Formalism of six functors

Functors on all (coherent) D-modules

Let $\pi : X \to Y$ be a morphism of irreducible algebraic varieties, and $d = \dim X - \dim Y$.

• Functors defined on the derived category of all D-modules: $\pi_*, \pi^!, \boxtimes$. The functor $\pi^!$ is $\pi^\bullet[d]$, where $\pi^\bullet$ is the inverse image of quasicoherent sheaves, i.e. it is obtained by introducing a flat connection on the sheaf-theoretic pullback (in the case of smooth varieties).

• The functors π_* and $\pi^!$ are compatible with compositions.

• The functors π_* and $\pi^!$ are compatible with base change. That is, if $\tau : S \to Y$, $W = X \times_Y S$, $\pi' : W \to S$ the lift of π and $\tau' : W \to X$ the lift of τ then $\tau^! \circ \pi_* = \pi'_* \circ (\tau')^!$.

• The functor $\mathbb{D}$ is defined on the derived category of coherent D-modules, and maps this category to its opposite. Moreover, $\mathbb{D}^2 = \mathrm{Id}$ (so $\mathbb{D}$ is an antiequivalence).

• The functor $\pi_! := \mathbb{D}\pi_*\mathbb{D}$ is defined on the coherent M such that $\pi_*\mathbb{D}M$ is coherent.

• The functor $\pi^* := \mathbb{D}\pi^!\mathbb{D}$ is defined on the coherent M such that $\pi^!\mathbb{D}M$ is coherent.

• If π is proper then π_* preserves the derived category of coherent D-modules, and on this category $\pi_!$ is defined and equals π_*. Also in this case $\pi_* = \pi_!$ is left adjoint to $\pi^!$ on the derived categories of coherent D-modules.

• If π is smooth then $\pi^!$ preserves the derived category of coherent D-modules, and on this category π^* is defined and equals $\pi^![-2d]$ (so for an etale map, in particular open embedding, $\pi^* = \pi^!$). Also, in this case π_* is right adjoint to $\pi^* = \pi^![-2d]$ on the derived categories of coherent D-modules. Finally, $\pi^![-d] = \pi^*[d]$ preserves the abelian category of coherent (and all) D-modules, and is exact.

• if π is a closed embedding then π_* is the derived functor of an exact functor on the abelian category, and preserves coherent D-modules.

• if π is affine then π_* is the derived functor of a right exact functor on the abelian category of D-modules.

• The functor $\boxtimes$ is compatible with the other functors in an obvious way.

Functors on holonomic D-modules

All six functors above are defined on the derived categories of holonomic D-modules. Moreover:

• $\mathbb{D}$ is an exact functor on the abelian category of holonomic D-modules to its opposite.

• $\pi_!$ is left adjoint to $\pi^!$ and π_* is right adjoint to π^*.

• if π is an open embedding then π_* is the sheaf-theoretic direct image, and it is the derived functor of a left exact functor. Similarly, $\pi_!$ is the derived functor of a right exact functor.