A Generalized McKay Correspondence

by

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McKay has defined a bijection between the set of all finite subgroups of $SL(2, \mathbf{C})$ and the set of all simply laced affine diagrams, which has several interesting properties [M]. This is called the McKay correspondence. In this note, we define a bijection between all pairs of finite subgroups G, H of $SL(2, \mathbf{C})$ which satisfy certain conditions and the set of all affine diagrams, which reduces to the McKay correspondence in case $G = H$. This is similar to, but not the same as, the correspondence defined by Slodowy [S, Appendix 3]. The motivation for this comes from attempts by the second author to generalize some work of Lusztig [L] on the representations of tame quivers.

Consider pairs of finite groups G, H which satisfy

$$(*) \qquad G' \subseteq H \subseteq G \subseteq SL(2, \mathbf{C}).$$

Here $G' = [G, G]$ denotes the commutator group of G.

Let $\mathrm{C}_G(H)$ denote the centralizer of H in G. We will be concerned with pairs G, H which satisfy (*) and one of the following (not mutually exclusive) conditions:

(i) $G = H \neq <1>$

(ii) $G \neq H$ and $\mathrm{C}_G(H) \subseteq H$.

(iii) $G \neq H$ and $\mathrm{Z}(G) = \mathrm{C}_G(G) \subseteq H$.

In each case, we will attach a quiver to the pair G, H which has certain properties. This will define a bijection between the set of all pairs which satisfy one of (i), (ii) or (iii) and the set of all affine diagrams. As is well known the set of affine diagrams coincides with the set of (connected) tame quivers. Table IV contains the list of all affine diagrams.

[1] The contents of this paper were part of the Hardy lecture of the London Mathematical Society given by the second author in the spring of 1993.

The finite subgroups of $SL(2, \mathbf{C})$ were classified in the 19^{th} century. They are:

$\tilde{\mathcal{A}}_5 = SL(2,5), |\tilde{\mathcal{A}}_5| = 120.$

$\tilde{\mathcal{A}}_4 = SL(2,3), |\tilde{\mathcal{A}}_4| = 24.$

$\tilde{S}_4$, the double cover of S_4 which contains a unique involution, $|\tilde{S}_4| = 48$.

Z_m, the cyclic subgroup of order m.

$Q_{4m} =< x, y \mid x^{2m} = 1, y^{-1}xy = x^{-1}, y^2 = x^m >$ for $m > 1, |Q_{4m}| = 4m$.

Each of these groups, except Z_m with m odd, necessarily contains a unique involution, -1. Observe that $Q'_{4m} = Z_m$.

Using this classification it is not difficult to list all the pairs which satisfy (i), (ii) or (iii).

If (*) is satisfied, the group G has a 2-dimensional representation by virtue of being contained in $SL(2, \mathbf{C})$. Let χ denote the character afforded by this representation.

Suppose that (i) or (ii) is satisfied. Let $\{\alpha_i\}$ denote the set of all characters of G induced by irreducible characters of H.

If (i) or (iii) is satisfied, let $\{\alpha_i\}$ denote the set of all restrictions to H of irreducible characters of G.

In any case, let $\Delta = \{d_{ij}\}$, where

$$\chi\alpha_i = \sum_i d_{ij}\alpha_j.$$

This is well defined in all cases since $(\chi\alpha_i)^G = \chi\alpha_i^G$ (α^G denotes the character of G induced by α).

Observe that if (i) is satisfied, then in both cases above $\{\alpha_i\}$ is the set of all irreducible characters of $G = H$ and Δ is the same in both cases.

In all cases $\{\alpha_i\}$ is an orthogonal set of characters of G, resp. H. Therefore

$$d_{ij} = (\chi\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$$

where (,) denotes the inner product with respect to G, resp. H. As χ is real valued

$$(\chi\alpha_i, \alpha_j) = (\chi, \bar{\alpha}_i\alpha_j) = (\chi, \alpha_i\bar{\alpha}_j) = (\chi\alpha_j, \alpha_i)$$

and so $d_{ij} = 0$ if and only if $d_{ji} = 0$.

Suppose that (i), (ii), or (iii) is satisfied for the pair G, H. Define a quiver as follows:

The vertices are in one-to-one correspondence with the set $\{\alpha_i\}$. The vertices corresponding to α_i and α_j are joined by an edge if and only if $d_{ij} \neq 0$, and hence $d_{ji} \neq 0$. If $d_{ij} \neq 1$ or $d_{ji} \neq 1$ we label the edge by the ordered pair (d_{ij}, d_{ji}) as follows

$$\underset{\alpha_j}{\circ}\overset{(d_{ij},d_{ji})}{\rule{2cm}{0.4pt}}\underset{\alpha_i}{\circ}$$

As usual, if $d_{ij} = d_{ji} = 1$, the label is omitted.

Furthermore, an integer n_i is attached to the vertex α_i so that the following is satisfied:

$$\Sigma n_i\alpha_i = \rho_G \qquad \text{in case (i) or (ii)}$$

$$\Sigma n_i\alpha_i = \rho_H \qquad \text{in case (iii).}$$

Here ρ_G, ρ_H respectively denote the character afforded by the regular representation of G, H respectively.

The following notation will be used to denote the quiver constructed above:

In case (i) it is $\Gamma(G)$.

In case (ii) it is $\Gamma(G, H, \uparrow)$.

In case (iii) it is $\Gamma(G, H, \downarrow)$.

If (ii) is satisfied then $\mathbf{C}_G(G) \subseteq \mathbf{C}_G(H) \subseteq H$ and so also (iii) holds. Thus there are two quivers $\Gamma(G, H, \uparrow)$ and $\Gamma(G, H, \downarrow)$ attached to the pair $H \subseteq G$ in this case. Let $\{\alpha_i\}, \{n_i\}$ correspond to $\Gamma(G, H, \uparrow)$ and $\{\alpha_i'\}, \{n_i'\}$ to $\Gamma(G, H, \downarrow)$. Then for each i

$$\alpha_i(1) = n_i'|G : H|, \qquad \alpha_i'(1) = n_i.$$

In case (i), $\alpha_i(1) = n_i$ for each i.

Table I contains a list of all the groups $G = H$ in case (i) and the corresponding quivers. This is the original McKay correspondence ($\tilde{A}_{12}$ is simply laced by fiat).

Table II, (resp. III) contains a list of all pairs G, H in case (ii), (resp. (iii)) and the corresponding quivers.

In each of tables II and III, m is an integer with $m > 1$.

The details can be checked in each case in a straightforward way and are left to the reader.

George Seligman has pointed out to us that the difference between cases (i) and (ii), and case (iii) also has a Lie theoretic interpretation. See [K, pp. 54-55].

TABLE I

G	Z_{n+1}	$\tilde{A}_5$	$\tilde{S}_4$	$\tilde{A}_4$	$Q_{4(n-2)}$
Γ	$\tilde{A}_n$	$\tilde{E}_8$	$\tilde{E}_7$	$\tilde{E}_6$	$\tilde{D}_n$

$(\tilde{A}_1 = \tilde{A}_{12})$

TABLE II

H	$\tilde{A}_4$	Q_8	Z_{2m}	Q_{4m}
G	$\tilde{S}_4$	$\tilde{A}_4$	Q_{4m}	Q_{8m}
$\Gamma(G, H, \uparrow)$	$\tilde{F}_{42}$	$\tilde{G}_{22}$	$\tilde{C}_m$	$\widetilde{BD}_{m+1}$

TABLE III

G	$\tilde{S}_4$	$\tilde{A}_4$	Q_{4m}	Q_{8m}	Q_{8m}	Q_8
H	$\tilde{A}_4$	Q_8	Z_{2m}	Q_{4m}	Z_{2m}	Z_2
$\Gamma(G, H, \downarrow)$	$\tilde{F}_{41}$	$\tilde{G}_{21}$	$\tilde{B}_m$	$\widetilde{CD}_{m+1}$	$\widetilde{BC}_m$	$\tilde{A}_{11}$

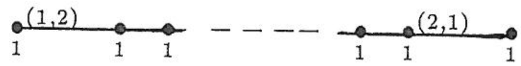
TABLE IV

AFFINE DIAGRAMS

$\tilde{A}_n$

$n \geq 2$

$\tilde{B}_n$

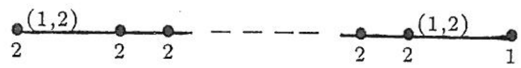


$n \geq 2$

$\tilde{C}_n$

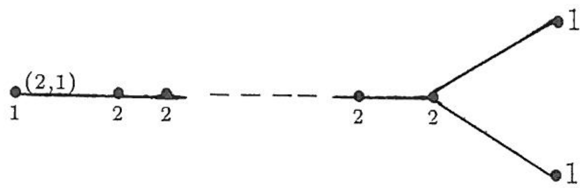
$n \geq 2$

$\widetilde{BC}_n$



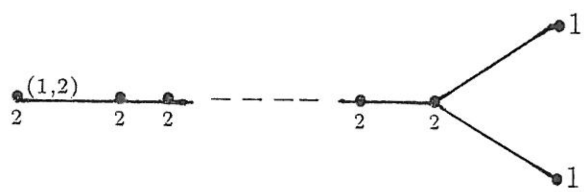
$n \geq 2$

$\widetilde{CD}_n$



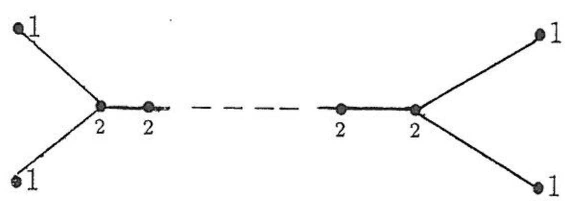
$n \geq 3$

$\widetilde{BD}_n$



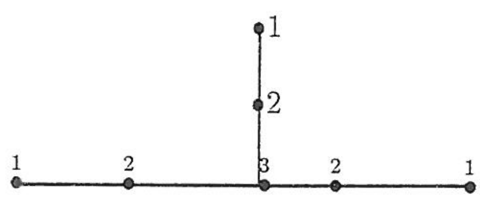
$n \geq 3$

$\tilde{D}_n$

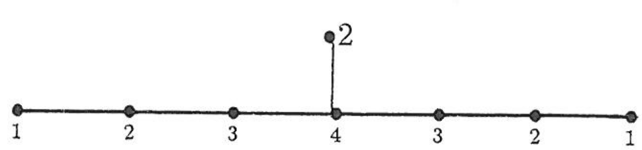


$n \geq 4$

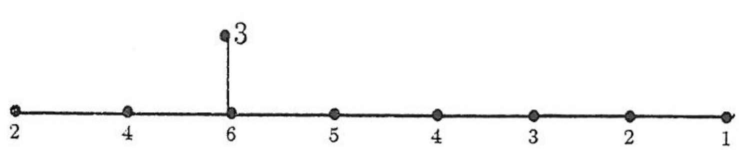
$\tilde{E}_6$



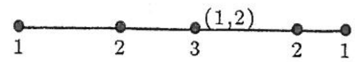
$\tilde{E}_7$



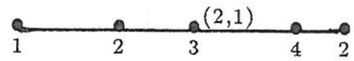
$\tilde{E}_8$



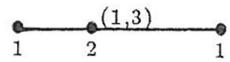
$\tilde{F}_{41}$



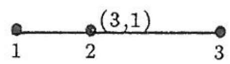
$\tilde{F}_{42}$



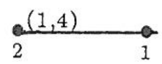
$\tilde{G}_{21}$



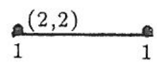
$\tilde{G}_{22}$



$\tilde{A}_{11}$



$\tilde{A}_{12}$



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