MASSIVE WAVES GRAVITATIONALLY BOUND TO STATIC BODIES

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Abstract. We show that, given any static spacetime whose spatial slices are asymptotically Euclidean (or, more generally, asymptotically conic) manifolds modeled on the large end of the Schwarzschild exterior, there exist stationary solutions to the Klein-Gordon equation having Schwartz initial data. In fact, there exist infinitely many independent such solutions. The proof is a variational argument, using some classical results regarding the highly excited states of the hydrogen atom (the "Rydberg series").

Contents

1. Introduction

In classical Newtonian gravity, massive particles can be bound to the gravitational potential-well generated by another body. Solutions to the Klein-Gordon equation

$$\Box U + m^2 U = 0 \tag{1}$$

serve as wavefunctions for massive scalar particles in relativistic quantum mechanics, so it is natural to expect that they can get "gravitationally bound" to the gravitational fields generated by astrophysical bodies. (Here, $\Box$ is the d'Alembertian of the spacetime, with the sign chosen so that the spatial Laplace-Beltrami operator is positive semidefinite.) One manifestation of this should be a lack of temporal decay. This line of reasoning should be taken with a grain of salt:

- in classical Newtonian gravity, the mass of a particle is irrelevant to its orbital motion, but solutions to the massless wave equation $\Box U = 0$ (on astrophysical spacetimes) do actually decay, specifically at a $\sim t^{-3}$ rate, a fact known as *Price's law* [Pri72a][Pri72b], and
- it has been predicted by physicists that, on the exact (exterior) Schwarzschild spacetime and some of its relatives, solutions to the Klein-Gordon equation also decay, but at a different rate, namely $\sim t^{-5/6}$ [HP98][KT01][KT02][BK04][KZM07][Bar+14].

We consider in this note a broad class of static spacetimes whose asymptotic structure is given by the *large end* of the Schwarzschild (or, more generally, Reissner-Nordström) exterior. A precise definition is below. One key example is a static spacetime whose spatial slices are isometric to the large end of the Schwarzschild exterior outside of some compact subset. The exact Schwarzschild exterior (and perturbations thereof) are excluded. The Schwarzschild exterior has two ends – the "large" end, where $r \to \infty$, and the horizon – whereas the *admissible* metrics considered here only have one. Admissible metrics appear in nature as the gravitational field configurations outside of spherically symmetric astrophysical bodies lacking the necessary density to form a black hole. Price's law applies to such spacetimes. In the present (in fact, greater) generality, this has been

Date: November 15, 2022 (Draft).

proven rigorously by Hintz [Hin21] — see also [DSS11][DSS12][MTT12][Tat13][MW21][Loo21]. On the other hand, confirming (or disconfirming) physicists' predictions regarding Klein-Gordon on exact Schwarzschild remains an open problem. In fact, proving even $o(1)$ decay remains an open problem.

Our goal is to prove:

Theorem 1.1. *Let $\mathbb{R}_t \times X$ denote an admissible spacetime. Then, for each $m > 0$, there exists an infinite sequence $\{E_n\}_{n=1}^\infty$ of $E_n \in (0, m^2)$ with $E_n \downarrow 0$ such that there exists, for each $n \in \mathbb{N}^+$, a Schwartz function*

$$u_n : X \to \mathbb{R}, \tag{2}$$

not identically 0, such that

$$U_n(t, -) = e^{it\sqrt{m^2 - E_n}} u_n \tag{3}$$

is a solution to the Klein-Gordon equation $\Box U + m^2 U = 0$. ■

So, on any admissible spacetime, there exist temporally non-decaying solutions to the Klein-Gordon equation. In contrast to the situation for massless waves, the temporal decay of massive waves on the exact Schwarzschild exterior (assuming that such decay does in fact occur) is *not* due to the asymptotic structure of the spacetime at the large end. Roughly speaking, massive waves with insufficient kinetic energy should not radiate away from a black hole. Rather, they should fall in towards the horizon. As the admissible spacetimes considered here look like the Schwarzschild exterior but lack a horizon, there is nowhere for the mass to go, and so solutions to the Klein-Gordon equation need not decay.

We start with the elementary observation that, given any stationary spacetime $\mathbb{R}_t \times X$, there exists a 1-parameter family

$$\{P(\sigma)\}_{\sigma \in \mathbb{C}} = \{P_m(\sigma)\}_{\sigma \in \mathbb{C}} \subset \mathrm{Diff}^2(X^\circ) \tag{4}$$

of 2nd order differential operators (depending on m) on X such that solutions $u \in \mathcal{D}'(X)$ to $P(\sigma)u = 0$ yield non-decaying solutions U to the Klein-Gordon equation. When our spacetime is not just stationary but actually static, then

$$P(\sigma) = P + m^2 - \sigma^2 \tag{5}$$

is the spectral family of an m-dependent (rescaled) Schrödinger operator $P = P(m)$ with a potential of the form

$$V_1 + V_2 = -\frac{\mathsf{M} m^2}{r} + V_2, \tag{6}$$

where V_2 is a short range potential depending on m and the metric of the spacetime. Thus, we have an *attractive* Coulomb potential proportional to the Schwarzschild mass $\mathsf{M} > 0$ and the Klein-Gordon mass-squared m^2. This (except, perhaps, for the fact that it is m^2 rather than m that shows up) should be unsurprising given the form of the potential in Newtonian gravity. (We are working here in "natural units" with respect to which the Newtonian gravitational constant is given by $G = 1/2$.) The low energy scattering theory of such operators was considered in [Sus22] — this corresponds to the $\sigma \to m^+$ limit. Here, we consider bound states with close to threshold energy, which instead involves the $\sigma \to m^-$ limit.

The operator P, with the L^2-based Sobolev space $H^2(X)$ as a domain, is self-adjoint with respect to the inner-product of a carefully chosen L^2-space on X (care required due to the rescaling in the definition of P), so the spectrum of P (defined accordingly) lies on the real axis. As is well known, even if $\mathsf{M} = 0$ or $\mathsf{M} < 0$,

$$\sigma(P) = \{-E_n\}_{n=1}^N \cup [0, \infty), \tag{7}$$

where $N \in \mathbb{N} \cup \{\infty\}$ and may be zero, and $E_1 > E_2 \cdots > 0$ is a strictly decreasing sequence of positive real numbers whose only possible accumulation point is zero. Each E_n is an eigenvalue of P, with a finite dimensional space of Schwartz eigenfunctions. Indeed, for $E \in \mathbb{C}$ with $E \notin [0, \infty)$, the differential operator $P+E$ is elliptic in Melrose's $\mathrm{Diff}^2_{\mathrm{sc}}(X)$ [Mel94][Mel95][Vas18], so analytic Fredholm theory applies there. When $\mathsf{M}>0$, we employ a standard (and quite elementary, though slightly tedious) variational argument in which the radially symmetric Hydrogen bound states are used as test functions in order to show that the number of linearly independent bound states is infinite. That is, $N=\infty$.

The choice of test functions deserves brief comment. In [Sus22], the ansatz

$$r^{-(n-1)/2}\Big(\sigma^2-m^2+\frac{\mathsf{M}m^2}{r}\Big)^{-1/4}\Bigg(\sqrt{\frac{r}{\mathsf{M}}\frac{\sigma^2-m^2}{2m^2-\sigma^2}}+\sqrt{1+\frac{r}{\mathsf{M}}\frac{\sigma^2-m^2}{2m^2-\sigma^2}}\Bigg)^{\pm i\mathsf{M}(2m^2-\sigma^2)(\sigma^2-m^2)^{-1/2}} \times \exp\Bigg[\pm ir\sqrt{\sigma^2-m^2+\frac{2\mathsf{M}m^2}{r}-\frac{\mathsf{M}\sigma^2}{r}}\Bigg] \quad (8)$$

was used (in a slightly different form) to study the $\sigma\to m^+$ limit of the limiting resolvents $R(\sigma\pm i0)$. We might consider using a similar ansatz as a test function in the $\sigma\to m^-$ limit, but we cannot avoid the singularity at $r=\mathsf{M}m^2/(m^2-\sigma^2)$. We could cut off our ansatz so that it is supported in the region $\{r\gg \mathsf{M}m^2/(m^2-\sigma^2)\}$, but because it is exponentially decaying as $r\to\infty$, we should expect $\langle u,P(\sigma)u\rangle_{L^2}$ to be dominated by the contribution from the cut-off, which breaks the variational argument. Thus, it seems we must take the essentially full Hydrogen wavefunctions as test functions.

On more general spacetimes than the static, horizon-free ones considered here, the family $\{P(\sigma)\}_{\sigma\in\mathbb{C}}$ is somewhat more complicated. For instance, on non-static stationary spacetimes $\mathbb{R}_t\times X$,

$$P(\sigma)=P+i\sigma Q+m^2-\sigma^2 \quad (9)$$

for some first-order differential operator Q on X° with real coefficients. Thus, $P(\sigma)$ is no longer a spectral family, and the techniques below no longer apply. The presence of an event horizon complicates matters further, as it obstructs appeals to Fredholm theory. This is most easily illustrated on the exact Schwarzschild spacetime, where the radial part $P_{\mathrm{rad}}(\sigma)$ of $P(\sigma)$ is

$$P_{\mathrm{rad}}(\sigma)=-\frac{\partial^2}{\partial r_*^2}-\frac{2}{r}\Big(1-\frac{\mathsf{M}}{r}\Big)\frac{\partial}{\partial r_*}+m^2-\sigma^2-\frac{\mathsf{M}m^2}{r} \quad (10)$$

with respect to the tortoise coordinate $r_*=r+\mathsf{M}\log(\mathsf{M}^{-1}r-1)$. Since the second-order term is the Laplacian on $\mathbb{R}_{r_*}$, it makes sense to analyze this ordinary differential operator in $\mathrm{Diff}_{\mathrm{sc}}(\mathbb{R}_{r_*})$. The large end of the spacetime corresponds to the $r_*\to\infty$ limit, where

$$m^2-\sigma^2-\frac{\mathsf{M}m^2}{r}=m^2-\sigma^2+O\Big(\frac{1}{r_*}\Big), \quad (11)$$

so $P_{\mathrm{rad}}(\sigma)$ is elliptic there, as an element of $\mathrm{Diff}_{\mathrm{sc}}(\mathbb{R}_{r_*})$, if $\sigma^2<m^2$. The horizon corresponds to the $r_*\to-\infty$ limit, in which

$$m^2-\sigma^2-\frac{\mathsf{M}m^2}{r}=-\sigma^2+O(e^{-r_*/\mathsf{M}}), \quad (12)$$

so $P_{\mathrm{rad}}(\sigma)$ is *not* elliptic there. In fact, on the exact Schwarzschild spacetime, P has no bound states, as can be shown by an elementary calculation involving Wronskians for the radial ODE. The test functions used in the variational argument below can be defined on the exact Schwarzschild exterior just as easily, but rather than conclude that $\sigma_{\mathrm{pp}}(P)\cap(-\infty,0)$ is infinite, we can only conclude that $\sigma(P)\cap(-\infty,0)$ is infinite, which is consistent with the continuous spectrum being

$$\sigma_{\mathrm{cont}}(P)=[-m^2,\infty) \quad (13)$$

and the pure-point spectrum being $\sigma_{\mathrm{pp}}(P)=\varnothing$.

2. Main argument

Fix $\delta > 0$. Consider a static Lorentzian spacetime of the form $(\mathbb{R}_t \times X, g)$, where X is a compact $d \in \mathbb{N}^+$ dimensional manifold-with-boundary and g is a Lorentzian metric of the form

$$g = -(1 + x\aleph + x^{1+\delta}\beth) \cdot \mathrm{d}t^2 + g_X, \tag{14}$$

where $\aleph \in \mathbb{R}$, $\beth \in S^0(X;\mathbb{R})$, and g_X is a (symbolic) asymptotically Riemannian conic metric on X that is classical to subleading order and symmetric to subleading order in the radial-radial direction, i.e. a Riemannian metric of the form

$$g_X = \Big(\frac{1}{x^4} - \frac{\mathsf{M}}{x^3}\Big)\,\mathrm{d}x^2 + \frac{h_{\partial X}}{x^2} + \frac{\Gamma_{1,\partial X} \odot \mathrm{d}x}{x^2} + \frac{h_{1,\partial X}}{x} + x^{1+\delta} h_X \tag{15}$$

with respect to some boundary collar $\iota : [0,\bar{x})_x \times \partial X \to X$, where $\bar{x} \in (0,\infty)$ and $x \in C^\infty(X;[0,\infty))$ denotes a boundary-defining function, and where the other terms are

- a Riemannian metric $h_{\partial X}$ on ∂X,
- a constant $\mathsf{M} \in \mathbb{R}$,
- a symbolic family of 1-forms $\Gamma_{1,\partial X} \in S^0([0,\bar{x})_x;\Omega^1(\partial X))$,
- a symbolic family of (not-necessarily positive semidefinite) symmetric 2-tensors $h_{1,\partial X} \in S^0([0,\bar{x})_x; C^\infty(\partial X; \operatorname{Sym}^2 T^*\partial X))$,

and a symbolic remainder

$$h_X \in S^0(\operatorname{Sym}{}^{\mathrm{sc}}T^*X). \tag{16}$$

We say that the given spacetime is *admissible* if, in addition to the requirements above, $\aleph < 0$. We refer to [Mel94][Sus22] for undefined notational conventions.

The condition that g is Lorentzian means that $1+x\aleph+x^{1+\delta}\beth > 0$ everywhere, so $(1+x\aleph+x^{1+\delta}\beth)^\alpha$ defines an element of $C^\infty(X;\mathbb{R}^+)$ for every $\alpha \in \mathbb{R}$. For the spacetimes of physical interest

$$\aleph = -\mathsf{M}, \tag{17}$$

although we do not enforce this relation. For Reissner-Nordström-like metrics, $\delta = 1$, and $\beth|_{\partial X}$ is constant, being related to the electric charge of the astrophysical body generating the gravitational field.

A straightforward calculation yields:

Proposition 2.1. *The d'Alembertian* $\Box = -|g|^{-1/2}\sum_{j,k=0}^d \partial_j(|g|^{1/2}g^{jk}\partial_k)$ *has the form*

$$\Box = \frac{1}{1 + x\aleph + x^{1+\delta}\beth}\frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{x^{1+\delta}\nabla\beth + (\aleph + (1+\delta)x^\delta\beth)\nabla x}{1 + x\aleph + x^{1+\delta}\beth} + \triangle, \tag{18}$$

where $\triangle$ *is the* positive semidefinite *Laplace-Beltrami operator of the Riemannian manifold* (X, g_X)*, which we consider as an operator on* $\mathbb{R}_t \times X$*. Near* ∂X*,* $\triangle$ *has the form*

$$\triangle = -\Big(1 - \frac{\mathsf{M}}{r}\Big)\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\triangle_{\partial X} - \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r}Q + S^{-1-\delta}\operatorname{Diff}^2_{\mathrm{sc}}(X;\mathbb{R}) \tag{19}$$

with respect to the given boundary collar, where $r = 1/x$*, where* $\triangle_{\partial X}$ *is the (positive semidefinite) Laplace-Beltrami operator of* $(\partial X, h_{\partial X})$ *and* $Q \in S^0\operatorname{Diff}^2_{\mathrm{sc}}(X;\mathbb{R})$ *has the form*

$$Q = \frac{1}{r}Q_\perp\frac{\partial}{\partial r} + \frac{1}{r^2}Q_\partial + \frac{1}{r}Q_{1,\partial} \tag{20}$$

for $Q_\perp, Q_{1,\partial} \in S^0([0,\bar{x})_x;\operatorname{Diff}^1(\partial X;\mathbb{R}))$ *and* $Q_\partial \in S^0([0,\bar{x})_x;\operatorname{Diff}^2(\partial X;\mathbb{R}))$*.* ■□

See [Sus22, Proposition 6.1] for details regarding the computation of $\triangle$.

Fix $V \in S^{-1-\delta}(X;\mathbb{R})$ and $m > 0$. Consider the rescaled Schrödinger operator $P = P_m$ on X° given by

$$P = (1 + x\aleph + x^{1+\delta}\beth)\triangle - \frac{1}{2}(x^{1+\delta}\nabla\beth + (\aleph + (1+\delta)x^\delta\beth)\nabla x) + V_{\mathrm{eff}}, \tag{21}$$

where $V_{\mathrm{eff}} \in x\mathbb{R} + x^{1+\delta}S^0(X;\mathbb{R})$ is given by $V_{\mathrm{eff}} = xm^2\aleph + x^{1+\delta}m^2\daleth + (1+x\aleph+x^{1+\delta}\daleth)V$. Observe that $\nabla\daleth \in S^{-1}\operatorname{Diff}^1_{\mathrm{sc}}(X)$ and $\nabla x \in x^2 S^0 \operatorname{Diff}^1_{\mathrm{sc}}(X)$.

At the level of sets, $L^2(X,\,\mathrm{dVol}_{g_X}) = L^2(X,(1+x\aleph+x^{1+\delta}\daleth)^{-1/2}\,\mathrm{dVol}_{g_X})$. We use '$\mathcal{S}(X)$' to denote the set of Schwartz functions on X, and we abbreviate $\mathcal{H} = L^2(X,(1+x\aleph+x^{1+\delta}\daleth)^{-1/2}\,\mathrm{dVol}_{g_X})$.

Let $P(E) = P + E$.

Proposition 2.2. *$P : H^2(X) \to L^2(X)$ defines a lower-semibounded self-adjoint operator on $L^2(X,(1+x\aleph+x^{1+\delta}\daleth)^{-1/2}\,\mathrm{dVol}_{g_X})$.* ■

Proof. Let $\tilde{P} = (1+x\aleph+x^{1+\delta}\daleth)^{+1/4}P(1+x\aleph+x^{1+\delta}\daleth)^{-1/4}$ denote the symbolic differential operator

$$\tilde{P}u = (1+x\aleph+x^{1+\delta}\daleth)^{+1/4}P((1+x\aleph+x^{1+\delta}\daleth)^{-1/4}u). \tag{22}$$

This has the form $\tilde{P} = (1+x\aleph+x^{1+\delta}\daleth)\triangle + W$ for $W \in S^0(X;\mathbb{R})$, so by the symmetry of $\triangle$ as a bilinear form on $L^2(X,\,\mathrm{dVol}_{g_X})$,

$$\int_X \frac{u_0^* \tilde{P} v_0 \,\mathrm{dVol}_{g_X}}{1+x\aleph+x^{1+\delta}\daleth} = \int_X \frac{(\tilde{P}u_0)^* v_0 \,\mathrm{dVol}_{g_X}}{1+x\aleph+x^{1+\delta}\daleth} \tag{23}$$

for all $u_0, v_0 \in \mathcal{S}(X)$. Consequently, for all $u, v \in \mathcal{S}(X)$,

$$\begin{aligned}\int_X \frac{u^* P v \,\mathrm{dVol}_{g_X}}{(1+x\aleph+x^{1+\delta}\daleth)^{1/2}} &= \int_X \frac{u^*}{(1+x\aleph+x^{1+\delta}\daleth)^{3/4}} \tilde{P}\Big[(1+x\aleph+x^{1+\delta}\daleth)^{1/4}v\Big]\,\mathrm{dVol}_{g_X} \\ &= \int_X \Big[(1+x\aleph+x^{1+\delta}\daleth)^{1/4}u\Big]^* \tilde{P}\Big[(1+x\aleph+x^{1+\delta}\daleth)^{1/4}v\Big] \quad \frac{\mathrm{dVol}_{g_X}}{1+x\aleph+x^{1+\delta}\daleth} \\ &= \int_X \Big[(1+x\aleph+x^{1+\delta}\daleth)^{1/4}v\Big]\Big[\tilde{P}((1+x\aleph+x^{1+\delta}\daleth)^{1/4}u)\Big]^* \frac{\mathrm{dVol}_{g_X}}{1+x\aleph+x^{1+\delta}\daleth} \\ &= \int_X \frac{v}{(1+x\aleph+x^{1+\delta}\daleth)^{3/4}}\Big[\tilde{P}((1+x\aleph+x^{1+\delta}\daleth)^{1/4}u)\Big]^* \,\mathrm{dVol}_{g_X} \\ &= \int_X \frac{(Pu)^* v \,\mathrm{dVol}_{g_X}}{(1+x\aleph+x^{1+\delta}\daleth)^{1/2}},\end{aligned} \tag{24}$$

which says that P defines a symmetric bilinear form $\mathcal{S}(X)^2 \to \mathbb{C}$. The same computations show that

$$\langle u, Pv\rangle_{\mathcal{H}} = \langle Pu, v\rangle_{\mathcal{H}} \tag{25}$$

for all $u \in \mathcal{S}(X)$ and $v \in L^2(X)$, where the left-hand side is defined as a distributional pairing: for all $v \in \mathcal{S}'(X)$ and $u \in \mathcal{S}(X)$, we write

$$\langle u, Pv\rangle_{\mathcal{H}} = Pv\Big(\frac{u^* \mathrm{d}\,\mathrm{Vol}_{g_X}}{(1+x\aleph+x^{1+\delta}\daleth)^{1/2}}\Big), \tag{26}$$

where $Pv : \mathcal{S}(X;|\Lambda^d|T^*X) \to \mathbb{C}$ is a tempered distribution.

In order to conclude that $P : \mathcal{S}(X) \to L^2(X)$ is essentially self-adjoint with respect to the $L^2(X,(1+x\aleph+x^{1+\delta}\daleth)^{-1/2}\,\mathrm{dVol}_{g_X})$ inner product, it suffices to check that

$$\overline{\mathrm{range}(P \pm i)} = L^2(X) \tag{27}$$

for both choices of sign [RS80, Chp. VIII, §2], where $\mathrm{range}(P \pm i) = \{Pu \pm iu : u \in \mathcal{S}(X)\}$. Let $\ker(P \mp i) = \{v \in \mathcal{S}'(X) : Pu = \pm iu\}$. For all $v \in L^2(X) \subset \mathcal{S}'(X)$, we have, via eq. (25),

$$\begin{aligned} v \in \mathrm{range}(P \pm i)^\perp &\iff \langle (P \pm i)u, v\rangle_{\mathcal{H}} = 0 \text{ for all } u \in \mathcal{S}(X) \\ &\iff \langle u, Pv\rangle_{\mathcal{H}} \mp \langle u, iv\rangle_{\mathcal{H}} = 0 \text{ for all } u \in \mathcal{S}(X) \\ &\iff (P \mp i)v = 0 \Rightarrow v \in \ker(P \mp i).\end{aligned} \tag{28}$$

So, $\operatorname{range}(P\pm i)^\perp \subseteq \ker(P\mp i)$. By elliptic regularity, $\ker(P\mp i)$ consists entirely of Schwartz functions. Thus, if $v\in\operatorname{range}(P\pm i)^\perp$, then

$$0=\langle v,(P\pm i)v\rangle_{\mathcal{H}}=\langle v,Pv\rangle_{\mathcal{H}}\pm i\|v\|^2_{\mathcal{H}}. \tag{29}$$

Since the first term on the right-hand side is real by symmetry (using the fact that v is Schwartz, so as to be able to appeal to the computations above), this forces $v=0$. So, $\operatorname{range}(P\pm i)^\perp=\{0\}$. Since

$$\overline{\operatorname{range}(P\pm i)}=(\operatorname{range}(P\pm i)^\perp)^\perp, \tag{30}$$

eq. (27) follows.

We now know that $P:\mathcal{S}(X)\to L^2(X)$ is essentially self-adjoint with respect to the $L^2(X,(1+x\aleph+x^{1+\delta}\daleth)^{-1/2}\,\mathrm{dVol}_{g_X})$ inner product. Let

$$\overline{P}:\mathcal{D}(\overline{P})\to L^2(X) \tag{31}$$

denote the closure of P. It remains only to observe that $\mathcal{D}(\overline{P})=H^2(X)$ and that $\overline{P}u$ is the result of applying the differential operator P to $u\in H^2(X)$.

- Since $P\in\mathcal{L}(H^2(X),L^2(X))$, any closure of $P:\mathcal{S}(X)\to L^2(X)$ contains $H^2(X)$ in its domain and acts on this domain in the expected way. So, $\mathcal{D}(\overline{P})\supseteq H^2(X)$, and $\overline{P}$ extends $P:H^2(X)\to L^2(X)$.
- We show that $P:H^2(X)\to L^2(X)$ is closed using the estimate

$$\|u\|_{H^2(X)}\preceq\|Pu\|_{L^2(X)}+\|u\|_{H^1(X)}+\|u\|_{L^2(X)}\preceq\|Pu\|_{L^2(X)}+\|u\|_{L^2(X)}, \tag{32}$$

  where the second inequality is deduced from the first via the interpolation estimate $\|u\|_{H^1(X)}\preceq\|Pu\|_{L^2(X)}+\|u\|_{L^2(X)}$. If $\{u_n\}_{n=0}^\infty\subset H^2(X)$ satisfies $u_n\to u$ in $L^2(X)$ for some $u\in L^2(X)$, and if $Pu_n\to v$ in $L^2(X)$ for some $v\in L^2(X)$, then

$$u_n\to u\in H^2(X), \tag{33}$$

  which also implies $v=Pu$.

Combining the previous two observations, we conclude that $\overline{P}=P$.

From the semidefiniteness of $\triangle$ on $L^2(X,\,\mathrm{dVol}_{g_X})$,

$$\int_X\frac{u^*Pu\,\mathrm{dVol}_{g_X}}{1+x\aleph+x^{1+\delta}\daleth}\geq(\inf W)\|u\|^2_{\mathcal{H}} \tag{34}$$

for all $u\in H^2(X)$. So, P is lower-semibounded. □

Proposition 2.3. *If $\aleph<0$, there exists some infinite sequence $\{v_n\}_{n=1}^\infty\subseteq C_c^\infty(X^\circ)$ such that*

- $\operatorname{supp}v_n\cap\operatorname{supp}v_{n'}=\varnothing$ *if* $n\neq n'$, *and*
- $\langle v_n,Pv_n\rangle_{L^2(X,(1+x\aleph+x^{1+\delta}\daleth)^{-1/2}\,\mathrm{dVol}_{g_X})}<0$ *for all* n.

■

Proof. By Proposition 2.1, there exists some $Q_0\in S^0\operatorname{Diff}^2_{\mathrm{sc}}(X)$ such that

$$P=-\Big(1-\frac{\mathsf{r}_0}{r}\Big)\frac{\partial^2}{\partial r^2}-\frac{d-1}{r}\frac{\partial}{\partial r}+\frac{m^2\aleph}{r}+\Big(1+\frac{\aleph}{r}\Big)\frac{\triangle_{\partial X}}{r^2}+\frac{1}{r}Q+\frac{1}{r^{1+\delta}}Q_0 \tag{35}$$

near ∂X, where $\mathsf{r}_0\in\mathbb{R}$ is defined by $\mathsf{r}_0=\mathsf{M}-\aleph$. We will work with v supported in the boundary collar, with respect to which we impose that v depends only on r. Then, $r^{-1}(\triangle_{\partial X}+Q)v=0$. Now let, for each $E>0$,

$$P_0(E)=-\Big(1-\frac{\mathsf{r}_0}{r}\Big)\frac{\partial^2}{\partial r^2}-\Big(\frac{d-1}{r}+\frac{\mathsf{r}_0(3-d)}{r^2}\Big)\frac{\partial}{\partial r}+E-\frac{\mathsf{Z}}{r}-\frac{(d^2-4d+3)}{4r^2}+\frac{\mathsf{r}_0(d^2-8d+15)}{4r^3}, \tag{36}$$

where $\mathsf{Z}=-m^2\aleph$. Thus, $Pv=P_0(E)v-Ev+r^{-(1+\delta)}Q_2v$ for some $Q_2\in S^0\operatorname{Diff}^2_{\mathrm{sc}}(X)$. Taking an inner product with v,

$$\langle v,Pv\rangle_{\mathcal{H}}=-E\|v\|^2_{\mathcal{H}}+\langle v,P_0(E)v\rangle_{\mathcal{H}}+\langle v,r^{-2}Q_2v\rangle_{\mathcal{H}}. \tag{37}$$

We can write the density $(1+x\aleph+x^{1+\delta}\beth)^{-1/2}\,\mathrm{dVol}_{g_X}$ near ∂X as $(1+x\aleph+x^{1+\delta}\beth)^{-1/2}\,\mathrm{dVol}_{g_X}\in r^{d-1}(1+S^{-1}(X))\,\mathrm{d}r\,\mathrm{dVol}_{h_{\partial X}}$. Thus, if $v(r)$ is supported in $(R,\infty)_r$ for R sufficiently large, which we denote by $R\gg 0$, we can estimate

$$(1+cR^{-1})\|r^{(d-1)/2}v\|^2_{L^2(R,\infty)}\leq\|v\|^2_{\mathcal{H}}\leq(1+CR^{-1})\|r^{(d-1)/2}v\|^2_{L^2(R,\infty)} \tag{38}$$

for some $c,C>0$. On the other hand,

$$|\langle v,P_0(E)v\rangle_{\mathcal{H}}|=O(\|r^{(d-1)/2}v\|_{L^2(R,\infty)}\|r^{(d-1)/2}P_0(E)v\|_{L^2(R,\infty)}). \tag{39}$$

Since Q_2 is bounded as a map $H^2(X)\to L^2(X)$, $|\langle v,r^{-2}Q_2v\rangle_{\mathcal{H}}|=O(R^{-2}\|r^{(d-1)/2}v\|^2_{H^2(R,\infty)})$.

Despite $P_0(E)$ not being uniformly elliptic as $E\to 0$, we can elementarily bound

$$\|r^{(d-1)/2}v\|^2_{H^2(R,\infty)}\preceq\|r^{(d-1)/2}P_0(E)v\|^2_{L^2(R,\infty)}+\|r^{(d-1)/2}v\|^2_{H^1(R,\infty)} \tag{40}$$

$$\preceq\|r^{(d-1)/2}P_0(E)v\|^2_{L^2(R,\infty)}+\varepsilon\|r^{(d-1)/2}v\|^2_{H^2(R,\infty)}+\varepsilon^{-1}\|r^{(d-1)/2}v\|^2_{L^2(R,\infty)} \tag{41}$$

for any $\varepsilon>0$, where the constants are independent of ε and $R\gg 0$, as well as $E\leq m^2$. Taking ε sufficiently small, we can absorb the second term on the right-hand side of eq. (41) into the left-hand side, yielding

$$\|r^{(d-1)/2}v\|^2_{H^2(R,\infty)}\preceq\|r^{(d-1)/2}P_0(E)v\|^2_{L^2(R,\infty)}+\|r^{(d-1)/2}v\|^2_{L^2(R,\infty)}. \tag{42}$$

Thus, assuming that v is nonzero, we can write

$$\|r^{(d-1)/2}v\|^{-2}_{L^2(R,\infty)}\langle v,Pv\rangle_{L^2(X,(1+x\aleph+x^{1+\delta}\beth)^{-1/2}\,\mathrm{dVol}_{g_X})}=-E+O\Big(\frac{E}{R}+\frac{1}{R^{1+\delta}}+\frac{\|r^{(d-1)/2}P_0(E)v\|_{L^2(R,\infty)}}{\|r^{(d-1)/2}v\|_{L^2(R,\infty)}}\Big), \tag{43}$$

where the constant is independent of $E\leq m^2$, $R\gg 0$, and v.

For $E\geq 0$, consider the ordinary differential operator $P_0(E)$ on $(\max\{0,\mathsf{r}_0\},\infty)$. The essentially unique solution $u=u[E]$ to $P_0(E)u=0$ decaying exponentially as $r\to\infty$ is given by

$$u[E](r)=Cr^{(3-d)/2}(r-\mathsf{r}_0)^{-1}e^{-\sqrt{E}(r-\mathsf{r}_0)}U\Big(-\frac{\mathsf{Z}-\mathsf{r}_0E}{2E^{1/2}},0,2E^{1/2}(r-\mathsf{r}_0)\Big), \tag{44}$$

where $U(a,b,z)$ denotes Tricomi's confluent hypergeometric function and $C=C[E]$ is an arbitrary nonzero factor.

For any four positive real numbers $\lambda<\lambda_0<\Lambda_0<\Lambda$, fix a function $\chi\in C^\infty_{\mathrm{c}}(\mathbb{R})$ that satisfies

- $\operatorname{supp}\chi\Subset(\lambda,\Lambda)$,
- $\chi(\rho)=1$ for all $\rho\in[\lambda_0,\Lambda_0]$, and
- $0\leq\chi(\rho)\leq 1$ for all $\rho\in\mathbb{R}$,

and let $\mathsf{R}_0=\max\{0,\mathsf{r}_0\}$. Let

$$v_\chi[E](r)=\chi\Big(\frac{2\mathsf{Z}}{E}\frac{1}{(r-\mathsf{R}_0)}\Big)u[E](r). \tag{45}$$

This is Schwartz and supported in $(\mathsf{R}_0+2\Lambda^{-1}E^{-1}\mathsf{Z},\mathsf{R}_0+2\lambda^{-1}E^{-1}\mathsf{Z})$, so the analysis above yields

$$\|r^{(d-1)/2}v_\chi\|^{-2}_{L^2(\mathsf{R}_0,\infty)}\langle v_\chi,Pv_\chi\rangle_{\mathcal{H}}=-E+O\Big(E^{1+\delta}+\frac{\|r^{(d-1)/2}P_0(E)v_\chi\|_{L^2(\mathsf{R}_0,\infty)}}{\|r^{(d-1)/2}v_\chi\|_{L^2(\mathsf{R}_0,\infty)}}\Big) \tag{46}$$

for E sufficiently close to zero. In the next section, we will produce a sequence of $E_n>0$ with $E_n\to 0$ such that

$$\frac{\|r^{(d-1)/2}P_0(E_n)v_\chi[E_n]\|_{L^2(\mathsf{R}_0,\infty)}}{\|r^{(d-1)/2}v_\chi[E_n]\|_{L^2(\mathsf{R}_0,\infty)}}=O(E_n^{3/2}) \tag{47}$$

as long as Λ_0 is sufficiently large and λ_0 is sufficiently small.

So, taking n_0 sufficiently large, $\{v_\chi[E_n]\}_{n=n_0}^\infty$ is a sequence of v's satisfying the second clause of the proposition. We can pass to a subsequence whose elements have disjoint support, so that the first clause is satisfied. □

Proposition 2.4. *If $\aleph < 0$, then there exist some infinite sequence $\{\sigma_n\}_{n=1}^\infty$ of $\sigma_n \in [0, m)$ such that*

(1) $\sigma_n \uparrow m$ as $n \to \infty$, and
(2) there exist $L^2(X, (1 + x\aleph + x^{1+\delta}\beth)^{-1/2}\,\mathrm{dVol}_{g_X})$-orthonormal $u_1, u_2, \cdots \in \mathcal{S}(X)$ such that $P(\sigma_n)u_n = 0$.

■

Proof. If $u \in \mathcal{S}'(X)$ satisfies $P(\sigma)u = 0$ for $\sigma \in [0, m)$, then $u \in \mathcal{S}(X)$, since $P(\sigma) = P + m^2 - \sigma^2$ is an elliptic element of the sc-calculus on X. So, we need only construct $u_1, u_2, \cdots$ as elements of $L^2(X)$, and then they are automatically Schwartz.

Via analytic Fredholm theory,

$$\sigma(P) \cap (-\infty, 0) = \sigma_{\mathrm{pp}}(P) \cap (-\infty, 0), \tag{48}$$

and $\sigma_{\mathrm{pp}}(P) \cap (-\infty, 0)$ has no accumulation points within $(-\infty, 0)$. So, $\sigma_{\mathrm{cont}}(P) \subseteq [0, \infty)$. (In fact, equality holds: $\sigma_{\mathrm{cont}}(P) = [0, \infty)$.) So (using the fact that P is lower-semibounded), we can conclude the proposition from the claim that $\sigma_{\mathrm{pp}}(P) \cap (-\infty, 0)$ is infinite.

Let

$$\mu_n = \begin{cases} \inf\{\|v\|_{\mathcal{H}}^{-2}\langle v, Pv\rangle_{\mathcal{H}} : v \in L^2(X)\backslash\{0\}\} & (n = 1), \\ \sup\{\inf\{\|v\|_{\mathcal{H}}^{-2}\langle v, Pv\rangle_{\mathcal{H}} : v \in \{\varphi_1, \cdots, \varphi_{n-1}\}^\perp\backslash\{0\}\} : \varphi_1, \ldots, \varphi_{n-1} \in L^2(X)\} & (n \geq 2), \end{cases} \tag{49}$$

for each $n \in \mathbb{N}^+$. From the previous proposition,

$$\mu_n \leq \max\{\|v_{n_0}\|_{\mathcal{H}}^{-2}\langle v_{n_0}, Pv_{n_0}\rangle_{\mathcal{H}} : n_0 = 1, \ldots, n\} < 0 = \inf \sigma_{\mathrm{cont}}(P) \tag{50}$$

for all n. Via the min-max version of the variational principle [RS78, Theorem XIII.1], we conclude that there exist infinitely many negative eigenvalues of P, counted with multiplicity. Via the ellipticity of $P + E$ for $E > 0$, each eigenvalue of P in $\sigma_{\mathrm{pp}}(P) \cap (-\infty, 0)$ has finite multiplicity, so we can conclude that $\sigma_{\mathrm{pp}}(P) \cap (-\infty, 0)$ is infinite. □

Since $P(\sigma)$ has real coefficients, we may take u_n to be $\mathbb{R}$-valued without loss of generality.

Finally, via one last straightforward calculation:

Proposition 2.5. *If $u \in \mathcal{S}(X)$ satisfies $P(E)u = 0$ for some $E > 0$, then the function $U : \mathbb{R}_t \times X \to \mathbb{C}$ given by*

$$U(t, -) = \begin{cases} e^{\pm it\sqrt{m^2 - E}}u & (E \leq m^2), \\ e^{\pm t\sqrt{E - m^2}}u & (E \geq m^2), \end{cases} \tag{51}$$

satisfies the Klein-Gordon equation $(\Box + m^2 + V)U = 0$, for either choice of sign. ■

Thus, if $u \neq 0$, then, choosing the sign appropriately in the $E \geq m^2$ case, U is a non-decaying solution to the Klein-Gordon equation on $(\mathbb{R}_t \times X, g)$.

3. Estimates for Rydberg wavefunctions

We now study the tempered solution $u[E] : (\mathsf{R}_0, \infty) \to \mathbb{R}$ of the ODE $P_0(E)u[E] = 0$, where

$$P_0(E) = -\Big(1 - \frac{\mathsf{r}_0}{r}\Big)\frac{\partial^2}{\partial r^2} - \Big(\frac{d-1}{r} + \frac{\mathsf{r}_0(3-d)}{r^2}\Big)\frac{\partial}{\partial r} + E - \frac{\mathsf{Z}}{r} - \frac{(d^2 - 4d + 3)}{4r^2} + \frac{\mathsf{r}_0(d^2 - 8d + 15)}{4r^3} \tag{52}$$

is the ordinary differential operator on (R_0,∞) introduced in the proof of Proposition 2.3. Here, $d\in\mathbb{N}^+$, and $\mathsf{r}_0\in\mathbb{R}$ is arbitrary, but $\mathsf{Z}>0$. For each $n\in\mathbb{N}^+$ such that $n^2\geq -\mathsf{r}_0\mathsf{Z}$, let

$$E_n = \begin{cases} \frac{\mathsf{Z}^2}{4n^2} & (\mathsf{r}_0=0), \\ \frac{1}{\mathsf{r}_0^2}(\mathsf{r}_0\mathsf{Z}+2n^2-2n\sqrt{n^2+\mathsf{r}_0\mathsf{Z}}) & (\mathsf{r}_0\neq 0), \end{cases} \tag{53}$$

so that $E_n>0$ satisfies the quadratic equation $2E_nn^2=(\mathsf{Z}-\mathsf{r}_0E_n)^2$, which means that the a-parameter in $U(a,b,z)$ in eq. (44) is $-n$. For any compact $K\subset\mathbb{R}_{\mathsf{r}_0}\times(0,\infty)_{\mathsf{Z}}$, $E_n=\mathsf{Z}^2/(4n^2)+O_K(\mathsf{r}_0\mathsf{Z}^3/n^4)$. for all $(\mathsf{r}_0,\mathsf{Z})\in K$. In particular, $E_n\downarrow 0$ as $n\to\infty$, for each individual r_0,Z. By choosing $C[E_n]$ appropriately, we can arrange that the function u defined by eq. (44) is

$$u[E_n](r) = r^{(3-d)/2}\psi_n(nE_n^{1/2}(r-\mathsf{r}_0)), \tag{54}$$

where

$$\psi_n(r)=\sqrt{\frac{1}{\pi n^5}}e^{-r/n}L^1_{n-1}\Big(\frac{2r}{n}\Big) \tag{55}$$

denotes the nth s-orbital *hydrogen wavefunction* [LL58, Chapter X][Hal13, §18.3]. Here, $L^1_n(z)=(n!)^{-1}z^{-1}e^z\frac{\mathrm{d}^n}{\mathrm{d}z^n}(e^{-z}z^{n+1})$ is a generalized Laguerre polynomial.

The coefficient in eq. (54) has been chosen for later convenience. For all $k\in\mathbb{N}$, we have $r^{k/2}\psi_n(r)\in L^2(\mathbb{R}^{\geq 0}_r)$, and the normalization is such that

$$\int_0^\infty 4\pi r^2\psi_n^2(r)\,\mathrm{d}r=1. \tag{56}$$

More generally, for any $k\in\mathbb{N}^+$,

$$\int_0^\infty r^k\psi_n^2(r)\,\mathrm{d}r=\frac{f_k(n^2)}{n^2} \tag{57}$$

for some polynomial f_k of degree $k-1$ with positive leading coefficient, which can be proven using the *Kramers-Pasternack* [Pas37][Kra57] recurrence relation. This computation can be found in many references. Thus, for any polynomial $g(r)\in\mathbb{R}[r]$ of degree $k\geq 1$ with positive leading coefficient,

$$cn^{2k-4}\leq\int_0^\infty g(r)\psi_n^2(r)\,\mathrm{d}r\leq Cn^{2k-4} \tag{58}$$

for n sufficiently large, for some g-dependent $c,C>0$. For example,

$$\int_0^\infty 4\pi r\psi_n^2(r)\,\mathrm{d}r=\frac{1}{n^2},\qquad \int_0^\infty 4\pi r^3\psi_n^2(r)\,\mathrm{d}r=\frac{3n^2}{2}. \tag{59}$$

The $k=0$ case of eq. (57) is degenerate, and requires a somewhat different argument, e.g. using *Pasternack's inversion relation*, which is also from [Pas37]. The result is

$$\int_0^\infty 4\pi\psi_n^2(r)\,\mathrm{d}r=\frac{1}{n^3}. \tag{60}$$

These moment formulas will be our main input to the calculations below, making up for partial analytical understanding of the operator $P_0(E)$ near $\{E=\mathsf{Z}/r\}$ in the $E\to 0$ limit. The key point is that the equations eq. (57), eq. (58), eq. (59), eq. (60) tell us (via Markov's inequality) something about the concentration of the probability measure $4\pi r^2\psi_n^2(r)\,\mathrm{d}r$ in the limit where $n\to\infty$.

The wavefunctions ψ_n for large n are known as *Rydberg states* in the physics literature, where they are used to model atomic and molecular electrons on the threshold of ionization. The $n\to\infty$ behavior of the generalized Laguerre polynomials L^1_{n-1} appearing in eq. (55) is very well understood, and we could, in principle, use this to get very precise asymptotic statements about $\psi_n(r)$ in the Rydberg limit. However, as this is a bit technically involved, and since an elementary argument suffices for the application above, we only carry out the elementary argument here. We summarize the upshot of the more precise analysis in Remark 3.1, but the proof is omitted.

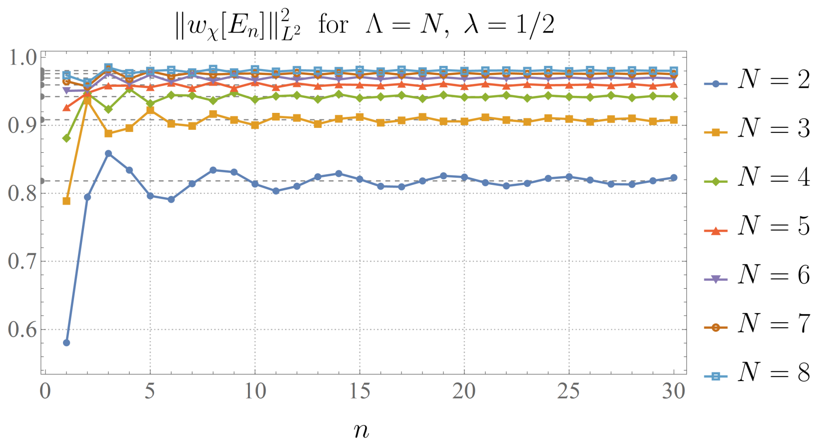

FIGURE 1. The L^2-norms $\|w_\chi[E_n]\|^2_{L^2(\mathsf{r}_0,\infty)}$ of the cut-off hydrogen wavefunctions $w_\chi[E_n]$, with $\chi = 1_{[\lambda,\Lambda]}$ an indicator function, versus n. The function for the 7 values $\Lambda \in \{2,\ldots,8\}$ are shown. The other parameters have been fixed at $\lambda = 1/2$, $\mathsf{Z} = 1$, and $\mathsf{r}_0 = .8$. Dashed horizontal lines, marking the values of the $n \to \infty$ limits according to Remark 3.1, have been drawn at each of the 7 vertical coordinates $2\pi^{-1}(N^{-1}(N-1)^{1/2} + \arctan((N-1)^{1/2}))$, for $N \in \{2,\ldots,8\}$.

Let χ be as in the previous section. We then consider

$$v_\chi[E](r) = \chi\Big(\frac{2\mathsf{Z}}{E}\frac{1}{r-\mathsf{R}_0}\Big)u[E](r), \tag{61}$$

for each $E > 0$. Additionally, set, for each $n \in \mathbb{N}^+$,

$$\begin{aligned} w_\chi[E_n](r) &= \sqrt{4\pi}(nE_n^{1/2})^{3/2}(r-\mathsf{r}_0)r^{(d-3)/2}v_\chi[E_n] \\ &= \sqrt{4\pi}(nE_n^{1/2})^{3/2}(r-\mathsf{r}_0)\chi\Big(\frac{2\mathsf{Z}}{E_n}\frac{1}{r-\mathsf{R}_0}\Big)\psi_n(nE_n^{1/2}(r-\mathsf{r}_0)). \end{aligned} \tag{62}$$

For some $n_0 = n_0(\mathsf{Z},\mathsf{r}_0) > 0$, we have, for all $n \geq n_0$, estimates

$$\|w_\chi[E]\|_{L^2(\mathsf{r}_0,\infty)} \preceq_{\mathsf{Z},\mathsf{r}_0,n_0} \|r^{(d-1)/2}v_\chi[E]\|_{L^2(\mathsf{R}_0,\infty)} \preceq_{\mathsf{Z},\mathsf{r}_0,n_0} \|w_\chi[E]\|_{L^2(\mathsf{r}_0,\infty)}, \tag{63}$$

so estimating $\|r^{(d-1)/2}v_\chi[E]\|_{L^2(\mathsf{R}_0,\infty)}$ amounts to estimating $\|w_\chi[E]\|_{L^2(\mathsf{r}_0,\infty)}$.

When χ is close to the indicator function $1_{[\lambda,\Lambda]}$ in a suitable norm and n is large, then the quantity

$$\begin{aligned} \|w_\chi[E_n]\|^2_{L^2(\mathsf{r}_0,\infty)} &= \int_{\mathsf{r}_0}^\infty 4\pi n^3E_n^{3/2}(r-\mathsf{r}_0)^2\chi\Big(\frac{2\mathsf{Z}}{E_n}\frac{1}{r-\mathsf{R}_0}\Big)^2\psi_n(nE_n^{1/2}(r-\mathsf{r}_0))^2\,\mathrm{d}r \\ &= \int_0^\infty 4\pi r^2\chi\Big(\frac{2\mathsf{Z}n}{E_n^{1/2}}\frac{1}{r-E_n^{1/2}(\mathsf{R}_0-\mathsf{r}_0)}\Big)^2\psi_n(r)^2\,\mathrm{d}r \end{aligned} \tag{64}$$

has, according to Born's rule, the following physical interpretation: it is (approximately) the probability that an electron in the s-orbital in the nth hydrogen shell appears in the annulus

$$\{n^2\Lambda^{-1} < r < n^2\lambda^{-1}\} \tag{65}$$

when the electron's position is measured. This annulus scales quadratically with n.

Proposition 3.1. *Fix $\varepsilon > 0$, and suppose that $\varepsilon \leq \Lambda_0$ and $\lambda_0 \leq \varepsilon^{-1}$. There exists some constant $C = C(\mathsf{Z},\mathsf{r}_0,\varepsilon) > 0$, depending Z, r_0, and ε, but nothing else, such that*

$$1 - \frac{1}{\Lambda_0} - \frac{3\lambda_0}{8} - \frac{C}{n^2} \leq \|w_\chi[E_n]\|^2_{L^2(\mathsf{r}_0,\infty)} \leq 1 \tag{66}$$

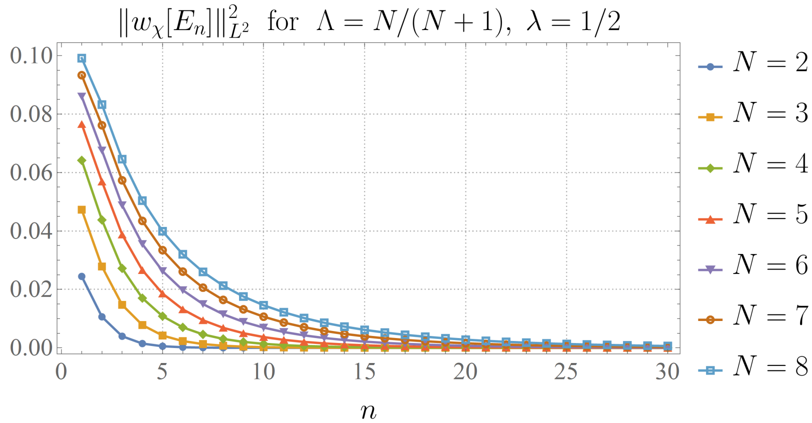

FIGURE 2. The quantities $\|w_\chi[E_n]\|^2_{L^2(\mathsf{r}_0,\infty)}$ versus n, but now for $\Lambda = N/(N+1) < 1$, $N \in \{2,\dots,8\}$. In contrast to the situation with $\Lambda > 1$, we see that the norms converge quickly (in fact, superpolynomially quickly, but we have not given the proof) to zero. Such values of Λ are therefore unsuitable for the variational argument.

for all $n \in \mathbb{N}^+$. ■

Proof. The upper bound in eq. (66) is just a consequence of eq. (56), eq. (64), and the assumption $\chi \le 1$. In order to get the lower bound, we split the right-hand side of eq. (66):

$$\|w_\chi[E_n]\|^2_{L^2(\mathsf{r}_0,\infty)} = 1 - \int_0^\infty 4\pi r^2\Big[1 - \chi\Big(\frac{2\mathsf{Z}n}{E_n^{1/2}}\frac{1}{r - E_n^{1/2}(\mathsf{R}_0 - \mathsf{r}_0)}\Big)^2\Big]\psi_n^2(r)\,\mathrm{d}r. \tag{67}$$

Since χ is identically equal to 1 on $[\lambda_0, \Lambda_0]$, and since $\chi \le 1$,

$$\int_0^\infty 4\pi r^2\Big[1 - \chi\Big(\frac{2\mathsf{Z}n}{E_n^{1/2}}\frac{1}{r - E_n^{1/2}(\mathsf{R}_0 - \mathsf{r}_0)}\Big)^2\Big]\psi_n^2(r)\,\mathrm{d}r \le I_1 + I_2, \tag{68}$$

where

$$I_1 = \int_0^{2\mathsf{Z}nE_n^{-1/2}\Lambda_0^{-1} + E_n^{1/2}(\mathsf{R}_0-\mathsf{r}_0)} 4\pi r^2\psi_n^2(r)\,\mathrm{d}r, \qquad I_2 = \int_{2\mathsf{Z}nE_n^{-1/2}\lambda_0^{-1} + E_n^{1/2}(\mathsf{R}_0-\mathsf{r}_0)}^\infty 4\pi r^2\psi_n^2(r)\,\mathrm{d}r. \tag{69}$$

We control these using two Markov bounds:

$$I_1 \le (2\mathsf{Z}nE_n^{-1/2}\Lambda_0^{-1} + E_n^{1/2}(\mathsf{R}_0 - \mathsf{r}_0))\int_0^\infty 4\pi r\psi_n^2(r)\,\mathrm{d}r = \frac{1}{n^2}(2\mathsf{Z}nE_n^{-1/2}\Lambda_0^{-1} + E_n^{1/2}(\mathsf{R}_0 - \mathsf{r}_0)), \tag{70}$$

and

$$I_2 \le \frac{1}{2\mathsf{Z}nE_n^{-1/2}\lambda_0^{-1} + E_n^{1/2}(\mathsf{R}_0 - \mathsf{r}_0)}\int_0^\infty 4\pi r^3\psi_n^2(r)\,\mathrm{d}r \le \frac{3n^2\lambda_0}{2}\frac{1}{(2\mathsf{Z}nE_n^{-1/2} + E_n^{1/2}\lambda_0(\mathsf{R}_0 - \mathsf{r}_0))}. \tag{71}$$

Combining these estimates, we get eq. (66). □

The key point is that, as long as Λ_0 is sufficiently large and λ_0 is sufficiently small,

$$\inf_{n\in\mathbb{N}^+}\|w_\chi[E_n]\|_{L^2(\mathsf{r}_0,\infty)} > 0. \tag{72}$$

The proof did not require that χ be differentiable; the same estimates (with $\lambda = \lambda_0$ and $\Lambda = \Lambda_0$) hold if $\chi = 1_{[\lambda,\Lambda]}$, and Figure 1 shows a plot of $\|w_\chi[E_n]\|^2_{L^2(\mathsf{r}_0,\infty)}$ versus n in this case.

Remark 3.1. For each $n \in \mathbb{N}^+$, let μ_n denote the probability measure on $[0,\infty)_{\hat{r}}$ whose density is given by $4\pi n^6 \hat{r}^2 \psi_n^2(n^2\hat{r})\,\mathrm{d}\hat{r}$. Using Erdélyi's uniform asymptotics for the Laguerre polynomials [Erd60a][Erd60b], it is possible to prove that these measures converge in law to the measure given by

$$\mu_\infty(\hat{r}) = \frac{2}{\pi}\Big(\frac{1}{\hat{r}} - 1\Big)^{-1/2} 1_{\hat{r}\in[0,1]}\,\mathrm{d}\hat{r}. \tag{73}$$

It follows that $\|w_\chi[E_n]\|^2_{L^2(\mathsf{r}_0,\infty)} \to \int_0^1 \chi(1/\hat{r})\mu_\infty(r)$ as $n\to\infty$. By the portmanteau theorem [Bil95, Theorem 25.8], this holds even if $\chi = 1_{[\lambda,\Lambda]}$, as long as $1 \notin \{\lambda,\Lambda\}$, in which case

$$\begin{aligned}\|w_\chi[E_n]\|^2_{L^2(\mathsf{r}_0,\infty)} &\to \frac{2}{\pi}\int_{\max\{0,1/\Lambda\}}^{\max\{0,1/\lambda\}} \Big(\frac{1}{\hat{r}}-1\Big)^{-1/2}\,\mathrm{d}\hat{r} \\ &= \begin{cases} 0 & (\Lambda > 1), \\ 2\pi^{-1}[\Lambda^{-1}\sqrt{\Lambda-1} + \arctan((\Lambda-1)^{1/2})] & (\Lambda > 1 > \lambda), \\ 2\pi^{-1}[L^{-1}\sqrt{L-1} + \arctan((L-1)^{1/2})]_{L=\lambda}^{L=\Lambda} & (\lambda > 1), \end{cases}\end{aligned} \tag{74}$$

as $n\to\infty$. Moreover, in the $\Lambda > 1$ case, the decay to 0 occurs at an $O(e^{-Cn(\Lambda-1)^{3/2}})$ rate, for some fixed $C > 0$ which can be computed explicitly. For the values of λ, Λ depicted in Figure 1, we have marked the quantity on the right-hand side of eq. (74) via dashed horizontal lines.

We also need to handle a derivative:

Proposition 3.2. *If λ_0 is sufficiently small and Λ_0 is sufficiently large, then*

$$\Big\| r^{(d-1)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)} \preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi} \frac{1}{n^2} \tag{75}$$

for sufficiently large $n \in \mathbb{N}^+$, where the constant depends on $d, \mathsf{Z}, \mathsf{r}_0, \chi$. ■

Proof. We have

$$\Big\| r^{(d-1)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)} = \int_{\mathsf{R}_0}^\infty \chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)^2 u'[E_n](r)^2 r^{d-1}\,\mathrm{d}r. \tag{76}$$

Integrating the right-hand side by parts, removing the derivative from one factor of $u'[E_n]$, yields $I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= -(d-1)\int_{\mathsf{R}_0}^\infty \chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{r}_0)}\Big)^2 u[E_n](r)u'[E_n](r)r^{d-2}\,\mathrm{d}r, \\ I_2 &= \frac{4\mathsf{Z}}{E_n}\int_{\mathsf{R}_0}^\infty \frac{1}{(r-\mathsf{R}_0)^2}\chi'\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u[E_n](r)u'[E_n](r)r^{d-1}\,\mathrm{d}r, \\ I_3 &= -\int_{\mathsf{R}_0}^\infty \chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)^2 u''[E_n](r)u[E_n](r)r^{d-1}\,\mathrm{d}r. \end{aligned} \tag{77}$$

We bound I_1 via Cauchy-Schwarz and Peter-Paul, getting, for sufficiently large $n \in \mathbb{N}^+$,

$$\begin{aligned} I_1 &\preceq_d \frac{1}{\varepsilon}\|r^{(d-3)/2}v_\chi[E_n]\|^2_{L^2(\mathsf{R}_0,\infty)} + \varepsilon\Big\| r^{(d-1)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)} \\ &\preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi} \frac{1}{\varepsilon n^4} + \varepsilon\Big\| r^{(d-1)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)}, \end{aligned} \tag{78}$$

for any $\varepsilon > 0$, where the constants in the bounds do not depend on ε. Similarly, for sufficiently large $n \in \mathbb{N}^+$,

$$\begin{aligned} I_2 &\preceq_{\mathsf{Z},\mathsf{r}_0,\chi} \frac{1}{\varepsilon E_n^2}\|r^{(d-5)/2}v_{\bar\chi}[E_n]\|^2_{L^2(\mathsf{R}_0,\infty)} + \varepsilon\Big\|r^{(d-1)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)} \\ &\preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi,\bar\chi} \frac{1}{\varepsilon n^4} + \varepsilon\Big\|r^{(d-1)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)}, \end{aligned} \tag{79}$$

where we have fixed $\bar\chi \in C_c^\infty((0,\infty);[0,1])$ that is identically equal to 1 on the support of χ. Finally,

$$\begin{aligned} I_3 &\preceq \frac{1}{\varepsilon}\|r^{(d-2)/2}v_{\chi}[E_n]\|^2_{L^2(\mathsf{R}_0,\infty)} + \varepsilon\Big\|r^{d/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u''[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)} \\ &\preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi} \frac{1}{\varepsilon n^2} + \varepsilon\Big\|r^{d/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u''[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)}. \end{aligned} \tag{80}$$

In order to bound the last term, we use the ODE $P_0(E_n)u[E_n] = 0$:

$$\begin{aligned} &\Big\|r^{d/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u''[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)} \preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi} \\ &\frac{1}{n^4}\|r^{d/2}v_{\chi}[E_n]\|^2_{L^2(\mathsf{R}_0,\infty)} + \|r^{(d-2)/2}v_{\chi}[E_n]\|^2_{L^2(\mathsf{R}_0,\infty)} + \Big\|r^{(d-2)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(r,\mathsf{R}_0)} \\ &\qquad \preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi} \Big\|r^{(d-1)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(r,\mathsf{R}_0)} + \frac{1}{n^2} \end{aligned} \tag{81}$$

for sufficiently large $n \in \mathbb{N}^+$. Combining the estimates above,

$$\begin{aligned} \Big\|r^{(d-1)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)} &\preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi} \Big(\frac{1}{\varepsilon}+\varepsilon\Big)\frac{1}{n^2} \\ &+ \varepsilon\Big\|r^{(d-1)/2}\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n](r)\Big\|^2_{L^2(\mathsf{R}_0,\infty)}, \end{aligned} \tag{82}$$

where the constant in the bound is independent of ε. Taking ε sufficiently small, we can absorb the second term on the right-hand side into the left-hand side to conclude the result. ☐

Proposition 3.3. *Given the setup above, there exists some $C > 0$ (depending on $d, \mathsf{Z}, \mathsf{r}_0, \chi$ and nothing else) such that $\|r^{(d-1)/2}P_0(E_n)v_\chi[E_n]\|^2_{L^2} \leq Cn^{-6}$ for sufficiently large $n \in \mathbb{N}^+$.* ■

Proof. We write $P_0(E_n)v_\chi[E_n] = v_1 + v_2 + v_3$, where

$$\begin{aligned} v_1 &= -\Big(1-\frac{\mathsf{r}_0}{r}\Big)\Big[\Big(\frac{4\mathsf{Z}^2}{E_n^2}\frac{1}{(r-\mathsf{R}_0)^4}\chi''\Big(\frac{2\mathsf{Z}}{E_n}\frac{1}{(r-\mathsf{R}_0)}\Big) + \frac{4\mathsf{Z}}{E_n}\frac{1}{(r-\mathsf{R}_0)^3}\chi'\Big(\frac{2\mathsf{Z}}{E_n}\frac{1}{(r-\mathsf{R}_0)}\Big)\Big)\Big]u[E_n](r), \\ v_2 &= \frac{2\mathsf{Z}}{E_n}\frac{1}{(r-\mathsf{R}_0)^2}\Big(\frac{d-1}{r} + \frac{\mathsf{r}_0(3-d)}{r^2}\Big)\chi'\Big(\frac{2\mathsf{Z}}{E_n}\frac{1}{(r-\mathsf{R}_0)}\Big)u[E_n](r), \\ v_3 &= 2\Big(1-\frac{\mathsf{r}_0}{r}\Big)\Big[\frac{2\mathsf{Z}}{E_n}\frac{1}{(r-\mathsf{R}_0)^2}\chi'\Big(\frac{2\mathsf{Z}}{E_n}\frac{1}{(r-\mathsf{R}_0)}\Big)\Big]u'[E_n](r). \end{aligned} \tag{83}$$

For sufficiently large n, we can bound, via the estimates above,

$$\begin{aligned}\|r^{(d-1)/2}v_1\|^2_{L^2(\mathsf{R}_0,\infty)}, \|r^{(d-1)/2}v_2\|^2_{L^2(\mathsf{R}_0,\infty)} &\preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi} \frac{1}{n^8}\|r^{(d-1)/2}v_{\bar\chi}[E_n]\|^2_{L^2(\mathsf{R}_0,\infty)} \\ &\preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi} \frac{1}{n^8},\end{aligned} \tag{84}$$

$$\begin{aligned}\|r^{(d-1)/2}v_3\|^2_{L^2(\mathsf{R}_0,\infty)} &\preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi} \frac{1}{n^4}\Big\|r^{(d-1)/2}\bar\chi\Big(\frac{2\mathsf{Z}}{E_n(r-\mathsf{R}_0)}\Big)u'[E_n]\Big\|^2_{L^2(\mathsf{R}_0,\infty)} \\ &\preceq_{d,\mathsf{Z},\mathsf{r}_0,\chi,\bar\chi} \frac{1}{n^6},\end{aligned} \tag{85}$$

where $\bar\chi$ is as in the proof of the previous proposition. To get the last estimate, we applied Proposition 3.2 with $\bar\chi$ in place of χ. Combining eq. (84) and eq. (85), we arrive at the conclusion of this proposition. □

Combining the propositions in this section, we get the estimate, eq. (47), needed previously.

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