^\dagger = 0,$$
$$Q_+\psi_+ = Q_-\psi_- = 0,\ Q_+\psi_+^\dagger = Q_-\psi_-^\dagger = -\frac{1}{2}\dot{\phi},$$

and the conjugate equations for $Q_\pm^\dagger$. Otherwise formulated, we have

$$Q_+ = \frac{1}{2}(Q - iQ_*),\ Q_- = \frac{1}{2}(Q + iQ_*),$$

where Q_* is as in Example 14.6.

Now suppose that M is a Riemannian manifold with an almost complex structure I which is orthogonal in the Riemannian metric g. Then we can define the operators $Q_\pm, Q_\pm^\dagger$ in the same way, but one can check (see [**QFS**, vol. 1, Theorem 3.46, p. 268]) that they only define $N = 4$ supersymmetry of the theory when two integrability conditions are satisfied:

1) the almost complex structure I on M is actually complex, and

2) the nondegenerate 2-form $\omega(v, u) := g(Iv, u)$ on M is symplectic, i.e., $d\omega = 0$.

These conditions are equivalent to the requirement that M is Kähler.

In quantum theory, recall that the Hilbert space $\mathcal{H}$ is the L^2 completion of the space of mixed square integrable differential forms $\Omega^\bullet(M)$, which on an almost complex manifold splits into a direct sum of spaces $\Omega^{p,q} = \Omega^{p,q}(M)$ of (p, q)-forms. Recall also that $Q = d$ is the de Rham differential, and $Q^\dagger = d^\dagger$. So if M is a complex manifold, then we have a splitting $Q = Q_+ + Q_-$, where $Q_+ = \partial, Q_- = \overline{\partial}$, with $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$, $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$, and we have

$$\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0,$$

and the conjugate equations. However, one can check that to have the equation

$$\partial\overline{\partial}^\dagger + \overline{\partial}^\dagger\partial = 0,$$

we need ω to be closed, i.e., a Kähler structure. Thus the quantum theory is $N = 4$ supersymmetric also precisely for Kähler M (and the supercharges depend on the Kähler structure).

Assume now that M is Kähler. Then we have

$$\partial\partial^\dagger + \partial^\dagger\partial = \overline{\partial\partial}^\dagger + \overline{\partial}^\dagger\overline{\partial} = -\frac{\Delta}{2},$$

where Δ is the Hodge Laplacian.

Recall that the *Dolbeault complex* is

$$0 \to \Omega^{p,0}(M) \to \Omega^{p,1}(M) \to \ldots \to \Omega^{p,\dim M}(M) \to 0,$$

where the differential is $\bar{\partial}$. The *Dolbeault cohomology* $H^{p,q}(M)$ is the degree q cohomology of this complex. By definition, it is the sheaf cohomology $H^q(M, \Omega^p_{M,\mathrm{hol}})$, where $\Omega^p_{M,\mathrm{hol}}$ is the sheaf of holomorphic p-forms on M; indeed, on a small ball the Dolbeault complex is an acyclic (soft) resolution of this sheaf.

Now assume that M is compact. Then, as in the Riemannian case, Theorem 14.7 implies that the Dolbeault cohomology $H^{p,q}(M)$ coincides with the space of $\eta \in \Omega^{p,q}(M)$ such that $\Delta\eta = 0$, i.e., the space of *harmonic* (p,q)*-forms*. Thus we obtain

Theorem 14.16 (Hodge theorem for Kähler manifolds)**.** *If M is a compact Kähler manifold, then every cohomology class in $H^{p,q}(M)$ has a unique harmonic representative in $\Omega^{p,q}(M)$. Thus we have the canonical Hodge decomposition*

$$H^i(M, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}(M).$$

Remark 14.17. As follows from Example 14.6, if a Euclidean space E is a quaternionic unitary space then the $N = 2$ supersymmetric theory of a particle in E in fact has $N = 8$ supersymmetry (see [**QFS**, v. 1, Theorem 3.46, p. 268]). It turns out that this supersymmetry extends to the curved case when M is a hyper-Kähler manifold. However, we will not discuss this here. We also note that flat models of Example 14.6 with larger amounts of supersymmetry do not admit curved generalizations preserving the supersymmetry.

14.8. Superspace formulations

A superspace formulation of a supersymmetric field theory is a formulation in which the spacetime manifold is upgraded to a supermanifold (the *superspacetime*) by adding some odd directions, and the fields of the theory are combined into *superfields*, which are functions (or, more generally, sections of various bundles) on this supermanifold. In such a formulation, supersymmetry becomes manifest, as it arises from symmetries of the super-spacetime.

Let us consider examples of superspace formulations. We start with the $N = 1$ supersymmetric mechanics. In this case, the super-spacetime is $\mathbb{R}^{1|1}$, with coordinates t (time) and θ (a new odd coordinate). Recall from Example 8.9 that we have odd vector fields D, D_* on $\mathbb{R}^{1|1}$ given by

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t},\ D_* = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}$$

such that

$$[D_*, D_*] = 2\frac{d}{dt} = -[D, D],\ [D_*, D] = 0.$$

Recall that the basic fields of our theory are a real boson $\phi(t)$ and a real fermion $\psi(t)$. Define the superfield $\Phi : \mathbb{R}^{1|1} \to \mathbb{R}$ given by

$$\Phi(t, \theta) := \phi(t) + \theta\psi(t),$$

and consider the Lagrangian

$$\mathcal{L}(\Phi) = -\frac{1}{2}\dot{\Phi}D\Phi = -\frac{1}{2}(\dot{\phi} + \theta\dot{\psi})(\psi - \theta\dot{\phi}) = -\frac{1}{2}\dot{\phi}\psi + \frac{1}{2}(\dot{\phi}^2 + \psi\dot{\psi})\theta.$$

It follows that

$$S := \int_{\mathbb{R}^{1|1}} \mathcal{L}(\Phi)dt(d\theta)^{-1} = \frac{1}{2}\int_{\mathbb{R}} (\dot{\phi}^2 + \psi\dot{\psi})dt, \tag{14.13}$$

exactly the action of $N = 1$ supersymmetric mechanics (Section 14.1). The supersymmetry of this theory is then simply given by the odd vector field $D_* = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}$ commuting with D, acting on functionals on the space $\mathrm{Maps}(\mathbb{R}^{1|1}, \mathbb{R})$.[65] Note that since $[D_*, D_*] = 2\frac{d}{dt}$, we have $[Q, Q] = -2\frac{d}{dt}$ (vector fields on $\mathbb{R}^{1|1}$ act dually on functionals on $\mathrm{Maps}(\mathbb{R}^{1|1}, \mathbb{R})$, which introduces a minus sign). It is easy to check that Q coincides with the supercharge constructed in Section 14.1.

A similar superspace formulation can be made for $N = 2$ supersymmetric mechanics. In this case we should consider the super-spacetime $\mathbb{R}^{1|2}$, with coordinates t (time) and θ_1, θ_2 (two new odd coordinates). Thus we have odd vector fields D_1, D_2, D_{1*}, D_{2*} on $\mathbb{R}^{1|2}$ given by

$$D_j = \frac{\partial}{\partial\theta_j} - \theta_j\frac{\partial}{\partial t},\ D_{j*} = \frac{\partial}{\partial\theta_j} + \theta_j\frac{\partial}{\partial t}$$

such that

$$[D_{j*}, D_{k*}] = 2\delta_{jk}\frac{d}{dt} = -[D_j, D_k],$$

and $[D_{j*}, D_k] = 0$. Define the superfield

$$\Phi(t, \theta_1, \theta_2) := \phi(t) + \theta_1\psi_1(t) + \theta_2\psi_2(t) + \theta_1\theta_2\eta(t),$$

where η is an auxiliary boson, and consider the Lagrangian

$$\mathcal{L}(\Phi) = \frac{1}{2}D_1\Phi D_2\Phi - V(\Phi).$$

The coefficient of $\theta_1\theta_2$ in $\mathcal{L}(\Phi)$ is

$$\frac{1}{2}(\dot{\phi}^2 + \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 + \eta^2) - V'(\phi)\eta + V''(\phi)\psi_1\psi_2,$$

so the action is

$$S = \int_{\mathbb{R}^{1,2}} \mathcal{L}(\Phi)dt(d\theta_1 d\theta_2)^{-1},$$

or in components

$$S = \int_{\mathbb{R}} \left(\frac{1}{2}(\dot{\phi}^2 + \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 + \eta^2) - V'(\phi)\eta + V''(\phi)\psi_1\psi_2\right)dt. \tag{14.14}$$

[65] For instance, the action of Q on homogeneous functionals on degree k is just the action of D_* on the dual symmetric power $(S^k \mathrm{Maps}(\mathbb{R}^{1|1}, \mathbb{R}))^*$, namely $(D_*F)(\phi_1, \dots, \phi_k) := -\sum_{i=1}^k F(\phi_1, \dots, D_*\phi_i, \dots, \phi_k)$.

The equations of motion imply that $\eta = V'(\phi)$, so substituting, we get

$$S = \int \left(\frac{1}{2}(\dot{\phi}^2 + \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 - V'(\phi)^2) + V''(\phi)\psi_1\psi_2\right) dt,$$

which coincides with the action defined by the supersymmetric Lagrangian (14.4) with $h(x) = V'(x)$. But the $N = 2$ supersymmetry is now manifest and does not require verification: it just comes from the fact that the supercharges are given by the formulas

$$Q_1 = L_{D_{1*}},\ Q_2 \coloneqq L_{D_{2*}}$$

and D_{1*}, D_{2*} commute with D_1, D_2, using that integral is invariant under Lie derivatives (Section 8.9).

Furthermore, the above superspace formulation of the $N = 1$ supersymmetric mechanics actually extends to the setting of Riemannian manifolds discussed in Section 14.5. In this case, the superfield Φ is simply a smooth map $\mathbb{R}^{1|1} \to M$, and the Lagrangian $-\frac{1}{2}\dot{\Phi}D\Phi$ makes perfect sense. This makes manifest the $N = 1$ supersymmetry in this theory, leading to the Atiyah-Singer theorem. Similarly, the superspace formulation of the $N = 2$ supersymmetric mechanics extends to the setting of Riemannian manifolds as discussed in Section 14.6, with the superfield Φ being a smooth map $\mathbb{R}^{1|2} \to M$. The Lagrangian $\frac{1}{2}D_1\Phi D_2\Phi$ makes perfect sense, which makes manifest the $N = 2$ supersymmetry in this theory, leading to Hodge theory of harmonic forms on compact Riemannian manifolds. Finally, if M is Kähler, then this theory admits a superspace formulation making manifest its $N = 4$ supersymmetry discussed in Section 14.7, leading to Hodge theory on compact Kähler manifolds. In this case, the superfield Φ is a smooth map $\mathbb{R}^{1|4} \to M$. However, there is no known superspace formulation making manifest the $N = 8$ supersymmetry when M is hyper-Kähler.

14.9. Off-shell and on-shell supersymmetry

The computation of the previous section raises a question about the significance of the auxiliary scalar field η, which was not present in the Lagrangian written "in components" and only appeared in the superspace formulation, just to be eliminated at once to return to the original Lagrangian. Is this η a red herring? Not quite.

To understand the meaning of η, we should discuss the notions of *off-shell* and *on-shell* (*super-*)*symmetry*.

Definition 14.18.

(i) A Lie superalgebra $\mathfrak{g}$ is said to act by an *off-shell symmetry* of classical field theory if it acts by derivations on its fields *before* imposing equations

of motion and preserves the action, i.e., preserves the Lagrangian up to total derivatives (=exact terms).

(ii) A Lie superalgebra $\mathfrak{g}$ is said to act by an *on-shell symmetry* of classical field theory if it acts by derivations on its fields *after* imposing equations of motion (i.e., by vector fields on the space of solutions of these equations) preserving the Hamiltonian (or, equivalently, the Lagrangian) and the Poisson bracket.[66]

Thus, an off-shell symmetry automatically defines an on-shell one, but **not** the other way around, in general. However, when one speaks of on-shell symmetry, one usually means that it does not exist off-shell.

In particular, we have notions of *off-shell and on-shell supersymmetry*.

Example 14.19. The $N = k$ supersymmetry (14.2) is off-shell for $k = 1$ but only on-shell for $k > 1$. Namely, in the $k = 1$ case (Example 14.2) we have $Q\phi = \psi$, $Q\psi = -\dot{\phi}$, so the equation $Q^2 = -\frac{d}{dt}$ holds without imposing the equations of motion, and we have

$$Q\mathcal{L} = \frac{1}{2}\dot{\psi}\dot{\phi} + \frac{1}{2}\psi\ddot{\phi} = \frac{d}{dt}\left(\frac{1}{2}\psi\dot{\phi}\right),$$

which is a total derivative. On the other hand, for $k > 1$ and $i \neq j$ we have

$$(Q_iQ_j + Q_jQ_i)\psi_i = -Q_j\dot{\phi} = -\dot{\psi}_j,$$

so for this to vanish, we have to impose the equation of motion $\dot{\psi}_j = 0$, hence $Q_j\dot{\psi}_j = \ddot{\phi} = 0$. But once we do, we have

$$Q_i\mathcal{L} = \frac{1}{2}\dot{\psi}_i\dot{\phi} + \frac{1}{2}\psi_i\ddot{\phi} = 0,$$

and Q_i preserves the Poisson bracket.

On the other hand, a theory with a superspace formulation has off-shell supersymmetry by design. This explains why the supersymmetry (14.2) for $k = 1$ is off-shell: the corresponding action (14.13) admits a superspace representation. Similarly, the action (14.14) has an off-shell $N = 2$ supersymmetry, but it requires the auxiliary field η. For example, if $V = 0$, we have for $j = 1, 2$,

$$Q_j\phi = \psi_j,\ Q_1\psi_1 = Q_2\psi_2 = -\dot{\phi},\ Q_1\psi_2 = -Q_2\psi_1 = \eta,\ Q_1\eta = -\dot{\psi}_2, Q_2\eta = \dot{\psi}_1.$$

One of the equations of motion is $\eta = 0$, so we can eliminate η (i.e., it is not a dynamical degree of freedom), but if we do so, the supersymmetry algebra relations will only hold if we also impose the equations of motion $\dot{\psi}_j = 0$ and $\ddot{\phi} = 0$, i.e., on-shell. Thus, the supersymmetry (14.2) for $k = 2$ is only on-shell, but can be upgraded to an off-shell one by introducing the auxiliary field η.[67]

[66] Here "shell" refers to the infinite-dimensional submanifold in the space of fields cut out by the equation of motion.

[67] We have already seen a similar appearance of an auxiliary field B in Section 13.7.

14.10. Supersymmetry in higher dimensions

Now let us discuss supersymmetry in higher-dimensional field theory. Similarly to $d = 1$, to have $N = 1$ supersymmetry in classical field theory in space-time dimension $d > 1$, we want an odd derivation Q of the (Poisson) algebra of observables satisfying the supersymmetry algebra relation $Q^2 = H$. But now we also have the Lorentz group symmetry, so conjugating Q by elements of this group yields new odd symmetries.

Consider first the Euclidean setting, where the Lorentz group is $\mathrm{SO}(d)$, and recall that the Hamiltonian H is the charge of the time translation symmetry. The time translation transforms in the tautological representation $\mathbb{R}^d$ of $\mathrm{SO}(d)$. So a supercharge Q should generate a complex representation Y of the Lie algebra $\mathfrak{so}(d)$ so that we have a surjective homomorphism of representations $\alpha : \mathrm{Sym}^2 Y \to \mathbb{C}^d$ resulting from the relation $Q^2 = H$. Moreover, since supersymmetry exchanges bosons and fermions, the spin-statistics theorem implies that Y should be not a representation of $\mathrm{SO}(d)$ but rather a representation of its double cover $\mathrm{Spin}(d)$ in which the central element ζ of $\mathrm{Spin}(d)$ acts by -1. The minimal choice of such Y is therefore a spin representation of $\mathrm{Spin}(d)$, so that the map α is induced by Clifford multiplication. More precisely, looking at the table in Section 10.4, we see that the minimal possibility for Y is $Y = S$, except:

1) $d = 8m + 5$ or $8m + 7$, where the minimal possibility is $2S$;

2) $d = 8m + 6$, where the minimal possibilities are S and $2S_\pm$;

3) $d = 8m + 2$, where the minimal possibility is S_+ or S_- (excluding $d = 2$, where S_+ is one dimensional and thus too small for α to be surjective, so we must use S).

However, we also have a condition that upon Wick rotation to Minkowski space, the representation Y (now of $\mathrm{Spin}_+(1, d-1)$) should have a real structure. This eliminates the possibility $Y = S$ for $d = 8m+6$ but imposes no other restrictions (see [**QFS**, p. 129]).

In particular, in low dimensions we obtain Table 14.1, where $\dim(Y)$ is the minimal number of real supercharges (and also for $d \neq 2$ the dimension of the smallest real spin representation).

Table 14.1

d	1	2	3	4	5	6	7	8	9	10	11
Y	S	S	S	S	$2S$	$2S_\pm$	$2S$	S	S	$S_\pm$	S
$\dim(Y)$	1	2	2	4	8	8	16	16	16	16	32

Thus the bottom row of the table gives the number of real supercharges for minimal supersymmetry in d dimensions.

This leads to the following definition.

Definition 14.20. An $N = 1$ supersymmetry in a d-dimensional classical field theory is a $\mathrm{Spin}(d)$-equivariant map $Y \to \mathrm{Der}(\mathcal{A})_1$ (where $\mathcal{A}$ is the algebra of observables and subscript 1 means odd derivations) such that for every $Q \in Y$, Q^2 acts by the corresponding translation symmetry $\alpha(Q)$. An $N = k$ supersymmetry is a collection of (super)commuting $N = 1$ supersymmetries.[68]

An $N = k$ supersymmetry in a quantum field theory is defined similarly, except that every $Q \in Y$ should act as an operator on the Hilbert space $\mathcal{H}$ of the theory.

In dimensions $d = 8m + 2$ and $d = 8m + 6$, where there is a possibility for chiral supersymmetries (involving only S_+ or S_-), physicists talk about $\mathcal{N} = (N_1, N_2)$ supersymmetry where there are N_1 chiral supersymmetries (involving only S_+) and N_2 anti-chiral ones (involving only S_-); so $N = N_1 + N_2$. Of course, an $\mathcal{N} = (N_1, N_2)$ supersymmetric theory becomes an $\mathcal{N} = (N_2, N_1)$ supersymmetric theory under orientation reversal.

Remark 14.21. For $d = 2$, an $\mathcal{N} = (1, 0)$ (or more generally, $\mathcal{N} = (N, 0)$) supersymmetric theory does not quite fall under our definition of supersymmetry, since the supercharges generate (in Minkowskian signature) only the left-moving translation $\partial_- = \partial_t - \partial_x$ but not the right-moving translation $\partial_+ = \partial_t + \partial_x$. For example, a minimal free $\mathcal{N} = (1, 0)$ supersymmetric theory has one real boson ϕ and one real (Majorana-Weyl) fermion ψ with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_+\phi\partial_-\phi + i\psi\partial_-\psi),$$

with the single supercharge Q acting by

$$Q\phi = \psi,\ Q\psi = i\partial_-\phi.$$

Still, even such partial supersymmetry, when present, is useful to consider.

Remark 14.22. Similarly to $d = 1$, many supersymmetric field theories in dimensions $d > 1$ have superspace formulations, in which the fields are combined into a superfield—a geometric object on the super-spacetime $\mathbb{R}^{d|\ell}$, where ℓ is the number of real supercharges. However, we will not discuss this here and instead refer the reader to [**QFS**, "Supersolutions"].

Let us now discuss how supersymmetry behaves under dimensional reduction. It is clear that all real supercharges automatically descend to the dimensionally reduced theory. On the other hand, irreducible spin representations can fall into several summands when restricted from $\mathrm{Spin}(d_1)$ to $\mathrm{Spin}(d_2)$. As

[68] Physicists call the case $N > 1$ *extended supersymmetry*.

a result, the amount of supersymmetry (the number N) often increases under dimensional reduction (although it may also stay the same). Namely, if $\ell(d)$ denotes the minimal number of supercharges in dimension d (last row of Table 14.1), then under reduction from dimension d_1 to dimension d_2, the number N (at least) multiplies by $\frac{\ell(d_1)}{\ell(d_2)}$. For example, an $N = 1$ supersymmetric theory in 10 dimensions would yield an $N = 1$ theory in $9, 8, 7$ dimensions, $N = 2$ theory in $6, 5$ dimensions, $N = 4$ theory in 4 dimensions, $N = 8$ theory in $3, 2$ dimensions, and $N = 16$ theory in 1 dimension. Thus a good way of constructing low dimensional theories with a large amount of supersymmetry is to obtain them by dimensional reduction from a theory with a smaller amount of supersymmetry (e.g., $N = 1$), but in higher dimension.

14.11. Supersymmetric sigma models

An example of a supersymmetric theory in higher dimensions is the 2-dimensional supersymmetric sigma model with Lagrangian being the 2-dimensional version of Lagrangian (14.10),

$$\mathcal{L} = \frac{1}{2}|\nabla\phi|^2 + \mathrm{Re}(\psi^\dagger, \mathbf{D}\psi) - \frac{1}{2}(\psi^\dagger, R(\psi, \psi^\dagger)\psi), \tag{14.15}$$

where ϕ is a map from $\mathbb{R}^2$ to a Riemannian manifold M and ψ is a section of the bundle $S \otimes \phi^* TM_{\mathbb{C}}$, with $\psi^\dagger$ being conjugate to ψ. This theory has an $\mathcal{N} = (1, 1)$-supersymmetry, defined similarly to the 1-dimensional case.

Moreover, a similar Lagrangian ([**QFS**, p. 268, formula (3.47)]) defines an $N = 1$ supersymmetric theory in three dimensions, but in this case the fermion ψ lives in $S \otimes_{\mathbb{R}} \phi^* TM$, where S is the irreducible 2-dimensional complex representation of $\mathrm{Spin}(3) = \mathrm{SU}(2)$ and does not fall into two complex conjugate components. The $\mathcal{N} = (1, 1)$-supersymmetric 2-dimensional sigma model is then obtained by dimensional reduction from three to two dimensions, and moreover it can be further reduced to one dimension to give the $N = 2$ supersymmetric mechanics on M (Lagrangian (14.10)).

Note that the $N = 1$ supersymmetric 3-dimensional sigma model and its reductions to lower dimensions have a supersymmetry preserving deformation involving a potential term (see [**QFS**, p. 282]). In the 1-dimensional case this deformation is defined by Lagrangian (14.4).

Furthermore, if M is Kähler, the sigma model has twice as much supersymmetry: $N = 2$ in dimension 3, $\mathcal{N} = (2, 2)$ in dimension 2, and $N = 4$ in dimension 1 (see Section 14.7). This is because the corresponding theory in four dimensions with similar Lagrangian (given by formula (3.48) in [**QFS**, p. 268]) has $N = 1$ supersymmetry. Upon reduction to lower dimensions this gives the aforementioned amounts of supersymmetry. As before, this theory admits a

supersymmetry preserving deformation involving a Landau-Ginzburg superpotential $W(z)$ (which is now a holomorphic function on M) that in one dimension reduces to the theory described by Lagrangian (14.7).

Finally, if M is hyper-Kähler, these theories have four times as much supersymmetry as in the general Riemannian case: $N = 4$ in dimension 3, $\mathcal{N} = (4,4)$ in dimension 2, and $N = 8$ in dimension 1. This is because the corresponding theory in six dimensions with similar Lagrangian (given by formula (3.49) in [**QFS**, p. 268]) has $\mathcal{N} = (1,0)$ supersymmetry. Upon reduction to lower dimensions, this gives the aforementioned amounts of supersymmetry.

However, as explained in Chapter 13, unless M is flat, above two dimensions these theories are not renormalizable and make sense only as classical field theories or low energy effective theories of quantum field theories which cannot be defined at high energies by a Lagrangian.

14.12. The super Poincaré group and supermultiplets

Now consider supersymmetric quantum field theories in $d > 1$ spacetime dimensions. The Hilbert space of states of such a theory carries a unitary action not just of the (centrally extended) Poincaré group $\widetilde{\mathbf{P}} := \mathrm{Spin}_+(1,d-1) \ltimes \mathbb{R}^d$ but also of the supercharges $Q \in Y$, where $Y = Y_{d,\mathcal{N}}$ is a real polyspin representation depending on the amount $\mathcal{N}$ of supersymmetry. Together they generate a supergroup called the (centrally extended) *super Poincaré group*, which we denote by $\widetilde{\mathbf{P}}_{\mathcal{N}}$. Its Lie algebra has the form $\mathfrak{p}_{\mathcal{N}} = \mathfrak{so}(1,d-1) \ltimes \mathfrak{q}_{d,\mathcal{N}}$, where $\mathfrak{q}_{d,\mathcal{N}} = \mathbb{R}^d \oplus Y$ with $\mathbb{R}^d$ central and the supercommutator $\alpha : \mathrm{Sym}^2 Y \to \mathbb{R}^d$ defined by Clifford multiplication. Thus $\widetilde{\mathbf{P}}_{\mathcal{N}} = \mathrm{Spin}_+(1,d-1) \ltimes \mathcal{Q}_{d,\mathcal{N}}$, where $\mathcal{Q}_{d,\mathcal{N}}$ is the supergroup attached to the Lie superalgebra $\mathfrak{q}_{d,\mathcal{N}}$.

To understand the particle content of supersymmetric QFT, we need to describe irreducible representations of the Lie superalgebra $\mathfrak{q}_{d,\mathcal{N}}$. In any such representation W, the even subalgebra $\mathbb{R}^d$, being central, acts by the character defined by an element $p \in \mathbb{R}^d$. Let $U_p := U_p(\mathfrak{q}_{d,\mathcal{N}})$ be the quotient of the universal enveloping algebra[69] $U(\mathfrak{q}_{d,\mathcal{N}})$ by the ideal generated by elements $x - (p,x), x \in \mathbb{R}^d$, where $(,)$ is the Minkowski inner product on $\mathbb{R}^d$. Thus W is an irreducible representation of U_p. If $p = 0$, then $U_p = \wedge Y$, so W is trivial. Thus assume that $p \neq 0$.

Proposition 14.23.

(i) *If p has mass $m > 0$ (i.e., $p^2 = m^2$), then U_p is the Clifford algebra* $\mathrm{Cl}(Y)$*, so W is the spin representation for this Clifford algebra of dimension $2^{\frac{\ell}{2}}$, where $\ell = \dim Y$ is the number of real supercharges.*

[69] The universal enveloping algebra of a Lie superalgebra $\mathfrak{g}$ is the quotient $U(\mathfrak{g})$ of the tensor algebra $T\mathfrak{g}$ by the relations $xy - (-1)^{p(x)p(y)}yx = [x,y]$ for homogeneous $x, y \in \mathfrak{g}$.

(ii) *If p is massless (i.e., $p^2 = 0$), then $U_p \cong \mathrm{Cl}(Y_+) \otimes \wedge Y_-$ for some decomposition $Y = Y_+ \oplus Y_-$ stable under the compact part $G_p \cong \mathrm{Spin}(d-2)$ of the little group fixing p. Consequently, if ℓ is divisible by 4, then W is the spin representation of $\mathrm{Cl}(Y_+)$ of dimension $2^{\frac{\ell}{4}}$, and if ℓ is not divisible by 4, then W is the spin representation of dimension $2^{\frac{\ell+2}{4}}$ (it is irreducible as a graded representation). The space Y_- acts on these representations by zero.*

Proof. The element p defines the Clifford multiplication operator

$$\Gamma(p) : Y \rightleftarrows Y' \cong Y^*,$$

see Section 10.3. To prove (i), we need to show that $\Gamma(p)$ is invertible in the massive case. But $\Gamma(p)^2 = m^2$ (as Γ defines an action of the Clifford algebra), so the statement follows. In the massless case, we have $\Gamma(p)^2 = 0$, so our job in (ii) is to show that the rank of the quadratic form on Y defined by p (i.e., the rank of the operator $\Gamma(p)$) is $\frac{\ell}{2}$. This is equivalent to saying that $\Gamma(p)$ is acyclic (its cohomology is zero). It suffices to prove this when $Y = \mathbb{S}$, where $\mathbb{S}$ is the spin representation of $\mathrm{Cl}(\mathbb{R}^d)$. We may assume that $p = (1, 1, 0, \dots, 0)$ and that $\mathrm{Cl}(\mathbb{R}^d)$ is generated by $\xi_0, \xi_1, \dots, \xi_{d-1}$ where $\xi_0^2 = 1$, $\xi_j^2 = -1$ for $j \geq 1$ and $\xi_i \xi_j = -\xi_j \xi_i$. Then $\Gamma(p)$ is the operator of multiplication by $\xi_0 + \xi_1$. Thus it suffices to prove the statement for $d = 2$, in which case it is easy. □

Now, since the Hilbert space of a supersymmetric QFT is a representation of $\widetilde{\mathbf{P}}_{\mathcal{N}}$, particles in such a theory live in positive energy irreducible unitary representations of $\widetilde{\mathbf{P}}_{\mathcal{N}}$ which are called *supermultiplets*. As in the even case, such representations are subdivided into massless and massive ones. Upon restriction to the usual Poincaré group $\widetilde{\mathbf{P}}$, these supermultiplets fall into ordinary 1-particle representations, which gives the *particle content* of the supermultiplet (see Section 10.12).

Massive supermultiplets $\mathcal{H}(X_m^+, \rho)$ are supported on the positive part X_m^+ of the hyperboloid $p^2 = m^2$ as representations of $\mathbb{R}^d$ (where $m > 0$) and are parametrized by irreducible graded representations ρ of the stabilizer $\mathrm{Spin}(d-1) \ltimes \mathcal{Q}_{d,\mathcal{N}}$ of the point $p = (m, 0, \dots, 0)$ such that $\mathbb{R}^d \subset \mathcal{Q}_{d,\mathcal{N}}$ acts by the corresponding character. Thus we have $\rho = \rho_0 \otimes W$, where ρ_0 is an irreducible representation of the little group $\mathrm{Spin}(d-1)$ (stabilizer in $\mathrm{Spin}_+(1, d-1)$ of p) and W is as in Proposition 14.23(i).

Similarly, massless supermultiplets $\mathcal{H}(X_0^+, \rho)$ are supported on the positive part X_0^+ of the cone $p^2 = 0$ as representations of $\mathbb{R}^d$ (with X_0^+ replaced by $X_0^{+\pm}$ for $d = 2$), and are parametrized by irreducible graded representations ρ of the stabilizer $(\mathrm{Spin}(d-2) \ltimes \mathbb{R}^{d-2}) \ltimes \mathcal{Q}_{d,\mathcal{N}}$ of the point $p = (1, 1, 0, \dots, 0)$ ($p = (1, 1)$ or $p = (1, -1)$ if $d = 2$), such that $\mathbb{R}^d \subset \mathcal{Q}_{d,\mathcal{N}}$ acts by the corresponding character. Thus we have $\rho = \rho_0 \otimes W$, where ρ_0 is an irreducible representation

of the compact part $\mathrm{Spin}(d-2)$ of the little group (stabilizer in $\mathrm{Spin}_+(1,d-1)$ of p) and W is as in Proposition 14.23(ii).

However, there is also a requirement that the theory should have a CPT symmetry (Section 10.11). Thus if a certain supermultiplet is not CPT-invariant, then it must come together with its conjugate partner.

14.13. R-symmetry

An important aspect of supersymmetric field theories is *R-symmetry*, which is a unitary CPT-compatible symmetry commuting with the Poincaré group and normalizing the supercharges. Thus the group H_R of R-symmetries is the maximal compact subgroup of the group $\mathrm{Aut}_{\mathrm{Spin}(d)}(Y)$. Hence this group for various amounts of supersymmetries in various dimensions is as shown in Table 14.2.

Table 14.2

d	1	2	3	4	5	6	7	8	9	10	11
H_R	$O(N)$	$O(N_1)$ $\times O(N_2)$	$O(N)$	$U(N)$	$\mathrm{USp}(2N)$	$\mathrm{USp}(2N_1)$ $\times \mathrm{USp}(2N_2)$	$\mathrm{USp}(2N)$	$U(N)$	$O(N)$	$O(N_1)$ $\times O(N_2)$	$O(N)$

Proposition 14.23 implies that the group H_R acts projectively on the Hilbert space $\mathcal{H}(M)$ of every supermultiplet M. This means that to every 1-dimensional subgroup $T = U(1)$ of H_R one can attach the R-charge

$$R_T : \mathcal{H}(M) \to \mathcal{H}(M),$$

which is defined up to shift by a real constant. If M is CPT-stable, this constant is fixed by the CPT invariance since R-charges are reversed by the CPT symmetry (which is anti-unitary). However, if M is not CPT-stable, the full Hilbert space of 1-particle states in $\mathcal{H}(M) \oplus \overline{\mathcal{H}(M)}$ with R-charge $R_T \oplus -\overline{R_T}$, and we have the freedom of shifting the first summand by a constant $c \in \mathbb{R}$ and simultaneously the second summand by $-c$.

We note, however, that in an actual supersymmetric quantum field theory the a priori R-symmetry group H_R can break down to a subgroup. For example, if the theory contains a gauge field A giving rise to a vector particle, then this field (and hence the particle) must transform trivially under any global symmetry, including R-symmetry, since A is a section of an affine bundle rather than a vector bundle over spacetime and acting by linear operators on A does not make gauge-invariant sense. For example, as we will see below, in the $N = 4$ supersymmetric gauge theory in $d = 4$ dimensions, this leads to the R symmetry breaking down from $U(4)$ to $\mathrm{SU}(4)$ (see Example 14.29).

14.14. Examples of supermultiplets

Consider now some examples of supermultiplets. Since we will simultaneously talk about representations of $\mathrm{Spin}(d)$, $\mathrm{Spin}(d-1)$, and $\mathrm{Spin}(d-2)$, we will mark the corresponding spinor and vector representations by the subscript indicating this dimension to avoid confusion.

Example 14.24 ([**QFS**, pp. 1287–1288])**.** Let us compute the particle content of the simplest $N = 1$ massive supermultiplets in four dimensions. Then $Y = S_{4+} \oplus S_{4-}$ is the 4-dimensional spin representation of $\mathrm{Spin}_+(1,3)$. Thus ρ is a representation of the semidirect product of the little group $\mathrm{Spin}(3)$ by the Clifford algebra $\mathrm{Cl}_4(\mathbb{C}) = \mathrm{Cl}(Y)$. This Clifford algebra has a unique 4-dimensional irreducible representation $\wedge S_3$ (as a $\mathrm{Spin}(3)$-equivariant representation), where $S_3 = S_{4\pm}|_{\mathrm{Spin}(3)}$. So if ρ_0 is trivial, then

$$\rho = \wedge S_3 = \mathbb{C} \oplus S_3 \oplus \mathbb{C},$$

i.e., the corresponding supermultiplet contains

- two real scalars ϕ_1, ϕ_2 (or one complex scalar $\phi = \phi_1 + i\phi_2$);
- one Dirac spinor $\psi \in S_3$.

The R-symmetry $U(1)$ acts on this multiplet by $\phi \mapsto z\phi$, $\overline{\phi} \mapsto z^{-1}\overline{\phi}$, $\psi \mapsto \psi$, $z \in U(1)$. This is called the $N = 1$ *massive chiral multiplet*.

The free (Euclidean) Lagrangian producing this multiplet involves a complex scalar ϕ and a Dirac spinor $\psi \in S_4 = S_{4+} \oplus S_{4-}$ and has the form

$$\mathcal{L} = \frac{1}{2}(|\nabla\phi|^2 - m^2|\phi|^2 + (\psi^\dagger, \mathbf{D}\psi) - m(\psi^\dagger, \psi)) \tag{14.16}$$

with the R-symmetry $U(1)_R$ rotating ϕ and acting trivially on ψ (there is also another global $U(1)$ symmetry commuting with supercharges and R-symmetry which rotates both ϕ and ψ).

On the other hand, if ρ_0 is the 2-dimensional spin representation S_3 of $\mathrm{Spin}(3)$, then $\rho = S_3 \otimes \wedge S_3$ decomposes over $\mathrm{Spin}(3)$ as

$$\rho = \mathbb{C} \oplus 2S_3 \oplus E_3,$$

where E_3 is the 3-dimensional vector representation (i.e., the adjoint representation) of $\mathrm{Spin}(3)$. Thus the corresponding supermultiplet contains

- one real scalar,
- two Dirac spinors,
- one vector.

The R-symmetry $U(1)$ acts on this multiplet by $z \mapsto 1$ on the scalar and vector and $z \mapsto z, z \to z^{-1}$ on the two spinors. This is called the $N = 1$ *massive vector multiplet*, since it is the simplest supermultiplet containing a massive vector.

Example 14.25 ([**QFS**, pp. 1287–1288])**.** Consider now $N = 1$ massless supermultiplets in four dimensions. We still have $Y = S_{4+} \oplus S_{4-}$ for $\mathrm{Spin}_+(1,3)$, and $Y_+ = S_{2+} \oplus S_{2-}$ over the compact part $\mathrm{Spin}(2)$ of the little group. Thus ρ is a representation of the semidirect product of $\mathrm{Spin}(2)$ by the Clifford algebra $\mathrm{Cl}_2(\mathbb{C}) = \mathrm{Cl}(Y_+)$. So $\rho = \rho_0 \otimes \wedge S_{2+}$. Thus, if $\rho_0 = \mathbb{C}$, then

$$\rho = \wedge S_{2+} = \mathbb{C} \oplus S_{2+},$$

which yields one real scalar (helicity 0) and one chiral Weyl spinor (helicity 1/2). However, the CPT symmetry requires that this multiplet should come with its conjugate partner, which contains a real scalar (helicity 0) and an antichiral Weyl spinor (helicity $-1/2$), which is obtained from $\rho_0 = S_{2-}$. Thus we get a multiplet containing

- two real scalars (or one complex scalar), helicity 0;
- two Weyl spinors, helicities $\frac{1}{2}$ ($\psi_+ \in S_{2+}$) and $-\frac{1}{2}$ ($\psi_- \in S_{2-}$) exchanged by the CPT symmetry (i.e., a Dirac spinor).

The R-symmetry $U(1)$ acts on this multiplet by $z \mapsto z$, $z \to z^{-1}$ on the scalars $\phi, \overline{\phi}$ and $z \mapsto 1$ on the spinors ψ_+, ψ_-, but here we can shift the R-charge by a constant as explained above. For example, another way to assign R-charges is to give $\phi, \overline{\phi}$ charge 0, ψ_+ charge 1 and ψ_- charge -1 (if $Q\phi = \psi_+$ where Q is a supercharge of R-charge 1, see Section 14.13). This is called the *massless chiral multiplet*, since it comes from a chiral spin representation S_{4+} in four dimensions. It is obtained from the massive chiral multiplet by taking the limit $m \to 0$, so is produced by Lagrangian (14.16) for $m = 0$.

Now assume that $\rho_0 = S_{2+}$. Then

$$\rho = S_{2+} \otimes \wedge S_{2+} = S_{2+} \oplus S_{2+}^{\otimes 2},$$

so after adding the conjugate to secure CPT-invariance we obtain a multiplet containing

- one vector, two components in $S_{2\pm}^{\otimes 2}$ of helicities ± 1;
- two Weyl spinors, helicities $\pm\frac{1}{2}$ (in $S_{2\pm}$).

The R-symmetry $U(1)$ acts on this multiplet by $z \mapsto 1$ on the vector and $z \mapsto z^{-1}$, $z \to z$ on the spinors (for a particular choice of the additive constant, fixed by the condition that the vector should be neutral for the R-symmetry). This is called the $N = 1$ *massless vector multiplet*. It is realized by the free (Euclidean) Lagrangian which involves a $U(1)$ gauge field A and a Dirac spinor $\lambda = (\lambda_+, \lambda_-)$ in the trivial $U(1)$-representation (called *gaugino*), which has the form

$$\mathcal{L} = \frac{1}{2}(|F_A|^2 - (\lambda, \mathbf{D}\lambda)) \tag{14.17}$$

with the R-symmetry rotating λ_+, λ_- in opposite directions and preserving the gauge field A.

Note that the theory defined by (14.17) generalizes to any gauge group G, with λ valued in the adjoint representation $\mathfrak{g}$ of G. If G is nonabelian, then this theory is interacting (in the Lagrangian we need to replace the Dirac operator $\mathbf{D}$ with the covariant Dirac operator $\mathbf{D}_A$). In this case at the level of fields we get $\dim G$ copies of the vector multiplet.

Example 14.26 ([**QFS**, p. 1288])**.** Consider now the $N = 2$ massive supermultiplet in four dimensions with $\rho_0 = \mathbb{C}$. Then

$$Y = 2S_{4+} \oplus 2S_{4-}$$

and $S_{4\pm}|_{\mathrm{Spin}(3)} = S_3$, so we have

$$\rho = \wedge(2S_3) = \mathbb{C} \oplus 2S_3 \oplus (\mathbb{C} \oplus (E_3 \oplus \mathbb{C}) \oplus \mathbb{C}) \oplus 2S_3 \oplus \mathbb{C},$$

where E_3 is the 3-dimensional vector representation of $\mathrm{Spin}(3)$. Thus this supermultiplet contains

- one vector,
- four Dirac spinors,
- five scalars.

The R-symmetry group in this case is $U(2)$, and it acts on $2S_3$ giving the representation $(1/2, 0) \otimes S_3$ of $U(2) \times \mathrm{Spin}(3)$. So taking into account the R-symmetry and its commutation relations with CPT, we see that ρ decomposes as

(14.18)

$$\rho = (-1,-1) \otimes \mathbb{C} \bigoplus (0,-1) \otimes S_3 \bigoplus ((1,-1) \otimes \mathbb{C} \oplus (0,0) \otimes E_3) \bigoplus (1,0) \otimes S_3 \bigoplus (1,1) \otimes \mathbb{C}.$$

This is the $N = 2$ *massive vector multiplet* in four dimensions.

Example 14.27 ([**QFS**, p. 1288])**.** Consider the simplest $N = 2$ massless supermultiplet in four dimensions. In this case $Y = W \otimes S_{4+} \oplus W^* \otimes S_{4-}$, where W, W^* are the vector and covector representation of the R-symmetry group $U(2)$. The restriction of $S_{4\pm}$ to $\mathrm{Spin}(2)$ is $S_{2+} \oplus S_{2-}$, so

$$Y_+ = W \otimes S_{2+} \oplus W^* \otimes S_{2-}.$$

Thus, taking into account CPT-invariance, we see that the simplest massless supermultiplet has $\rho_0 = \det^{-1/2} \otimes S_{2-}$ and

$$\rho = \det^{-1/2} \otimes S_{2-} \otimes \wedge(W \otimes S_{2+})$$

as $U(2) \times \mathrm{Spin}(2)$-representations. This yields the decomposition of ρ by helicity as shown in Table 14.3.

Table 14.3

helicity	$-1/2$	0	$1/2$
$U(2)$-representation	$\det^{-1/2}$	$\det^{-1/2} \otimes W$	$\det^{-1/2} \otimes \wedge^2 W = \det^{1/2}$

Thus we see that the R-symmetry group is in fact the double cover $U(1) \times \mathrm{SU}(2)$ of $U(2)$ (i.e., $U(2)$ acts only projectively) and the particle content is as follows:

• two Weyl spinors (one chiral of helicity $1/2$ transforming in $\det^{1/2}$ and one anti-chiral of helicity $-1/2$ transforming in $\det^{-1/2}$);

• two scalars (helicity 0, transforming in $\det^{-1/2} \otimes W$).

This is the $N = 2$ *massless hypermultiplet* in four dimensions.

Example 14.28 ([**QFS**, p. 1288])**.** Consider now the $N = 2$ massless supermultiplet in four dimensions corresponding to $\rho_0 = S_{2-}^{\otimes 2}$. Then the induced representation is

$$\rho_- = S_{2-}^{\otimes 2} \otimes \wedge(W \otimes S_{2+}).$$

This yields the decomposition of ρ_- by helicity shown in Table 14.4.

Table 14.4

helicity	-1	$-1/2$	0
$U(2)$-representation	$\mathbb{C}$	W	det

The CPT-invariance implies that this representation must come with its anti-partner $\rho_+ = \rho_-^*$, i.e., $\rho = \rho_+ \oplus \rho_-$. Thus this supermultiplet has the decomposition by helicity that is shown in Table 14.5.

Table 14.5

helicity	-1	$-1/2$	0	$1/2$	1
$U(2)$-representation	$\mathbb{C}$	W	$\det \oplus \det^{-1}$	W^*	$\mathbb{C}$

So the particle content is as follows:

• one vector in the trivial $U(2)$-representation (two components of helicities 1 and -1);

• four Weyl spinors (two chiral of helicity $1/2$ transforming in W^* and two anti-chiral of helicity $-1/2$ transforming in W);

• two scalars (helicity 0, transforming in the $U(2)$ representations det and $\det^{-1}$).

This is the $N = 2$ *massless vector multiplet* in 4 dimensions.

Example 14.29. Consider the simplest $N = 4$ massless supermultiplet in four dimensions. In this case $Y = W \otimes S_{4+} \oplus W^* \otimes S_{4-}$, where W, W^* are the vector and covector representation of the R-symmetry group $\mathrm{SU}(4)$. Thus $Y_+ = W \otimes S_{2+} \oplus W^* \otimes S_{2-}$, so taking into account CPT-invariance, we see that the simplest massless supermultiplet has $\rho_0 = S_{2-}^{\otimes 2}$ (half-vector representation) and

$$\rho = S_{2-}^{\otimes 2} \otimes \wedge(W \otimes S_{2+})$$

as $\mathrm{SU}(4) \times \mathrm{Spin}(2)$-representations. This yields the decomposition of ρ by helicity as shown in Table 14.6.

Table 14.6

helicity	-1	$-1/2$	0	$1/2$	1
$\mathrm{SU}(4)$-representation	$\mathbb{C}$	W	$\wedge^2 W$	$\wedge^3 W = W^*$	$\wedge^4 W = \mathbb{C}$

Thus the particle content is as follows:

- one vector (two components of helicities 1 and -1);
- eight Weyl spinors (four chiral of helicity $1/2$ transforming under $\mathrm{SU}(4)$ in W^* and four anti-chiral of helicity $-1/2$ transforming in W);
- six scalars (helicity 0, transforming in $\wedge^2 W$).

And what happens to the central $U(1) \subset U(4)$ which acts on the supersymmetry algebra? The supercharges with R-charge 1 under this $U(1)$ shift one position to the right in Table 14.6, so the R-charges of the two components of the vector would have to be m and $m + 4$ for some m. But these two components come from a gauge field, so must be neutral under R-symmetry. This means that this $U(1)$ symmetry must break down to a proper subgroup $\mathbb{Z}/r \subset U(1)$, which acts on the two components of the vector by χ^m and χ^{m+4} for some m, where χ is a primitive character of $\mathbb{Z}/r$. Since the vector must be neutral under R-symmetry, we have $\chi^m = \chi^{m+4} = 1$, hence $\chi^4 = 1$ and r divides 4. But $\mathbb{Z}/4$ already acts as part of $\mathrm{SU}(4)$. So we see that the a priori $U(4)$-symmetry breaks down to $\mathrm{SU}(4)$.

This is the $N = 4$ *massless vector multiplet* in four dimensions.

Now, what happens for $N > 4$ in the massless case, in four dimensions? In this case the simplest supermultiplet would be the $\mathrm{Spin}(2)$-representation $\rho = S_{2-}^{\otimes[\frac{N+1}{2}]} \otimes \wedge(2[\frac{N+1}{2}]S_{2+})$ which contains $S_{2+}^{[\frac{N+1}{2}]}$, of spin $\frac{1}{2}[\frac{N+1}{2}] \geq \frac{N}{4}$, so $N > 4$ in a nonfree theory is forbidden by the Weinberg-Witten theorem which prohibits interacting massless particles of spin > 1 (see Section 10.13).

This is a prototypical example showing that interacting QFT cannot have too much supersymmetry, due to restrictions coming from the Weinberg-Witten theorem (and its versions in dimensions $d \neq 4$): a large amount of supersymmetry would imply the presence of particles of high spin, which this theorem disallows. In fact, the larger the dimension of the theory, the smaller the possible amount of supersymmetry. Even if one includes *gravitons* and *gravitinos* (particles of spin 2 and $\frac{3}{2}$ arising in nonrenormalizable low energy theories coming from string theory), the largest dimension of a supersymmetric theory is 11 (11-*dimensional supergravity*, which is believed to be the low energy effective theory of *M-theory*), and it can only have $N = 1$ supersymmetry; with larger dimension or supersymmetry there will inevitably be particles of spin > 2 which do not arise in physically meaningful interacting theories. For a conventional QFT with particles of spin ≤ 1, the largest dimension for an interacting supersymmetric quantum field theory is 6, and the largest amount of supersymmetry such a theory can have is $N = 2$ ($\mathcal{N} = (2,0)$ or $(1,1)$).

Example 14.30. Let us compute the particle content of the simplest $\mathcal{N} = (1,0)$ massive supermultiplet in dimension $d = 6$. The little group is $\mathrm{Spin}(5)$ and $Y = 2S_{6+}$ is the 8-dimensional polyspin representation of $\mathrm{Spin}_+(1,5)$ which restricts to $2S_5$ for $\mathrm{Spin}(5)$. Thus as a $\mathrm{Spin}(5)$-representation,

$$\rho = \wedge S_5 = \mathbb{C} \oplus S_5 \oplus (\mathbb{C} \oplus E_5) \oplus S_5 \oplus \mathbb{C},$$

where $E_5 = \wedge^2 S_5/\mathbb{C}$ is the 5-dimensional vector representation. It follows that this multiplet contains

- one vector;
- two Dirac spinors; and
- three scalars.

The R-symmetry group in this case is $\mathrm{USp}(2) = \mathrm{SU}(2)$. Taking it into account, we obtain the following decomposition as $\mathrm{SU}(2) \times \mathrm{Spin}(5)$-representations:

$$\rho = \mathbb{C} \otimes E_5 \oplus W \otimes S_5 \oplus S^2 W \otimes \mathbb{C},$$

where W is the 2-dimensional irreducible representation of $\mathrm{SU}(2)$.

This is the $\mathcal{N} = (1,0)$ *massive vector multiplet* in six dimensions.

Note that upon reduction to four dimensions, this supermultiplet becomes the $N = 2$ massive vector multiplet of Example 14.26, and correspondingly the restriction of ρ from $\mathrm{Spin}(5) \times \mathrm{SU}(2)$ to $\mathrm{Spin}(3) \times \mathrm{SU}(2)$ decomposes as the restriction of (14.18) from $\mathrm{Spin}(3) \times U(2)$ to $\mathrm{Spin}(3) \times \mathrm{SU}(2)$. Note also that as the mass becomes 0, the massive vector splits into a massless vector and a scalar (as expected by counting physical degrees of freedom, see Section 10.12 and Exercise 10.24).

Example 14.31. Let us now compute the particle content of the simplest $\mathcal{N} = (1,0)$ massless supermultiplet in dimension $d = 6$. The compact part of the little group is $\mathrm{Spin}(4)$, $Y = 2S_+$ is the 8-dimensional polyspin representation of $\mathrm{Spin}_+(1,5)$ which restricts to $2S_+ \oplus 2S_-$ for $\mathrm{Spin}(4)$, and $Y_+ = 2S_+$. Thus as a $\mathrm{Spin}(4)$-representation,

$$\rho = \wedge S_+ = \mathbb{C} \oplus S_+ \oplus \mathbb{C}.$$

It follows that this supermultiplet contains

- one chiral spinor; and
- two scalars.

The R-symmetry group in this case is still $\mathrm{USp}(2) = \mathrm{SU}(2)$. Taking it into account, we obtain the following decomposition as $\mathrm{SU}(2) \times \mathrm{Spin}(4)$-representations:

$$\rho = W \otimes \mathbb{C} \oplus \mathbb{C} \otimes S_+.$$

However, in a quantum field theory this multiplet must come with an anti-chiral partner, since a single spin representation S_+ of $\mathrm{Spin}_+(1,5)$ has no real structure. Thus altogether we get the $\mathcal{N} = (1,0)$ *massless hypermultiplet* with the following structure:

- one chiral and one anti-chiral spinor, both neutral for the R-symmetry group $\mathrm{SU}(2)$;
- four scalars transforming in two copies of the 2-dimensional irreducible representation W of $\mathrm{SU}(2)$.

This multiplet is produced by the Lagrangian with a quaternion-valued scalar field ϕ and a symplectic Majorana-Weyl spinor field ψ (with eight components transforming in S_{6+} viewed as a real representation, which is the real form of $2S_{6+}$) having the form

$$\mathcal{L} = \frac{1}{2}(|\nabla\phi|^2 + (\psi^\dagger, \mathbf{D}\psi)),$$

where $\mathrm{SU}(2)_R$ leaves ψ invariant and acts on ϕ by right multiplication by unit quaternions. Note that here we also have another $\mathrm{SU}(2)$ global symmetry that commutes with the supercharges and the R-symmetry—it acts by left multiplication of ϕ and ψ by unit quaternions (using that $\mathrm{End}_{\mathbb{R}\,\mathrm{Spin}_+(1,5)}(S_{6+}) = \mathbb{H}$).

Example 14.32. Let us now compute the particle content of the simplest $\mathcal{N} = (1,1)$ massless supermultiplet in dimension $d = 6$. The compact part of the little group is $\mathrm{Spin}(4)$, and $Y = 2S_{6+} \oplus 2S_{6-}$ is the 16-dimensional polyspin representation of $\mathrm{Spin}_+(1,5)$ which restricts to $4S_{4+} \oplus 4S_{4-}$ for $\mathrm{Spin}(4)$. Also we have $Y_+ = 2S_{4+} \oplus 2S_{4-}$ Thus as a $\mathrm{Spin}(4)$-representation,

$$\rho = \wedge(S_{4+} \oplus S_{4-}) = \mathbb{C} \oplus (S_{4+} \oplus S_{4-}) \oplus (\mathbb{C} \oplus E_4 \oplus \mathbb{C}) \oplus (S_{4+} \oplus S_{4-}) \oplus \mathbb{C},$$

where $E_4 = S_{4+} \otimes S_{4-}$ is the vector representation of $\mathrm{Spin}(4)$. It follows that this supermultiplet contains

- one vector;
- four spinors (two chiral and two anti-chiral); and
- four scalars.

The R-symmetry group in this case is $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Taking it into account, we obtain the following decomposition as $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{Spin}(4)$-representations:

$$\rho = (\tfrac{1}{2}, \tfrac{1}{2}) \otimes \mathbb{C} \oplus (\tfrac{1}{2}, 0) \otimes S_{4+} \oplus (0, \tfrac{1}{2}) \otimes S_{4-} \oplus (0,0) \otimes E_4.$$

Note that upon reduction to four dimensions, this supermultiplet becomes the $N = 4$ massless vector multiplet of Example 14.29, and correspondingly the restriction of ρ from $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{Spin}(4)$ to $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{Spin}(2)$ decomposes as the restriction of the representation of $\mathrm{SU}(4) \times \mathrm{Spin}(2)$ defined in Table 14.6 to $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{Spin}(2)$.

Example 14.33. Let us compute the particle content of the simplest $\mathcal{N} = (2,0)$ massless supermultiplet in dimension $d = 6$. The compact part of the little group is $\mathrm{Spin}(4)$, and $Y = 4S_{6+}$ is the 16-dimensional polyspin representation of $\mathrm{Spin}_+(1,5)$ which restricts to $4S_{4+} \oplus 4S_{4-}$ for $\mathrm{Spin}(4)$. Also we have $Y_+ = 4S_{4+}$. Thus as a $\mathrm{Spin}(4)$-representation,

$$\rho = \wedge(2S_{4+}) = \mathbb{C} \oplus 2S_{4+} \oplus (\mathbb{C}^3 \oplus \mathrm{Sym}^2 S_{4+}) \oplus 2S_{4+} \oplus \mathbb{C}.$$

It follows that this supermultiplet contains

- a *self-dual 2-form* spin 1 particle, i.e., one valued in the 3-dimensional representation $\mathrm{Sym}^2 S_{4+}$ of self-dual 2-forms for $\mathrm{Spin}(4)$;
- four chiral spinors valued in S_{4+}; and
- five scalars.

The R-symmetry group in this case is $\mathrm{USp}(4) \cong \mathrm{Spin}(5)$. Taking it into account, we obtain the following decomposition as $USp(4) \times \mathrm{Spin}(4)$-representations:

$$\rho = \mathbf{5} \otimes \mathbb{C} \oplus \mathbf{4} \otimes S_{4+} \oplus \mathbf{1} \otimes \mathrm{Sym}^2 S_{4+},$$

where $\mathbf{1}$, $\mathbf{4}$, $\mathbf{5}$ are the irreducible $\mathrm{USp}(4)$-representations of the corresponding dimensions.

This is the so-called $\mathcal{N} = (2,0)$ *tensor multiplet* in six dimensions (as it contains a self-dual 2-form, which is a rank 2 tensor).

Note that upon reduction to four dimensions, this supermultiplet becomes the $N = 4$ massless vector multiplet of Example 14.29, and correspondingly the restriction of ρ from $\mathrm{USp}(4) \times \mathrm{Spin}(4)$ to $\mathrm{USp}(4) \times \mathrm{Spin}(2)$ decomposes as

the restriction of the representation of $\mathrm{SU}(4) \times \mathrm{Spin}(2)$ defined by Table 14.6 to $\mathrm{USp}(4) \times \mathrm{Spin}(2)$.

Example 14.34. Let us compute the particle content of the simplest $N = 1$ massless supermultiplet in dimension $d = 10$. The compact part of the little group is $\mathrm{Spin}(8)$ and $Y = S_{10+}$ is the 8-dimensional spin representation of $\mathrm{Spin}_+(1,9)$ which restricts to $S_{8+} \oplus S_{8-}$ for $\mathrm{Spin}(8)$. Thus as a $\mathrm{Spin}(8)$-representation,

$$\rho = \wedge S_{8-} = S_{8+} \oplus E_8;$$

this follows from the standard fact that for even m the simple spin representation of $\mathrm{Cl}(E_m)$ decomposes over $\mathrm{Spin}(m)$ as $S_{m+} \oplus S_{m-}$, and triality for $\mathrm{Spin}(8)$, i.e., the action of the group S_3 on its Dynkin diagram.

Thus the particle content of this theory is

- one vector;
- one chiral spinor.

This is called the $N = 1$ massless *vector multiplet* in ten dimensions. It is realized by the Lagrangian of $U(1)$ gauge theory with gauge field A and gaugino λ given by (14.17).

By dimensional reduction of this multiplet, we obtain various massless vector multiplets in lower dimensions. For example, for $d = 6$ we get the $\mathcal{N} = (1,1)$ massless vector multiplet with one vector, four spinors (two chiral and two anti-chiral) and four scalars, and for $d = 4$ we obtain the $N = 4$ massless vector multiplet with one vector, eight spinors (four chiral and four anti-chiral) and six scalars described above.

Exercise 14.35.

1. Compute the particle content of the simplest $N = 1$ massless supermultiplet ($\rho_0 = \mathbb{C}$) in various dimensions ≤ 10.

2. Do the same for $N = 2$ massless and $N = 1$ massive supermultiplets in dimensions ≤ 6.

14.15. Balance of physical degrees of freedom

If the superdimension of ρ is $(m|n)$, we say that the supermultiplet corresponding to ρ has m bosonic and n fermionic *physical degrees of freedom*. From the above description we see that for $d \geq 2$, the superdimension of ρ is always $(n|n)$ for some n, i.e., in every supermultiplet there is an *equal number of bosonic and fermionic physical degrees of freedom* that are interchanged by the supercharges.[70] This is a fundamental property of supersymmetric quantum field theories.

[70] Here we do not consider partially supersymmetric theories such as the one of Remark 14.21.

If a quantum field theory has a Lagrangian description (at low energies), we can count massless physical degrees of freedom just by looking at the Lagrangian. For example, a (real) scalar contributes one even degree of freedom, a vector (e.g., a $U(1)$-gauge field) contributes $d-2$ even degrees of freedom, and a spinor valued in a spin representation Y contributes $\frac{1}{2}\dim Y$ odd degrees of freedom. The factor $\frac{1}{2}$ has to do with the fact that the equations of motion for spinors are first rather than second order.

Example 14.36. Consider the free $U(1)$-gauge theory with gauge field A and a Y-valued fermion (gaugino) λ in the trivial $U(1)$-representation given by (14.17). For which spacetime dimensions d is the theory defined by this Lagrangian supersymmetric?

Let's count physical degrees of freedom. We have $d-2$ even ones and $\frac{1}{2}\dim Y$ odd ones. So the balance of even and odd degrees of freedom yields

$$d-2=\frac{\dim Y}{2}.$$

Looking at Table 14.1, we see that the only solutions are $d=3,4,6,10$, and in this case Y is minimal, so there is a possibility for at most $N=1$ supersymmetry. One can check that this $N=1$ supersymmetry indeed exists, and we have discussed the corresponding supersymmetric theories in the previous section.

Thus upon reduction to lower dimensions, we obtain theories with larger amounts of supersymmetry, as explained above.

On the contrary, if $d=1$ (quantum mechanics), where there is no notion of 1-particle states, the count of physical degrees of freedom does not apply, and there is no simple relation between the number of bosonic and fermionic fields in a supersymmetric theory. For example, as we have seen, the theory with $N=1$ supersymmetry (14.1) has an equal number of bosons and fermions, while the theory with $N=2$ supersymmetry (14.3) has twice as many fermions as bosons.

14.16. Central charges of the supersymmetry algebra

We have discussed that in a classical field theory with $\mathcal{N}$ supersymmetries, the algebra of observables carries a Lorentz-equivariant action of the supersymmetry algebra $\mathfrak{q}_{d,\mathcal{N}}$, and likewise for quantum theory, the only difference being that the algebra of quantum observables is noncommutative. The algebra $\mathfrak{q}_{d,\mathcal{N}}$ should also act compatibly on the quantum states. However, since quantum states are unit vectors in the Hilbert space $\mathcal{H}$ *defined up to a phase factor*, i.e., points of the projective space $\mathbb{P}\mathcal{H}$, the action of $\mathfrak{q}_{d,\mathcal{N}}$ is in general only projective, rather than linear (see Section 11.4). In other words, what acts linearly

on $\mathcal{H}$ is a certain central extension of the Lie superalgebra $\mathfrak{q}_{d,\mathcal{N}}$. This central extension has the form

$$\hat{\mathfrak{q}}_{d,\mathcal{N}} = Z \oplus \mathfrak{q}_{d,\mathcal{N}} = Z \oplus \mathbb{R}^d \oplus Y,$$

where Z is an even vector space consisting of central elements. Since this extension should be Lorentz-equivariant and Z must commute with the Poincaré group, the commutator in $\hat{\mathfrak{q}}_{d,\mathcal{N}}$ is determined by a $\mathrm{Spin}_+(1,d-1)$-invariant map $\wedge^2 Y \to Z$. Thus the most universal case is when $Z = (\wedge^2 Y)_{\mathrm{Spin}_+(1,d-1)}$ is the space of coinvariants. In the case of $N = 1$ supersymmetry, this space is zero, so the possibility of central extension is only relevant for extended supersymmetry, $N > 1$.

For example, consider $N = 2$ supersymmetry in $d = 4$ dimensions (see [**QFS**, pp. 1288–1290]). Thus $Y = 2S_{4+} \oplus 2S_{4-}$, so $Z = \mathbb{C}_+ \oplus \mathbb{C}_-$, where $\mathbb{C}_\pm = \wedge^2 S_{4\pm}$, with basis $z_+, z_- = z_+^\dagger$. Thus we have eight supercharges $Q_{j,\pm}, Q_{j,\pm}^\dagger$, $j = 1, 2$ acting on $\mathcal{H}$ which satisfy the usual commutation relations (coming from Definition 14.20) except that we have

$$[Q_{1+}, Q_{2-}] = z_+, \ [Q_{2+}, Q_{1-}] = -z_+$$

and the conjugate equations (instead of zero on the right-hand side in the usual case).

Consider a unitary representation of this algebra in which the translation group $\mathbb{R}^4$ acts by the character $(m, 0, 0, 0)$. The reality condition requires that in this representation z_+ acts by some number $z \in \mathbb{C}$ and z_- acts by $\overline{z}$. Thus, setting $\mathcal{Q}_\theta := Q_{1+} - e^{i\theta} Q_{2-}^\dagger$, we have

$$\begin{aligned}[\mathcal{Q}_\theta, \mathcal{Q}_\theta^\dagger] &= [Q_{1+} - e^{i\theta} Q_{2-}^\dagger, Q_{1+}^\dagger + e^{-i\theta} Q_{2-}] \\ &= [Q_{1+}, Q_{1+}^\dagger] + [Q_{2-}, Q_{2-}^\dagger] - e^{-i\theta} z - e^{i\theta} \overline{z} = 2(H - \mathrm{Re}(e^{-i\theta} z)),\end{aligned}$$

where H is the Hamiltonian. So taking $\theta = \arg(z)$, we get

$$[\mathcal{Q}_\theta, \mathcal{Q}_\theta^\dagger] = 2(H - |z|). \tag{14.19}$$

But $[\mathcal{Q}_\theta, \mathcal{Q}_\theta^\dagger] = \mathcal{Q}_\theta \mathcal{Q}_\theta^\dagger + \mathcal{Q}_\theta^\dagger \mathcal{Q}_\theta \geq 0$, so we obtain

Proposition 14.37 (Bogomolnyi-Prasad-Sommerfield (BPS) bound)**.** *In an $N = 2$ supersymmetric QFT in four dimensions with central charge $z \in \mathbb{C}$, we have $H \geq |z|$.*

Thus the lowest possible energy of a state in this theory is $|z|$.

Definition 14.38. A state with energy exactly $|z|$ is called a *BPS state*.

It is easy to see that if $z = 0$, then a BPS state Ψ is invariant under all supercharges. If $z \neq 0$, this is no longer the case, but we have

Proposition 14.39. *If $z \neq 0$, then a BPS state Ψ is invariant under half of the supercharges:*

$$(Q_{1+} - e^{i\theta}Q_{2-}^\dagger)\Psi = (Q_{1+}^\dagger - e^{-i\theta}Q_{2-})\Psi$$
$$= (Q_{2+} + e^{i\theta}Q_{1-}^\dagger)\Psi = (Q_{2+}^\dagger + e^{-i\theta}Q_{1-})\Psi = 0.$$

Proof. By (14.19), $[\mathcal{Q}_\theta, \mathcal{Q}_\theta^\dagger]\Psi = 0$, hence

$$(\Psi, [\mathcal{Q}_\theta, \mathcal{Q}_\theta^\dagger]\Psi) = \|\mathcal{Q}_\theta\Psi\|^2 + \|\mathcal{Q}_\theta^\dagger\Psi\|^2 = 0,$$

which implies the first two equalities $\mathcal{Q}_\theta\Psi = \mathcal{Q}_\theta^\dagger\Psi = 0$. The proof of the second two equalities is analogous, switching indices $1, 2$ and z with $-z$. □

It is not obvious that BPS states exist, but in interesting $N = 2$ supersymmetric theories they do, and they play a key role in understanding the low-energy behavior of the theory. So let us describe the irreducible representations of the centrally extended super Poincaré group generated by BPS states.

To this end we need to describe irreducible unitary representations of the Lie superalgebra $\hat{\mathfrak{q}}_{4,2}$ with central charge $z \neq 0$. As in the case $z = 0$, in any such representation W, the even subalgebra $\mathbb{R}^d$, being central, acts by the character defined by an element $p \in \mathbb{R}^d$, and the BPS bound implies that $p^2 \geq |z|^2$. Let $U_{p,z} = U_{p,z}(\mathfrak{q}_{4,2})$ be the quotient of the universal enveloping algebra $U(\hat{\mathfrak{q}}_{4,2})$ by the ideal generated by elements

$$x - (p, x),\ x \in \mathbb{R}^d,\ z_+ - z,\ z_- - \overline{z}.$$

Thus W is an irreducible representation of $U_{p,z}$. The following proposition is a generalization of Proposition 14.23 for $d = 4$ and $N = 2$ (with analogous proof).

Proposition 14.40.

(i) *If p has mass $m > |z|$ (i.e., $p^2 = m^2$), then $U_{p,z}$ is the Clifford algebra $\mathrm{Cl}(2S_{4+} \oplus 2S_{4-})$, so W is the 16-dimensional irreducible spin representation of this algebra.*

(ii) *If p has mass $|z|$ (i.e., $p^2 = |z|^2$), then*

$$U_{p,z} \cong \mathrm{Cl}(S_{4+} \oplus S_{4-}) \otimes \wedge(S_{4+} \oplus S_{4-}).$$

Consequently, W is the 4-dimensional irreducible spin representation of $\mathrm{Cl}(S_{4+} \oplus S_{4-})$. The second copy of $S_{4+} \oplus S_{4-}$ acts on this representation by zero.

Thus we see that irreducible representations of the extended super Poincaré group generated by BPS states produce $N = 2$ massive multiplets of mass $m = |z| > 0$ which have the same content as the corresponding $N = 1$ massive multiplets for $z = 0$. Thus they are smaller than ordinary $N = 2$ massive

multiplets (which is made possible by a nontrivial central charge). Such massive multiplets are therefore called *short*, as opposed to *long* ones, which are the usual $N = 2$ massive multiplets in the $z = 0$ case (also seen in the centrally extended case for masses $m > |z|$).

As $z \to 0$, a short multiplet degenerates into one or more $N = 2$ massless multiplets. For example, the *short vector multiplet*, which has the same content as the $N = 1$ massive vector multiplet of Example 14.24, degenerates as $m = |z| \to 0$ to the $N = 2$ massless vector multiplet of Example 14.28. Namely, the massive vector for $m > 0$ breaks into a massless vector and a scalar. Note that the $U(2)_R$-symmetry is broken by the central charge to $\mathrm{SU}(2)$, so all scalars are neutral under the R-symmetry.

Conversely, such a collection of massless multiplets present classically can acquire mass through the mechanism of *dynamical mass generation* (see Section 13.9) and turn into a short massive multiplet, with the central charge turning on. But the mass of this multiplet will then be tied to the value of the central charge by the BPS condition $m = |z|$.

This story generalizes to other d and $N \geq 2$, but we will not discuss this generalization here and refer the reader to more advanced texts about supersymmetric field theories with extended supersymmetry.

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