MITTAG-LEFFLER TYPE SUMS ASSOCIATED WITH ROOT SYSTEMS

PAVEL ETINGOF AND ERIC RAINS

1. The main theorem

Let G be a simply connected simple algebraic group over $\mathbb{C}$. Let $\mathfrak{g} = \mathrm{Lie}G$, $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, W the Weyl group, $Q, P, Q^\vee, P^\vee$ the root, weight, coroot and coweight lattices, $\rho \in \mathfrak{h}^*$ the half-sum of positive roots, $P_+ \subset P$ the set of dominant integral weights, V_λ the irreducible representation of G with highest weight $\lambda \in P_+$, and k a positive integer. Consider the function on $\mathfrak{h}$ given by

$$f(x) = \prod_{\alpha>0} \frac{\sin \pi\alpha(x)}{\pi\alpha(x)},$$

where the product is taken over positive roots of G.

Let $Z \subset G$ be the center and $\xi : Z \to \mathbb{C}^*$ a character. Since $Z = P^\vee/Q^\vee$, we may view ξ as a character of $P^\vee$ which is trivial on $Q^\vee$. Define the function

$$F_{k,\xi}(x) := \sum_{a\in P^\vee} \xi(a) f^k(x + a).$$

(If $G = SL_2$ and $k = 1$ then the sum is not absolutely convergent, and should be understood in the sense of principal value). Thus, the meromorphic function

$$M_{k,\xi}(x) := \frac{F_{k,\xi}(x)}{\prod_{\alpha>0} \pi^{-k} \sin^k \pi\alpha(x)}$$

has a Mittag-Leffler type decomposition

$$M_{k,\xi}(x) = \sum_{a\in P^\vee} \frac{(-1)^{2k\rho(a)}\xi(a)}{\prod_{\alpha>0} \alpha^k(x)}.$$

For example, for $G = SL_2$, we have

$$F_{1,1}(x) = 1, F_{1,-1}(x) = \cos \pi x,$$

which gives the classical Mittag-Leffler decompositions

$$\pi \cot \pi x = \sum_{n\in\mathbb{Z}} \frac{1}{x + n}, \quad \frac{\pi}{\sin \pi x} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{x + n}.$$

The goal of this note is to prove the following theorem.

Theorem 1.1. *The function $F_{k,\xi}$ is a W-invariant trigonometric polynomial on the maximal torus $T = \mathfrak{h}/Q^\vee$ of G, which is a nonnegative rational linear combination of irreducible characters of G.*

For $G = SL_n$ and ξ being a character of order 2, this theorem was conjectured by R. Coquereaux and J.-B. Zuber ([CZ], Conjecture 1 in Subsection 2.2).

The proof is elementary, and all the techniques we use are well known.

2. Proof of the main theorem

2.1. **Contraction of representations.** We start with the following general fact.

Proposition 2.1. *([Li1], Proposition 3) Let (V, ρ_V) be a rational representation of G, and N a positive integer. Let V_N be the direct sum of all the weight subspaces of V of weights divisible by N. Then the action of T on V_N given by $t \circ v := \rho_V(t^{1/N})v$ extends to an action of G.[1] In other words, the function*

$$\chi_{V,N}(x) := \sum_{\lambda \in P} \dim V[N\lambda] e^{2\pi i \lambda(x)}$$

is a nonnegative linear combination of irreducible characters of G. Namely, the multiplicity of χ_λ in $\chi_{V,N}(x)$ equals the multiplicity of $V_{N\lambda+(N-1)\rho}$ in $V \otimes V_{(N-1)\rho}$.

Proof. Littelmann proves this proposition via his path model as an illustration of its use, but we give a more classical proof using the Weyl character formula. We have to show that the integral

$$I := \int_{\mathfrak{h}_\mathbb{R}/Q^\vee} \sum_{\lambda \in P} \dim V[N\lambda] e^{-2\pi i \lambda(x)} \chi_\lambda(x) |\Delta(x)|^2 dx$$

is nonnegative, where $\Delta(x)$ is the Weyl denominator, since the multiplicity in question is $I/|W|$.

Denoting the character of V by χ_V, we have

$$I = \int_{\mathfrak{h}_\mathbb{R}/Q^\vee} \int_{\mathfrak{h}_\mathbb{R}/Q^\vee} \sum_{\lambda \in P} \overline{\chi_V(y)} e^{2\pi i N\lambda(y)} e^{-2\pi i \lambda(x)} \chi_\lambda(x) |\Delta(x)|^2 dy dx =$$

$$\int_{\mathfrak{h}_\mathbb{R}/Q^\vee} \int_{\mathfrak{h}_\mathbb{R}/Q^\vee} \overline{\chi_V(y)} \delta(x - Ny) \chi_\lambda(x) |\Delta(x)|^2 dy dx =$$

$$\int_{\mathfrak{h}_\mathbb{R}/Q^\vee} \overline{\chi_V(y)} \chi_\lambda(Ny) |\Delta(Ny)|^2 dy.$$

Using the Weyl character formula, we then have

$$I = \int_{\mathfrak{h}_\mathbb{R}/Q^\vee} \overline{\chi_V(y)} \left(\sum_{w \in W} (-1)^w e^{2\pi i (w(\lambda+\rho), Ny)} \right) \overline{\Delta(Ny)} dy =$$

[1] Note that $\rho_V(t^{1/N})v$ is independent on the choice of the N-th root $t^{1/N}$.

$$\int_{\mathfrak{h}_{\mathbb{R}}/Q^\vee} \overline{\chi_V(y)} \frac{\sum_{w\in W}(-1)^w e^{2\pi i(w(\lambda+\rho),Ny)}}{\Delta(y)} \frac{\overline{\Delta(Ny)}}{\overline{\Delta(y)}} |\Delta(y)|^2 dy =$$

$$\int_{\mathfrak{h}_{\mathbb{R}}/Q^\vee} \overline{\chi_V(y)} \chi_{N\lambda+(N-1)\rho}(y) \frac{\overline{\Delta(Ny)}}{\overline{\Delta(y)}} |\Delta(y)|^2 dy.$$

Now recall that $\frac{\Delta(Ny)}{\Delta(y)} = \chi_{(N-1)\rho}(y)$. Thus we get

$$I = \int_{\mathfrak{h}_{\mathbb{R}}/Q^\vee} \overline{\chi_V(y)\chi_{(N-1)\rho}(y)} \chi_{N\lambda+(N-1)\rho}(y) |\Delta(y)|^2 dy,$$

i.e., $I/|W|$ is the multiplicity of $V_{N\lambda+(N-1)\rho}$ in $V \otimes V_{(N-1)\rho}$, as desired. □

Remark 2.2. 1. If N is odd (and coprime to 3 for G of type G2), then Proposition 2.1 has a nice representation-theoretic interpretation. Namely, if q is a root of unity of order N and G_q the corresponding Lusztig quantum group, then there is an exact *contraction functor* $F : \mathrm{Rep}\, G_q \to \mathrm{Rep}\, G$ which at the level of P-graded vector spaces transforms V into V_N with weights divided by N (see [GK] and references therein). Proposition 2.1 is then obtained by applying the functor F to a Weyl module.

2. Suppose that G is not simply laced. Normalize the inner product on $\mathfrak{h}^*$ so that long roots have squared length 2. This inner product identifies $\mathfrak{h}$ with $\mathfrak{h}^*$ so that $\alpha_i^\vee$ map to $2\alpha_i/(\alpha_i,\alpha_i)$. Note that $2/(\alpha_i,\alpha_i)$ is an integer, so under this identification $Q^\vee \subset Q$, hence $P^\vee \subset P$. Let $V_N' \subset V_N$ be the span of the weight subspaces of V of weights belonging to $NP^\vee$ with weights divided by N. Then, analogously to Proposition 2.1, V_N' extends to a representation of the Langlands dual Lie algebra $\mathfrak{g}^L$, with a similar descrition of multiplicities ([Li1], Proposition 4). Note that this statement is nontrivial even if $\mathfrak{g}^L \cong \mathfrak{g}$, since the arrow on the Dynkin diagram is reversed. This also has a representation-theoretic interpretation similar to (1), see [Li1], Section 3, [GK].

3. As explained in [Li1], Proposition 2.1 generalizes to symmetrizable Kac-Moody algebras (both our proof and that of [Li1] can be straightforwardly extended to this case). So does the non-simply laced version of Proposition 2.1 given in (2) and the above representation-theoretic interpretations, see [Li2].

2.2. **Duisetmaat-Heckman measures and proof of Theorem 1.1.** Now recall ([GLS]) that for each dominant $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ we can define the Duistermaat-Heckman measure $DH_\lambda(\mu)d\mu$ on $\mathfrak{h}_{\mathbb{R}}^*$, which is the direct image of the Liouville measure on the coadjoint orbit of λ. This measure is supported on the convex hull of the Weyl group orbit $W\lambda$, and its Fourier transform is given by the formula

$$\mathcal{F}(DH_\lambda)(x) = \frac{\sum_{w\in W}(-1)^w e^{2\pi i(w\lambda,x)}}{\prod_{\alpha>0} 2\pi i\alpha(x)}. \tag{1}$$

For simplicity assume that λ is regular and $G \neq SL_2$. Then DH_λ is absolutely continuous with respect to the Lebesgue measure (i.e., the density function $DH_\lambda(\mu)$ is continuous). Then it is known ([GLS]) that if $\mu_N \in P, \lambda_N \in P_+$ are sequences such that $\mu_N/N \to \mu, \lambda_N/N \to \lambda$ as $N \to \infty$ and $\lambda_N - \mu_N \in Q$ then

$$\lim_{N\to\infty} \frac{\dim V_{\lambda_N}[\mu_N]}{N^{|R_+|}} = DH_\lambda(\mu), \tag{2}$$

where R_+ is the set of positive roots. Note that equation (1) follows immediately from equation (2) and the Weyl character formula.

Proposition 2.3. *Let $\lambda_1, ..., \lambda_k \in \mathfrak{h}_\mathbb{R}^*$ be regular dominant weights. Then the trigonometric polynomial*

$$\sum_{\mu\in P} (DH_{\lambda_1} * ... * DH_{\lambda_k})(\mu) e^{2\pi i \mu(x)}$$

(where $*$ denotes convolution of measures) is a linear combination of irreducible characters of G with nonnegative real coefficients.

Proof. First assume that λ_i are rational, and let d be their common denominator. Then, taking the limit as $N \to \infty$ in Proposition 2.1 with $V = V_{N\lambda_1} \otimes ... \otimes V_{N\lambda_k}$ and N divisible by d, we obtain the desired statement. Now the general case follows from the facts that rational weights are dense in $\mathfrak{h}_\mathbb{R}$ and $DH_\lambda(\mu)$ is continuous in λ. □

Now Theorem 1.1 follows from equation (1) and Proposition 2.3 by taking $\lambda_1, ..., \lambda_k = \rho$ and noting that by the Weyl denominator formula

$$\frac{\sum_{w\in W} (-1)^w e^{2\pi i (w\rho, x)}}{\prod_{\alpha>0} 2\pi i \alpha(x)} = \prod_{\alpha>0} \frac{\sin \pi\alpha(x)}{\pi\alpha(x)} = f(x). \tag{3}$$

The rationality of the coefficients follows from the rationality of the values of the convolution power $(DH_\rho)^{*k}$ at rational points.

Remark 2.4. It follows from (3) that the measure DH_ρ is the convolution of uniform measures on $[-\alpha/2, \alpha/2]$ over all positive roots α.

2.3. **The characters occuring in $F_{k,\xi}$.** Let us now discuss which irreducible characters can occur in the decomposition of $F_{k,\xi}$. Let us view ξ as an element of P/Q. Clearly, if χ_λ occurs in $F_{k,\xi}$ then the central character of the representation V_λ must be $\xi = k\rho - \lambda \mod Q$. If so, then, as shown above, the multiplicity of χ_λ in $F_{k,\xi}$ is $(DH_\rho)^{*k}(\lambda)$. Since this density is continuous and supported on the k times dilated convex hull B of the orbit $W\rho$, we see that if χ_λ occurs then λ has to be strictly in the interior of B.

Let $m_i(\xi)$ be the smallest strictly positive number such that $m_i(\xi) = (\xi, \omega_i^\vee)$ in $\mathbb{R}/\mathbb{Z}$, where $\omega_i^\vee$ are the fundamental coweights, and let $\beta_\xi := \sum_i m_i(\xi)\alpha_i \in P$. Then we get

Proposition 2.5. *The character χ_λ occurs in $F_{k,\xi}$ if and only if $\xi = k\rho - \lambda \mod Q$ and $(\lambda, \omega_i^\vee) < k(\rho, \omega_i^\vee)$ for all i. Moreover, in presence of the first condition, the second condition is equivalent to the inequality $\lambda \leq k\rho - \beta_\xi$.*

We also have

Proposition 2.6. *The weight $\rho - \beta_\xi$ (and hence $k\rho - \beta_\xi$ for all $k \geq 1$) is dominant.*

Proof. We need to show that for all i we have $(\rho - \beta_\xi, \alpha_i^\vee) \geq 0$. Since $\rho - \beta_\xi$ is integral, it suffices to show that $(\rho - \beta_\xi, \alpha_i^\vee) > -1$, i.e., $(\beta_\xi, \alpha_i^\vee) < 2$. But $(\beta_\xi, \alpha_i^\vee) = 2m_i + \sum_{j \neq i} m_j(\alpha_j, \alpha_i^\vee) < 2$, since $0 < m_j \leq 1$ and $(\alpha_j, \alpha_i^\vee) \leq 0$ for all $j \neq i$ and is strictly negative for some j. $\square$

Corollary 2.7. *$F_{k,\xi} = \sum_{\mu \leq k\rho - \beta_\xi} C_{k,\xi}(\mu)\chi_\mu$, where $C_{k,\xi}(\mu) \in \mathbb{Q}_{>0}$. In particular, the leading term is a multiple of $\chi_{k\rho - \beta_\xi}$.*

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References

[CZ] R. Coquereaux and J.-B. Zuber, From orbital measures to Littlewood-Richardson coefficients and hive polytopes, arXiv:1706.02793.

[GLS] V. Guillemin, E. Lerman and S. Sternberg, Symplectic fibrations and multiplicity diagrams, Cambridge University Press, 1996.

[GK] M. Gros and T. Kaneda, Un scindage du morphisme de Frobenius quantique, Ark. Mat. Volume 53, Number 2 (2015), 271—301.

[Li1] P. Littelmann, The path model, the quantum Frobenius map and the Standard Monomial theory, Theory, in "Algebraic Groups and Their Representations, (R. Carter and J. Saxl, eds.), Kluwer Academic Publishers (1998).

[Li2] P. Littelmann, Contracting modules and standard monomial theory for symmetrizable Kac-Moody algebras, JAMS,v. 11(3), 1998, p. 551—567.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

E-mail address: `etingof@math.mit.edu`

Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA

E-mail address: `rains@caltech.edu`