A STRENGTHENED ORLICZ–PETTIS THEOREM VIA ITÔ–NISIO

ETHAN SUSSMAN

Abstract. In this note we deduce a strengthening of the Orlicz–Pettis theorem from the Itô–Nisio theorem. The argument shows that given any series in a Banach space which isn't summable (or more generally unconditionally summable), we can *construct* a (coarse-grained) subseries with the property that – under some appropriate notion of "almost all" – almost all further subseries thereof fail to be weakly summable. Moreover, a strengthening of the Itô–Nisio theorem by Hoffmann-Jørgensen allows us to replace 'weakly summable' with 'τ-weakly summable' for appropriate topologies τ weaker than the weak topology. A treatment of the Itô–Nisio theorem for admissible τ is given.

Contents

1. Introduction 1
2. Measurability 6
3. Proof of Itô–Nisio 8
4. Proof of Orlicz–Pettis 13
Acknowledgements 14
Appendix A. Admissible topologies 14
References 15

1. Introduction

Let $\mathscr{X}$ denote a Banach space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Call a subset $\tau\subseteq 2^{\mathscr{X}}$ an *admissible topology* on $\mathscr{X}$ if

(1) it is an LCTVS[1]-topology on $\mathscr{X}$ identical to or weaker than the norm (a.k.a. strong) topology under which the norm-closed unit ball $\mathbb{B}=\{x\in\mathscr{X}:\|x\|\leq 1\}$ is τ-closed, and
(2) if $\mathscr{X}$ is *not* separable, then τ is at least as strong as the weak topology.

Cf. [Hof74], from which the separable case of this definition arises. By the Hahn-Banach separation theorem, if τ is an admissible topology then the τ-weak topology (a.k.a. $\sigma(\mathscr{X},\mathscr{X}^*_\tau)$-topology) is also admissible (see Lemma A.1).

Besides the norm topology itself, which is trivially admissible (and uninteresting below), the most familiar example of an admissible topology on $\mathscr{X}$ is the weak topology. Many others arise in functional analysis. For example, given a compact Riemannian manifold M, for most function spaces $\mathscr{F}$ it is the case that the $\sigma(\mathscr{F},C^\infty(M))$-topology (a.k.a. the topology of distributional convergence) is admissible. An even weaker typically admissible topology is that on $\mathscr{F}$ generated by the functionals $\langle -,\varphi_n\rangle:\mathscr{D}'(M)\to\mathbb{C}$ for $\varphi_0,\varphi_1,\varphi_2,\cdots$ the eigenfunctions of the Laplacian.

Denote by $\mathscr{X}^{\mathbb{N}}$ the vector space of all $\mathscr{X}$-valued sequences $\{x_n\}_{n=0}^\infty\subseteq\mathscr{X}$. In the usual way, we identify such sequences with $\mathscr{X}$-valued formal series (and denote accordingly). We say that a formal series $\sum_{n=0}^\infty x_n\in\mathscr{X}^{\mathbb{N}}$ is "τ-summable" if $\sum_{n=0}^N x_n\in\mathscr{X}$ converges as $N\to\infty$ in $\mathscr{X}_\tau$.

Date: October 20th, 2022 (Last update). July 1st, 2021 (Preprint).
2020 *Mathematics Subject Classification.* 46B09, 60B05.
Key words and phrases. Itô–Nisio, Orlicz–Pettis, Gaussian noise.

[1]By 'LCTVS' we mean a *Hausdorff* locally convex topological vector space, so we follow the conventions in [Rud73].

Consider the following (slightly generalized) version of the Orlicz–Pettis theorem [Orl29]:

Theorem 1.1. *Suppose that τ is an admissible topology on $\mathscr{X}$. If $\sum_{n=0}^\infty x_n \in \mathscr{X}^{\mathbb{N}}$ fails to be unconditionally summable in the norm topology, then*

- *there exist some $\epsilon_0, \epsilon_1, \epsilon_2, \cdots \in \{-1,+1\}$ such that the sequence $\Sigma(\{\epsilon_n\}_{n=0}^\infty) = \{\Sigma_N\}_{N=0}^\infty$ defined by*
$$\Sigma_N = \sum_{n=0}^N \epsilon_n x_n \tag{1}$$
does not τ-converge as $N \to \infty$ to any element of $\mathscr{X}$, and
- *there exist some $\chi_0, \chi_1, \chi_2, \cdots \in \{0,1\}$ such that the sequence $S(\{\epsilon_n\}_{n=0}^\infty) = \{S_N\}_{N=0}^\infty$ defined by*
$$S_N = \sum_{n=0}^N \chi_n x_n, \tag{2}$$
does not τ-converge as $N \to \infty$ to any element of $\mathscr{X}$.

In particular, this applies if $\sum_{n=0}^\infty x_n$ is not summable in the norm topology. ■

Remark. From the formulas

$$\Sigma_N(\{\epsilon_n\}_{n=0}^N) = S_N(\{2^{-1}(1+\epsilon_n)\}_{n=0}^N) - S_N(\{2^{-1}(1-\epsilon_n)\}_{n=0}^N) \tag{3}$$
$$S_N(\{\chi_n\}_{n=0}^N) = 2^{-1}\Sigma_N(\{1\}_{n=0}^N) + 2^{-1}\Sigma_N(\{2\chi_n - 1\}_{n=0}^N), \tag{4}$$

we deduce that $\Sigma(\{\epsilon_n\}_{n=0}^\infty)$ is τ-convergent for all $\{\epsilon_n\}_{n=0}^\infty \in \{-1,+1\}^{\mathbb{N}}$ if and only if $S(\{\chi_n\}_{n=0}^\infty)$ is τ-convergent for all $\{\chi_n\}_{n=0}^\infty \in \{0,1\}^{\mathbb{N}}$. We will phrase the discussion below in terms of whichever of $\Sigma(-), S(-)$ is convenient, but this equivalence should be kept in mind.

See Proposition 2.5 for the probabilistic version of this remark. ■

Example. Let M be a compact Riemannian manifold and $\mathscr{F} \subseteq \mathscr{D}'(M)$ be a function space on M. Let τ denote the topology generated by the functionals $\langle -, \varphi_n \rangle_{L^2(M)}$, where $\varphi_0, \varphi_1, \varphi_2, \cdots$ denote the eigenfunctions of the Laplace-Beltrami operator. Suppose that τ is admissible. This holds, for example, if $\mathscr{F}$ is an L^p-based Sobolev space for $p \in [1, \infty)$.

Then, for any $\{x_n\}_{n=0}^\infty \subseteq \mathscr{F}$, the formal series $\sum_{n=0}^\infty x_n$ is unconditionally summable in $\mathscr{F}$ (in norm) if and only if

$$\sum_{n=0}^\infty |\langle x_n, \varphi_m \rangle| < \infty \tag{5}$$

for all $m \in \mathbb{N}$ and, for all $\{\chi_n\}_{n=0}^\infty \subseteq \{0,1\}$, there exists an element $S(\{\chi_n\}_{n=0}^\infty) \in \mathscr{F}$ whose mth Fourier coefficient is given by

$$\langle S(\{\chi_n\}_{n=0}^\infty), \varphi_m \rangle = \sum_{n=0}^\infty \chi_n \langle x_n, \varphi_m \rangle. \tag{6}$$

■

We focus on Banach spaces – as opposed to more general LCTVSs – for simplicity. Most of the considerations below apply equally well to Fréchet spaces. There is a long history of variants of the Orlicz–Pettis theorem for various sorts of TVSs [Die77]. A short proof of the Orlicz–Pettis theorem for Banach spaces can be found in [BP58], and a textbook presentation can be found in [Meg98]. The proof below has much in common with a probabilistic proof [Die84] based on the Bochner integral (due to Kwapień).

The proof below is nonconstructive, in the following sense: upon being given a formal series $\sum_{n=0}^\infty x_n \in \mathscr{X}^{\mathbb{N}}$ which fails to be unconditionally summable, we do not construct any particular sequence $\{\epsilon_n\}_{n=0}^\infty \subseteq \{-1,+1\}$ such that $\Sigma(\{\epsilon_n\}_{n=0}^\infty) \subseteq \mathscr{X}$ fails to converge in $\mathscr{X}_\tau$, or any particular

$\{\chi_n\}_{n=0}^\infty \subseteq \{0,1\}$ such that $S(\{\chi_n\}_{n=0}^\infty) \subseteq \mathscr{X}$ fails to converge in $\mathscr{X}_\tau$. All proofs of the Orlicz–Pettis theorem seem to be nonconstructive in this regard. We do, however, construct a function

$$\mathcal{E} : \{\{x_n\}_{n=0}^\infty \in \mathscr{X}^{\mathbb{N}} \text{ not unconditionally summable}\} \to 2^{\{-1,+1\}^{\mathbb{N}}}, \tag{7}$$

such that, when $\{x_n\}_{n=0}^\infty$ is not unconditionally summable, $\Sigma(\{\epsilon_n\}_{n=0}^\infty)$ and $S(\{2^{-1}(1-\epsilon_n)\}_{n=0}^\infty)$ both fail to be τ-summable for $\mathbb{P}_{\mathrm{Coarse}}$-almost all sequences $\{\epsilon_n\}_{n=0}^\infty \in \mathcal{E}$, where

$$\mathbb{P}_{\mathrm{Coarse}} : \mathrm{Borel}(\{-1,+1\}^{\mathbb{N}})|_{\mathcal{E}(\{x_n\}_{n=0}^\infty)} \to [0,1] \tag{8}$$

is a probability measure on the subspace σ-algebra

$$\mathrm{Borel}(\{-1,+1\}^{\mathbb{N}})|_{\mathcal{E}(\{x_n\}_{n=0}^\infty)} = \{S \cap \mathcal{E}(\{x_n\}_{n=0}^\infty) : S \in \mathrm{Borel}(\{-1,+1\}^{\mathbb{N}})\}. \tag{9}$$

So, while the proof is nonconstructive, it is only just. Put more colorfully, the proof follows the "hay in a haystack" philosophy familiar from applications of the probabilistic method to combinatorics [AS16]: using an appropriate sampling procedure, we choose a random subseries and show that – with "high probability" (which in this case means probability one) – it has the desired property.

Precisely, letting $\mathbb{P}_{\mathrm{Haar}}$ denote the Haar measure on the Cantor group $\{-1,+1\}^{\mathbb{N}} \cong \mathbb{Z}_2^{\mathbb{N}}$ [Die84] (which is a compact topological group under the product topology, by Tychonoff's theorem):

Theorem 1.2 (Probabilist's Orlicz–Pettis Theorem)**.** *Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function such that $|f^{-1}(\{n\})| < \infty$ for all $n \in \mathbb{N}$. If $\mathcal{T} \subseteq \mathbb{N}$ is infinite and satisfies*

$$\limsup_{n\to\infty, n\in\mathcal{T}} \Big\| \sum_{n_0 \in f^{-1}(\{n\})} x_{n_0} \Big\| > 0, \tag{10}$$

then it is the case that, for $\mathbb{P}_{\mathrm{Haar}}$-almost all $\{\epsilon_n\}_{n=0}^\infty \in \{-1,+1\}^{\mathbb{N}}$, the formal series

$$\sum_{n=0, f(n)\in\mathcal{T}}^\infty \epsilon_{f(n)} x_n \in \mathscr{X}^{\mathbb{N}}, \qquad \sum_{n=0, f(n)\in\mathcal{T}}^\infty \frac{1}{2}(1-\epsilon_{f(n)}) x_n \in \mathscr{X}^{\mathbb{N}} \tag{11}$$

both fail to be τ-summable. ■

The relation to Orlicz–Pettis is as follows. If $\sum_{n=0}^\infty x_n \in \mathscr{X}^{\mathbb{N}}$ is not unconditionally summable, then we can find some pairwise disjoint, finite subsets $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \cdots \subseteq \mathbb{N}$ such that

$$\inf_{N\in\mathbb{N}} \Big\| \sum_{n\in\mathcal{N}_N} x_n \Big\| > 0. \tag{12}$$

We can then choose some $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = f(m)$ if and only if either $n = m$ or $n, m \in \mathcal{N}_N$ for some $N \in \mathbb{N}$. Thus, if we set $\mathcal{T} = \mathbb{N}$, eq. (10) holds. Appealing to Theorem 1.2, we conclude that, for $\mathbb{P}_{\mathrm{Haar}}$-almost all $\{\epsilon_n\}_{n=0}^\infty$, the formal series

$$\sum_{n=0}^\infty \epsilon_{f(n)} x_n \in \mathscr{X}^{\mathbb{N}}, \qquad \sum_{n=0}^\infty \frac{1}{2}(1-\epsilon_{f(n)}) x_n \in \mathscr{X}^{\mathbb{N}} \tag{13}$$

both fail to be τ-summable. Theorem 1.1, therefore, follows from Theorem 1.2. The connection with eq. (7), eq. (8) is that we can choose f such that $\mathcal{E}$ is the set of $\{\epsilon_n\}_{n=0}^\infty \in \{-1,+1\}^{\mathbb{N}}$ such that $\epsilon_n = \epsilon_m$ whenever $f(n) = f(m)$, and $\mathbb{P}_{\mathrm{Coarse}}$ is $\mathbb{P}_{\mathrm{Haar}}$ conditioned on the event that $\{\epsilon_n\}_{n=0}^\infty \in \mathcal{E}$.

Remark. The Haar measure on the Cantor group is the unique measure on $\mathrm{Borel}(\{-1,+1\}^{\mathbb{N}}) = \sigma(\{\epsilon_n\}_{n=0}^\infty)$ such that if we define $\epsilon_n : \{-1,+1\}^{\mathbb{N}} \to \{-1,+1\}$ by $\epsilon_n : \{\epsilon'_m\}_{m=0}^\infty \mapsto \epsilon'_n$, the random variables $\epsilon_0, \epsilon_1, \epsilon_2, \cdots$ are i.i.d. Rademacher random variables. ■

Remark. It suffices to prove the theorems above when $\mathscr{X}$ is separable. Indeed, if $\mathscr{X}$ is not separable and $\mathscr{Y}$ denotes the norm-closure of the span of $x_0, x_1, x_2, \cdots \in \mathscr{X}$, then, for any $\{\lambda_n\}_{n=0}^\infty \subseteq \mathbb{K}$,

$$\tau\text{-}\lim_{N\to\infty} \sum_{n=0}^N \lambda_n x_n \tag{14}$$

exists in $\mathscr{X}$ if and only if it exists in $\mathscr{Y}$. (This is a consequence of the requirement that τ be at least as strong as the weak topology, so the limit in eq. (14) is also a weak limit. Norm-closed convex subsets of $\mathscr{X}$ are weakly closed by Hahn-Banach, so this implies that $\mathscr{Y}$ is τ-closed.)

The subspace topology on $\mathscr{Y} \hookrightarrow \mathscr{X}_\tau$ is admissible, and $\mathscr{Y}$ is separable, so we can deduce Theorem 1.1 and Theorem 1.2 for $\mathscr{X}$ from the same theorems for $\mathscr{Y}$. ■

Remark. If $\mathscr{X}$ is not separable and τ not at least as strong as the weak topology, then the conclusions of these theorems may fail to hold, even if the norm-closed balls in $\mathscr{X}$ are τ-closed. As a simple counterexample, let $\mathscr{X} = L^\infty[0,1]$, and let τ be the $\sigma(L^\infty, L^1)$-topology. This being a weak-$*$ topology, the norm-closed balls are τ-closed (and even τ-compact). Let

$$\Sigma_N(t) = t^N, \tag{15}$$

$x_n(t) = \Sigma_n(t) - \Sigma_{n-1}(t)$ for $n \geq 1$, $x_0(t) = \Sigma_0(t)$. Then, the series $\sum_{n=0}^\infty x_n$ is τ-subseries summable, being τ-summable to the identically zero function. But, Σ_N does not converge to zero uniformly, so $\sum_{n=0}^\infty x_n$ is not strongly summable. ■

Remark. When $\mathscr{X}$ is separable, it suffices to consider the case when τ is the topology generated by a countable norming set of functionals. Recall that a subset $\mathcal{S} \subseteq \mathscr{X}_\tau^*$ is called *norming* if

$$\|x\| = \sup_{\Lambda \in \mathcal{S}} |\Lambda x| \tag{16}$$

for all $x \in \mathscr{X}$. We can scale the members of a norming subset to get another norming subset whose members Λ satisfy $\|\Lambda\|_{\mathscr{X}^*} = 1$, and this generates the same topology. If τ is admissible, then (by the Hahn-Banach theorem and separability) there exists a countable norming subset $\mathcal{S} \subseteq \mathscr{X}_\tau^*$ (see Lemma A.2). Whenever $\mathcal{S} \subseteq \mathscr{X}_\tau^*$ is a countable norming subset, the $\sigma(\mathscr{X}, \mathcal{S})$-topology is admissible as well (see Lemma A.3), and identical with or weaker than τ. ■

It is not necessary to consider probability spaces other than

$$(\{-1,+1\}^{\mathbb{N}}, \mathrm{Borel}(\{-1,+1\}^{\mathbb{N}}), \mathbb{P}_{\mathrm{Haar}}), \tag{17}$$

but it will be convenient to have a bit more freedom. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space on which i.i.d. Bernoulli random variables

$$\chi_0, \chi_1, \chi_2, \cdots : \Omega \to \{0,1\} \tag{18}$$

are defined. For example,

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\{-1,+1\}^{\mathbb{N}}, \mathrm{Borel}(\{-1,+1\}^{\mathbb{N}}), \mathbb{P}_{\mathrm{Haar}}), \tag{19}$$

in which case we set $\chi_n = (1/2)(1-\epsilon_n)$. Given this setup and given a formal series $\sum_{n=0}^\infty x_n \in \mathscr{X}^{\mathbb{N}}$, we can construct a random formal subseries $S : \Omega \to \mathscr{X}^{\mathbb{N}}$ by

$$S(\omega) = \sum_{n=0}^\infty \chi_n(\omega) x_n. \tag{20}$$

This is a measurable function from Ω to $\mathscr{X}^{\mathbb{N}}$ when $\mathscr{X}$ is separable (see Lemma 2.1)

Suppose that $\mathscr{X}$ is separable. Given any Borel subset $\mathtt{P} \subseteq \mathscr{X}^{\mathbb{N}}$ the probability $\mathbb{P}(S^{-1}(\mathtt{P})) \in [0,1]$ of the "event" $S \in \mathtt{P}$ is well-defined. Given some "property" $\mathtt{P}$ – which we identify with a not-necessarily-Borel subset $\mathtt{P} \subseteq \mathscr{X}^{\mathbb{N}}$ – that a formal series may or may not possess, to say that almost all subseries of $\sum_{n=0}^\infty x_n$ have property $\mathtt{P}$ means that there exists some $F \in \mathcal{F}$ with

$$\mathbb{P}(F) = 1 \tag{21}$$

and $\omega \in F \Rightarrow S(\omega) \in \mathtt{P}$. In this case, we say that S has the property $\mathtt{P}$ for $\mathbb{P}$-almost all ω. (Note that we do not require $S^{-1}(\mathtt{P}) \in \mathcal{F}$, although this is automatic if $\mathtt{P}$ is Borel, and can be arranged by passing to the completion of $\mathbb{P}$.) Analogous locutions will be used for random formal series generally. If $\mathtt{P}$ is Borel then $S(\omega)$ will have the property $\mathtt{P}$ for $\mathbb{P}$-almost all $\omega \in \Omega$ if and only if $\mathbb{P}(S^{-1}(\mathtt{P})) = 1$.

In order to prove the theorems above, we use the following variant of a theorem of Itô and Nisio [IN68] refined by Hoffmann-Jørgensen [Hof74]:

Theorem 1.3. *Suppose that τ is an admissible topology on $\mathscr{X}$. Let*

$$\gamma_0, \gamma_1, \gamma_2, \cdots : \Omega \to \{-1, +1\} \tag{22}$$

be independent, symmetric random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. If $\mathscr{X}$ is a Banach space and $\{x_n\}_{n=0}^\infty \in \mathscr{X}^\mathbb{N}$, the following are equivalent:

(I) for $\mathbb{P}$-almost all $\omega \in \Omega$, $\sum_{n=0}^\infty \gamma_n(\omega) x_n$ is summable in $\mathscr{X}$,
(II) for $\mathbb{P}$-almost all $\omega \in \Omega$, $\sum_{n=0}^\infty \gamma_n(\omega) x_n$ is τ-summable, i.e. summable in $\mathscr{X}_\tau$.

Moreover, whether or not the conditions above hold depends only on $\{x_n\}_{n=0}^\infty$ and the laws of each of $\gamma_0, \gamma_1, \gamma_2, \cdots$. ■

This result is essentially contained in [Hof74], but, since our formulation is slightly different, we present a proof in §3 below.

See [Hyt+16] for a modern account of the Itô–Nisio result in the case when τ is the weak topology. Our proof follows theirs.

A special case of this theorem was stated in [Sus22], and the proof was sketched. This paper fills in some details of that sketch.[2]

Remark. We will refer to Theorem 1.3 as "the Itô–Nisio theorem," with the following three caveats:

- Unlike in the usual Itô–Nisio theorem, we do not discuss convergence in probability.
- The result is often stated with general Bochner-measurable symmetric and independent random variables $x_n(\omega) : \Omega \to \mathscr{X}^\mathbb{N}$ in place of $\gamma_n(\omega) x_n$. (A $\mathscr{X}$-valued random variable X will be called *symmetric* if X and $-X$ are equidistributed, i.e. have the same law.[3]) In fact, Theorem 1.3 implies the more general version via a rerandomization argument.
- Itô and Nisio only consider the case when τ is the weak topology, the generalization to admissible τ being the result of [Hof74].

■

Remark. A strengthening of the Itô–Nisio result in the case when $\mathscr{X}$ does not admit an isometric embedding $c_0 \hookrightarrow \mathscr{X}$ is essentially contained – and explicitly conjectured – in [Hof74]. The proof is due to Kwapień [Kwa74]. If (and only if) $\mathscr{X}$ does not admit an isometric embedding $c_0 \hookrightarrow \mathscr{X}$, then (I), (II) in Theorem 1.3 are equivalent to

(III) for almost all $\omega \in \Omega$, $\sup_{N \in \mathbb{N}} \|\sum_{n=0}^N \epsilon_n(\omega) x_n\| < \infty$.

(The event described above, that of "uniform boundedness," is also measurable. See Lemma 2.2.)

Recall that – by the uniform boundedness principle – the weak convergence of a sequence $\{X_N\}_{N=0}^\infty \subseteq \mathscr{X}$ implies that $\sup_N \|X_N\| < \infty$, so (II) implies (III) when τ is the weak topology. Condition (I) obviously implies (III), so by the Itô–Nisio theorem (once we've proven it), (II) implies (III) for any admissible τ. The converse obviously does not hold if $\mathscr{X}$ admits an isometric embedding $c_0 \hookrightarrow \mathscr{X}$. ■

Remark. By Lemma 2.2, the events described in (I), (III) above are measurable, and so, Theorem 1.3 is a statement about their probabilities. If $\mathscr{X}$ is separable and τ is the topology generated by a countable norming collection of functionals, the event in (II) is measurable as well. It is a consequence of Theorem 1.3 that, if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, then (II) is measurable regardless. ■

An outline for the rest of this note is as follows:

[2] See [Sus22, Thm. 3.11]. The statement there involves convergence in probability, but the proof in §3 below applies.

[3] Note that, if $\mathbb{K} = \mathbb{C}$, this convention differs from some in the literature, in particular [Hyt+16, Definition 6.1.4]. (We use 'symmetric' when they would use 'real-symmetric.')

- In §2, we fill in some measure-theoretic details related to the main line of argument.
- We prove the Itô–Nisio theorem in §3 using a version of the standard argument based on uniform tightness and Lévy's maximal inequality.
- Using Theorem 1.3, we prove the probabilist's Orlicz–Pettis theorem in §4

2. Measurability

Let $\mathscr{X}$ be an arbitrary separable Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let τ be an admissible topology on it. Below, $\epsilon_0, \epsilon_1, \epsilon_2, \cdots$ will be as in Theorem 1.3, i.i.d. Rademacher random variables $\Omega \to \{-1, +1\}$. Similarly, $\chi_0, \chi_1, \chi_2, \cdots$ will be i.i.d. uniformly distributed $\Omega \to \{0, 1\}$.

Lemma 2.1. *The function $S : \Omega \to \mathscr{X}^{\mathbb{N}}$ defined by eq. (20) is measurable with respect to the Borel σ-algebra* $\mathrm{Borel}(\mathscr{X}^{\mathbb{N}})$*, so it is a well-defined random formal $\mathscr{X}$-valued series.* ■

Proof. The Borel σ-algebra of a countable product of separable metric spaces agrees with the product $\mathcal{P}$ of the Borel σ-algebras of the individual factors [Kal02, Lemma 1.2]. So, $\mathrm{Borel}(\mathscr{X}^{\mathbb{N}}) = \sigma(\mathrm{eval}_n : n \in \mathbb{N}) = \mathcal{P}$, where

$$\mathrm{eval}_n : \mathscr{X}^{\mathbb{N}} \to \mathscr{X} \tag{23}$$

is shorthand for the map $\sum_{n=0}^{\infty} x_n \mapsto x_n$. To deduce that S is Borel measurable, we just observe that it is measurable with respect to the σ-algebra $\sigma(\mathrm{eval}_n : n \in \mathbb{N})$, since $\mathrm{eval}_n \circ S(\omega) = \chi_n(\omega) x_n$. □

Let $\mathtt{P}_{\mathrm{I}}, \mathtt{P}_{\mathrm{II}}, \mathtt{P}_{\mathrm{III}} \subseteq \mathscr{X}^{\mathbb{N}}$ denote the sets of (I) strongly summable formal series, (II) τ-summable formal series, and (III) bounded formal series, respectively. In other words,

$$\mathtt{P}_{\mathrm{I}} = \{\{x_n\}_{n=0}^{\infty} \in \mathscr{X}^{\mathbb{N}} : \lim_{N\to\infty} \textstyle\sum_{n=0}^{N} x_n \text{ exists in } \mathscr{X}\}, \tag{24}$$

$$\mathtt{P}_{\mathrm{II}} = \{\{x_n\}_{n=0}^{\infty} \in \mathscr{X}^{\mathbb{N}} : \tau\!-\!\lim_{N\to\infty} \textstyle\sum_{n=0}^{N} x_n \text{ exists in } \mathscr{X}_\tau\}, \tag{25}$$

$$\mathtt{P}_{\mathrm{III}} = \{\{x_n\}_{n=0}^{\infty} \in \mathscr{X}^{\mathbb{N}} : \sup_{N\in\mathbb{N}} \|\textstyle\sum_{n=0}^{N} x_n\| < \infty\}. \tag{26}$$

Likewise, given a countable norming subset $\mathcal{S} \subseteq \mathscr{X}_\tau^*$, let

$$\mathtt{P}_{\mathrm{II}'} = \mathtt{P}_{\mathrm{II}'}(\mathcal{S}) = \{\{x_n\}_{n=0}^{\infty} \in \mathscr{X}^{\mathbb{N}} : \mathcal{S}\!-\!\lim_{N\to\infty} \textstyle\sum_{n=0}^{N} x_n \text{ exists in } \mathscr{X}_{\sigma(\mathscr{X},\mathcal{S})}\} \tag{27}$$

denote the set of $\mathcal{S}$-weakly summable formal $\mathscr{X}$-valued series.

Lemma 2.2. $\mathtt{P}_{\mathrm{I}}, \mathtt{P}_{\mathrm{II}'}, \mathtt{P}_{\mathrm{III}} \in \mathrm{Borel}(\mathscr{X}^{\mathbb{N}})$. *Consequently, given any random formal series $\Sigma : \Omega \to \mathscr{X}^{\mathbb{N}}$, $\Sigma^{-1}(\mathtt{P}_i) \in \mathcal{F}$ for each $i \in \{\mathrm{I}, \mathrm{II}', \mathrm{III}\}$.* ■

Proof. For each $M, N \in \mathbb{N}$, the function $\mathfrak{N}_{N,M} : \mathscr{X}^{\mathbb{N}} \to \mathbb{R}$ given by

$$\mathfrak{N}_{N,M}(\{x_n\}_{n=0}^{\infty}) = \Big\| \sum_{n=M}^{N} x_n \Big\| \tag{28}$$

satisfies $\mathfrak{N}_{N,M}^{-1}(S) \in \mathcal{P}$ for all $S \in \mathrm{Borel}(\mathbb{R})$. Therefore, $\mathtt{P}_{\mathrm{III}} = \cup_{R\in\mathbb{N}} \cap_{N\in\mathbb{N}} \mathfrak{N}_{N,0}^{-1}([0, R])$ is in $\mathcal{P}$, as is

$$\mathtt{P}_{\mathrm{I}} = \bigcap_{R\in\mathbb{N}^+} \bigcup_{M\in\mathbb{N}} \bigcap_{N\geq M} \mathfrak{N}_{N,M}^{-1}([0, 1/R]). \tag{29}$$

Let $\mathscr{X}_0 \subseteq \mathscr{X}$ denote a dense countable subset. *Claim*: a sequence $\{X_N\}_{N=0}^{\infty} \subseteq \mathscr{X}$ converges $\mathcal{S}$-weakly if and only if for each rational $\varepsilon > 0$ there exists $X_\approx = X_\approx(\varepsilon) \in \mathscr{X}_0$ such that for each $\Lambda \in \mathcal{S}$ there exists a $N_0 = N_0(\varepsilon, \Lambda) \in \mathbb{N}$ such that

$$|\Lambda(X_N - X_\approx)| < \varepsilon \tag{30}$$

for all $N \geq N_0$.

- Proof of 'only if:' if $X_N \to X$ $\mathcal{S}$-weakly, then, for each $\varepsilon > 0$, choose $X_\approx = X_\approx(\varepsilon) \in \mathscr{X}_0$ such that $\|X - X_\approx\| < \varepsilon/2$, and for each $\Lambda \in \mathcal{S}$ choose $N_0(\varepsilon, \Lambda)$ such that $|\Lambda(X_N - X)| < \varepsilon/2$ for all $N \geq N_0$.

  Since the elements of $\mathcal{S}$ have operator norm at most one, $|\Lambda(X - X_\approx)| < \varepsilon/2$.

  Combining these two inequalities, eq. (30) holds for all $N \geq N_0$.
- Proof of 'if:' suppose we are given $X_\approx(\varepsilon)$ with the desired property. First, observe that $\{X_\approx(1/N)\}_{N=1}^\infty$ is Cauchy. Indeed, it follows from the definition of the $X_\approx(\varepsilon)$ that $|\Lambda(X_\approx(\varepsilon) - X_\approx(\varepsilon'))| < \varepsilon + \varepsilon'$ for all $\Lambda \in \mathcal{S}$, which implies (since $\mathcal{S}$ is norming) that $\|X_\approx(\varepsilon) - X_\approx(\varepsilon')\| \leq \varepsilon + \varepsilon'$. So, by the completeness of $\mathscr{X}$, there exists some $X \in \mathscr{X}$ such that
$$\lim_{N\to\infty} X_\approx(1/N) = X. \tag{31}$$
  We now need to show that, as $N \to \infty$, $X_N \to X$ $\mathcal{S}$-weakly. Indeed, given any $\Lambda \in \mathcal{S}$ and $M \in \mathbb{N}^+$,
$$|\Lambda(X_N - X)| \leq |\Lambda(X_N - X_\approx(1/M))| + |\Lambda(X - X_\approx(1/M))|. \tag{32}$$
  Given any $\varepsilon > 0$, pick M such that $1/M < \varepsilon/2$ and such that $\|X_\approx(1/M) - X\| < \varepsilon/2$. Since the elements of $\mathcal{S}$ have operator norm at most one, $|\Lambda(X - X_\approx(1/M))| < \varepsilon/2$. By the hypothesis of this direction, we can choose $N_0 = N_0(\varepsilon, \Lambda)$ sufficiently large such that $|\Lambda(X_N - X_\approx(1/M))| < 1/M < \varepsilon/2$ for all $N \geq N_0$. Therefore, $|\Lambda(X_N - X)| < \varepsilon$ for all $N \geq N_0$. It follows that $X_N \to X$ $\mathcal{S}$-weakly.

We therefore conclude that
$$\mathtt{P}_{\mathrm{II}'} = \bigcap_{\varepsilon>0,\varepsilon\in\mathbb{Q}} \bigcup_{X_\approx\in\mathscr{X}_0} \bigcap_{\Lambda\in\mathcal{S}} \bigcup_{M\in\mathbb{N}} \bigcap_{N\geq M} \{\{x_n\}_{n=0}^\infty : |\Lambda(X_N - X_\approx)| < \varepsilon\} \tag{33}$$
is in $\mathcal{P}$ as well, where $X_N = x_0 + \cdots + x_{N-1}$, which depends measurably on $\{x_n\}_{n=0}^\infty$.

□

Remark. We do not address the question of when $\mathtt{P}_{\mathrm{II}}$ is Borel. Even when $\mathscr{X}_\tau^*$ is not second countable, it can be the case that $\mathtt{P}_{\mathrm{II}} \in \mathcal{P}$. For example, if $\mathscr{X} = \ell^1(\mathbb{N})$, then sequential weak convergence is equivalent to sequential strong convergence [Car05, Theorem 6.2], and hence $\mathtt{P}_{\mathrm{I}} = \mathtt{P}_{\mathrm{II}}$. ■

Let $\pi_N : \mathscr{X}^\mathbb{N} \to \mathscr{X}^\mathbb{N}$ denote the left-shift map $\sum_{n=0}^\infty x_n \mapsto \sum_{n=0}^\infty x_{n+N}$. Let $\pi_N^*\mathcal{P} = \{\pi_N^{-1}(S) : S \in \mathcal{P}\}$.

Lemma 2.3. *Let $\mathtt{P}_{\mathrm{I}}, \mathtt{P}_{\mathrm{II}'}, \mathtt{P}_{\mathrm{III}}$ be as above. Then*
$$\mathtt{P}_{\mathrm{I}}, \mathtt{P}_{\mathrm{II}'}, \mathtt{P}_{\mathrm{III}} \in \mathcal{T}, \tag{34}$$
where $\mathcal{T} \subseteq \mathrm{Borel}(\mathscr{X}^\mathbb{N})$ is the "tail σ-algebra" $\mathcal{T} = \cap_{N\in\mathbb{N}}\pi_N^*\mathcal{P}$. Consequently, given any $\mathbb{K}$-valued random variables $\lambda_0, \lambda_1, \lambda_2, \dots : \Omega \to \mathbb{K}$, the random formal series $\Sigma : \Omega \to \mathscr{X}^\mathbb{N}$ given by $\Sigma(\omega) = \sum_{n=0}^\infty \lambda_n(\omega) x_n$ is such that
$$\Sigma^{-1}(\mathtt{P}_i) \in \cap_{N\in\mathbb{N}}\sigma(\{\lambda_n\}_{n=N}^\infty) \tag{35}$$
for each $i \in \{\mathrm{I}, \mathrm{II}', \mathrm{III}\}$. ■

Proof. Clearly, $\pi_N^{-1}(\mathtt{P}_i) = \mathtt{P}_i$ for each $i \in \{\mathrm{I}, \mathrm{II}', \mathrm{III}\}$. By Lemma 2.2, we can therefore conclude that $\mathtt{P}_i \in \mathcal{T}$. If Σ is as above, then $\Sigma^* \circ \pi_N^*\mathcal{P} \subseteq \sigma(\{\lambda_n\}_{n=N}^\infty)$. Since $\Sigma^{-1}(\mathtt{P}_i)$ is in the left-hand side for each $N \in \mathbb{N}$, eq. (35) follows. □

Proposition 2.4. *Let $f : \mathbb{N} \to \mathbb{N}$ satisfy $|f^{-1}(\{n\})| < \infty$ for all $n \in \mathbb{N}$. Suppose that $\lambda_0, \lambda_1, \lambda_2, \dots : \Omega \to \mathbb{K}$ are independent random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and consider the random formal series $\Sigma : \Omega \to \mathscr{X}^\mathbb{N}$ given by*
$$\Sigma(\omega) = \sum_{n=0}^\infty \lambda_{f(n)}(\omega) x_n. \tag{36}$$

Then $\mathbb{P}(\Sigma^{-1}(\mathtt{P})) = \mathbb{P}[\Sigma \in \mathtt{P}] \in \{0,1\}$ *for any element* $\mathtt{P} \in \mathcal{T}$*, and in particular for the sets* $\mathtt{P}_i$ *for each* $i \in \{\mathrm{I}, \mathrm{II}', \mathrm{III}\}$. ■

Proof. Since $\lambda_0, \lambda_1, \lambda_2, \cdots$ are now assumed to be independent, that $\mathbb{P}[\Sigma \in \mathtt{P}] \in \{0,1\}$ follows immediately from the Kolmogorov zero-one law [Dur19, Theorem 2.5.3]. By Lemma 2.3, this applies to $\mathtt{P}_{\mathrm{I}}, \mathtt{P}_{\mathrm{II}'}, \mathtt{P}_{\mathrm{III}}$. □

Proposition 2.5. *Let* $f : \mathbb{N} \to \mathbb{N}$ *satisfy* $|f^{-1}(\{n\})| < \infty$ *for all* $n \in \mathbb{N}$*. Suppose that* $\mathtt{P} \subseteq \mathscr{X}^{\mathbb{N}}$ *is a* $\mathbb{K}$*-subspace and that* $\zeta_0, \zeta_1, \zeta_2, \cdots : \Omega \to \mathbb{K}$ *are a collection of symmetric, independent* $\mathbb{K}$*-valued random variables.*

Then, letting $\Sigma, S : \Omega \to \mathscr{X}^{\mathbb{N}}$ *denote the random formal series*

$$\Sigma(\omega) = \sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n \quad \textit{and} \quad S(\omega) = \sum_{n=0}^{\infty} \chi_{f(n)}(\omega) x_n, \tag{37}$$

where $\chi_n = 2^{-1}(1-\zeta_n)$*, the following are equivalent: (*$*$*)* $\Sigma \in \mathtt{P}$ *for* $\mathbb{P}$*-almost all* $\omega \in \Omega$ and $\sum_{n=0}^{\infty} x_n \in \mathtt{P}$*, (*$**$*)* $S \in \mathtt{P}$ *for* $\mathbb{P}$*-almost all* $\omega \in \Omega$*. Consequently, if* $\mathtt{P} \in \mathcal{T}$*, by Proposition 2.4 the following are equivalent: (*$*'$*)* $\Sigma \notin \mathtt{P}$ *for* $\mathbb{P}$*-almost all* $\omega \in \Omega$ or $\sum_{n=0}^{\infty} x_n \notin \mathtt{P}$ *and (*$**'$*)* $S \notin \mathtt{P}$ *for* $\mathbb{P}$*-almost all* $\omega \in \Omega$*.* ■

This is essentially an immediate consequence of eq. (3), eq. (4), *mutatis mutandis.*

Proof. First suppose that ($*$) holds. In particular, $\sum_{n=0}^{\infty} x_n \in \mathtt{P}$. Then, since $\mathtt{P}$ is a subspace of $\mathscr{X}^{\mathbb{N}}$,

$$\sum_{n=0}^{\infty} \chi_{f(n)}(\omega) x_n = -\frac{1}{2} \sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n + \frac{1}{2} \sum_{n=0}^{\infty} x_n \tag{38}$$

is in $\mathtt{P}$ if $\sum_{n=0}^{\infty} \zeta_n(\omega) x_n$ is. By assumption, this holds for $\mathbb{P}$-almost all $\omega \in \Omega$, and so we conclude that ($**$) holds.

Conversely, suppose that ($**$) holds, so that $S(\omega) \in \mathtt{P}$ for all ω in some some subset $F \in \mathcal{F}$ with $\mathbb{P}(F) = 1$. Clearly, the two formal series $S, S' : \Omega \to \mathscr{X}^{\mathbb{N}}$,

$$S(\omega) = \sum_{n=0}^{\infty} \chi_{f(n)}(\omega) x_n \quad \text{and} \quad S'(\omega) = \sum_{n=0}^{\infty} (1 - \chi_{f(n)}(\omega)) x_n \tag{39}$$

are equidistributed. We deduce that $S'(\omega) \in \mathtt{P}$ for almost all $\omega \in \Omega$, i.e. that there exists some $F' \in \mathcal{F}$ with $\mathbb{P}(F') = 1$ such that $S'(\omega) \in \mathtt{P}$ whenever $\omega \in F'$. This implies, since $\mathtt{P}$ is a subspace of $\mathscr{X}^{\mathbb{N}}$, that the random formal series

$$S(\omega) + S'(\omega) = \sum_{n=0}^{\infty} x_n \tag{40}$$

$$S(\omega) - S'(\omega) = -\sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n \tag{41}$$

are both in $\mathtt{P}$ for all $\omega \in F \cap F'$. Since $\mathbb{P}(F \cap F') = 1$, it is the case that $F \cap F' \neq \varnothing$, and so we conclude that $\sum_{n=0}^{\infty} x_n \in \mathtt{P}$. Likewise, $\sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n \in \mathtt{P}$ for almost all $\omega \in \Omega$. □

Proposition 2.5 applies in particular to the sets $\mathtt{P}_{\mathrm{I}}, \mathtt{P}_{\mathrm{II}'}, \mathtt{P}_{\mathrm{III}}$. We will not discuss $\mathtt{P}_{\mathrm{III}}$ further, but the preceding results are useful for the treatment of the Jørgensen–Kwapień and Bessaga–Pełczyński theorems along the lines of §4.

3. Proof of Itô–Nisio

Let $\mathscr{X}$ be a separable Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We now give a treatment, via the method in [Hyt+16], of the particular variant of the Itô–Nisio theorem stated in Theorem 1.3.

The key result allowing the generalization from the weak topology to all admissible topologies is:

Proposition 3.1. *If τ is an admissible topology on $\mathscr{X}$, then* $\operatorname{Borel}(\mathscr{X}) = \operatorname{Borel}(\mathscr{X}_\tau)$. ■

Proof. The inclusion $\operatorname{Borel}(\mathscr{X}) \supseteq \operatorname{Borel}(\mathscr{X}_\tau)$ is an immediate consequence of the assumption that τ is weaker than or identical to the norm topology, so it suffices to prove that $\operatorname{Borel}(\mathscr{X}_\tau)$ contains a collection of sets that generate $\operatorname{Borel}(\mathscr{X})$ as a σ-algebra. Consider the collection

$$\mathcal{B} = \{x + \lambda\mathbb{B} : x \in \mathscr{X}, \lambda \in \mathbb{R}^{\geq 0}\} \subseteq \operatorname{Borel}(\mathscr{X}) \tag{42}$$

of all norm-closed balls in $\mathscr{X}$. Since $\mathscr{X}$ is separable, the collection of all open balls generates $\operatorname{Borel}(\mathscr{X})$, and each open ball $x + \lambda\mathbb{B}^\circ$, $x \in \mathscr{X}, \lambda > 0$, is a countable union

$$x + \lambda\mathbb{B}^\circ = \bigcup_{N\in\mathbb{N}, 1/N<\lambda} (x + (\lambda - 1/N)\mathbb{B}) \tag{43}$$

of closed balls, so the closed balls generate $\operatorname{Borel}(\mathscr{X})$. Since τ is an LCTVS topology, once we know that $\mathbb{B}$ is τ-closed, the same holds for all other norm-closed balls. Because τ is admissible, the elements of $\mathcal{B}$ are τ-closed, so $\mathcal{B} \subseteq \operatorname{Borel}(\mathscr{X}_\tau)$. □

Suppose now that τ is admissible, and suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which symmetric, independent random variables $\gamma_0, \gamma_1, \gamma_2, \cdots : \Omega \to \mathbb{K}$ are defined.

Proposition 3.2. *Suppose that $\sum_{n=0}^\infty \gamma_n(\omega)x_n$ converges in $\mathscr{X}_\tau$ for $\mathbb{P}$-almost all $\omega \in \Omega$, so that we may find some $F \in \mathcal{F}$ with $\mathbb{P}(F) = 1$ such that*

$$\Sigma_\infty(\omega) = \tau\text{-}\lim_{N\to\infty} \sum_{n=0}^N \gamma_n(\omega)x_n \tag{44}$$

exists for all $\omega \in F$. Set $\Sigma_\infty(\omega) = 0$ for all $\omega \in \Omega\backslash F$. Then, Σ_∞ is a well-defined $\mathscr{X}$-valued random variable. ■

Proof. We want to prove that Σ_∞ is measurable with respect to $\mathcal{F}$ and $\operatorname{Borel}(\mathscr{X})$. By Proposition 3.1 and Lemma A.1, $\operatorname{Borel}(\mathscr{X}) = \operatorname{Borel}(\mathscr{X}_\tau) = \operatorname{Borel}(\sigma(\mathscr{X}, \mathscr{X}_\tau^*)) = \sigma(\mathscr{X}_\tau^*)$, so it suffices to check that $\Lambda \circ \Sigma_\infty$ is a measurable $\mathbb{K}$-valued function for each $\Lambda \in \mathscr{X}_\tau^*$. Certainly,

$$\Lambda \circ \tilde{\Sigma}_N(\omega) = 1_{\omega\in F}\Lambda \circ \Sigma_N(\omega) = \begin{cases} \Sigma_N(\omega) & (\omega \in F) \\ 0 & (\omega \in \Omega\backslash F) \end{cases} \tag{45}$$

is measurable. Consequently, $\Lambda \circ \Sigma_\infty = \lim_{N\to\infty} \Lambda \circ \tilde{\Sigma}_N$ is the limit of measurable $\mathbb{K}$-valued random variables and, therefore, measurable. □

Proposition 3.3. *Consider the setup of Proposition 3.2. For each $N \in \mathbb{N}$, the $\mathscr{X}$-valued random variables Σ_∞ and $\Sigma_\infty - 2\Sigma_N$ are equidistributed.* ■

Proof. Denote the laws $\Sigma_\infty, \Sigma_\infty - 2\Sigma_N$ by $\mu, \lambda_N : \operatorname{Borel}(\mathscr{X}) \to [0, 1]$, respectively. The measures μ, λ_N are uniquely determined by their Fourier transforms $\mathcal{F}\mu, \mathcal{F}\lambda_N : \mathscr{X}_\tau^* \to \mathbb{C}$,

$$\mathcal{F}\mu(\Lambda) = \int_\Omega e^{-i\Lambda\Sigma_\infty(\omega)}\, \mathrm{d}\mathbb{P}(\omega) = \int_{\mathscr{X}} e^{-i\Lambda x}\, \mathrm{d}\mu(x), \tag{46}$$

where $\mathcal{F}\lambda_N$ is defined analogously. For each $\Lambda \in \mathscr{X}_\tau^*$, $\Lambda(\Sigma_\infty - \Sigma_N)$ and $\Lambda(\Sigma_N)$ are clearly independent, and $\Lambda(\Sigma_N)$ is equidistributed with $-\Lambda(\Sigma_N)$, so

$$\begin{aligned}\mathcal{F}\mu(\Lambda) = \int_\Omega e^{-i\Lambda\Sigma_\infty(\omega)}\,\mathrm{d}\mathbb{P}(\omega) &= \int_\Omega e^{-i\Lambda(\Sigma_\infty(\omega)-\Sigma_N(\omega))}e^{-i\Lambda\Sigma_N(\omega)}\,\mathrm{d}\mathbb{P}(\omega) \\ &= \Big(\int_\Omega e^{-i\Lambda(\Sigma_\infty(\omega)-\Sigma_N(\omega))}\,\mathrm{d}\mathbb{P}(\omega)\Big)\Big(\int_\Omega e^{-i\Lambda\Sigma_N(\omega)}\,\mathrm{d}\mathbb{P}(\omega)\Big) \\ &= \Big(\int_\Omega e^{-i\Lambda(\Sigma_\infty(\omega)-\Sigma_N(\omega))}\,\mathrm{d}\mathbb{P}(\omega)\Big)\Big(\int_\Omega e^{+i\Lambda\Sigma_N(\omega)}\,\mathrm{d}\mathbb{P}(\omega)\Big) \\ &= \int_\Omega e^{-i\Lambda(\Sigma_\infty(\omega)-\Sigma_N(\omega))}e^{+i\Lambda\Sigma_N(\omega)}\,\mathrm{d}\mathbb{P}(\omega) \\ &= \int_\Omega e^{-i\Lambda(\Sigma_\infty(\omega)-2\Sigma_N(\omega))}\,\mathrm{d}\mathbb{P}(\omega) = \mathcal{F}\lambda_N(\Lambda).\end{aligned} \tag{47}$$

Hence the Fourier transforms of μ, λ_N agree, and we conclude that Σ_∞ and $\Sigma_\infty - 2\Sigma_N$ are equidistributed. □

The proof is identical to the standard one, except we need to know that the law of an $\mathscr{X}$-valued random variable is uniquely determined by the restriction of its Fourier transform (a.k.a. "characteristic functional") from $\mathscr{X}^*$ to $\mathscr{X}_\tau^*$, for any admissible τ. The proof of this fact for τ the strong or weak topologies, which is just the proof that a finite Borel measure on $\mathscr{X}$ is uniquely determined by the Fourier transform of its law, is given in [Hyt+16, E.1.16, E.1.17]. The general statement follows from analogous reasoning: the finite-dimensional version (i.e. finite Borel measures on $\mathbb{R}^d$ are identifiable with particular tempered distributions, and are, therefore, uniquely determined by their Fourier transforms), the Dynkin π-λ theorem (which implies that a finite measure is uniquely determined by its restriction to any π-system which generates the σ-algebra on which the measure is defined [Dur19, Theorem A.1.5]), and Proposition 3.1.

Another way to prove the proposition is to show that Σ_∞ agrees, almost everywhere, with the composition of the random formal series $\sum_{n=0}^\infty \gamma_n(-)x_n : \Omega \to \mathscr{X}^{\mathbb{N}}$ and $\Sigma_{\infty,\mathrm{Uni}} : \mathscr{X}^{\mathbb{N}} \to \mathscr{X}$,

$$\Sigma_{\infty,\mathrm{Uni}}\Big(\sum_{n=0}^\infty x_n\Big) = \begin{cases} \mathcal{S}-\lim_{N\to\infty}\sum_{n=0}^N x_n & (\sum_{n=0}^\infty x_n \in \mathtt{P}_{\mathrm{II}'}), \\ 0 & (\text{otherwise}), \end{cases} \tag{48}$$

where $\mathcal{S} \subseteq \mathscr{X}_\tau^*$ is a countable norming collection of functionals and $\mathtt{P}_{\mathrm{II}'}$ is as in §2. By the results in §2, $\Sigma_{\infty,\mathrm{Uni}} : \mathscr{X}^{\mathbb{N}} \to \mathscr{X}$ is Borel measurable. Thus, we can form the pushforward under it of the law of the formal series $\sum_{n=0}^\infty \gamma_n(-)x_n$. The initial claim, then, is that the law of Σ_∞ is this pushforwards. Likewise, the pushforwards of the law of the random formal series

$$\omega \mapsto -\sum_{n=0}^N \gamma_n(\omega)x_n + \sum_{n=N+1}^\infty \gamma_n(\omega)x_n \in \mathscr{X}^{\mathbb{N}} \tag{49}$$

is the law of $\Sigma_\infty - 2\Sigma_N$. Since the random formal series eq. (49) is equidistributed with the original, we deduce that Σ_∞ and $\Sigma_\infty - 2\Sigma_N$ are equidistributed as well.

Recall that an $\mathscr{X}$-valued random variable $X : \Omega \to \mathscr{X}$ is called *tight* if for every $\varepsilon > 0$ there exists a norm-compact set $K \subseteq \mathscr{X}$ such that $\mathbb{P}[X \notin K] \le \varepsilon$. By an elementary argument, every $\mathscr{X}$-valued random variable is tight [Hyt+16, Proposition 6.4.5]. A family $\mathcal{X}$ of $\mathscr{X}$-valued random variables is called *uniformly tight* if we can choose the same $K = K(\varepsilon)$ for every $X \in \mathcal{X}$, i.e. if for each $\varepsilon > 0$ there exists some norm-compact $K \subseteq \mathscr{X}$ such that $\mathbb{P}[X \notin K] \le \varepsilon$ holds for all $X \in \mathcal{X}$. If $\mathcal{X}$ is uniformly tight, then

$$\mathcal{X} - \mathcal{X} = \{X_1 - X_2 : X_1, X_2 \in \mathcal{X}\} \tag{50}$$

is uniformly tight as well, a fact which is used below. (The map $\Delta : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ given by $(x, y) \mapsto x - y$ is continuous. If $K \subseteq \mathscr{X}$ is compact, then $K \times K$ is a compact subset of $\mathscr{X} \times \mathscr{X}$.

Its image $\Delta(K \times K) = K - K$ under Δ is, therefore, also compact. By a union bound,

$$\mathbb{P}[X_1 - X_2 \notin \Delta(K \times K)] \leq \mathbb{P}[X_1 \notin K] + \mathbb{P}[X_2 \notin K]. \tag{51}$$

See [Hyt+16, Lemma 6.4.6].)

To complete the proof of the Itô–Nisio theorem, we use Lévy's maximal inequality [Hyt+16, Proposition 6.1.12][4]:

Proposition 3.4 (Lévy's maximal inequality)**.** *Let $\mathscr{X}$ be a separable Banach space over $\mathbb{K}$. Let $x_0, x_1, x_2, \cdots$ be independent symmetric $\mathscr{X}$-valued random variables. Then, setting $\Sigma_N = \sum_{n=0}^N x_n$,*

$$\mathbb{P}[(\exists N_0 \in \{0, \cdots, N\}) \|\Sigma_{N_0}\| \geq R] \leq 2\mathbb{P}[\|\Sigma_N\| \geq R] \tag{52}$$

for all $N \in \mathbb{N}$ and real $R > 0$. ■□

Proposition 3.5. *Suppose that $\sum_{n=0}^\infty \gamma_n(\omega) x_n$ converges in $\mathscr{X}_\tau$ for $\mathbb{P}$-almost all $\omega \in \Omega$, and let Σ_∞ denote the $\mathscr{X}$-valued random variable constructed in the statement of Proposition 3.2. Then*

$$\Sigma_\infty(\omega) = \lim_{N \to \infty} \sum_{n=0}^N \gamma_n(\omega) x_n \tag{53}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. ■

The limit here is taken in the strong topology.

Proof. The proof is split into three parts. We first show that it suffices to show that $\Sigma_N \to \Sigma_\infty$ in probability, where $\Sigma_N = \sum_{n=0}^N \gamma_n(\omega) x_n$, i.e. that

$$\lim_{N \to \infty} \mathbb{P}[\|\Sigma_\infty - \Sigma_N\| > \varepsilon] = 0 \tag{54}$$

for all $\varepsilon > 0$. This part of the argument uses Lévy's inequality. We then establish (via a standard trick) the uniform tightness of $\{\Sigma_N\}_{N=0}^\infty$. The third step involves showing that, if Σ_N fails to converge to Σ_∞ in probability, then, with positive probability, Σ_N fails to converge to Σ_∞ in $\mathscr{X}_\tau$. Under our assumption to the contrary, we can then conclude that $\Sigma_N \to \Sigma_\infty$ in probability, which by the first part of the argument completes the proof of the proposition.

(1) Suppose that $\lim_{N \to \infty} \mathbb{P}[\|\Sigma_\infty - \Sigma_N\| > \varepsilon] = 0$ for all $\varepsilon > 0$. We want to prove that $\Sigma_N \to \Sigma_\infty$ $\mathbb{P}$-almost surely. It suffices to prove that $\{\Sigma_N\}_{N=0}^\infty$ is $\mathbb{P}$-almost surely Cauchy, since then by the completeness of $\mathscr{X}$ it converges strongly $\mathbb{P}$-almost surely to some random limit $\Sigma'_\infty : \Omega \to \mathscr{X}$. Since the τ topology is weaker than (or identical to) the strong topology and Hausdorff, $\Sigma'_\infty = \Sigma_\infty$ $\mathbb{P}$-almost surely.

By the triangle inequality, for any $M, M', N \in \mathbb{N}$, $\|\Sigma_M - \Sigma_{M'}\| \leq \|\Sigma_M - \Sigma_N\| + \|\Sigma_{M'} - \Sigma_N\|$. Therefore, by a union bound,

$$\mathbb{P}\Big[\bigcup_{M, M' \geq N} \|\Sigma_M - \Sigma_{M'}\| \geq \varepsilon\Big] \leq 2\mathbb{P}\Big[\bigcup_{M \geq N} \|\Sigma_M - \Sigma_N\| \geq \varepsilon/2\Big]. \tag{55}$$

By the countable additivity of $\mathbb{P}$ and by Lévy's maximal inequality,

$$2\mathbb{P}\Big[\bigcup_{M \geq N} \|\Sigma_M - \Sigma_N\| \geq \varepsilon/2\Big] = \lim_{N' \to \infty} 2\mathbb{P}\Big[\bigcup_{N' \geq M \geq N} \|\Sigma_M - \Sigma_N\| \geq \varepsilon/2\Big] \tag{56}$$

$$\leq \lim_{N' \to \infty} 4\mathbb{P}\Big[\|\Sigma_{N'} - \Sigma_N\| \geq \varepsilon/2\Big]. \tag{57}$$

[4] The statement there uses strict inequalities for the events, but the version for nonstrict inequalities follows by the countable additivity of $\mathbb{P}$.

Consequently,

$$\mathbb{P}\Big[\bigcup_{\varepsilon>0}\bigcap_{N=0}^{\infty}\bigcup_{M,M'\geq N}\|\Sigma_M-\Sigma_{M'}\|\geq\varepsilon\Big]=\lim_{\varepsilon\to 0^+}\lim_{N\to\infty}\mathbb{P}\Big[\bigcup_{M,M'\geq N}\|\Sigma_M-\Sigma_{M'}\|\geq\varepsilon\Big] \\ \leq 4\lim_{\varepsilon\to 0^+}\lim_{N\to\infty}\lim_{N'\to\infty}\mathbb{P}[\|\Sigma_{N'}-\Sigma_N\|\geq\varepsilon/2]. \tag{58}$$

By the triangle inequality and a union bound,

$$\mathbb{P}[\|\Sigma_{N'}-\Sigma_N\|\geq\varepsilon/2]\leq\mathbb{P}[\|\Sigma_\infty-\Sigma_N\|\geq\varepsilon/4]+\mathbb{P}[\|\Sigma_{N'}-\Sigma_\infty\|\geq\varepsilon/4]. \tag{59}$$

It follows from the assumption that $\Sigma_N\to\Sigma_\infty$ in probability that

$$\lim_{N\to\infty}\lim_{N'\to\infty}\mathbb{P}[\|\Sigma_{N'}-\Sigma_N\|\geq\varepsilon/2]=0. \tag{60}$$

Consequently, the right-hand side and thus left-hand side of eq. (58) are zero. The event on the left-hand side of eq. (58) is the event that the sequence $\{\Sigma_N\}_{N=0}^\infty$ fails to be Cauchy, so the preceding argument shows that $\{\Sigma_N(\omega)\}_{N=0}^\infty$ is Cauchy for $\mathbb{P}$-almost all $\omega\in\Omega$.

(2) By Proposition 3.3, Σ_∞ and $\Sigma_\infty-2\Sigma_N$ are equidistributed, for each $N\in\mathbb{N}$. For any $\varepsilon>0$, by the (automatic) tightness of Σ_∞ there is a norm-compact subset $K\subseteq\mathscr{X}$ such that $\mathbb{P}[\Sigma_\infty\notin K]<\varepsilon$. Let $L=(1/2)(K-K)$, which is also compact. Then, by a union bound,

$$\mathbb{P}[\Sigma_N\notin L]\leq\mathbb{P}[\Sigma_\infty\notin K]+\mathbb{P}[\Sigma_\infty-2\Sigma_N\notin K]=2\mathbb{P}[\Sigma_\infty\notin K]<2\varepsilon. \tag{61}$$

We conclude that $\{\Sigma_N\}_{N=0}^\infty$ is uniformly tight.

Also, since Σ_∞ is tight, the family $\mathcal{X}=\{\Sigma_N\}_{N=0}^\infty\cup\{\Sigma_\infty\}$ is uniformly tight, which implies that the family $\{\Sigma_\infty-\Sigma_N\}_{N=0}^\infty\subseteq\mathcal{X}-\mathcal{X}$ is uniformly tight. Consequently, there exists for each $\varepsilon>0$ a norm-compact subset $K_0=K_0(\varepsilon)\subseteq\mathscr{X}$ such that

$$\mathbb{P}[(\Sigma_\infty-\Sigma_N)\notin K_0(\varepsilon)]\leq\varepsilon \tag{62}$$

for all $N\in\mathbb{N}$.

(3) Suppose that Σ_N does not converge to Σ_∞ in probability, so that there exist some $\varepsilon,\delta>0$ and some subsequence $\{\Sigma_{N_k}\}_{k=0}^\infty\subseteq\{\Sigma_N\}_{N=0}^\infty$ such that

$$\mathbb{P}[\|\Sigma_\infty-\Sigma_{N_k}\|>\varepsilon]\geq\delta \tag{63}$$

for all $k\in\mathbb{N}$. Consider the set $K_0=K_0(\delta/2)$ defined in eq. (62), so that $\mathbb{P}[(\Sigma_\infty-\Sigma_N)\notin K_0]\leq\delta/2$ for all $N\in\mathbb{N}$. Then, combining this inequality with the inequality eq. (63), $\mathbb{P}[(\Sigma_\infty-\Sigma_{N_k})\in K_0\backslash\varepsilon\mathbb{B}]\geq\delta/2$ for all $k\in\mathbb{N}$. It follows that the quantity

$$\mathbb{P}[(\Sigma_\infty-\Sigma_{N_k})\in K_0\backslash\varepsilon\mathbb{B}\text{ i.o.}]=\mathbb{P}[\cap_{K\in\mathbb{N}}\cup_{k\geq K}(\Sigma_\infty-\Sigma_{N_k})\in K_0\backslash\varepsilon\mathbb{B}] \tag{64}$$

$$=\lim_{K\to\infty}\mathbb{P}[\cup_{k\geq K}(\Sigma_\infty-\Sigma_{N_k})\in K_0\backslash\varepsilon\mathbb{B}] \tag{65}$$

(where "i.o." means for infinitely many k) is bounded below by $\delta/2$ and is in particular positive. So, for ω in some set of positive probability, there exists an ω-dependent subsequence $\{N'_\kappa(\omega)\}_{\kappa=0}^\infty=\{N_{k_\kappa}(\omega)\}_{\kappa=0}^\infty$ such that $\Sigma_\infty(\omega)-\Sigma_{N'_\kappa}(\omega)\in K_0\backslash\varepsilon\mathbb{B}$ for all $\kappa\in\mathbb{N}$.

Since K_0 is a compact subset of a metric space, it is sequentially compact, so by passing to a further subsequence we can assume without loss of generality that $\Sigma_\infty(\omega)-\Sigma_{N'_\kappa}(\omega)$ converges strongly to some ω-dependent $\Delta(\omega)\in\mathscr{X}$, for ω in some subset of positive probability. But, for such ω, $\|\Delta(\omega)\|\geq\varepsilon$ necessarily, so $\Delta(\omega)\neq 0$. Since τ is weaker than or identical to the strong topology,

$$(\Sigma_\infty(\omega)-\Sigma_{N'_\kappa}(\omega))\to\Delta(\omega)\neq 0 \tag{66}$$

in $\mathscr{X}_\tau$ for such ω. Since τ is Hausdorff, $\Sigma_N(\omega)$ does not τ-converge to $\Sigma_\infty(\omega)$ as $N\to\infty$.

We conclude that (53) holds for $\mathbb{P}$-almost all $\omega\in\Omega$ under the hypotheses of the proposition. $\square$

It is clear that which of the cases in Theorem 1.3 hold depends only on $\{x_n\}_{n=0}^\infty$ and the laws of the random variables $\gamma_0,\gamma_1,\gamma_2,\cdots$.

4. Proof of Orlicz–Pettis

Let $\mathscr{X}$ be a separable Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let τ be an admissible topology on it.

Proposition 4.1. *Suppose that $\zeta_0, \zeta_1, \zeta_2, \dots : \Omega \to \mathbb{K}$ are a collection of symmetric, independent $\mathbb{K}$-valued random variables such that, for some infinite $\mathcal{T} \subseteq \mathbb{N}$,*

$$\mathbb{P}[\exists \varepsilon > 0 \text{ s.t. } |\zeta_n| > \varepsilon \text{ for infinitely many } n \in \mathcal{T}] = 1. \tag{67}$$

Suppose further that $\{X_n\}_{n=0}^\infty \in \mathscr{X}^{\mathbb{N}}$ is some sequence satisfying

$$\inf_{n \in \mathcal{T}} \|X_n\| > 0. \tag{68}$$

Then, for any $\mathcal{T}_0 \subseteq \mathbb{N}$ such that $\mathcal{T}_0 \supseteq \mathcal{T}$, it is the case that, for $\mathbb{P}$-almost all $\omega \in \Omega$, the sequence $\{\Sigma_N(\omega)\}_{N=0}^\infty$ given by

$$\Sigma_N(\omega) = \sum_{n=0, n \in \mathcal{T}_0}^{N} \zeta_n(\omega) X_n \tag{69}$$

fails to τ-converge as $N \to \infty$. Therefore, the random formal series $\Sigma : \Omega \to \mathscr{X}^{\mathbb{N}}$ defined by $\Sigma(\omega) = \sum_{n=0}^\infty 1_{n \in \mathcal{T}_0} \zeta_n(\omega) X_n$ satisfies $\Sigma(\omega) \notin \mathtt{P}_{\mathrm{II}}$ for $\mathbb{P}$-almost all $\omega \in \Omega$. ■

Proof. By Proposition 2.4 and the inclusion $\mathtt{P}_{\mathrm{II}'} \supset \mathtt{P}_{\mathrm{II}}$ (where $\mathtt{P}_{\mathrm{II}'}$ is as in §2), it suffices to prove that it is not the case that $\Sigma(\omega) = \sum_{n=0}^\infty 1_{n \in \mathcal{T}_0} \zeta_n(\omega) X_n$ is $\mathbb{P}$-almost surely $\mathcal{S}$-weakly summable, where $\mathcal{S} \subseteq \mathscr{X}_\tau^*$ is a countable collection of norming functionals. Suppose, to the contrary, that Σ were almost surely $\mathcal{S}$-weakly summable. By the Itô–Nisio theorem, this would imply that $\{\Sigma_N(\omega)\}_{N=0}^\infty$ converges strongly for $\mathbb{P}$-almost all $\omega \in \Omega$. But, the conjunction of eq. (67) and $\inf_{n \in \mathcal{T}} \|X_n\| > 0$ implies instead that $\{\Sigma_N(\omega)\}_{N=0}^\infty$ almost surely *fails* to converge strongly. □

Proposition 4.2. *Let $f : \mathbb{N} \to \mathbb{N}$. If it is the case that*

$$\tau - \lim_{N \to \infty} \sum_{n=0}^{N} \epsilon_{f(n)}(\omega) x_n \tag{70}$$

exists for $\mathbb{P}$-almost all $\omega \in \Omega$, then, for any subset $\mathcal{T} \subseteq \mathbb{N}$,

$$\tau - \lim_{N \to \infty} \sum_{n=0, f(n) \in \mathcal{T}}^{N} \epsilon_{f(n)}(\omega) x_n \tag{71}$$

exists for $\mathbb{P}$-almost all $\omega \in \Omega$. ■

Proof. Let

$$\epsilon'_n = \begin{cases} \epsilon_n & (n \notin \mathcal{T}) \\ -\epsilon_n & (n \in \mathcal{T}). \end{cases} \tag{72}$$

We can now consider the random formal series

$$\sum_{n=0}^\infty (\epsilon'_{f(n)} - \epsilon_{f(n)}) x_n = \sum_{n=0}^\infty \epsilon'_{f(n)} x_n - \sum_{n=0}^\infty \epsilon_{f(n)} x_n \tag{73}$$

$$= 2 \sum_{n=0, f(n) \in \mathcal{T}}^{\infty} \epsilon_{f(n)} x_n. \tag{74}$$

The two random formal series on the right-hand side of eq. (73) are equidistributed, so, under the hypothesis of the proposition, both are τ-summable for $\mathbb{P}$-almost all $\omega \in \Omega$. Thus, the formal series on the right-hand side of eq. (74) is $\mathbb{P}$-almost surely τ-summable. □

We deduce Theorem 1.2 (and thus Theorem 1.1) as a corollary of the previous two propositions. We prove the slightly strengthened claim that, for $\mathbb{P}_{\mathrm{Haar}}$-almost all $\{\epsilon_n\}_{n=0}^\infty \in \{-1,+1\}^{\mathbb{N}}$, the formal series in eq. (11) both fail to even be $\mathcal{S}$-weakly summable. By Proposition 2.5, we just need to show that it is *not* the case that, for $\mathbb{P}_{\mathrm{Haar}}$-almost all $\{\epsilon_n\}_{n=0}^\infty \in \{-1,+1\}^{\mathbb{N}}$, the formal series

$$\sum_{n=0,f(n)\in\mathcal{T}}^{\infty} \epsilon_{f(n)} x_n \in \mathscr{X}^{\mathbb{N}} \tag{75}$$

is $\mathcal{S}$-weakly summable. Suppose, to the contrary, that it is $\mathcal{S}$-weakly summable for $\mathbb{P}_{\mathrm{Haar}}$-almost all $\{\epsilon_n\}_{n=0}^\infty$. Owing in part to the assumption that $|f^{-1}(\{n\})| < \infty$ for all $n \in \mathbb{N}$ (along with eq. (10)), there exists a $\mathcal{T}_0 \subseteq \mathcal{T}$ such that

- $f : f^{-1}(\mathcal{T}_0) \to \mathbb{N}$ is monotone and
- $\inf_{n\in\mathcal{T}_0} \| \sum_{n_0 \in f^{-1}(\{n\})} x_{n_0} \| > 0$.

By the previous proposition, $\sum_{n=0,f(n)\in\mathcal{T}_0}^\infty \epsilon_{f(n)} x_n \in \mathscr{X}^{\mathbb{N}}$ is $\mathcal{S}$-weakly summable $\mathbb{P}$-almost surely. Since $f|_{f^{-1}(\mathcal{T}_0)}$ is monotone, we deduce that

$$\sum_{n=0,n\in\mathcal{T}_0}^{\infty} \epsilon_n \Big[\sum_{n_0\in f^{-1}(\{n\})} x_{n_0} \Big] \in \mathscr{X}^{\mathbb{N}} \tag{76}$$

is $\mathcal{S}$-weakly summable $\mathbb{P}$-almost surely. However, this contradicts Proposition 4.1.

Acknowledgements

This work was partially supported by a Hertz fellowship. The author would like to thank the reviewer for their comments.

Appendix A. Admissible topologies

Let $\mathscr{X}$ denote a Banach space over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, and let τ be an admissible topology on it.

Lemma A.1. *The τ-weak topology, a.k.a. the $\sigma(\mathscr{X},\mathscr{X}^*_\tau)$-topology, is admissible.* ■

Proof.

(1) The τ-weak topology is an LCTVS-topology on $\mathscr{X}$ [Rud73, §3.10, §3.11] identical to or weaker than the norm topology.

For each $\Lambda \in \mathscr{X}^*_\tau$ and closed interval $I \subseteq [-\infty,+\infty]$, let $C_{\Lambda,I}$ denote the τ-weakly closed subset (I) $C_{\Lambda,I} = \Lambda^{-1}(I)$ if $\mathbb{K}=\mathbb{R}$ or (II) $C_{\Lambda,I} = \Lambda^{-1}(\{z \in \mathbb{C} : \Re z \in I\})$ otherwise. By the Hahn-Banach theorem, $\mathscr{X}^*_\tau$ is not empty — picking any $\Lambda \in \mathscr{X}^*_\tau \subseteq \mathscr{X}^*$, there exists some closed interval I such that $C_{\Lambda,I} \supseteq \mathbb{B}$, so we can form the intersection

$$\tilde{\mathbb{B}} = \bigcap_{\substack{\Lambda\in\mathscr{X}^*_\tau, I\subseteq[-\infty,+\infty] \\ C_{\Lambda,I}\supseteq\mathbb{B}}} C_{\Lambda,I}. \tag{77}$$

This is a τ-weakly closed set containing $\mathbb{B}$. If $x \notin \mathbb{B}$, we can apply the Hahn-Banach separation theorem [NB11, Thm. 7.8.6] to the sets $\{x\}$ and $\mathbb{B}$ to get some $\Lambda \in \mathscr{X}^*_\tau$ such that $\Re\Lambda x > 1$ and $\Re\Lambda x_0 < 1$ for all $x_0 \in \mathbb{B}$. Then, since $\mathbb{B}$ is closed under multiplication by -1, $\Re\Lambda x_0 \in (-1,+1)$ for all $x_0 \in \mathbb{B}$, which means that $C_{\Lambda,[-1,+1]}$ appears on the right-hand side of eq. (77).

Since $x \notin C_{\Lambda,[-1,+1]}$, we get $x \notin \tilde{\mathbb{B}}$. We conclude that $\tilde{\mathbb{B}} = \mathbb{B}$ and, therefore, that the latter is τ-weakly closed.

(2) If $\mathscr{X}$ is not separable, then τ is at least as strong as the weak topology. Since the weak topology of the weak topology is just the weak topology [Rud73, §3.10, §3.11] – that is, $\sigma(\mathscr{X},\mathscr{X}^*_{\mathrm{w}}) = \sigma(\mathscr{X},\mathscr{X}^*)$, where $\mathscr{X}_{\mathrm{w}} = \sigma(\mathscr{X},\mathscr{X}^*)$ – the τ-weak topology is at least as strong as the weak topology.

Thus, the τ-weak topology is admissible. □

Lemma A.2. *If $\mathscr{X}$ is separable, there exists a countable norming subset $\mathcal{S} \subseteq \mathscr{X}_\tau^*$.* ■

Proof. Let $\{x_n\}_{n=0}^\infty$ denote a dense subset of $\mathscr{X}\backslash\{0\}$. By [NB11, Thm. 7.8.6], there exists for each $n \in \mathbb{N}$ and each $R \in (0, \|x_n\|)$ an element $\Lambda_{n,R} \in \mathscr{X}_\tau^*$ such that $\Re\Lambda_{n,R}x_n > 1$ and $\Re\Lambda_{n,R} < 1$ on the closed ball $R\mathbb{B}$ (which is τ-closed by admissibility). Since $R\mathbb{B}$ is closed under multiplication by phases,

$$\|\Lambda_{n,R}x\| < 1 \tag{78}$$

for all $x \in R\mathbb{B}$. Thus, $\|\Lambda_{n,R}\|_{\mathscr{X}^*} \leq 1/R$. It follows that $1 < \Re\Lambda_{n,R}x_n < |\Lambda_{n,R}x_n| \leq \|x_n\|/R$, so $\lim_{R\uparrow\|x_n\|} |\Lambda_{n,R}x_n| = 1$.

Now let $\mathcal{S}$ be the set of all functionals of the form $R\Lambda_{n,R}$ for R of the form $\|x_n\| - 1/m$ for $m \in \mathbb{N}^+$ sufficiently large such that $1/m < \|x_n\|$. Then, it is straightforward to check that $\mathcal{S}$ is a norming subset, and $\mathcal{S}$ is countable. □

Cf. [Car05, Lemma 6.7].

Lemma A.3. *If $\mathscr{X}$ is separable and $\mathcal{S} \subseteq \mathscr{X}_\tau^*$ is a norming subset, then the $\sigma(\mathscr{X}, \mathcal{S})$-topology is admissible.* ■

Proof. We can assume without loss of generality that, if $\mathbb{K} = \mathbb{C}$, $e^{i\theta}\Lambda \in \mathcal{S}$ whenever $\Lambda \in \mathcal{S}$ and $\theta \in \mathbb{R}$. By [Rud73, Thm. 3.10], the $\sigma(\mathscr{X}, \mathcal{S})$-topology is an LCTVS topology, and it is no stronger than the norm topology. Consider

$$\tilde{\mathbb{B}} = \bigcap_{\substack{\Lambda\in\mathcal{S}, I\subseteq[-\infty,+\infty] \\ C_{\Lambda,I}\supseteq\mathbb{B}}} C_{\Lambda,I}, \tag{79}$$

which is a $\sigma(\mathscr{X}, \mathcal{S})$-closed set containing $\mathbb{B}$. If $x \notin \mathbb{B}$, then there exists some $\Lambda \in \mathcal{S}$ such that $|\Re\Lambda x| \in (1, \|x_n\|]$. Since $\mathcal{S}$ is norming, $\|\Lambda\|_{\mathscr{X}^*} \leq 1$, so $C_{\Lambda,[-1,+1]}$ appears on the right-hand side of eq. (79). But,

$$x \notin C_{\Lambda,[-1,+1]}, \tag{80}$$

so $x \notin \tilde{\mathbb{B}}$.

We conclude that $\tilde{\mathbb{B}} = \mathbb{B}$, so $\mathbb{B}$ is $\sigma(\mathscr{X}, \mathcal{S})$-closed. □

References

[AS16] N. Alon and J. H. Spencer. *The Probabilistic Method*. Wiley Series in Discrete Mathematics and Optimization. Fourth ed. John Wiley & Sons, Inc., 2016 (cit. on p. 3).

[BP58] C. Bessaga and A. Pełczyński. "On bases and unconditional convergence of series in Banach spaces". *Studia Math.* 17 (1958), 151–164. DOI: 10.4064/sm-17-2-151-164 (cit. on p. 2).

[Car05] N. Carothers. *A Short Course on Banach Space Theory*. London Mathematical Society Student Texts 64. Cambridge University Press, 2005 (cit. on pp. 7, 15).

[Die77] P. Dierolf. "Theorems of the Orlicz–Pettis-type for locally convex spaces". *Manuscripta Math.* 20.1 (1977), 73–94. DOI: 10.1007/BF01181241 (cit. on p. 2).

[Die84] J. Diestel. "The Orlicz-Pettis Theorem". *Sequences and series in Banach spaces*. Graduate Texts in Mathematics 92. Springer-Verlag, 1984, 24–31. DOI: 10.1007/978-1-4612-5200-9_4 (cit. on pp. 2, 3).

[Dur19] R. Durrett. *Probability—Theory and Examples*. Cambridge Series in Statistical and Probabilistic Mathematics 49. Fifth ed. Cambridge University Press, 2019. DOI: 10.1017/9781108591034 (cit. on pp. 8, 10).

[Hof74] J. Hoffmann-Jørgensen. "Sums of independent Banach space valued random variables". *Studia Math.* 52 (1974), 159–186. DOI: 10.4064/sm-52-2-159-186 (cit. on pp. 1, 5).

[Hyt+16] T. Hytönen, J. Van Neerven, M. Veraar, and L. Weis. *Analysis in Banach Spaces. Volume II: Probabilistic Methods and Operator Theory*. A Series of Modern Surveys in Mathematics 67. Springer, 2016. DOI: 10.1007/978-3-319-69808-3 (cit. on pp. 5, 8, 10, 11).

[IN68] K. Itô and M. Nisio. "On the convergence of sums of independent Banach space valued random variables". *Osaka Math. J.* 5 (1968), 35–48. URL: http://projecteuclid.org/euclid.ojm/1200692040 (cit. on p. 5).

[Kal02] O. Kallenberg. *Foundations of Modern Probability*. Probability and its Applications. Second ed. Springer-Verlag, 2002. DOI: 10.1007/978-1-4757-4015-8 (cit. on p. 6).

[Kwa74] S. Kwapień. "On Banach spaces containing c_0". *Studia Math.* 52 (1974), 187–188 (cit. on p. 5).

[Meg98] R. Megginson. *An Introduction to Banach Space Theory*. Graduate Texts in Mathematics 183. Springer Science & Business Media, 1998. DOI: 10.1007/978-1-4612-0603-3 (cit. on p. 2).

[NB11] L. Narici and E. Beckenstein. *Topological Vector Spaces*. Pure and Applied Mathematics 296. Second ed. CRC Press, 2011 (cit. on pp. 14, 15).

[Orl29] W. Orlicz. "Beiträge zur theorie der orthogonalentwicklungen II". *Studia Math.* 1 (1929), 241–255. URL: http://eudml.org/doc/216979 (cit. on p. 2).

[Rud73] W. Rudin. *Functional Analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., 1973 (cit. on pp. 1, 14, 15).

[Sus22] E. Sussman. "The microlocal irregularity of Gaussian noise". *Studia Math.* 266 (2022), 1–54. arXiv: 2012.07084 [math.SP] (cit. on p. 5).

Email address: ethanws@mit.edu

Department of Mathematics, Massachusetts Institute of Technology, Massachusetts 02139-4307, USA