HYDROGEN-LIKE SCHRÖDINGER OPERATORS AT LOW ENERGIES

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Abstract. Consider a Schrödinger operator on an asymptotically Euclidean manifold $X$ of dimension at least two, and suppose that the potential is of attractive Coulomb-like type. Using Vasy’s second 2nd-microlocal approach, “the Lagrangian approach,” we analyze – uniformly, all the way down to $E = 0$ – the output of the limiting resolvent $R(E \pm i\varepsilon) = \lim_{\varepsilon \to 0^+} R(E \pm i\varepsilon)$. The Coulomb potential causes the output of the low-energy resolvent to possess oscillatory asymptotics which differ substantially from the sorts of asymptotics observed in the short-range case by Guillarmou, Hassell, Sikora, and (more recently) Hintz and Vasy. Specifically, the compound asymptotics at low energy and large spatial scales are more delicate, and the resolvent output is smooth all the way down to $E = 0$. In fact, we will construct a compactification of $(0, 1] \times X$ on which the resolvent output is given by a specified (and relatively complicated) function that oscillates as $r \to \infty$ times something smooth. As a corollary, we get complete and compatible asymptotic expansions for solutions to the scattering problem as functions of both position and energy, with a transitional regime.

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1. Introduction

In [Mel94], Melrose introduced the programme of understanding the limiting absorption principle on asymptotically conic (a.k.a. asymptotically Euclidean) manifolds (denoted $X$ below) from the microlocal point of view, initially for the Laplacian and then for Schrödinger operators more generally [Mel94, §16]. The case of fixed energy $E > 0$ was dealt with first by Melrose [Mel94], then by Hassell & Vasy [HV99] and – using a more modern approach – Vasy [Vas21a] again, while uniform estimates in the high energy (a.k.a. “semiclassical,” $E \to \infty$) limit have been established by Vasy & Zworski [VZ00] and Vasy [Vas21a, §5]. More recently, the low energy $E \to 0^+$ behavior has been understood to a highly satisfactory degree in the works of Guillarmou, Hassell, and Sikora.
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[110x687][GHS13] and Vasy [Vas21b][Vas21c] — see also Hintz [Hin21, §2]. (Complementarily, Guillarmou and Hassell [GH08][GH09] consider the $E \to 0^-$ limit of the resolvent kernel.) While the previous results applied to all Schrödinger operators (with the semiclassical results holding under standard dynamical assumptions), the low energy results proven so far in this level of generality apply only to Schrödinger operators without Coulomb-like (a.k.a. long range) terms. Indeed, a Coulomb-like potential has a serious effect on the asymptotics of formal Schrödinger eigenfunctions in the low energy limit — this is true in the context of Euclidean potential scattering, and the general case inherits that complexity.

In this paper, we study the case of an attractive Coulomb-like potential using the framework of [Vas21a][Vas21c], Vasy’s 2nd second microlocal approach (the “Lagrangian approach”). This is in contrast to his 1st second microlocal approach to the low energy limit, pursued in [Vas21b], which utilized variable order Sobolev spaces (as previously used in e.g. [Vas18, Proposition 5.28] for the $E > 0$ case). The use of the term “Lagrangian” belies the fact that the framework of Lagrangian (more properly Legendrian) distributions does not appear explicitly in this approach — in fact this is precisely the point: instead of “module regularity [HMV04, §6][Gel+19] for a Lagrangian submanifold” — which implies extra regularity outside of that submanifold via some elliptic estimates — we need only consider $b$-Sobolev regularity. We use a conjugation to move what would otherwise have been the energy-dependent Legendrian submanifold for which module regularity (in this case essentially the Sommerfeld radiation condition) is established to the zero section of the scattering cotangent bundle (which can be blown up to the fibers of the $b$-cotangent bundle over the boundary). The upshot is that the Sommerfeld radiation condition, in one of its forms, is replaced by a condition regarding $b$-Sobolev regularity. This is convenient for getting estimates which are uniform in $E$ because, unlike the relevant notion of module regularity prior to conjugation (see e.g. [HMV04][Gel+19]), in which the relevant test module depends on $E$, the relevant notion of $b$-regularity (as stated in [Vas21a, Theorem 1.1]) does not depend so much on $E$. (The sharpest form of [Vas21a, Theorem 1.1] is phrased using $sc,b$-Sobolev regularity, which is a form of module regularity — the key point here is that the relevant test module is $E$-independent.)

We will apply similar considerations to the study of the $E \to 0^+$ limit of the limiting resolvents $P((E \pm i0)^{1/2})^{-1} = (P(0) - E \mp i0)^{-1}$ of the Schrödinger operator

$$P(0) = \Delta_g - Zx + V,$$  

where $V$ is short-range and $Z > 0$, so the total potential $W = -Zx + V$ is of attractive Coulomb-like type. Here $x \in C^\infty(X;\mathbb{R}^\geq 0)$ is a boundary defining function (bdf), e.g. $1/\langle r \rangle$ when $X$ is asymptotically Euclidean, $r$ denoting the Euclidean radial coordinate. (Note: we follow the convention of parametrizing the spectral family $\{P(0) - E\}_{E \geq 0}$ of $P(0)$ in terms of $\sigma = E^{1/2}$, so we write $P(\sigma) = P(0) - \sigma^2$.) Singular versions of the operator eq. (1) first appeared in Schrödinger’s model of atomic hydrogen – or more generally hydrogenic ions – hence the operators we consider could also be called “hydrogen-like.” In eq. (1), $\Delta_g \geq 0$ is the positive semidefinite Laplacian associated with an asymptotically conic metric. The conjugated perspective complicates the family of operators under consideration (see §3) – more so than in the case $Z = 0$, when the total potential is short-range – but it greatly facilitates the derivation of low energy asymptotics. See below for a heuristic discussion (and §6 for details). It should be noted that although we work with general asymptotically conic manifolds, our results are new even on exact Euclidean space. The existing literature on low energy asymptotics in the presence of a Coulomb-like term is quite sparse — the only previous treatments the author is aware of are [Yaf82][Nak94][FS04][Ski13]. In comparison to
these earlier works, we require more of our potential, but the payoff is a complete understanding of asymptotics at spatial infinity.

We comment briefly on our focus on the case \( E > 0 \) and \( Z > 0 \). Considering the spectral family of a not-necessarily-attractive Coulomb-like Schrödinger operator at energy \( E \in \mathbb{R} \setminus \{0\} \) and with attractivity strength \( Z \in \mathbb{R} \setminus \{0\} \) (the atomic number for the case of an electron orbiting an atomic nucleus, using appropriately natural units): out of the four cases (I) \( E > 0, Z > 0 \) (attractive, scattering near zero energy), (II) \( E < 0, Z > 0 \) (attractive, ellipticity near zero energy), (III) \( E > 0, Z < 0 \) (repulsive, scattering near zero energy), (IV) \( E < 0, Z < 0 \) (repulsive, ellipticity near zero energy), the first and fourth are the most tractable, as evidenced by the state of the literature on similar problems in exact Euclidean space — see [Yaf82][Nak94] for work on cases (I) and (III) and [FS04][Sk13] for case (I). In the more difficult case (II), for example, one can encounter an infinite sequence of bound states, as in the hydrogen atom — see [Ivr19, Theorem 11.6.7]. Case (III), which has been studied partially in [Yaf82], is expected to be intermediate between (II) and (I) in terms of difficulty. While our focus is on (I), the pseudodifferential technology developed yields an easy treatment of case (IV).

The jumping off point for our analysis is the proof of a symbolic estimate, Theorem 5.12, of \( u \in S'(X) \) in terms of \( f = P(E^{1/2})u \) that is uniform down to \( E = 0 \). The estimate is structurally similar to the combined radial point and propagation estimates proven for \( E > 0 \) in [Me94, §8]. We will formulate the estimate in a “second microlocal” framework akin to that in [Vas21a], as this dovetails with the conjugated perspective, but a similar estimate can be articulated using function spaces analogous to the somewhat more standard variable order sc-Sobolev spaces [Vas13][Vas18][Gel+19]. To illustrate the idea, we rewrite the operator \( P(\sigma) \) in terms of the coordinate \( \hat{x} = x/\sigma^2 \), which is appropriate for homogenizing the spectral parameter and Coulomb potential. To the relevant order, \( \sigma^{-2}P(\sigma) \) is given in terms of \( \hat{x}, \sigma > 0 \), the latter of which we suggestively rename ‘\( h \),’ by

\[
\hat{P}(h) = -\hbar^2(\hat{x}^2 \partial_{\hat{x}}^2) + \hbar \hat{x}^2 \triangle_{\partial X} - 1 - Z \hat{x},
\]

which we consider as a 1-parameter family of operators on the exact cone \([0, \infty)_\hat{x} \times \partial X\). Note that \( \sigma = h \) appears on the right-hand side of eq. (2) as an effective semiclassical parameter (somewhat surprisingly, since semiclassical problems typically arise in the study of the high energy regime rather than the low energy regime of interest here), and so \( \hat{P} \) can be studied as a semiclassical operator on an exact cone. This differs from the conic problem arising in [Wan06][GH08][GHS13][Vas21c], which has no semiclassical parameter. The qualitative features of the semiclassical family defined by eq. (2) are as follows:

- At the “large” end of the cone, \( \{ \hat{x} = 0 \} \), \( \hat{P}(h) \) is of real principal type for each \( h > 0 \). This holds regardless of the sign of \( Z \) and reflects the dynamical structure of \( P(\sigma) \) for \( \sigma > 0 \).
- At \( \{ \hat{x} = \infty \} \), we use \( \hat{x}^{-1/2} \) as a bdf, and then \( \hat{P}(h) \) is of real principal type there too for each \( h > 0 \). This behavior depends on the sign of \( Z \).
- At \( h = 0 \), one has real principal type propagation from \( \hat{x} = 0 \) to \( \hat{x} = \infty \) (at finite frequencies), though this is of secondary importance below.

The situation changes completely if \( Z < 0 \) or if \( E < 0 \). Then, eq. (2) is to be replaced by

\[
\hat{P}(h) = -\hbar^2(\hat{x}^2 \partial_{\hat{x}}^2) + \hbar \hat{x}^2 \triangle_{\partial X} \pm 1 \pm Z \hat{x},
\]

where the first sign is that of \( -E \) and the second sign is that of \( -Z \). Each possible pair of signs corresponding to one of the four cases (I), (II), (III), (IV) above. If \( E < 0 \), then eq. (3) is elliptic at the large end of the cone, and if \( Z < 0 \) then eq. (3) is elliptic at the small end of the cone.
Figure 1. The mwc $X^{\text{sp}}_{\text{res}} = \{0, \infty\}_E \times \{0\} \times \partial X; \frac{1}{2}\}$, with bdfs $\varrho_{\text{bf}00} = Zx/(\sigma^2 + Zx)$, $\varrho_{\text{tf}00} = (\sigma^2 + Zx)^{1/2}$, and $\varrho_{\text{zf}00} = \sigma^2/(\sigma^2 + Zx)$ in terms of $E^{1/2} = \sigma$.

This is $X^{\text{sp},0}_{\text{res}} = \{0, \infty\}_E \times \{0\} \times \partial X$ with the smooth structure at the front face of the blow-up modified. The interior of the mwc is to the upper-left of the drawn boundary. (Degrees of freedom associated with $\partial X$ omitted from the diagram.)

Even for the case when $X$ is one-dimensional, one subtlety of the cases (II) $E < 0$, $Z > 0$ and (III) $E > 0$, $Z < 0$ is understanding precisely the transition from ellipticity to nonellipticity that occurs at $h = 0$ and $\hat{x} = 1/|Z|$. This subtlety arises, for example, in physicists’ treatment of the WKB approximation, where Airy functions are used to patch quasimodes in the classically allowed and classically forbidden regions — see [SN17] for a standard treatment at a physicist’s level of rigor. In case (II), the full partial differential operator eq. (3) has an additional subtlety: its Hamiltonian flow has closed loops corresponding to classical Keplerian orbits. See [Ivr19, §11.6] for a discussion of the effects of this on eigenvalue counting.

Zooming back from the boundary, we will analyze $P(\sigma)$ on a mwc ("mwc" standing for "manifold-with-corners," in the sense of Melrose’s school [Mel92]) which we will denote $X^{\text{sp}}_{\text{res}} = \{0, \infty\}_E \times \{0\} \times \partial X; \frac{1}{2}\}$, depicted in Figure 1. This is the result of modifying the smooth structure of $X^{\text{sp},0}_{\text{res}} = \{0, \infty\}_E \times \{0\} \times \partial X$ at the front face of the blow-up so that $(E + x)^{1/2}$ becomes a bdf. One can analyze $P(\sigma)$ by quantizing the Lie algebra $\mathcal{V}_{\text{sc,leC}}(X)$ of smooth vector fields on $X^{\text{sp}}_{\text{res}}$ that

(I) are tangent to the level sets of $E$,

(II) lie in $\varrho_{\text{bf}00} \varrho_{\text{tf}00} C^\infty(X^{\text{sp}}_{\text{res}}) \otimes \mathcal{V}_b(X) \subset \mathcal{V}_E(X^{\text{sp}}_{\text{res}})$,

where $\varrho_{\text{bf}00}, \varrho_{\text{tf}00}$ are as in Figure 1 and $\mathcal{V}_E(X^{\text{sp}}_{\text{res}})$ is the Lie algebra of smooth vector fields on $X^{\text{sp}}_{\text{res}}$. The quotient algebra $\mathcal{V}_{\text{sc,leC}}(X)/\varrho_{\text{bf}00} \varrho_{\text{tf}00} \mathcal{V}_{\text{sc,leC}}(X)$ is commutative, so the corresponding $\Psi$DO calculus (which is closely related to the leC-calculus discussed below) is under symbolic control. This allows us to prove a half-Fredholm estimate involving variable order "sc,leC"-Sobolev spaces that is uniform down to zero energy. The second-microlocal estimate Theorem 5.12 is a sharper version of this.

Although it is not our most general result (as the main propositions of §5, §6 require less classicality), we will prove the following “main” theorem. We state it using some standard or semistandard terminology, which, if not standard, is recalled in §1.1 below. The special case of Euclidean potential scattering off of an attractive asymptotically Coulomb-like potential is partially stated in Corollary 1.3, Corollary 1.4, and Corollary 1.5 using only completely standard terminology.
**Theorem 1.1.** Suppose that \((X, \iota, g_0)\) is an exactly conic manifold of dimension \(\dim X = n \geq 2\), and let \(x \in C^\infty(X)\), \(x : X \rightarrow [0, \infty)\), denote a compatible boundary-defining-function \((bdf)\), so that, near \(\partial X\),

\[
g_0 = \frac{dx^2}{x^4} + g_{\partial X} \frac{d\sigma^2}{x^2}
\]

for some Riemannian metric \(g_{\partial X}\) on \(\partial X\). Let \(g\) denote a fully classical asymptotically conic metric on \(X\), which in this context means a Riemannian metric on \(X^\circ = X \setminus \partial X\) which near \(\partial X\) has the form

\[
g = g_0 + a_{00} \frac{dx^2}{x^3} + \frac{1}{x} \left( \frac{\Gamma_{1,0X} \circ dx}{x^2} + \frac{h_{1,0X}}{x} + x^2 C^\infty(X; \text{sc} \text{Sym}^2 T^*X) \right)
\]

for some \(a_{00} \in \mathbb{R}\), \(\Gamma_{1,0X} \in \Omega^1(\partial X)\), and \(h_{1,0X} \in C^\infty(\partial X; \text{Sym}^2 T^* \partial X)\). Given \(Z > 0\) and \(V \in x^2 C^\infty(X; \mathbb{R})\), consider the Schrödinger operator

\[
P(0) = \Delta_g - Zx + V : \mathcal{S}'(X) \rightarrow \mathcal{S}'(X),
\]

where \(\Delta_g\) is the (positive semidefinite) Laplacian. For each \(E > 0\), let

\[
R(E \pm i0) = R(E \pm i0; Z) : \mathcal{S}(X) \rightarrow \mathcal{S}'(X)
\]

denote the limiting resolvent from above or below the spectrum — cf. Melrose [Mel94, §14]. (That is, for any \(f \in \mathcal{S}(X)\), \(u_\pm = R(E \pm i0)f\) is the unique solution to \(P(0)u = Eu + f\) satisfying the weak Sommerfeld radiation condition [Mel94, §11].) Set

\[
\Phi(x; \sigma) = \frac{1}{x} \sqrt{\sigma^2 + Zx} - \frac{1}{\sigma} (Z - \sigma^2 a_{00}) \arcsinh \left( \frac{1}{x^{1/2} (Z - \sigma^2 a_{00})^{1/2}} \right)
\]

for all \(\sigma > 0\) such that \(\sigma^2 a_{00} < Z\).

Then, for any Schwartz function \(f \in \mathcal{S}(X)\), there exists some some \(u_{0,\pm} \in C^\infty(X_{\text{res}}^\infty \cap \{Z > Ea_{00}\})\) such that the family \(u_\pm = \{u_\pm(-; E^{1/2}) = R(E \pm i0) f\}_{E > 0} \subset \mathcal{S}'(X)\) can be written for \(E\) such that \(Z > Ea_{00}\) as

\[
u_\pm = e^{\pm i\Phi(x; E^{1/2})} x^{(n-1)/2} (E + Zx)^{-1/4} u_{0,\pm}.
\]

Moreover, \(R(E = 0; Z \pm i0) : \mathcal{S}(X) \rightarrow \mathcal{S}'(X)\) is well-defined (e.g. as a strong limit of \(R(E = 0; Z \pm i\epsilon)\) as \(\epsilon \to 0^+\)), and we can write \(u_{\pm}(--; 0) = R(E = 0; Z \pm i0) f\) as

\[
u_{\pm}(--; 0) = e^{\pm i\Phi(x; 0)} x^{(n-1)/2} (Zx)^{-1/4} u_{0,\pm}(--; 0)
\]

(\(\Phi(--; 0)\) being defined by removing the removable singularity of eq. (9) at \(\sigma = 0\)), where

\[
u_{0,\pm}(--; 0) \in C^\infty(X_{1/2})
\]

is the restriction of \(u_{0,\pm}\) to \(Z = \text{cl}\{\sigma = 0, x > 0\} \subset X_{\text{res}}^\infty\). Thus, defining \(u_{\pm}(--; E^{1/2})\) either as \(R(E \pm i0) f\) for \(E > 0\) or \(R(E = 0; Z \pm i0) f\) for \(E = 0\), the formula eq. (10) holds for all \(E \geq 0\) sufficiently small.

Furthermore, the map \(\mathcal{S}(X) \ni f \mapsto u_{0,\pm} \in C^\infty(X_{\text{res}}^\infty \cap \{Z > Ea_{00}\})\) is continuous with respect to the topologies of \(\mathcal{S}(X)\) and \(C^\infty(X_{\text{res}}^\infty \cap \{Z > Ea_{00}\})\).

For \(z > 0\), by \(\arcsinh(z)\) we mean \(\log(z + (1 + z^2)^{1/2})\).

**Remark 1.** Theorem 1.1 applies equally well for \(\sigma\)-dependent forcing \(f \in C^\infty([0, \infty)_x^\infty; \mathcal{S}(X))\) as can be proven using the argument in §6.
In a standard manner - cf. [Mel94, §15] - we can deduce from the conjunction of (1) an asymptotic summation argument, (2) Theorem 1.1 (strengthened slightly by the preceding remark), and (3) the other results in the body of the paper below the following:

Corollary 1.2. Consider the setup of Theorem 1.1, and fix $E_0 > 0$ such that $E_0 a_{00} < Z$. If $\alpha \in C^\infty(\partial X)$, then there exist $A, B \in C^\infty(X^\text{sp}_{\text{res}})$ such that $A|_{bf} = \alpha \circ \pi$ (where $\pi : tf \cup bf \to \partial X$ denotes the restriction to $tf \cup bf$ of the composition of the blowdown map $X^\text{sp}_{\text{res}} \to [0, \infty)_\sigma \times X$ and the projection $[0, \infty)_\sigma \times X \to X$) and

$$u = e^{-i\Phi(x;E^{1/2})}x^{(n-1)/2}(E + Zx)^{-1/4}A + e^{+i\Phi(x;E^{1/2})}(E + Zx)^{-1/4}x^{(n-1)/2}B$$

solves the Schrödinger-Helmholtz equation $\triangle_x u - Z xu + Vu = Eu$ for all $E \in [0, E_0]$. Moreover, $u$ is the unique solution in $\{E \leq E_0\}$ with this property, and $A, B$ are unique in $\{E \leq E_0\}$ modulo

$$\bigotimes_{k=0}^{\infty} \otimes_{\text{bf}} C^\infty(X^\text{sp}_{\text{res}}) = \bigcap_{k \in \mathbb{N}} \bigotimes_{\text{bf}} C^k(X^\text{sp}_{\text{res}}).$$

This shows that $\exp(\pm i\Phi(x;E^{1/2}))x^{(n-1)/2}(E + Zx)^{-1/4}$ serves as a notion of “incoming/outgoing spherical wave” in the presence of an attractive Coulomb-like potential that makes sense all the way down to $E = 0$. In Corollary 1.2, $\alpha$ serves as a notion of “incoming data.” The proof actually constructs the asymptotic expansion of $A$ at $tf \cup bf$. A natural question, which we do not investigate here, is whether or not $B|_{bf} = \beta \circ \text{bdn}$ for some $\beta \in C^\infty([0, \infty)_\sigma \times \partial X)$, where $\text{bdn} : X^\text{sp}_{\text{res}} \to [0, \infty)_\sigma \times X$ is the blowdown map. Physically, Corollary 1.2 describes the scattering of nonrelativistic electrons off of a hydrogenic nucleus, or alternatively nonrelativistic Bhaba scattering.

Compare the following with [Nak94, Cor. 1.5][FS04, Eq. 1.2]:

Corollary 1.3 (Asymptotics at $zf$). For each $l \in \mathbb{R}$, let $S^l(\mathbb{R}^n) = S^l_{1,0}(\mathbb{R}^n)$ denote the space of symbols of order $l$. Fix $n \geq 2$. Suppose that $W \in C^\infty(\mathbb{R}^n)$ is a classical symbol of order $-1$, so that there exist (unique) $W_0, W_1, W_2, W_3, \ldots \in C^\infty(S^{n-1})$ such that

$$W(x) - (1 - \chi(r)) \sum_{k=0}^{K} \frac{W_k(x/r)}{r^{k+1}} \in S^{-K-2}(\mathbb{R}^n),$$

for each $K \in \mathbb{N}$, where $r = ||x||$ and $\chi \in C^\infty_c(\mathbb{R})$ is identically equal to one in some neighborhood of the origin. Suppose further that $W$ is attractive and Coulomb-like, meaning that $W_0 = -Z$ for some constant $Z > 0$. Consider the Schrödinger-Helmholtz operator

$$P(\sigma) = \triangle - \sigma^2 + W$$

for $\sigma \geq 0$, where $\triangle = -\partial^2_{x_1} + \cdots + \partial^2_{x_n}$ is the positive semidefinite Laplacian. Given $f \in S(\mathbb{R}^n)$ (where $S(\mathbb{R}^n)$ denotes the set of Schwartz functions) let

$$u_{\pm}(x; \sigma) = R(\sigma^2 \pm i0)f(x)$$

denote the output of the limiting resolvent $R(\sigma^2 \pm i0) = \lim_{\epsilon \to 0^+} R(\sigma^2 \pm i\epsilon)$ applied to $f$ for $\sigma > 0$.

Then, there exist functions $w_{\pm,0}, w_{\pm,1}, w_{\pm,2}, \ldots \in C^\infty(\mathbb{R}^n)$ such that, for each $K \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$u_{\pm}(x; \sigma) = \sum_{k=0}^{K} w_{\pm,k}(x)\sigma^{2k} + O_{\epsilon,f,K}(\sigma^{2K+2})$$


as $\sigma \to 0^+$, where the $O_{x,f,K}(\sigma^{2K+2})$ term is uniformly bounded (i.e. by $C_{K,f,K}\sigma^{2K+2}$) in compact subsets $K \subseteq \mathbb{R}^n$ of $x \in \mathbb{R}^n$. In fact, for each $K \in \mathbb{N}$, there exists some $K \in \mathbb{R}$ such that $|O_{x,f,K}(\sigma^{2K+2})| \leq C_{K,f,K}(r)^{\kappa K} \sigma^{2K+2}$ for some $C_{f,K,o} > 0$.

Moreover, $w_{\pm,0}$ solves the PDE $P(0)w_{\pm,0} = f$. □

(Of course, the $O_{x,f,K}(\sigma^{2K+2})$ error also depends on $W$, though we do not write this dependence explicitly. The same applies to the other errors in Corollary 1.4, Corollary 1.5 below.) Corollary 1.3 applies in particular to any $W \in C^\infty(\mathbb{R}^n)$ which is equal, outside of some compact set, to

$$-Z/r + \sum_{j=2}^{J} r^{-j}W_{j}(x/r) + W_{\infty},$$

for $W_{j} \in C^\infty(S^{n-1}), J \in \mathbb{N}$, and $W_{\infty} \in S(\mathbb{R}^n)$. Thus, while we impose significant restrictions on the radial behavior of the potential (and the potential’s regularity) in order to get full asymptotic expansions, there are no symmetry requirements on $W_{1}, W_{2}, \cdots$. We do, however, require that $W_{0} = -Z$ is constant, so the Coulomb-like part of $W$ is required to be spherically symmetric. We remark that Proposition 6.3 (see also Remark 4) allows us to study more general symbolic $W$, but then instead of full asymptotic expansions we get only partial asymptotic expansions together with symbolic estimates for the remainders. We also remark that the classical symbols of order $-2$ on $\mathbb{R}^n$ are precisely those of the form $V = (r)^{-2}U$ for $U$ a smooth function on the radial compactification $\mathbb{R}^n = \mathbb{R}^{+}$ of $\mathbb{R}^n$, this being diffeomorphic to a closed ball, so Corollary 1.3 applies to many potentials that are not of the form eq. (19), e.g. $W = -(r)^{-1}$.

The $K$ in Corollary 1.3 are increasing in $K$. In a sense made precise by Theorem 1.1, the asymptotic expansion eq. (18) can be refined into an asymptotic expansion in powers of $E = \sigma^2/(\sigma^2 + 1/r)$ (with a complicated oscillatory prefactor) whose error terms are uniformly bounded without a loss of decay as $K$ increases. The non-uniform expansion eq. (18) already stands in stark contrast with the situation when $W$ is short range, where instead one only has e.g. in the case $n = 3$ (according to [Hin21, Theorem 3.1])

$$u_{\pm}(x;\sigma) = w_{\pm,0}(x) + w_{\pm,1}(x)\sigma + O_{x,f}(\sigma^{2} \log \sigma).$$

(20)

This is sharp — the singular $\sigma^{2} \log \sigma$ term is the source of the main term in Price’s law.

Given the setup of Corollary 1.3, the known large-$r$ asymptotic expansion of $u_{\pm}(x;\sigma) = R(\sigma^{2} \pm i0)f$ is [Mel94]:

- there exist functions $t_{\pm,0}, t_{\pm,1}, t_{\pm,2}, \cdots \in C^{\infty}(S^{n-1} \times \mathbb{R}^{+})$ such that, for each $\sigma > 0$ and nonzero $x \in \mathbb{R}^{n}$,

$$u_{\pm}(x;\sigma) = r^{-(n-1)/2}e^{\pm i \sigma r + \frac{i}{2} \epsilon} \left[ \sum_{k=0}^{K} t_{\pm,k}(x/r, \sigma)r^{-k} + O_{\sigma,f,K}(r^{-(K+1)}) \right]$$

(21)

for each $K \in \mathbb{N}$, where the $O_{\sigma,f,K}(r^{-(K+1)})$ term is uniformly bounded in compact subsets $K \subseteq \mathbb{R}^{+}$ worth of $\sigma$.

See e.g. Proposition A.3 for the Whittaker case, where the $t_{\pm}(\sigma)$ are written down explicitly. Note that the existence of the expansion eq. (21) is contained in Theorem 1.1 (from the asymptotic expansion at $bf^{2} \subseteq B_{E,\eta}$). Note that the $r^{\pm i 2/\kappa}$ term in eq. (21) is singular as $\sigma \to 0^+$, which renders eq. (21) unsuitable to study the low energy limit. The situation is ameliorated in the following way — according to Theorem 1.1, eq. (21) admits a repackaging that is uniform down to $\sigma = 0$:
Corollary 1.4 (Asymptotics at f). Consider the setup of Corollary 1.3. Let \( \varrho = (\sigma^2 r + 1)^{-1} \), so that \( r = (1 - \varrho) \varrho^{-1} \sigma^{-2} \). There exist functions \( \tau_{\pm,0}, \tau_{\pm,1}, \tau_{\pm,2}, \ldots \in C^\infty(\mathbb{S}^{n-1} \times (0,1]) \) such that, for any \( K \in \mathbb{N} \) and \( \sigma > 0 \),

\[
u_{\pm}\left(\frac{1}{\varrho} \varrho^{-1} \sigma^{-2}; \varrho, \sigma\right) = \varrho^{(n-1)/2} \sigma^{-2} \exp\left( \pm i \frac{1 - \varrho}{\varrho} \right) \left(1 + \frac{Z \varrho}{1 - \varrho}\right)^{1/2} \times \left(1 + \frac{1 - \varrho}{Z \varrho}\right)^{1/2} + O_{f,K}(\varrho^{K+1}) \tag{22} \]

holds for every \( \varrho \in \mathbb{S}^{n-1} \) as \( \varrho \to 0^+ \), where \( O_{f,K}(\varrho^{K+1}) \) is uniformly bounded by \( \varrho^{K+1}(1 - \varrho)^{(n-1)/2} L^\infty(\mathbb{R}_x \setminus \mathbb{B}^n \times [0, \varrho]) \) for every \( \varrho > 0 \).

Note that, for \( \sigma \) bounded away from zero, \( \varrho \in C^\infty([0,1]/r) \) and

\[
\exp\left( \frac{Z}{1 - \varrho} \left(1 + \frac{Z \varrho}{1 - \varrho}\right)^{1/2} \right) \left(1 + \frac{1 - \varrho}{Z \varrho}\right)^{1/2} + O_{f,K}(\varrho^{K+1}) \]

uniformly, so Corollary 1.4 does imply eq. (21).

Finally, we have the following result regarding joint asymptotics as \( r \to \infty, \sigma^2 \to 0 \) together, with the product \( \varsigma^2 = r \sigma^2 \) fixed:

Corollary 1.5 (Asymptotics at tf). Consider the setup of Corollary 1.3. There exist functions \( v_{\pm,0}, v_{\pm,1}, v_{\pm,2}, \ldots \in C^\infty(\mathbb{S}^{n-1} \times [0,\infty)) \) such that, for every \( K \in \mathbb{N} \) and \( \varsigma > 0 \),

\[
u_{\pm}(x; \varsigma) = r^{-2n/4} e^{\pm i r^{1/2} \sqrt{\varsigma^2 + Z}} \left(1 + \frac{Z^2}{\varsigma^2} \right) e^{i Z r^{1/2}/\varsigma} \left[ \sum_{k=0}^{K} v_{\pm,k}(x^r, \varsigma^2) \right] + O_{f,K}(r^{(K+1)/2}) \tag{24} \]

as \( r \to \infty \), where \( O_{f,K}(r^{-(K+1)/2}) \) is uniformly bounded by \( r^{-(K+1)/2} \) in compact subsets worth of \( \varsigma \in [0,\infty) \).

Moreover, \( \rho^{k/2} v_{\pm,k}(\theta,1/\rho) \) is smooth in \( \rho \in [0,\infty) \), for each \( \theta \in \mathbb{S}^{n-1} \). Thus, for \( \varsigma \geq 1 \), \( v_{\pm,k}(\theta, \varsigma^2) = O_{f,K}(\varsigma^2) \).

We emphasize that these compound asymptotics are in a different regime than the one relevant to the short-range case (this being the regime of \( r \to \infty \) for \( \sigma r \) fixed, not \( r \sigma^2 \) fixed) — see Remark 5.

Statements analogous to Corollary 1.3, Corollary 1.4, and Corollary 1.5 apply to the A,B in Corollary 1.2. A few remarks regarding Theorem 1.1:

Remark 2. The apparent singularity of \( u_{\pm} \) in Theorem 1.1 when \( Z = \sigma^2 a_{00} \) is fictitious — we can write

\[
u_{\pm} = e^{\pm i r^{1/2} \sigma^{n-1/2} Z} v_{0,\pm} \tag{25} \]

for \( v_{0,\pm} : (0,\infty) \sigma \times X \to \mathbb{C} \) smooth. In terms of \( v_{0,\pm}, u_{0,\pm} = e^{\pm i \Phi(E + Z x)^{1/4} e^{\pm i r^{1/2} \sigma^{n-1/2} Z} Z} v_{0,\pm} \). Thus, \( v_{0,\pm} \) is singular when \( Z = \sigma^2 a_{00} \). The left-hand side of eq. (10) is smooth for all \( \sigma > 0 \), but we have written it as a product of two functions which both have singularities when \( Z = \sigma^2 a_{00} \). The particular form of \( \Phi \) in Theorem 1.1 is needed to get uniform estimates down to \( \sigma = 0 \), but it introduces a fictitious singularity at some positive \( \sigma \) when \( a_{00} > 0 \).
Remark 3. Some explicit bounds for $u_{0,\pm}$ in the b-Sobolev spaces

$$H^m_{b,\pm}(X) = \{u \in S'(X) : Lu \in L^2(X, g) \text{ for all } L \in \Psi^m_{b,\pm}(X)\}$$

will be found in §6. We are indexing the b-Sobolev spaces such that $H^0_{b,0}(X)$ is equal to $L^2(X, g)$ rather than $L^2(X, g_b)$ for some b-metric $g_b$. This is the (somewhat nonstandard) convention followed in [Vas21a][Vas21c], and it is convenient here for the same reasons. See [Mel93][Vas18] for a pedagogical introduction to the b-calculus.

We write $X_{1/2}$ to denote the mwc canonically diffeomorphic to $X$ over the interior, with the smooth structure at the boundary modified so that $x_{1/2} = 2^{-1/2}x_{1/2}$ becomes a bdf (the factor of $2^{-1/2}$ being a convenient choice). (Note that, in terms of the mwb $X$, the mwb $X_{1/2}$ is canonically defined without needing to fix a choice of bdf for $X$, unlike the case for what Wunsch calls $X_q$ in [Wun99, §4]. However, since we are fixing a boundary-collar once and for all, this is not a crucial point.) The b-Sobolev spaces are convenient to work with in part because (except for indexing) they do not depend on whether we use $x_{1/2}$ or $x$ as a bdf — that is, at the level of sets

$$H^m_{b,\pm}(X) = H^m_{b,2}(X_{1/2}),$$

with an equivalence at the level of Banach spaces. A refined estimate of $u_{0,\pm}$ in terms of the “leC-Sobolev spaces”

$$H^{m,s,\pm}_{b,\pm,\ell}(X) = \{H^{m,s,\pm}_{b,\pm,\ell}(X)(\sigma)\}_{\sigma \geq 0}$$

is given in Proposition 6.9. The leC-Sobolev spaces play the same role here as the sc,b,res-Sobolev spaces in [Vas21c, Theorem 1.1]. (Note: we will omit the comma between “sc,b,".) Although it may be natural to develop doubly- second-microlocalized Sobolev spaces $H^{m,s,\pm}_{sc,b}(X)$, refining both $H^{m,\pm}_{scb}(X)$ and $H^{m,\pm}_{scb,2}(X_{1/2})$, we will not do so here, since the leC-Sobolev spaces suffice for the proof of Theorem 1.1.

Remark 4. We also have analogues of Theorem 1.1 dropping the classicality assumptions regarding the metric and the potential. If $g$ is a symbolically asymptotically conic metric in the sense below (precisely the sort of metric considered in [Vas21a]) and $V \in S^{-3/2-\delta}(X) = x^{3/2+\delta}S^0(X)$ for $\delta > 0$, the conclusion of the theorem holds except that “$u_{0,\pm} \in C^\infty(X_{res}^{sp})$” is replaced by the weaker

$$u_{0,\pm} \in C^0(X_{res}^{sp}),$$

$$\chi u_{0,\pm} \in C^\infty(X_{res}^{sp}) \text{ for all } \chi \in C_c^\infty(X^o).$$

Moreover, if $g$ is classical to $\alpha_1$st order, $\alpha_1 > 1$, and $V$ is classical to $\alpha_2 > 3/2$ order (so, for instance, $V \in x^{2}C^\infty(X) + S^{-3/2-\delta}x(\chi)$ is classical to $3/2 + \delta_4$th order, and any sc-metric in the sense of [Mel94] is classical to all orders, while the metrics considered by Vasy in [Vas21a] are classical to $> 1$ order), then, setting $\delta_1 = \min\{\alpha_1 - 1, \alpha_2 - 1\}$ and $\delta_0 = \min\{\alpha_1 - 1, \alpha_2 - 3/2\}$, we have

$$u_{0,\pm} \in C[\min\{\delta_1, 2\delta_0\}^{-1}](X_{res}^{sp}).$$

In fact (Proposition 6.3), we have

$$u_{0,\pm} \in C^\infty(X_{res}^{sp}) + A_{loc}^{(0,0),\delta_1,2\delta_0,\delta_0}(X_{res}^{sp}) \subset A_{loc}^{(0,0),\delta_1,0,0}(X_{res}^{sp}) \subset A_{loc}^{(0,0),\delta_1,0,0}(X_{res}^{sp}),$$

where $A_{loc}^{(0,0),\delta_1,0,0}(X_{res}^{sp})$ is the Fréchet space of conormal distributions on $X_{res}^{sp}$ that have partial polyhomogenous expansions at each of $bf, tf, zf$, with merely conormal remainders at order
Remark 6. When

• \( g = g_0 \) is an exactly conic metric and

• \( f, V \in C^\infty_c(X^0) \),

the radial dependence of \( u_\pm \) near infinity can be solved for (up to a multiplicative constant) exactly via separation of variables. When separating variables, we need only solve the “radial ODE,” which ends up being a Whittaker ODE for \( \sigma > 0 \) and a Bessel ODE for \( \sigma = 0 \). This motivating example is discussed in Appendix §A, where the consequences of Theorem 1.1 for solutions to Whittaker’s ODE are discussed. The necessity of the factor of \( (\sigma^2 + Zx)^{-1/4} \) in eq. (10) can be read off of the asymptotics of the Bessel functions (or can alternatively be deduced from the structure of the radial ODE, with slightly more work). See eq. (685). Even in this exactly conic case, which was studied in [Yaf82], Theorem 1.1 (or at least the part specifying the existence of a full asymptotic expansion) seems to be novel.

Remark 7. At first, it may seem somewhat paradoxical that at zero energy an attractive (rather than repulsive) Coulomb-like Schrödinger operator has scattering behavior. The paradox is resolved if we realize that, if a classical particle traveling in an attractive Coulomb force field has zero energy, then it must have more kinetic energy than if the force field were absent (the point being that “zero energy” refers to energy as measured with the potential present, not without).
It may also seem somewhat surprising that the presence of the attractive Coulomb potential allows us to avoid the b-analysis of the Laplacian in [Vas21c], but such considerations are expected to be necessary in order to understand the $E \to 0^-$ limit. Roughly speaking, the presence of an attractive Coulomb potential has moved the locus of b-analysis infinitesimally negativelywards – more specifically negativelywards on the front face of $[\mathbb{R}_E \times X; \{0\} \times \partial X]$, to its intersection with the lift of $\{E = -ZX\}$ – where it is irrelevant to the $E \to 0^+$ limit studied here.

The proof of Theorem 1.1, which is a special case of Proposition 6.3, is spread throughout §6, with a key estimate proven in §5 (and other lemmas appearing in §2 and §3). The conormality component of the theorem and smoothness at $z f^\circ$ are deduced in Proposition 6.15. From there, classicality is deduced via an inductive argument using an explicit parametrix for the “b,leC-normal operator” of the conjugated spectral family — see Proposition 6.16. Under only partial classicality assumptions, the inductive step of the argument can only be carried out finitely many times, and the upshot is Proposition 6.3. The analysis in §6 is essentially the analysis of a family of ODEs and applies equally well in the $n = 1$ case. The analysis in §5 is a uniform version of the sc-analysis of the Laplacian carried out in [Mel94][Vas21a], where we rely on the presence of the attractive Coulomb potential to prevent a full degeneration (from the perspective of the b-calculus) of the spectral family to the Laplacian at zero energy.

We note four deficiencies of our treatment, which can form the basis for further work:

1. While we discuss the output and mapping properties of the low energy resolvent, we do not discuss its Schwartz kernel (as Guillarmou, Hassell, and Sikora [GHS13] do in the case of a short-range potential).

2. We only treat the case when the Coulomb-like potential has the form $ZX$ for constant $Z > 0$, while more general $Z \in C^\infty(\partial X; \mathbb{R}^+)$ (and $a_{00} \in C^\infty(\partial X; \mathbb{R})$) may be of interest [Mel94, §14] (if $Z \in C^\infty(\partial X; \mathbb{R}^+)$, a change of coordinates from $x$ to $x_0 = Zx$ does not necessarily preserve the form of the metric we require, so we cannot just reduce the case of nonconstant $Z$ to the constant case via a change of bdf).

3. It should be possible to study the resolvent for $\sigma$ in a cone $\{\sigma \in \mathbb{C} : \arg(\sigma) \in [0, \theta]\}$ for $\theta \in (0, \pi/2)$, in accordance with the work of Skibsted and Fournais [FS04][Ski13] in the Euclidean case – see [Ski13, Theorem 1.2] in particular – thus giving a version of the limiting absorption principle that is uniform down to zero energy (as opposed to the ad hoc use of $Z \pm i\theta$ in Theorem 1.1 to describe the zero energy limit), and

4. It should be possible to extend Theorem 1.1 to the case where the Coulomb potential and manifold possess conic singularities, so as to handle an exact Coulomb potential on $[\mathbb{R}^d; \{0\}]$ as a special case. Indeed, blowing up the origin of Euclidean space and using $r$ as a bdf, the Coulomb potential $Z/r$ and spectral term $\sigma^2$ are both lower order than the Euclidean Laplacian with respect to the b-calculus at $r = 0$ in terms of both decay and regularity.) The exact 1D model problem discussed in §A is an example of this.

Along with some analysis of the $E < 0$ case, the present work can serve as input to the study of the Klein-Gordon equation on (not necessarily axially symmetric) asymptotically Schwarzschild spacetimes (away from the event horizon) in the spirit of Hintz’s recent treatment of Price’s law [Hin21] on asymptotically subextremal Kerr spacetimes. This problem was the original motivation for the present work. Some heuristic investigations in this direction have been undertaken by physicists [Det80][HP98][KT01][KT02][BK04][KZM07][Bar+12][Bar+14], usually in the case of a spherically or axially symmetric black hole spacetime (e.g. Schwarzschild, Reissner-Nordström, Kerr, etc.). Indeed, [KT01] treat their problem (pointwise temporal decay rates) using an uncontrolled
approximation of the radial part of their PDE as a Whittaker ODE, this being equivalent to the radial part of the time-independent Schrödinger equation in an attractive Coulomb potential. This problem can be solved exactly in terms of special functions (see §A), but we do not need to do so in order to understand the asymptotics. Moreover, while the asymptotics for fixed $E > 0$ can be read off of the exact solution (using the large argument expansions of Whittaker functions), the exact solution does not even help much in understanding the $E \to 0$ limit. Instead of relying on the Whittaker approximation, it is possible to exactly solve the Klein-Gordon equation on the Schwarzschild exterior (at the level of individual spherical harmonics) using confluent Heun functions [Bar+14, §IV], but despite this it is still open to rigorously establish temporal decay at a polynomial rate (as this involves knowing the asymptotics of the confluent Heun functions in some regime where the argument and a parameter are both taken to infinity). Moreover, it is not known how to control the sum of spherical harmonics to handle the case where the initial data is not spherically symmetric.

So, even in the few cases where special functions yield an exact solution to the PDE (with radially symmetric initial data) via separation of variables, the problem of understanding temporal decay rates remains. And, of course, for more complicated — in particular, more realistic — spacetimes no exact solution is possible. A more robust analysis is therefore necessary. For other mathematical work on Price’s law and similar problems, see [DSS11][DSS12][MTT12][Tat13][Mor19][MW21]. We will also consider (symbolically) asymptotically conic metrics, which — according to our use of the term “asymptotically” — are Riemannian metrics endowed with the induced metric. Here $\kappa = 1/|r|$, $r$ being the Euclidean radial coordinate. See [Mel94, §I].) We will also consider (symbolically) asymptotically conic metrics, which — according to our use of the term “asymptotically” — are Riemannian metrics $g$ on $X^\circ$ of the form

$$g - g_0 \in \mathcal{C}^\infty(X; \text{ScSym}^2 T^* X) + S^{-1-\delta}(X; \text{ScSym}^2 T^* X)$$

for some $\delta > 0$. Hence, $g$ is conic up to a classical subleading term — decaying faster than $g_0$ by one order of $x$ — and a merely symbolic error decaying slightly faster than that. More generally, for

\begin{equation}
\tau : \hat{X} = [0, \bar{x})_x \times \partial X_0 \to \partial X_0
\end{equation}

(The “boundary-collar”) which is a diffeomorphism between the cylinder $[0, \bar{x})_x \times \partial X_0$ for some $\bar{x} > 0$ and a neighborhood of $\partial X_0$. Note that $\tau$ satisfies $\text{id}|_{\partial X_0} = \tau(0, -)$. We notationally conflate $X$ with $X_0$. We also conflate the image of $\tau$ with $[0, \bar{x})_x \times \partial X_0$. (This is an alternative, if a somewhat crude one, to Melrose’s use in [Mel94] of the inward pointing normal bundle of $\partial X_0$.) Given an exactly conic manifold $X$, a Riemannian metric $g_0$ on $X^\circ$ is called an exactly conic metric (this notion being defined relative to $\tau$) if

\begin{equation}
g_0 \in \mathcal{C}^\infty(X; \text{ScSym}^2 T^* X), \quad g_0 = x^{-4} dx^2 + x^{-2} g_{\partial X} \text{ near } \partial X.
\end{equation}

Below, we write $x : X \to [0, \infty)$ to denote a globally defined boundary-defining function, compatible with the boundary-collar $\tau$ in the sense that $x(\tau(x_0, y)) = x_0$ for all $x_0 \in [0, \bar{x})$. (Euclidean space fits into this framework, where by “Euclidean space” we mean the radial compactification of $\mathbb{R}^d$, endowed with the induced metric. Here $x = 1/|r|$, $r$ being the Euclidean radial coordinate. See [Mel94, §I].) We will also consider (symbolically) asymptotically conic metrics, which — according to our use of the term “asymptotically” — are Riemannian metrics $g$ on $X^\circ$ of the form

1.1. Preliminaries and Outline. Recall that an exactly conic manifold $X$ (according to one of several essentially equivalent definitions) consists of the following data: an arbitrary (smooth, connected) compact manifold-with-boundary $X_0$, a Riemannian metric $g_{\partial X} \in \mathcal{C}^\infty(X; \text{Sym}^2 T^* \partial X_0)$ on the closed manifold $\partial X_0$, and an embedding

\begin{equation}
t : \hat{X} = [0, \bar{x})_x \times \partial X_0 \to X_0
\end{equation}

We remark finally that there does not appear to be much overlap between the present work and similarly titled physics papers, e.g. [BOW67][BT67][BTO67][MB67][Mac67].
α₁ > 1, we will say that the asymptotically conic metric \( g \) is “classical to \( α₁ \)th order” if eq. (36) can be strengthened to
\[
g - g₀ \in xC^{∞}(X; {^a}⁡\text{Sym}^2T^{*}X) + S^{-α₁}(X; {^a}⁡\text{Sym}^2T^{*}X).
\] (37)

We will call \( g \) “(fully) classical” if the symbolic term in eq. (36) can be dropped entirely, i.e. if eq. (37) holds for all \( α₁ > 1 \). (Any sc-metric in the sense of [Mel94] is fully classical in this sense.)

The term asymptotically conic manifold will be used to refer to a conic manifold equipped with an asymptotically conic metric, so a triple \((X, \imath, g)\). This is a more general notion than that of an “sc-manifold” [Mel94][Mel95][JS99][HW05], which are also sometimes called asymptotically conic — besides allowing a symbolic term in eq. (36), we also allow a classical \( O(x^{-3})dx^2 \) term (see \( a₀₀ \) in Theorem 1.1). (Rewriting the metric in terms of the tortoise coordinate \( x_\ast \), the \( O(x^{-3})dx^2 \) term can be removed, but \( x_\ast \) is not a smooth function of \( x \) and so \( x_\ast \) is not a bd of the mbw \( X \).)

We will assume that this term is constant on \( \partial X \). Thus, the metrics we consider are Riemannian metrics on \( X^\circ \) of the form
\[
g = g₀ + a₀₀ \frac{dx^2}{x^3} = \frac{\Gamma_{1,∂X} ⊗ dx}{x^2} + \frac{h_{1,∂X}}{x} + x²C^{∞}(X; {^a}⁡\text{Sym}^2T^{*}X) + S^{-α₁}(X; {^a}⁡\text{Sym}^2T^{*}X) \] (38)

for some \( a₀₀ \in \mathbb{R}, 1\)-form \( \Gamma_{1,∂X} ∈ \Omega^1(∂X) \), and symmetric 2-cotensor \( h_{1,∂X} ∈ C^{∞}(∂X; \text{Sym}^2T^{*}∂X) \).

A fully classical \( g \) is then of the form
\[
g = g₀ + a₀₀ \frac{dx^2}{x^3} + \frac{\Gamma_{1,∂X} ⊗ dx}{x^2} + \frac{h_{1,∂X}}{x} + x²C^{∞}(X; {^a}⁡\text{Sym}^2T^{*}X) \] (39)

near \( ∂X \).

Before we specialize to the case of spectral families of Schrödinger operators for the proof of Theorem 1.1, we study in §3, §5 smooth families \( \{P(σ)\}_{σ ≥ 0} ⊂ \text{Diff}^2(X^\circ) \) of elliptic 2nd-order differential operators on \( X^\circ \) of the form
\[
P(σ) = P₀(σ) + P₁(σ) + P₂(σ), \] (40)

where, for each \( i = 1, 2, 3, P_i = \{P_i(σ)\}_{σ ≥ 0} ⊂ \text{Diff}^2(X^\circ) \) is a family of differential operators (depending smoothly on \( σ^2 ≥ 0 \), all the way down to \( σ = 0 \)), which near \( ∂X \) can be written as

- \( P₀(σ) = -(1 + xₐ₀₀(σ))(x²∂ₓ)² + x²Δₓ + x³[a(σ) + n - 1]∂ₓ - Zx - σ² \) (41)

for
- \( a₀₀ ∈ C^{∞}([0, ∞)ₐ₀₀; \mathbb{R}) \), \( Z > 0 \) satisfying the attractivity condition
- \( Z - σ²a₀₀(σ) > 0 \) (42)

for each \( σ ≥ 0 \),
- \( a ∈ C^{∞}([0, ∞)ₐ; \mathbb{R}) \).

- \( P₁(σ) = \sum_{j=1}^{J} (x^4P₁,j(σ)b_j(x; σ)∂ₓ + x³b'_j(x; σ)P₀X,j(σ) + x²b''_j(x; σ)Q₀X,j(σ)) \) (43)

for some \( J ∈ \mathbb{N} \), where
- \( b_j(-; σ), b'_j(-; σ), b''_j(-; σ) ∈ S₀(X) \), more specifically
- \( b_j, b'_j, b''_j ∈ C^{∞}([0, ∞)ₐ; S₀(X)) \), (44)
\[ -P_{\perp,j}(\sigma), Q_{\partial X,j}(\sigma) \in \text{Diff}^1(\partial X) \text{ and } P_{\partial X,j}(\sigma) \in \text{Diff}^2(\partial X) \text{ for each } \sigma \geq 0, \text{ so that} \]
\[ P_{\perp,j}, Q_{\partial X,j} \in C^\infty([0, \infty)_{\sigma^2}; \text{Diff}^1(\partial X)), \]
\[ P_{\partial X,j} \in C^\infty([0, \infty)_{\sigma^2}; \text{Diff}^2(\partial X)), \] (45)

for each \( j = 1, \ldots, J \),

\[ P_2 \in C^\infty([0, \infty)_{\sigma^2}; S \text{Diff}_{\text{scb}}^{2,1-\delta,-3/2-\delta}(X)) \text{ for some } \delta > 0, \]

Here \( \text{Diff}_{\text{scb}}^{m,s,l}(X) = \text{Diff}_{\text{sc}}^m(X) \cap \text{Diff}_{\text{sc}}^{s,l}(X) \) and \( S \text{Diff}_{\text{scb}}^{m,s,l}(X) = S \text{Diff}_{\text{sc}}^m(X) \cap S \text{Diff}_{\text{sc}}^{s,l}(X) \).

We may take \( \delta < 1/2 \), as this will be useful in reducing casework later on. Note that we are requiring smoothness with respect to \( E = \sigma^2 \), rather than with respect to \( \sigma \), in compact subsets of \( X^0 \). This is not strictly necessary for the analysis in §5, for which even smoothness with respect to \( \sigma \) is mostly unnecessary, but since we use smoothness in \( E \) in §6 we will work with it from the outset and develop \( \Psi DO \) calculi accordingly in §2. We will say that \( P_1 \) is “classical to \( \beta_1 \text{th order}” \) for \( \beta_1 > 0 \) if we can replace eq. (44) by

\[ b_j, b_j', b_j'' \in C^\infty([0, \infty)_{\sigma^2}; C^\infty(X)) + C^\infty([0, \infty)_{\sigma^2}; x^{\beta_1} S^0(X)). \] (47)

Similarly, we say that \( P_2 \) is “classical to \( (\beta_2, \beta_3)\text{th order}” \) for \( \beta_2 > \beta_3 > 0 \) if

\[ P_2 \in C^\infty([0, \infty)_{\sigma^2}; \text{Diff}_{\text{scb}}^{2,1-\delta,-3/2-\delta}(X) + S \text{Diff}_{\text{scb}}^{2,1-1-\beta_2,-3/2-\beta_2}(X) + x^{3/2+\beta_3} S^0(X)) \]
\[ = C^\infty([0, \infty)_{\sigma^2}; \text{Diff}_{\text{scb}}^{2,1-\delta,-3/2-\delta}(X) + S \text{Diff}_{\text{scb}}^{2,1-1-\beta_2,-3/2-\beta_2}(X) + x^{3/2+\beta_3} S^0(X)) \] (48)

(note that \( \text{Diff}_{\text{scb}}^{2,-2}(X) = \text{Diff}_{\text{scb}}^{2,-2}(X) \)). We could work with even finer measures of classicality, but for the proof of Theorem 1.1 (and eventually Proposition 6.3) it suffices to keep track of \( \beta_1, \beta_2, \beta_3 \).

We restrict attention to the case of constant \( Z \), though the case of variable \( Z \) may also be of interest. (The \( x^3(n-1)\partial_x \) term in eq. (43) is merely conventional and can be parametrized away upon redefining \( a \), but it is convenient because \( x^2 \partial_x - x(n-1)/2 \) is formally anti- self-adjoint with respect to the \( L^2 = L^2([0, x] \times \partial X, g_0) = L^2_{\text{sc}}([0, x] \times \partial X) \) inner product.) The “main” and “subleading” terms of \( P \) are collected in \( P_0 \), with the classical components being collected in \( P_0 \in \text{Diff}_{\text{sc}}(X) \). Although subleading by only one order, the terms in \( P_1 \) are all negligible in the analysis of §5, as are the terms in \( P_2 \) (which is why they are allowed to be symbolic). (The key point here is that the principal symbol of \( P_1 \) vanishes to one additional order at the radial sets of \( P \)’s Hamiltonian flow because of the tangential derivatives. On the other hand, \( P_2 \) is just lower order.) Note that we are only requiring \( P(\sigma), P_0(\sigma) \) to be elliptic in the traditional sense: the operators above can be considered as sc-operators, but they will be nonelliptic in the sc-calculus. The ‘‘S’’ in front of \( S \text{Diff} \) indicates that the differential operators therein have coefficients which are required only to be conormal functions on \( X \), hence are “symbolic.” We do not concern ourselves with uniformity as \( \sigma \to \infty \), this already being handled in [Vas21a, §5] for a suitably large class of operators.

By “attractive Coulomb-like Schrödinger operator (on \( X \),” we mean an operator of the form

\[ P(0) = \Delta g - Z x + V(x) \] (49)

for some asymptotically conic metric \( g \) on \( X \), \( Z > 0 \), and real-valued \( V \in S^{-3/2-\delta}(X) \) for \( \delta > 0 \). The spectral family of \( P(0) \) is the family \( \{P(\sigma) = P(0) - \sigma^2\}_{\sigma \geq 0} \). The spectral families of attractive Coulomb-like Schrödinger operators are therefore included in the class of families we study here if we restrict attention to sufficiently small \( \sigma \) such that the attractivity condition eq. (42) is satisfied. See Proposition 6.2. From the explicit formula for the Laplacian in local coordinates, we see that \( a_{00} \) is the restriction to \( \partial X \) of \( -x^3(\partial g)_{00} \), where \( \partial g = g - g_0 \). Although it is only obvious upon changing coordinates from \( x \) to \( x/(1 + x a_{00}) \) – see §A – a nonzero value of \( a_{00} \) has essentially
the same effect as a nonzero Coulomb potential on the asymptotics of solutions to Helmholtz’s equation for positive $\sigma$, and it turns out that eq. (42) – taking into account both the actual Coulomb potential and the effective Coulomb potential from $a_{00}$ – is the correct notion of “attractivity” in this generality. Note that if $P$ is the spectral family of a Coulomb-like Schrödinger operator with $Z > 0$ on an asymptotically conic manifold, then we can modify $P$ by cutting off $a_{00}$ outside of some sufficiently small neighborhood of $\sigma = 0$ such that the resulting family of operators satisfies the attractivity condition. We only care about behavior in a neighborhood of $\sigma = 0$, so this does not restrict applicability.

For $\sigma \gg 0$, attractive Coulomb-like operators do not need to be distinguished from non-attractive Coulomb-like operators as far as the limiting absorption principle is concerned, except with regards to the precise logarithmic corrections to spherical waves. The difference matters only in the zero energy. Clearly, eq. (52) paints an incomplete picture of the oscillations of solutions to eq. (50) as $x \to 0^+$, while the latter oscillate like

$$\exp \left( \pm i \frac{\sigma}{x} x^{\mp \frac{i}{2}} \right) \exp \left( \pm i \left( \frac{\sigma}{x} - \frac{Z}{2\sigma} \log x \right) \right)$$

as $x \to 0^+$, while the former oscillate like $\exp(\pm 2iZ^{1/2}/x^{1/2})$. While the leading order term of the phase in eq. (52) is suppressed as $\sigma \to 0^+$, the logarithmic correction $\mp i(Z/\sigma) \log x$ blows up as $\sigma \to 0^+$. This, along with $O_x(1)$, $O_x(x)$, etc. contributions which were omitted from eq. (52) (and include terms like $1/\sigma$, $1/\sigma^2$, $\cdots$), end up contributing as $\sigma \to 0^+$ to the phase $\pm 2iZ/2^{1/2}$ relevant at zero energy. Clearly, eq. (52) paints an incomplete picture of the oscillations of solutions to eq. (50) in the $\sigma \to 0^+$ limit. This is the first indication that the analysis of the low energy limit will need to be done on a resolution of $[0, \infty)_{\sigma} \times X$. We refer the reader to §A for a further discussion of the ODE case.

The “conjugated perspective” involves studying the conjugated spectral family $\tilde{P} = \{\tilde{P}(\sigma)\}_{\sigma \geq 0} \subset Diff^2(X^\circ)$,

$$\tilde{P}(\sigma) = \exp(-i\Phi)P(\sigma)\exp(+i\Phi) : S'(X) \ni u \mapsto \exp(-i\Phi)(P(\sigma)(\exp(+i\Phi)u)),$$

where the “phase” $\Phi = \Phi(-; \sigma)$ is an appropriate function on $[0, \infty)_{\sigma} \times X$ such that $\exp(+i\Phi)$ captures the asymptotics of outgoing solutions $u$ to the PDE $Pu = f$ to some desired level of precision. (Thus, we deal with the ‘+’ case of Theorem 1.1, the ‘−’ case being analogous.) At the level of the phase space of the sc-calculus, this conjugation corresponds to a symplectomorphism moving one of the two sets of radial points (the “selected radial set”) to the zero section of the
sc-fibers, which upon second microlocalization gets blown up to the fibers of $bT^*_p X$, the phase space of the b-calculus over $\partial X$ (see [Vas21a]). The following choice of $\Phi$ is sufficiently detailed:

$$
\Phi(x; \sigma) = \frac{1}{x} \sqrt{\sigma^2 + Z x - \sigma^2 a_{00} x} + \frac{1}{\sigma} \left( \frac{1}{\sigma^2 a_{00}} \right) \arcsinh \left( \frac{\sigma}{x^{1/2} (Z - \sigma^2 a_{00})^{1/2}} \right) - \frac{i}{2} a \log x,
$$

(extended from $\sigma > 0$ to $\sigma = 0$ by continuity), hence the appearance of $\Phi$ in Theorem 1.1 (for which we have $a = 0$). The square roots in eq. (54) are well-defined by the attractivity condition eq. (42).

See the beginning of §3 for a motivation of eq. (54). Given compact $K \subset (0, \infty)$, we can write, for $\sigma \in K$,

$$
\Phi = \sigma x^{-1} - \frac{Z - \sigma^2 a_{00}}{2\sigma} \log x - \frac{i}{2} a \log x + O_K(1)
$$

as $x \to 0^+$. This is in accordance with eq. (52), [Vas21a, Theorem 1.1]: $\Phi$ differs from Vasy’s phase modulo a logarithmic plus smooth correction for each individual $\sigma > 0$. The $O_K$ term in eq. (55), when formally expanded around $\sigma = 0^+$, contains terms proportional to $\sigma^{-2N}$ for all $N \in \mathbb{N}^+$. The estimate eq. (55) is therefore not uniform as $\sigma \to 0$. Similarly, while we can write $(\sigma^2 + Z x)^{1/2} = \mathcal{Z}^{1/2} x^{1/2} (1 + \sigma^2 / 2Zx + O(\sigma^4/Z^2x^2))$ for $x$ bounded away from zero, this obviously cannot be used to understand the asymptotics of outgoing solutions to $Pu = f$ for any $\sigma > 0$. Hence, the precise form of $(\sigma^2 + Z x)^{1/2}$ in eq. (54) (and the precise form of the other terms appearing in eq. (54)) is important (at least modulo functions which are well-behaved on $X^\text{sp}_{\text{res}}$). This is the second indication that we will need to resolve the $x, \sigma \to 0^+$ regime, and apparently in doing so we had ought to resolve the ratio $x/\sigma^2$.

This is accomplished by the mwc $X^\text{sp}_{\text{res}}$. While, as already remarked upon in Remark 5, this resolution is reminiscent of that used in [Vas21c][Hin21] in order to resolve the low energy behavior of interest there, it performs a somewhat different role here; theirs was used to study the degeneration of $\Delta_g - \sigma^2$ as $\sigma \to 0^+$ to an elliptic element

$$
\Delta_g \in \text{Diff}^2_{b,-2}(X)
$$

of the b-calculus, but

$$
\Delta_g - Z x \in \text{Diff}^2_{b,-1}(X)
$$

is not elliptic in the b-calculus, hence the oscillating solutions to the ODE eq. (51) noted above. Instead, we use the resolution to interpolate between Vasy’s analysis in [Vas21a] of $\Delta_g - \sigma^2 - Z x$ performed in $\Psi_{\text{sch}}(X)$ for $\sigma > 0$ and the analysis of the “zero-energy operator” $\Delta_g - Z x$ performed in $\Psi_{\text{sch}}(X_{1/2})$ along similar lines (see §4). According to the previous paragraph, what makes the $\Phi$ in Theorem 1.1 the “right” choice is the asymptotics of $\Phi$ near the boundary $\partial X^\text{sp}_{\text{res}}$ of this resolution (or more accurately near $\partial X^\text{sp}_{\text{res}} \setminus x^2 = \text{bf} \cup \text{tf}$). Indeed, for $\sigma = 0$, we can write

$$
\Phi = 2 \frac{Z}{\sqrt{x}} - \frac{i}{2} a \log x.
$$

Hence, eq. (54) interpolates between the oscillations eq. (55) seen in solutions of the ODE eq. (50) at positive energy and the oscillations eq. (58) seen in solutions of the ODE eq. (51) at zero energy.

We refer to §3 for a further discussion of the conjugated operator family and §4 for a discussion of the situation at zero energy.

As our main technical tool, we situate the family $\tilde{P} = \{ \tilde{P}(\sigma) \}_{\sigma \geq 0}$ in a pseudodifferential calculus,

$$
\Psi_{\text{leC}}(X) = \bigcup_{m,s,c,t,\ell \in \mathbb{R}} \Psi_{\text{leC}}^{m,s,c,t,\ell}(X),
$$

(59)
which we will call the “leC-calculus” (“low energy Coulomb”-calculus, for lack of a better name) the elements of which can be interpreted as particular families of $b$-$\Psi$DOs on $X$. The calculus comes with a refined symbol calculus tailored to the problem at hand. Compared to the calculus of $b$-$\Psi$DOs with parameters (i.e. $\Psi_b(X)$-valued symbols on some parameter space), the symbol calculus here is refined in the sense of being second-microlocalized à la Vasy (so as to keep track both of $b$-decay and $s$-decay orders) and “resolved at the corner” (so as to keep track of the asymptotic regime when both $\alpha^2 \to 0$ and $x \to 0$ at compatible rates). The corresponding symbols are conormal functions on the “leC-phase space,” an iterated blow-up of $[0, \infty)_\sigma \times b^\infty X$. This mwc has six boundary faces – $df$, $sf$, $bf$, $tf$, $zf$ – and is described in the next section. In eq. (59), $M,N$ (where $(\text{order at fiber infinity, df})$, $\Psi$, $scb$, $\psi$, $X$) to write an intermediate estimate of the form

$$M,N \quad (\text{order at fiber infinity, df}), \quad s \quad \text{is the sc-decay order at positive energy (order at sf)}, \quad \varsigma \quad \text{is the sc-decay order at zero energy (order at ff)}, \quad l \quad \text{is the b-decay order at positive energy (order at bf)}, \quad \ell \quad \text{is the b-decay order at zero energy (order at tf)}. \quad \text{We remark that the scattering calculus with respect to } x^2 \quad (\text{rather than } x^{1/2}) \quad \text{has been used by Wunsch [Wun99][HW05], who called it the “quadratic scattering calculus.” The leC-calculus is discussed in §2.}

We now say a word about our usage of the symbol ‘$\preceq$’ (the usage of ‘$\succeq$’ being analogous). When stating a proposition involving an estimate, we will be explicit about the dependence of the constants involved on parameters. In order to avoid a proliferation of symbols denoting different but unimportant constants, when proving a proposition of the form

- for all $p_1 \in P_2, \ldots, p_M \in P_M$, there exists a constant $C(p_1, \ldots, p_M) > 0$ such that $r_1(p_1, \ldots, p_N) \leq C(p_1, \ldots, p_M) r_2(p_1, \ldots, p_N)$ for all $p_{M+1} \in P_{M+1}, \ldots, p_N \in P_N$, (where $M,N \in \mathbb{N}$, $N \geq M, P_2, \ldots, P_N$ are some sets, $r_i : P_2 \times \cdots \times P_N \to \mathbb{R}$ for $i = 1, 2$) we will write an intermediate estimate of the form

$$r_3(p_1, \ldots, p_N) \leq C'(p_1, \ldots, p_M) r_4(p_1, \ldots, p_N) \quad (60)$$

as $r_3 \preceq r_4$, with the key point being that, according to eq. (60), $C'$ depends only on the parameters that $C$ depends on (so that the estimate eq. (60) is “uniform” in $p_{M+1}, \ldots, p_N$). All constants below depend on the geometric data in the setup of Theorem 1.1, so we will not be explicit about that dependence.

2. The leC-Calculus

We now turn to our discussion of the leC-calculus. This calculus is, in many ways, similar to Vasy’s second microlocalized calculi $\Psi_{\text{sch}}(X) = \cup_{m,s,l} \Psi_{\text{sch},m,s,l}^m(X)$ [Vas21a, §2] and $\Psi_{\text{sch, res}}(X) = \cup_{m,s,l} \Psi_{\text{sch, res},m,s,l}^m(X)$ [Vas21c, §3], with the main novel feature of the leC-calculus (besides the relatively unimportant alteration of the smooth structure at $\sigma = 0$) being another resolution of the phase space (which we actually carry out before resolving the scattering face for $\sigma > 0$). This can be seen at a glance, comparing Figure 3 to Figure 4, [Vas21c, Figure 4]. The zero face $zf$ of the leC-phase space is identifiable with the phase space $scbT X_{1/2}$ of the calculus $\Psi_{\text{sch}}(X_{1/2})$, while for $\sigma_0 > 0$ the $\{\sigma = \sigma_0\}$ cross-section of the leC-phase space is identifiable with the phase space $scbT X$ of the scb-calculus. Thus, the leC-calculus interpolates between these two calculi as $\sigma \to 0^+$, as $\Psi_{\text{sch, res}}(X)$ interpolates between $\Psi_{\text{sch}}(X)$ and $\Psi_b(X)$ in the same limit.

The leC-phase space $\text{leC}^\infty X$ is introduced in §2.1, along with corresponding algebras of symbols. Calculi are discussed in §2.2, and the corresponding leC-Sobolev spaces (which are really families of scb-Sobolev spaces) are discussed in §2.3. As many of the results in this section are either consequences of standard results for the b-calculus or derivable via similar arguments, some details are omitted. Still, we’ve made an effort to give a relatively complete list of the results needed later,
and in the process we give the leC analogues of some standard arguments. In this section we mainly
write \( \lambda = \sigma^2 \) in place of the parameter \( E \) used in the introduction.

When we write \( \Psi_b(X) \), and more generally when we refer to \( "b\)-\( \Psi \)DOs, we mean the conormal (a.k.a. “symbolic”) \( b \)-algebra \([Vas18, Definition 5.15]\), rather than the closely related, slightly smaller calculus defined in \([Mel93, Definition 4.22]\). The latter calculus has symbols that have some additional classicality at the boundary. Our convention follows \([Vas21a]\). In \([Vas21b]\), the notation \( \Psi_{bc}(X) \) is used instead.

2.1. Phase Spaces and Symbols. Recall that we can identify \( bT^*X \) over the boundary collar
with \( [0, \bar{x}] \times \mathbb{R}_{\xi_b} \times (T^*\partial X)_{\eta_b} \), via the map
\[
(0, \bar{x}) \times \mathbb{R}_{\xi_b} \times (T^*\partial X)_{\eta_b} \ni (x, \xi_b, \eta_b) \mapsto \xi_b \frac{dx}{x} + \text{plr}^*(\eta_b),
\]
(61)
where \( \text{plr} = \pi_r \circ \iota^{-1} : \iota(\bar{X}) \rightarrow \partial X \). This defines a diffeomorphism between \( [0, \bar{x}] \times \mathbb{R}_{\xi_b} \times (T^*\partial X)_{\eta_b} \) and \( bT^*X \).

Let
\[
b^{, sp}T^*X = [[0, \infty)_\lambda \times bT^*X; \{ \lambda = x = 0 \}]
= [[0, \infty)_\lambda \times bT^*X; \{ 0 \} \times b\partial X] \tag{62}
\]
denote the phase space of the resolved calculus of 1-parameter families of \( b \)-\( \Psi \)DOs \([Vas21c, Figure 4]\), and letting \( \beta : b^{, sp}T^*X \rightarrow [0, \infty)_\lambda \times bT^*X \) denote the blowdown map, let
\[
zf_{00} = \text{cl} \beta^{-1}(\{ \lambda = 0, x > 0 \}),
\]
\[
tf_{00} = \beta^{-1}(\{ \lambda = 0, x = 0 \}),
\]
\[
bf_{00} = \text{cl} \beta^{-1}(\{ \lambda > 0, x = 0 \}),
\]
and
\[
df_{00} = \text{cl}((\partial^{sp}b\partial X) \setminus (zf_{00} \cup tf_{00} \cup bf_{00})) \tag{64}
\]
denote its (closed) boundary faces. For each \( f \in \{ zf_{00}, tf_{00}, bf_{00}, df_{00} \} \), let \( \vartheta_{0,f} \in C^\infty(b^{, sp}T^*X; [0, \infty)) \) denote a bdf of the respective face, which we can take to be equal near \( \{ x = 0 \} \) to
\[
\vartheta_{0,zf_{00}} = \frac{\lambda}{\lambda + x}, \quad \vartheta_{0,tf_{00}} = \lambda + \xi_b \frac{x}{\lambda + \xi_b}, \quad \vartheta_{0,bf_{00}} = \frac{x}{\lambda + \xi_b}, \quad \vartheta_{0,df_{00}} = (1 + \lambda^2 + \eta_b^2)^{-1/2} \tag{65}
\]
(defined initially in the interior of \( b^{, sp}T^*X \), these then extending to smooth functions on \( b^{, sp}T^*X \).)
(Below, we conflate smooth functions on mwc with their restrictions to the interior when such a conflation does not cause trouble.) In eq. (65) and below, we write \( \xi_b \) for the \( b \)-cofiber coordinate dual to \( x \), and \( \eta_b \) for \( \eta_b \in T^*\partial X \).

There exists a unique mwc \( b,leC T^*X = [[0, \infty)_\lambda \times bT^*X; \{ 0 \} \times b\partial X; \{ 1/2 \}] = [b^{, sp}T^*X; tf_{00}; \{ 1/2 \}] \), depicted in Figure 2, with the following properties:

1. as a set, \( b,leC T^*X \) is equal to \( b^{, sp}T^*X \) (a convenient convention),
2. \( b,leC T^*X \) has the same smooth structure as \( b^{, sp}T^*X \) away from \( tf_{00} \), so that if \( \varphi \in C^\infty(b^{, sp}T^*X) \) is supported away from \( tf_{00} \), then \( \varphi \in C^\infty(b,leC T^*X) \); moreover,
3. \( b,leC T^*X \) has four faces, equal as sets to \( zf_{00}, tf_{00}, bf_{00}, df_{00} \), for which \( \vartheta_{0,zf_{00}}, \vartheta_{0,tf_{00}}, \vartheta_{0,bf_{00}}, \vartheta_{0,df_{00}} \) serve as bdfs (respectively).
We will refer to \( b, leC T^* X \) as the \( b, leC \)-phase space. See Figure 2.

We will refer to the bdfs of \( b, leC T^* X \) as \( q_{bf00} = q_{0,0,0,0} \), \( q_{t0} = q_{0,0,0,0}^{1/2} \), and \( q_{df00} = q_{0,0,0,0} \). Thus, in terms of \( \sigma = \lambda^{1/2} \),

\[
q_{t0} = \frac{x}{\sigma^2 + Zx}, \quad q_{bf00} = \frac{x}{\sigma^2 + Zx}, \quad q_{df00} = \frac{\sigma^2}{\sigma^2 + Zx}.
\]

(66)

Note that the notions of zeroth order conormality on \( b, leC T^* X \) and \( b, sp T^* X \) agree, as does the notion of smoothness at \( z_{f00} \). For each \( m, l, \ell \in \mathbb{R} \), we let

\[
S_{m,l,\ell}^0(b, leC T^* X) = \bigcup_{m,l,\ell \in \mathbb{R}} S_{m,l,\ell}^0(b, leC T^* X),
\]

(67)

\[
S_{m,l,\ell}^0(b, leC T^* X) = A_{\sigma}^{-m, -l, -\ell, (0,0)}(b, leC T^* X),
\]

(68)

where we are enumerating the faces of the \( b, leC \)-phase space in the order \( df_{00}, bf_{00}, tf_{00}, zf_{00} \). Thus, \( m \) is the order at \( df_{00} \), \( l \) is the order at \( bf_{00} \), \( \ell \) is the order at \( tf_{00} \), and the order at \( zf_{00} \) is just zero (and we have a full Taylor series there, with the terms in the Taylor series elements of \( A_{\sigma}^{-m, -\ell, (0,0)}(b, leC T^* X) \)).

The ‘loc’ subscript in “\( A_{\sigma} \)” refers to the fact that we do not require \( L^\infty \) bounds in the \( \sigma \to \infty \) direction. That is, we only have bounds in compact subsets (which can include boundary points) of the mwe \( b, leC T^* X \), which is only noncompact because of the \( \sigma \to \infty \) direction. We also define

\[
S_{m,l,\ell}^0(b, leC T^* X) = q_{df00}^m q_{bf00}^l q_{t0}^{-m} C^\infty(b, sp T^* X) \subset S_{m,l,\ell}^0(b, leC T^* X).
\]

(69)

Given \( a \in S_{m,l,\ell}^0(b, leC T^* X) \), we may restrict \( a \) to \( z_{f00} \), giving an element \( a(\cdot, 0) \in S_{m,l,\ell}^0(b, leC T^* X) = S_{m,l,\ell}^0(b, leC T^* X) \).

Also define

\[
S_{m,l,\ell}^0(b, leC T^* X) = \bigcup_{m, l, \ell \in \mathbb{R}} S_{m,l,\ell}^0(b, leC T^* X).
\]

(70)

whenever \( l_0 \geq l \) and \( 2\nu + 2l_0 \geq \ell \). Consequently, if \( l_0 \geq l, \ell/2 \),

\[
S_{m,l,\ell}^0(b, leC T^* X) \subseteq A_0^0([0, \infty)_\sigma : S_{m,l,\ell}^0(b, leC T^* X)).
\]

(71)

**Lemma 2.1.** For any \( \ell \in \mathbb{R} \), \( (\sigma^2 + Zx)^{-\ell/2} \) is \( C^0([0, \infty)_\sigma : S_{m,l,\ell}^0([0, \infty)) \) for any \( \varepsilon > 0 \).

**Proof.** Continuity at \( \sigma > 0 \) is clear. Let \( l = \max[\ell/2 + \varepsilon] \). It suffices to restrict attention to \( x < \bar{x} \), so we have to prove that, for each \( k \in \mathbb{N} \), \( x^l(x\partial_x)^k(\sigma^2 + Zx)^{-\ell/2} \to x^l(x\partial_x)^k(\sigma^2 + Zx)^{-\ell/2} |_{\sigma = 0} \) in \( L^\infty([0, \bar{x}]) \). We compute

\[
x^l(x\partial_x)^k(\sigma^2 + Zx)^{-\ell/2} = x^l \sum_{j=0}^k c_{j,k} x^j (\sigma^2 + Zx)^{-\ell/2 - j}
\]

(72)

for some \( c_{j,k}(Z) \in \mathbb{R} \). Observe that \( x^{\max[0,\ell/2]}(\sigma^2 + Zx)^{-\ell/2 - j} \in L^\infty([0, \bar{x}] \times [0, 1]) \) for every \( j \in \mathbb{N} \). Consequently, if \( \ell > 0 \), the extra factor of \( x^\varepsilon \) in eq. (72) in conjunction with the uniform convergence as \( \sigma \to 0^+ \) of \( x^{j+l/2}(\sigma^2 + Zx)^{-\ell/2 - j} \to Z^{-\ell - j} \) in compact subsets of \( (0, \bar{x}]_x \) implies that

\[
x^l \sum_{j=0}^k c_{j,k} x^j (\sigma^2 + Zx)^{-\ell/2 - j} \to x^l \sum_{j=0}^k c_{j,k} Z^{-\ell/2 - j}
\]

(73)

uniformly in all of \( [0, \bar{x}] \) as \( \sigma \to 0^+ \). If \( \ell \leq 0 \), then we can write \( l = \ell/2 + \Delta \) for some \( \Delta > 0 \), so the same analysis applies. □
Proposition 2.2. If \( a \in S_{b,\text{leC}}^{m,l,\ell}(X) \) and \( l_0 \in \mathbb{R} \) satisfies \( l_0 \geq l \) and \( l_0 > \ell/2 \), then \( \{a(-;\sigma)\}_{\sigma \geq 0} \in C^0([0,\infty)_\sigma; S_b^{m,l,0}(X)) \).

Proof. We first reduce to the case \( m, l, \ell = 0 \):

- For any \( m, l, \ell \in \mathbb{R} \),

\[
x^{-l}(\sigma^2 + Zx)^{l-\ell/2}g_{df00}^{-m} \in C^0([0,\infty)_\sigma; S_b^{m,l,1}(X))
\]

if \( l_1 \geq l \), \( l_1 > \ell/2 \) by the previous lemma.

Any \( a \in S_{b,\text{leC}}^{m,l,\ell}(X) \) can be written as \( a = x^{-l}(\sigma^2 + Zx)^{l-\ell/2}g_{df00}^{-m}a_0 \) for \( a_0 \in S_{b,\text{leC}}^{0,0,0}(X) \), so if we know that \( a_0 \in C^0([0,\infty)_\sigma; S_b^{0,\ell,0}(X)) \) for \( \varepsilon > 0 \) then we can conclude that

\[
a \in C^0([0,\infty)_\sigma; S_b^{m,l,1}(X))C^0([0,\infty)_\sigma; S_b^{0,\ell,0}(X)) = C^0([0,\infty)_\sigma; S_b^{m,l,\ell+\varepsilon}(X)).
\]

Taking \( l_1 < l_0 \) and \( \varepsilon \in (0, l_0 - l_1) \), we get \( a \in C^0([0,\infty)_\sigma; S_b^{m,\ell,\varepsilon}(X)) \).

To prove the proposition in the case \( m, l, \ell = 0 \):

- Continuity at \( \sigma > 0 \) is clear, so we only need to check that, for \( a \in S_{b,\text{leC}}^{0,0,0}(X) \), \( a(-;\sigma) \to a(-;0) \) in \( S_b^{0,\varepsilon}(X) \) for every \( \varepsilon > 0 \). Note that \( La \in L^\infty([0,1]_x \times [0,\bar{x}]_x \times \partial X) \) for any \( L \in \text{Diff}_b(X) \). Since, as \( \sigma \to 0^+ \), \( La \to La|_{\sigma=0} \) uniformly in compact subsets of \( (0,\bar{x}]_x \times \partial X \), we can conclude that

\[
\lim_{\sigma \to 0^+} x^\varepsilon L a = x^\varepsilon L a|_{\sigma=0}
\]

uniformly in all of \( [0,\bar{x}]_x \times \partial X \). Thus, \( a(-;\sigma) \to a(-;0) \) in \( S_b^{0,\varepsilon}(X) \).

We now introduce the full leC- phase space \( \text{leC}^* T^* X \). This is the mwc gotten from \( S_{b,\text{leC}}^* T^* X \) by first blowing up the edge \( df_{00} \cap tf_{00} \), resulting in a mwc with five faces – \( df_0, ff_0, bf_0, tf_0, zf_0 \) (Figure 2, right), where \( ff_0 \) is the front face of the blow up – with bdfs

\[
\theta df_0 = \frac{\partial df_0}{\partial df_{00} + \partial tf_{00}}, \quad \theta ff_0 = \frac{\partial df_0}{\partial df_{00} + \partial tf_{00}}, \quad \theta tf_0 = \frac{\partial tf_{00}}{\partial df_{00} + \partial tf_{00}},
\]

Figure 2. The phase spaces \( b,\text{leC}^* T^* X \) (left, cf. [Vas21c, Fig. 1]) and \( [b,\text{leC}^* T^* X; df_{00} \cap tf_{00}] \) (right), with the degrees of freedom associated with \( T^* \partial X \) omitted. (In other words, if we were to consider the case \( \dim X = 1 \), then the figures above would depict the phasesss.) For simplicity, we only depict the \( \xi_b > 0 \) half of phase space.

\begin{equation}
\tag{77}
\end{equation}
At the level of sets (and at the level of it is important to understand the residual operators the leC-calculus in the rest of the paper (i.e. §3, §5, §6) could be replaced by references to the leC-that we enforce classicality at §2.2.2 and the full leC-calculus in §2.2.3. Since the b,leC-calculus is essentially pointwise multiplication of symbols is jointly continuous with respect to the relevant topologies.

\[ S_{\text{leC}}(X) = \bigcup_{m,s,l,\ell \in \mathbb{R}} S^m_{\text{leC}}(X), \]

\[ S^m_{\text{cl,leC}}(X) = \bigcup_{m,s,l,\ell \in \mathbb{R}} S^m_{\text{cl,leC}}(X). \]

At the level of sets (and at the level of \( \mathbb{C} \)-algebras), \( S_{\text{leC}}(X) \) is equal to \( S_{b,\text{leC}}(X) = \bigcup_{m,l,\ell \in \mathbb{R}} S^m_{\text{leC}}(X) \), but the filtration above presents \( S_{\text{leC}}(X) \) as a multigraded \( \mathbb{C} \)-algebra. The isomorphism

\[ \times : S^m_{\text{leC}}(X) \rightarrow S^m_{\text{leC}}(X) \]

of vector spaces allows us to consider each \( S^m_{\text{leC}}(X) \) as a Fréchet space, and likewise for \( S^m_{\text{cl,leC}}(X) \). The \( \mathbb{C} \)-algebras \( S_{\text{leC}}(X) \) and \( S_{\text{cl,leC}}(X) \) are then multigraded Fréchet algebras, as pointwise multiplication of symbols is jointly continuous with respect to the relevant topologies.

Observe that if \( a \in S^m_{\text{leC}}(X) \) then \( a(-;0) \in S^m_{\text{scb}}(X_{1/2}) \). For \( \sigma > 0 \), \( a(-;\sigma) \in S^m_{\text{scb}}(X) \).

2.2. Calculi. After recalling some preliminary notions in §2.2.1, we discuss the b,leC-calculus in §2.2.2 and the full leC-calculus in §2.2.3. Since the b,leC-calculus is essentially \( \Psi_{b,\text{leC}}(X) \), except that we enforce classicality at \( z_{b,0} \) (which introduces no complications), and since all references to the b,leC-calculus in the rest of the paper (i.e. §3, §5, §6) could be replaced by references to the leC-calculus with only notational complications, we will only sketch the arguments in §2.2.2. (However, it is important to understand the residual operators \( \Psi_{\text{leC}}(X) \), as these are the “non-symbolic”
Figure 4. The phase space of Vasy’s resolved calculus. Cf. [Vas21c, Figure 4] (which also depicts the $\xi < 0$ half of this phase space).

parts of leC-operators. But – once again – this is essentially $\Psi_{\sp,\res}^{-\infty, l, \ell} (X)$ except for additional classicality at the “zero face” of Vasy’s double space.)

2.2.1. $\Psi_{\sp,\res}(X)$. We now recall the notion of the $b$-calculus with conormal dependence on parameters: for any (compact) $\mr{mb} X$ and a (connected) $\mr{mc} M$ (the “parameter space”), we have a multigraded $\mathbb{C}$-algebra

$$\Psi_{b; M}(X) = \bigcup_{m, l \in \mathbb{R}} \Psi_{b; M}^{m, l}(X),$$

$$\Psi_{b; M}^{m, l}(X) = A_0^0(M; \Psi_{b}^{m, l}(X)),$$

the members of which are the families \{\(A_\lambda\)\}_{\lambda \in M^o} of $b$-PDOs on $X$ depending conormally on a parameter $\lambda \in M^o$ (as above, the ‘loc’ refers to the fact that we only require uniform bounds in compact subsets of $M$). So, letting $C^\infty \cap L_\loc^\infty (M; \Psi_{b}^{m, l}(X))$ denote the Fréchet space of smooth maps $M^o \to \Psi_{b}^{m, l}(X)$ that are (but whose derivatives are not necessarily) uniformly bounded with respect to each Fréchet seminorm of $\Psi_{b}^{m, l}(X)$ in every compact subset of $M$,

$$\Psi_{b; M}^{m, l}(X) = \{A_\bullet \in C^\infty \cap L_\loc^\infty (M; \Psi_{b}^{m, l}(X)) : [L A_\bullet]_{a.e.} \in L_\loc^\infty (M; \Psi_{b}^{m, l}(X)) \ \forall L \in \Diff_b(M)\}. \quad (83)$$

Note that each $\Psi_{b; M}^{m, l}(X)$ is a Fréchet space, and $\Psi_{b; M}(X)$ can be regarded as a multigraded Fréchet algebra. Relevant to the study of spectral families of operators is the case $M = [0, \infty)$, In this case, we write “sp” in place of “; M” in the notation.

In [Vas21c, §3], Vasy (using slightly different notation) defines a particular “refinement” of $\Psi_{b,\sp}(X) = \Psi_{b; M}(X)$,

$$\Psi_{b,\sp,\res}(X) = \Psi_{b,\sp}(X), \quad \Psi_{b,\sp,\res} (X) = \bigcup_{m, l, \ell \in \mathbb{R}} \Psi_{b,\sp,\res}^{m, l, \ell}(X),$$

a multigraded $\mathbb{C}$-algebra which is equal, at the level of $\mathbb{C}$-algebras, to $\Psi_{b,\sp}(X)$, but with a 3-parameter multigrading (and associated symbol calculus) such that

- $\Psi_{b,\sp,\res}^{m, l, \ell}(X) \subset \Psi_{b,\sp}^{m, \max(l, \ell)}(X)$
- and $(\lambda + x)^{-\ell}$, considered as a multiplication operator, is in $\Psi_{b,\sp,\res}^{0, 0, l}(X)$.

The three indices $m, l, \ell$ in $\Psi_{b,\sp,\res}^{m, l, \ell}(X)$ keep track of three notions of order, roughly the “differential order” $m$, the $b$-decay order away from zero energy $l$ – that is at $bf_00$ – and the $b$-decay order $\ell$ at
Define \( \sigma_{b,sp,\text{res}}^{m,l,\ell} : \Psi_{b,sp,\text{res}}(X) \to A_{\text{loc}}^{m,-l,-\ell,0}(b,sp,T^*X)/A_{\text{loc}}^{m+1,-l,-\ell,0}(b,sp,T^*X) \) by
\[
\sigma_{b,sp,\text{res}}^{m,l,\ell}(a) = (\lambda + x)^{-\ell} \sigma_{b,sp}^{m,l}((\lambda + x)^{-\ell} a)
\] (85)
for \( a \in S_{b,sp,\text{res}}^{m,l,\ell}(X) \), where \( \sigma_{b,sp}(a) \) denotes the b-principal symbol map applied \( \lambda \)-wise to \( a \) considered as an element of the family b-algebra \( \Psi_{b,sp}(X) \). Then, for all \( m, l, \ell, \ell' \in \mathbb{R} \),
\[
0 \to \Psi_{b,sp,\text{res}}^{m-1,l,\ell}(X) \hookrightarrow \Psi_{b,sp,\text{res}}^{m,l,\ell}(X) \xrightarrow{\sigma_{b,sp,\text{res}}^{m,l,\ell}} A_{\text{loc}}^{m,-l,-\ell,0}(b,sp,T^*X)/A_{\text{loc}}^{m+1,-l,-\ell,0}(b,sp,T^*X) \to 0
\] (86)
is a short exact sequence and, for all \( m', l', \ell' \in \mathbb{R} \),
\[
\sigma_{b,sp,\text{res}}^{m,l,\ell}(A)\sigma_{b,sp,\text{res}}^{m',l',\ell'}(B) = \sigma_{b,sp,\text{res}}^{m+m',l+l',\ell+\ell'}(AB)
\] (87)
\[
\{\sigma_{b,sp,\text{res}}^{m,l,\ell}(A), \sigma_{b,sp,\text{res}}^{m',l',\ell'}(B)\} = -i\sigma_{b,sp,\text{res}}^{m+m'-1,l+l'+\ell+\ell'}([A, B]).
\] (88)
for all \( A \in \Psi_{b,sp,\text{res}}^{m,l,\ell}, B \in \Psi_{b,sp,\text{res}}^{m',l',\ell'} \). We will compute Poisson brackets using the convention that momentum derivatives of the first entry have positive sign. (The sign in eq. (88) depends on the choice of sign used in the Fourier transform used in defining the calculus.) The bdfs \((\sigma^2 + Zx)^{1/2}\) and \(x/(\sigma^2 + Zx)\) of \( X_{sp}^b\), considered as multiplication operators, are representatives of their own principal symbols:
\[
\sigma_{b,sp,\text{res}}^{0,0,0}(x^{-\ell}(\sigma^2 + Zx)^{1/2}) = x^{-\ell}(\sigma^2 + Zx)^{1/2} \mod S_{b,sp,\text{res}}^{-1,0,0}(X)
\] (89)
\[
\sigma_{b,sp,\text{res}}^{0,0,\ell}(x^{-\ell}(\sigma^2 + Zx)^{-\ell}) = (\sigma^2 + Zx)^{-\ell} \mod S_{b,sp,\text{res}}^{-1,0,\ell}(X).
\] (90)
More generally, if \( a \in S_{b,sp,\text{res}}^{m,l,\ell}(X) \), then \( a \in \sigma_{b,sp,\text{res}}^{m,l,\ell}(a) \).

It is very convenient to make use of a “(left) quantization” map (right quantization working equally well):
\[
\text{Op} : S_b(X) \to \Psi_b(X)
\] (91)
\[
: S_{b}^{m,l}(X) \to \Psi_{b}^{m,l}(X),
\] (92)
discussed e.g. in [Vas18] among other places, given by the left quantization of symbols in local coordinates. This will be noncanonical, depending on a choice of atlas on \( X \), among other things. While not surjective (missing out on the remainder term \( R' \) in [Vas18, Definition 5.15]), it will be modulo \( \Psi_{b,\infty}^{-1,0,0}(X) \).

Applied \( \lambda \)-wise to an element of \( \mathcal{A}^0([0, \infty)_\lambda; S_b^{m,l}(X)) \), the result is an element of \( \Psi_{b,sp}^{m,l}(X) = \mathcal{A}^0([0, \infty)_\lambda; \Psi_{b}^{m,l}(X)) \). Some elementary properties of Op which we can arrange are:

- for any \( f \in \cup_{l \in \mathbb{R}} \mathcal{A}^l(X) \),
\[
\text{Op}(f) = f
\] (93)
(this property distinguishing left quantization from right), where the \( f \) on the right-hand side denotes the multiplication operator \( u \mapsto fu \),
- Op is \( \mathbb{C} \)-linear,
- eq. (92) is continuous for any \( m, l \in \mathbb{R} \),
- \( \sigma_{b}^{m,l}(\text{Op}(a)) = a \mod S_{b}^{m-1,l}(X) \) for all \( a \in S_{b}^{m,l}(X) \).

(Equation (93) holds for the calculus \( \Psi_\infty \) under left quantization. Since the Schwartz kernel of the multiplication operator \( u \mapsto fu \) is supported on the diagonal, it is unaltered by the cutoff \( \psi(t - t') \) in [Vas18, Definition 5.15]. As a consequence, eq. (93) holds also for \( \Psi_b \).)
2.2.2. \( \Psi_{b, \text{leC}}(X) \). Let \( \Psi_{b, \text{leC}}^{-\infty, l, \ell}(X) \) denote the elements of \( \Psi_{b, \text{sp,res}}^{-\infty, l, \ell/2}(X) \) whose Schwartz kernels are smooth at the face \( cI\{\sigma = 0, x' > 0, x > 0\} \) of the double space \( X^{2b, \text{sp,res}} \) [Vas21c, Figure 2] (with the terms in the Taylor series being elements of \( \Psi_{b}^{-\infty, l}(X_{1/2}) \)):

\[
\text{SK} \Psi_{b, \text{leC}}^{-\infty, l, \ell}(X) = A_{\text{loc}}^{-l, -\ell/2, -\infty, -\infty, (0,0)}(X_{2b, \text{sp,res}}),
\]

where we are listing the boundary faces of \( X^{2b, \text{sp,res}} \) in the order \( \text{bf} = cI\{x, x' = 0, \sigma > 0\}, \text{tf} = \{x, x' = 0, \sigma = 0\}, \text{lf} = \{x = 0, x' > 0, \sigma > 0\}, \text{rb} = cI\{x' = 0, x > 0, \sigma > 0\}, \text{zf} = cI\{\sigma = 0, x > 0, x' > 0\} \). Thus, \( \Psi_{b, \text{leC}}^{-\infty, l, \ell}(X) \) inherits from eq. (94) a Fréchet space structure, and it can be shown that

\[
\Psi_{b, \text{leC}}^{-\infty, -\infty, \infty}(X) = \bigcup_{l, \ell \in \mathbb{R}} \Psi_{b, \text{leC}}^{-\infty, l, \ell}(X)
\]

is then a multigraded Fréchet space. (So operator composition defines a jointly continuous map \( \Psi_{b, \text{leC}}^{-\infty, l, \ell}(X) \times \Psi_{b, \text{leC}}^{-\infty, l', \ell'}(X) \to \Psi_{b, \text{leC}}^{-\infty, l+l', \ell+\ell'}(X) \) for all \( l, \ell, l', \ell' \in \mathbb{R} \).

An argument similar to that used to prove Proposition 2.2 yields:

**Proposition 2.3.** Given an element \( K \in \text{SK} \Psi_{b, \text{leC}}^{-\infty, l, \ell}(X) \), if \( l_0 \geq l \) and \( l_0 > \ell/2 \), then it is the case that \( \{K(-; \sigma)\}_{\sigma \geq 0} \in C^0([0, \infty), \text{SK} \Psi_{b}^{m, l_0}(X)) \) for any \( m \in \mathbb{R} \).

Consequently, elements of \( \Psi_{b, \text{leC}}^{-\infty, -\infty, \infty}(X) \) can be considered as continuous families of \( b \)-\( \Psi \)DOs indexed either by \( \mathbb{R}_+^\ast \) or by \([0, \infty), \sigma \). We now define, for each \( m, l, \ell \in \mathbb{R} \),

\[
\Psi_{b, \text{leC}}^{m, l, \ell}(X) = \text{Op}(S_{b, \text{leC}}^{m, l, \ell}(X)) + \Psi_{b, \text{leC}}^{-\infty, l, \ell}(X).
\]

Thus, by eq. (71), if \( l_0 \geq l, \ell/2 \), then \( \Psi_{b, \text{leC}}^{m, l, \ell}(X) \subseteq \Psi_{b, \text{sp}}^{m, l_0}(X) \). In addition:

**Proposition 2.4.** If \( a = \{a(-; \sigma)\}_{\sigma \geq 0} \in S_{b, \text{leC}}^{m, l, \ell}(X) \), then, if \( l_0 \geq l \) and \( l_0 > \ell/2 \),

\[
\text{Op}(a(-; \sigma))_{\sigma \geq 0} \in C^0([0, \infty), \Psi_{b}^{m, l_0}).
\]

Consequently, if \( A = \{A(\sigma)\}_{\sigma \geq 0} \in \Psi_{b, \text{leC}}^{m, l, \ell}(X) \), then there exists some \( A(0) \in \Psi_{b}^{m, l_0}(X) \) such that \( \{A(\sigma)\}_{\sigma \geq 0} \in C^0([0, \infty), \Psi_{b}^{m, l_0}) \).

**Proof.** Using the continuity of \( \text{Op} : S_b^{m, l_0}(X) \to \Psi_{b}^{m, l_0}(X) \), the first statement follows from Proposition 2.2. The second statement follows from the first in conjunction with Proposition 2.3.

Since \( \text{Op} \) is linear, \( \Psi_{b, \text{leC}}^{m, l, \ell}(X) \) is a vector space, and it inherits a Fréchet space structure from \( S_{b, \text{leC}}^{m, l, \ell}(X) \) and \( \Psi_{b, \text{leC}}^{-\infty, l, \ell}(X) \), more specifically the quotient topology associated to the definitional surjection

\[
S_{b, \text{leC}}^{m, l, \ell}(X) \times \Psi_{b, \text{leC}}^{-\infty, l, \ell}(X) \to \Psi_{b, \text{leC}}^{m, l, \ell}(X).
\]

From the definition of \( \Psi_{b, \text{sp,res}}^{m, l, \ell/2}(X) \) given in [Vas21c, §3], \( \Psi_{b, \text{leC}}^{m, l, \ell}(X) \subset \Psi_{b, \text{sp,res}}^{m, l, l/2}(X) \). Just as the set of classical Kohn-Nirenberg \( \Psi \)DOs is a subalgebra of the calculus of all Kohn-Nirenberg \( \Psi \)DOs,

\[
\Psi_{b, \text{leC}}(X) = \bigcup_{m, l, \ell \in \mathbb{R}} \Psi_{b, \text{leC}}^{m, l, \ell}(X)
\]

is a subalgebra of \( \Psi_{b, \text{sp,res}}(X) \), with composition of \( \Psi \)DOs defining jointly continuous products

\[
\Psi_{b, \text{leC}}^{m, l, \ell}(X) \times \Psi_{b, \text{leC}}^{m', l', \ell'}(X) \to \Psi_{b, \text{leC}}^{m+m', l+l', \ell+\ell'}(X)
\]
for all $m, l, \ell, m', l', \ell' \in \mathbb{R}$. The key observation here, in addition to the continuity of $\Psi_{b, \text{leC}}^{m,l,\ell}(X) \times \Psi_{b, \text{leC}}^{-m,l,\ell'}(X) \to \Psi_{b, \text{leC}}^{-m',l,\ell'}(X)$ for all $m, l, \ell, m', l', \ell' \in \mathbb{R}$, is that the reduction formula for full symbols in local coordinates respects classicality at $z f_{00}$.

From $\sigma_{b, \text{sp}, \text{res}}$, we get a set $\{\sigma_{b, \text{sp}, \text{res}}^{m,l,\ell}\}_{m,l,\ell \in \mathbb{R}}$ of maps $\sigma_{b, \text{leC}}^{m,l,\ell} : \Psi_{b, \text{leC}}^{m,l,\ell}(X) \to S_{b, \text{leC}}^{m-1,l,\ell}(X)$ such that

$$0 \to \Psi_{b, \text{leC}}^{m-1,l,\ell}(X) \hookrightarrow \Psi_{b, \text{leC}}^{m,l,\ell}(X) \twoheadrightarrow S_{b, \text{leC}}^{m,l,\ell}(X) \to 0$$

is a short exact sequence and such that

$$\sigma_{b, \text{leC}}^{m,l,\ell}(A) \sigma_{b, \text{sp}, \text{res}}^{m',l',\ell'}(B) = \sigma_{b, \text{leC}}^{m+m',l+l',\ell+\ell'}(AB)$$

for all $A \in \Psi_{b, \text{leC}}^{m,l,\ell}, B \in \Psi_{b, \text{leC}}^{m',l',\ell'}$, with each of eq. (101), eq. (102), eq. (103) following from each of eq. (86), eq. (87), eq. (88) respectively.

Let $\mathcal{A}_{0,0}^{0,0}(X_{\text{sp}})$ denote the set of distributions on $X_{\text{sp}}$ which are conormal to all boundaries and smooth at $z f$ (in particular smooth everywhere except possibly at $t f, b f$).

**Proposition 2.5.** For any $f \in \mathcal{A}_{0,0}^{0,0}(X_{\text{sp}})$ and $l, \ell \in \mathbb{R}$, the multiplication operator given by multiplication by $x^{-l}(\sigma^2 + Z x)^{l-\ell/2} f(x; \sigma)$ defines an element of $\Psi_{b, \text{leC}}^{0,0}(X)$.

**Proof.** The given multiplication operator $M = \{M(\sigma)\}_{\sigma > 0}$ is given by $M(\sigma) = \text{Op}(x^{-l}(\sigma^2 + Z x)^{l-\ell/2} f(x; \sigma))$ (using eq. (93) for each individual $\sigma > 0$), so the proposition follows from $f \in S_{b, \text{leC}}^{0,0}(X)$. \hfill \Box

### 2.2.3. $\Psi_{\text{leC}}(X)$

In order to define the full leC-calculus, we will use the following properties of Op:

- $\text{Op}(a) \in \Psi_{b, \text{leC}}^{m,l,\ell}(X)$ (if and) only if $a \in S_{b, \text{leC}}^{m,l,\ell}(X)$.

- Whenever $\text{Op}(a) \in \Psi_{b, \text{leC}}^{m,l,\ell}(X)$,

- there exists a function $\sharp : S_{b, \text{leC}}^{m,l,\ell}(X)^2 \to S_{b, \text{leC}}^{m,l,\ell}(X)$ (given by the “reduction formula” for $\Psi_{\infty}$, related to $\Psi_b$ via [Vas18, §6]) such that, for any $m, m', s, s', \varsigma, \varsigma', l, l', \ell, \ell' \in \mathbb{R}$, $a \in S_{\text{leC}}^{m,\varsigma,l,\ell}(X)$ and $b \in S_{\text{leC}}^{m',\varsigma',l',\ell'}(X)$,

$$a \sharp b \in S_{\text{leC}}^{m+m'+s+s',\varsigma+\varsigma',l+l',\ell+\ell'}(X),$$

$$\text{esssupp}_{\text{leC}}(a \sharp b) \subseteq \text{esssupp}_{\text{leC}}(a) \cap \text{esssupp}_{\text{leC}}(b),$$

$$\text{Op}(a) \text{Op}(b) = \text{Op}(a \sharp b) + E,$$

for some $E \in \Psi_{b, \text{leC}}^{-m,l,\ell}(X)$ which depends continuously on $a, b$. Moreover,

$$a \sharp b = a b \mod S_{\text{leC}}^{m+m'-1,s+s'-1,\varsigma+\varsigma'-1,l+l',\ell+\ell'}(X),$$

$$a \sharp b - b \sharp a = i\{a, b\} \mod S_{\text{leC}}^{m+m'-2,s+s'-2,\varsigma+\varsigma'-2,l+l',\ell+\ell'}(X)$$

for all such $a, b$.

- there exists another continuous ($\mathbb{C}$-antilinear) function $\flat : S_{b, \text{leC}}^{m,l,\ell}(X) \to S_{b, \text{leC}}^{m,l,\ell}(X)$ (which can also be written in local coordinates in terms of the reduction formula) such that for all $a \in S_{\text{leC}}^{m,\varsigma,l,\ell}(X)$,

$$\text{Op}(a) = \text{Op}(a)^* + E$$
At the level of vector spaces, this is just
\[ ba = a^* \mod S_{b,\leq C}^{m-1,s-1,\varsigma-1,l,\ell}(X) \] (111)
\[ \text{esssupp}_{\leq C}(ba) = \text{esssupp}_{\leq C}(a). \] (112)
Here, for \( s \in S_{b,\leq C}(X) \), \( \text{esssupp}_{\leq C}(s) \) consists of those points in \( df \cup sf \cup tf \) failing to possess a neighborhood in which \( s \) vanishes to infinite order at the boundary of the \( \leq C \)-phase space.

As in [Vas21a][Vas21c], these properties follow from the relation between \( \Psi_{b}(X) \) and \( \Psi_{\infty}(\mathbb{R}^n) \), as explained in [Vas18, §6], and the basic properties of \( \Psi_{\infty}(\mathbb{R}^n) \) (in particular the reduction formula), for which the standard reference is [Hör07]. It is crucial for us that \( \sharp \) satisfies the equations eq. (106), eq. (108), eq. (109) above and not just the weaker \( \leq C \)-analogues. This fundamental fact can be read off of the reduction formula for full symbols in local coordinates, in terms of which \( \sharp \) can be written.

We can now define, for each \( m, s, \varsigma, l, \ell \in \mathbb{R}, \)
\[ \Psi_{\leq C}^{m,s,\varsigma,l,\ell}(X) = \text{Op}(S_{\leq C}^{m,s,\varsigma,l,\ell}(X)) + \Psi_{\leq C}^{-\infty,l,\ell} \subseteq \Psi_{\leq C}^{-\infty,l,\ell}. \] (113)
Evidently, eq. (113) endows \( \Psi_{\leq C}^{m,s,\varsigma,l,\ell}(X) \) with a topology, so that it becomes a Fréchet space. Consider the graded vector space
\[ \Psi_{\leq C}(X) = \bigcup_{m, s, \varsigma, l, \ell \in \mathbb{R}} \Psi_{\leq C}^{m,s,\varsigma,l,\ell}(X). \] (114)
At the level of vector spaces, this is just \( \Psi_{b,\leq C}(X) \).

Moreover, since \( S_{\leq C}^{m,m+l,m+l,\ell,\ell}(X) = S_{\leq C}^{m,l,\ell}(X) \) at the level of sets, \( \Psi_{\leq C}^{m,m+l,m+l,\ell,\ell}(X) = \Psi_{\leq C}^{m,l,\ell}(X) \) for all \( m, l, \ell \in \mathbb{R}. \)

**Proposition 2.6.** \( \Psi_{\leq C}(X) \) is a multigraded \( \mathbb{C} \)-algebra: for any \( m, m', s, s', \varsigma, \varsigma', l, l', \ell, \ell' \in \mathbb{R}, \)
\( A \in \Psi_{\leq C}^{m,s,\varsigma,l,\ell}(X) \) and \( B \in \Psi_{\leq C}^{m',s',\varsigma',l',\ell'}(X), \)
\[ AB \in \Psi_{\leq C}^{m+m',s+s',\varsigma+\varsigma',l+l',\ell+\ell'}(X). \] (115)

**Proof.** We can write \( A = \text{Op}(a) + E \) and \( B = \text{Op}(b) + F \) for \( a \in S_{\leq C}^{m,s,\varsigma,l,\ell}, b \in S_{\leq C}^{m',s',\varsigma',l',\ell'}, E \in \Psi_{\leq C}^{-\infty,l,\ell}, F \in \Psi_{\leq C}^{-\infty,l',\ell'}. \) Thus,
\[ AB = \text{Op}(a) \text{Op}(b) + E \text{Op}(b) + \text{Op}(a)F + EF \] (116)
\[ = \text{Op}(a\sharp b) + E \text{Op}(b) + \text{Op}(a)F + EF + G \] (117)
for some \( G \in \Psi_{\leq C}^{-\infty,l'+l',\ell+\ell'}. \) Since \( a \in S_{b,\leq C}^{M,l,\ell}, b \in S_{b,\leq C}^{M',l',\ell'} \) for \( M = \max\{m, s - l, \varsigma - \ell\} \) and \( M' = \max\{m', s' - l', \varsigma' - \ell'\}, \)
\[ \text{Op}(a) \in \Psi_{b,\leq C}^{M,l,\ell}(X), \quad \text{Op}(b) \in \Psi_{b,\leq C}^{M',l',\ell'}(X), \] (118)
which implies that \( E \text{Op}(b), \text{Op}(a)F \in \Psi_{b,\leq C}^{-\infty,l'+l',\ell+\ell'}, \) and likewise \( EF \in \Psi_{b,\leq C}^{-\infty,l+l',\ell+\ell'}. \)

Since \( a\sharp b \in S_{\leq C}^{m+m',s+s',\varsigma+\varsigma',l+l',\ell+\ell'} \), we deduce that eq. (115) holds. \( \square \)

In fact (as can be proven using finite order truncations of the reduction formula), operator composition defines a jointly continuous map
\[ \Psi_{\leq C}^{m,s,\varsigma,l,\ell}(X) \times \Psi_{\leq C}^{m',s',\varsigma',l',\ell'}(X) \to \Psi_{\leq C}^{m+m',s+s',\varsigma+\varsigma',l+l',\ell+\ell'}(X), \] (119)
so we can say that the leC-calculus is a multigraded Fréchet algebra.

**Lemma 2.7.** Suppose that \( A \in \Psi_{\text{leC}}^{m,s,\varsigma,l,\ell}(X) \) can be written either as \( A = \text{Op}(a_1) + E_1 \) or \( A = \text{Op}(a_2) + E_2 \) for some symbols \( a_1, a_2 \in S_{\text{leC}}^{m,s,\varsigma,l,\ell}(X) \) and \( E_1, E_2 \in \Psi_{\text{b-leC}}^{-\infty,\ell,\ell} \).

Then \( a_1 - a_2 \in S_{\text{b-leC}}^{-\infty,\ell,\ell}(X) \).

**Proof.** By the linearity of \( \text{Op} \), \( \text{Op}(a_1 - a_2) \in \Psi_{\text{b-leC}}^{-\infty,\ell,\ell}(X) \). Then, using the \( \sigma_{\text{b-leC}} \)-short exact sequence and the property eq. (104) of \( \text{Op} \) for all \( N \in \mathbb{N} \),

\[
0 = \sigma_{\text{b-leC}}^{-\infty,\ell,\ell}(\text{Op}(a_1 - a_2)) = a_1 - a_2 \mod S_{\text{b-leC}}^{-N,\ell,\ell}(X),
\]

which means that \( a_1 - a_2 \in S_{\text{b-leC}}^{-N,\ell,\ell}(X) \).

Thus, for any \( A \in \Psi_{\text{leC}}^{m,s,\varsigma,l,\ell}(X) \), we get notions

\[
\text{Ell}_{\text{leC}}^{m,s,\varsigma,l,\ell}(A) = \text{Ell}_{\text{leC}}^{m,s,\varsigma,l,\ell}(a) \subset \text{df} \cup \text{sf} \cup \text{ff},
\]

\[
\text{Char}_{\text{leC}}^{m,s,\varsigma,l,\ell}(A) = \text{Char}_{\text{leC}}^{m,s,\varsigma,l,\ell}(a) = \text{df} \cup \text{sf} \cup \text{ff} \setminus \text{Ell}_{\text{leC}}^{m,s,\varsigma,l,\ell}(a),
\]

\[
\text{WF}_{\text{leC}}^{m,s,\varsigma,l,\ell}(A) = \text{esssupp}_{\text{leC}}(a),
\]

for any \( a \in S_{\text{leC}}^{m,s,\varsigma,l,\ell} \) with \( A = \text{Op}(a) + E \) for \( E \in \Psi_{\text{b-leC}}^{-\infty,\ell,\ell}(X) \). All three are subsets of \( \text{df} \cup \text{sf} \cup \text{ff} \).

Another useful notion, which we will only apply to \( A \in \bigcap_{t \in \mathbb{R}} \Psi_{\text{leC}}^{m,s,\varsigma,-\infty,-\infty}(X) = \Psi_{\text{leC}}^{m,s,\varsigma,-\infty,-\infty}(X) \) is

\[
\text{WF}_{\text{leC}}^{m,s,\varsigma,l,\ell}(A) = \bigcap_{t \in \mathbb{R}} \text{WF}_{\text{leC}}^{m,s,\varsigma,l,\ell}(A).
\]

Observe that if \( a \in S_{\text{leC}}^{m,s,\varsigma,-\infty,-\infty}(X) \), then \( \text{Op}(A) \in \Psi_{\text{leC}}^{m,s,\varsigma,-\infty,-\infty}(X) \) (tautologically) and \( \text{WF}_{\text{leC}}^{m,s,\varsigma,l,\ell}(A) = \text{esssupp}_{\text{leC}}(a) \), which is disjoint from \( \text{bf} \cup \text{tf} \).

In addition, we get the leC- principal symbol maps

\[
\{\sigma_{\text{leC}}^{m,s,\varsigma,l,\ell}\}_{m,s,\varsigma,l,\ell \in \mathbb{R}}, \quad \sigma_{\text{leC}}^{m,s,\varsigma,l,\ell} : \Psi_{\text{leC}}^{m,s,\varsigma,l,\ell}(X) \to S_{\text{leC}}^{m,s,\varsigma,l,\ell}(X)/S_{\text{leC}}^{m-1,s-1,\varsigma-1,l,\ell}(X),
\]

\[
\sigma_{\text{leC}}^{m,s,\varsigma,l,\ell}(A) = a \mod S_{\text{leC}}^{m-1,s-1,\varsigma-1,l,\ell} \quad \text{for any } a \in S_{\text{leC}}^{m,s,\varsigma,l,\ell} \text{ with } A = \text{Op}(a) + E \text{ for } E \in \Psi_{\text{b-leC}}^{-\infty,\ell,\ell}.
\]

Unlike \( \text{Op} \), which is not canonical and depends on a particular choice of local coordinate charts, these notions are all canonical.

**Proposition 2.8.** For every \( m,s,\varsigma,l,\ell \in \mathbb{R} \), we have a short exact sequence

\[
0 \to \Psi_{\text{leC}}^{m-1,s-1,\varsigma-1,l,\ell}(X) \to \Psi_{\text{leC}}^{m,s,\varsigma,l,\ell}(X) \to S_{\text{leC}}^{m,s,\varsigma,l,\ell}(X)/S_{\text{leC}}^{m-1,s-1,\varsigma-1,l,\ell}(X) \to 0,
\]

where the second-to-last map is \( \sigma_{\text{leC}}^{m,s,\varsigma,l,\ell} \).

**Proof.** The surjectivity of \( \sigma_{\text{leC}}^{m,s,\varsigma,l,\ell} \) follows from the properties of \( \text{Op} \) listed above.

If, on the other hand, \( A \in \Psi_{\text{leC}}^{m,s,\varsigma,l,\ell}(X) \) satisfies \( \sigma_{\text{leC}}^{m,s,\varsigma,l,\ell}(A) = 0 \), then \( A = \text{Op}(a) + E \) for \( a \in S_{\text{leC}}^{m-1,s-1,\varsigma-1,l,\ell} \) and \( E \in \Psi_{\text{b-leC}}^{-\infty,\ell,\ell} \). Then, by the definition of \( S_{\text{leC}}^{m-1,s-1,\varsigma-1,l,\ell}(X) \), \( A \in \Psi_{\text{leC}}^{m-1,s-1,\varsigma-1,l,\ell}(X) \).

**Proposition 2.9.** For every \( m,s,\varsigma,l,\ell, m', s', \varsigma', l', \ell' \in \mathbb{R} \) and pair of \( A \in \Psi_{\text{leC}}^{m,s,\varsigma,l,\ell}(X) \) and \( B \in \Psi_{\text{leC}}^{m',s',\varsigma',l',\ell'}(X) \),

\[
\sigma_{\text{leC}}^{m,s,\varsigma,l,\ell}(A)\sigma_{\text{leC}}^{m',s',\varsigma',l',\ell'}(B) = \sigma_{\text{leC}}^{m+m',s+s',\varsigma+\varsigma',l+l',\ell+\ell'}(AB).
\]

Moreover, \( \sigma_{\text{leC}}^{m+m'-1,s+s'-1,\varsigma+\varsigma'-1,l+l'+\ell+\ell'}([A,B]) \) is equal to

\[
i\{a,b\} \mod S_{\text{leC}}^{m+m'-2,s+s'-2,\varsigma+\varsigma'-2,l+l'+\ell+\ell'}(X),
\]

where \( a \) and \( b \) are symbols.
where \( a, b \) are any representatives of \( \sigma_{a,b}^{m,s,l,t} (A) \) and \( \sigma_{a,b}^{m',s',l',t'} (B) \).

**Proof.** Write \( A = \text{Op}(a) + E \) and \( B = \text{Op}(b) + F \) for \( a \in S_{m,s,l,t}^{m',s',l',t'} (X) \), \( b \in S_{m,s,l,t}^{m',s',l',t'} (X) \), \( E \in \Psi_{c}^{-\infty,l,t} (X) \), \( F \in \Psi_{c}^{-\infty,l',t'} (X) \). Then

\[
\sigma_{a,b}^{m+m',s+s',l+l',t+t'} (AB) = \sigma_{a,b}^{m+m',s+s',l+l',t+t'} (\text{Op}(a) \text{Op}(b)) = \sigma_{a,b}^{m+m'-1,s+s'-1,l+l',t+t'} (\text{Op}(a \text{Op}(b))) = a^g b \text{ mod } \sigma_{a,b}^{m+m'-1,s+s'-1,l+l',t+t'} (X) = ab \text{ mod } S_{m,s,l,t}^{s+s',l+l',t+t'} (X)
\]

The proof of eq. (127) is similar. \( \square \)

**Proposition 2.10.** Suppose that \( L \in S \text{Diff}_{\text{scb}}^{m,l}(X) \), \( m \in \mathbb{N}, s, l, \in \mathbb{R} \). Then, the constant family \( \{ L(x) \}_{x > 0}, L(x) = L \), (which we conflate with \( L \)) defines an element of \( \Psi_{c}^{m,s,l,t} (X) \).

Given any \( f \in A^{0,0,0,0}(X_{\text{res}}) \) and \( l_0, \ell_0 \in \mathbb{R} \), \( x/(\sigma^2 + Zx)^{l_0} (\sigma^2 + Zx)^{\ell_0}/2 fL \) defines an element of \( \Psi_{c}^{m,s-l_0,\ell_0,\ell_0,2-\ell_0} (X) \). \( \square \)

**Proof.** First consider the case when \( s = m + l \), so that \( L \in S \text{Diff}_{\text{scb}}^{m,l}(X) \). Then,

\[
L \in \Psi_{c}^{b,m,l,2l}(X) = \Psi_{c}^{m,m+l,m+2l,2l}(X) = \Psi_{c}^{m,s,l,t}(X).
\]

To handle the general case, we note that we may write any \( L \in S \text{Diff}_{\text{scb}}^{m,l}(X) \) – where now \( m, s, l \in \mathbb{R} \) are arbitrary – as

\[
L = \sum_{j \geq 0} L_j
\]

for \( L_j \in S \text{Diff}_{\text{d}}^{j,\min \{s-j,l\}}(X) \subset S \text{Diff}_{\text{scb}}^{j,\min \{s-j,l\}}(X) \subset S \text{Diff}_{\text{scb}}^{m,l}(X) \). (Indeed, it suffices to construct such a decomposition on \( 0, x \times \partial X \), where \( L_j \) can be written as a linear combination of elements of \( S(X) \partial_{\mu}^j \text{Diff}^j(\partial X) \) for \( j_1 + j_2 = j, j_1, j_2 \in \mathbb{N} \).) Since

\[
L_j \in \Psi_{c}^{j,\min \{s-j,l\},2 \min \{s-j,l\}}(X) = \Psi_{c}^{j,\min \{s+j,l\},2 \min \{s+j,l\}}(X)
\]

we deduce that

\[
L \in \Psi_{c}^{m,\min \{s,l+m\},\min \{s,l\},2 \min \{s,l\}}(X) \subset \Psi_{c}^{m,s,l,t}(X).
\]

From Proposition 2.5, for any \( f \in A^{0,0,0,0}(X_{\text{res}})(X_{\text{res}}) \) and \( l_0, \ell_0 \in \mathbb{R} \), the multiplication operator \( x^{l_0}(\sigma^2 + Zx)^{-l_0+\ell_0/2} f(x; \sigma) \) defines an element of \( \Psi_{c}^{0,-l_0,-\ell_0,0}(X) = \Psi_{c}^{0,-l_0,-\ell_0,0}(X) \). The second statement of the proposition therefore follows from the first via an application of Proposition 2.6. \( \square \)

For each \( m, s, \varsigma, l, \ell \in \mathbb{R} \), we let \( S \text{Diff}_{\text{scb}}^{m,s,\varsigma,l,\ell}(X) \) denote the set of elements of \( \Psi_{c}^{m,s,\varsigma,l,\ell}(X) \) which are families of differential operators (all of which arise from the construction in Proposition 2.10). Likewise, we let

\[
S \text{Diff}_{\text{leC}}^{m,s,\varsigma,l,\ell}(X) \subset S \text{Diff}_{\text{leC}}^{m,s,\varsigma,l,\ell}(X)
\]

denote the subset of families of differential operators which can be written as a linear combination of elements of \( \text{Diff}_{\text{leC}}(X) \) times classical symbols on \( X_{\text{res}}^{0} \).
The elliptic parametrix construction, applied to the leC-calculus, yields the following: for any \( m, s, \varsigma, l, \ell \in \mathbb{R} \) and (totally) elliptic \( A \in \Psi^{m,s,\varsigma,l,\ell}_{\text{leC}}(X) \) — that is, \( A \) with
\[
\text{Ell}^{m,s,\varsigma,l,\ell}_{\text{leC}}(A) = \text{df} \cup \text{sf} \cup \text{tf}
\] (134)
— there exists, for each \( N \in \mathbb{N} \), some \( B \in \Psi^{-m,-s,-\varsigma,-l,-\ell}_{\text{leC}}(X) \) such that \( AB - 1, BA - 1 \in \Psi^{-N,0,0}_{\text{leC}}(X) \).

To elaborate, the leC-symbol calculus analogue of the construction of left and right parametrices via Neumann series yields \( B_L, B_R \) such that \( AB_R - 1, B_L A - 1 \in \Psi^{-N,0,0}_{\text{leC}}(X) \). Then, setting \( E_L = B_L A - 1 \) and \( E_R = AB_R - 1 \),
\[
B_L = B_L(AB_R - E_R) = (E_L + 1)B_R - B_L E_R = B_R + (E_L B_R - B_L E_R),
\] (135)
so taking either \( B = B_L \) or \( B = B_R \), both of \( AB - 1, BA - 1 \in \Psi^{-N,0,0}_{\text{leC}}(X) \) hold. Hence we do not need to distinguish left vs. right parametrices.

**Lemma 2.11.** Given any \( m, s, \varsigma, l, \ell \in \mathbb{R} \), \( A \in \Psi^{m,s,\varsigma,l,\ell}_{\text{leC}}(X) \) and \( \alpha \in \text{df} \cup \text{sf} \cup \text{ff} \), \( \alpha \notin \text{WF}^{d,l}_{\text{leC}} \) only if there exists some \( B \in \Psi^{0,0,0,0}_{\text{leC}}(X) \) that is elliptic at \( \alpha \) and satisfies \( AB, BA \in \Psi^{-\infty,-\infty,-\infty,l,\ell}_{\text{leC}}(X) \).

The same statement applies if we replace \( \alpha \) with the intersection of a finite union of closed balls with \( \text{df} \cup \text{sf} \cup \text{ff} \).

**Proof.** We write \( A = \text{Op}(a) \) for \( a \in \mathcal{S}^{\infty,\infty,\infty,l,\ell}_{\text{leC}}(X) \), so that \( \text{WF}^{d,l}_{\text{leC}}(A) = \text{esssupp}_{\text{leC}}(a) \).

Given \( B \in \Psi^{0,0,0,0}_{\text{leC}}(X) \), \( B = \text{Op}(b), b \in \mathcal{S}^{0,0,0,0}_{\text{leC}}(X) \), define \( E \in \Psi^{-\infty,-\infty,-\infty,l,\ell}_{\text{leC}}(X) \) by
\[
AB = \text{Op}(a \check{\ast} b) + E.
\] (136)
If \( \alpha \notin \text{esssupp}_{\text{leC}}(a) \), then (since essential supports are closed) we can choose \( b \in \mathcal{S}^{0,0,0,0}_{\text{leC}}(X) \) that is identically equal to one in a neighborhood of \( \alpha \) but supported away from \( \text{esssupp}_{\text{leC}}(a) \), so that
\[
\text{esssupp}_{\text{leC}}(a) \cap \text{esssupp}_{\text{leC}}(b) = \emptyset.
\]
Then, by eq. (106), \( a \check{\ast} b \) has empty essential support, which implies that \( AB \in \Psi^{-\infty,-\infty,-\infty,l,\ell}_{\text{leC}}(X) \).

That handles \( AB \), and \( BA \) is analogous. If we replace \( \alpha \) in the previous argument with the intersection of a finite union of closed balls with \( \text{df} \cup \text{sf} \cup \text{ff} \), the argument goes through the same. \( \square \)

We briefly discuss uniform families of leC-\( \Psi \)DOs (i.e. one-parameter families of b-\( \Psi \)DOs which are uniformly bounded in a sense appropriate for the leC-calculus). For each \( m, s, \varsigma, l, \ell \in \mathbb{R} \cup \{-\infty\} \),
\[
\Psi^{m,s,\varsigma,l,\ell}_{\text{leC}}(X) = \bigcap_{m', m'' \geq m, \ldots, \ell' \geq \ell} \Psi^{m',s',\varsigma',l',\ell'}_{\text{leC}}(X)
\] (137)
is a Fréchet space, so for any nonempty interval \( I \subset \mathbb{R} \), we have a Fréchet space \( L^\infty(I; \Psi^{m,s,\varsigma,l,\ell}_{\text{leC}}(X)) \) whose elements are a.e. equivalence classes of measurable functions \( I \to \Psi^{m,s,\varsigma,l,\ell}_{\text{leC}}(X) \) which are uniformly bounded with respect to each of our countably many Fréchet seminorms on the codomain. For \( I \) closed, we can safely confine an element of \( \mathcal{A}^0(I; \Psi^{m,s,\varsigma,l,\ell}_{\text{leC}}(X)) \), which we can consider as a smooth family \( \{A_t\}_{t \in I} \), with the corresponding element of \( L^\infty(I; \Psi^{m,s,\varsigma,l,\ell}_{\text{leC}}(X)) \).

For \( A = \{A_t\}_{t \in I} \subset L^\infty(I; \Psi^{m,s,\varsigma,l,\ell}_{\text{leC}}(X)) \), we define subsets \( \text{WF}^{d,l}_{L^\infty,\text{leC}}(A) \subset \text{df} \cup \text{sf} \cup \text{ff} \) by stipulating that a point \( \alpha \in \text{df} \cup \text{sf} \cup \text{ff} \) does not lie in
- \( \text{WF}^{d,l}_{L^\infty,\text{leC}}(A) \) if and only if there exists some \( B \in \Psi^{0,0,0,0}_{\text{leC}}(X) \) that is elliptic at \( \alpha \) and satisfies \( \{BA_t\}_{t \in I} \subset L^\infty(I; \Psi^{-\infty,-\infty,-\infty,-\infty,-\infty,\ell}_{\text{leC}}(X)) \),
- \( \text{WF}^{d,l}_{L^\infty,\text{leC}}(A) \) if and only if there exists some \( B \in \Psi^{0,0,0,0}_{\text{leC}}(X) \) that is elliptic at \( \alpha \) and satisfies \( \{BA_t\}_{t \in I} \subset L^\infty(I; \Psi^{-\infty,-\infty,-\infty,-\infty,\ell}_{\text{leC}}(X)) \).
A uniform version of the elliptic parametrix construction goes through. We likewise have a notion of \( \text{esssupp}_{L^\infty,\text{leC}}(a) \) for \( a \in L^\infty(I; S_m^{s,\varsigma,l,\ell}(X)) \). One definition is that \( a \in \text{df} \cup \text{sf} \cup \text{ff} \) is not in \( \text{esssupp}_{L^\infty,\text{leC}}(a) \) if and only if there exists some \( b \in S_m^{0,0,0,0}(X) \) that is elliptic at \( a \) and satisfies \( ab \in S_m^{\infty,-\infty,-\infty,-\infty}(X) \). Quantizing: for any interval \( I \subset \mathbb{R} \) and any \( m, s, \varsigma, l, \ell \in \mathbb{R} \), given \( a \in L^\infty(I; S_m^{s,\varsigma,l,\ell}(X)) \), letting \( A = \{ \text{Op}(A_t) \}_{t \in I} \),

\[
\text{WF}^{\ell,\ell}_{L^\infty,\text{leC}}(A) = \text{esssupp}_{L^\infty,\text{leC}}(a).
\] (138)

2.3. Sobolev Spaces. For each \( m, s, l \in \mathbb{R} \), let \( H_m^{s,l}(X) \) denote the Sobolev space of differential order \( m \), sc-decay order \( s \), and b-decay order \( l \) associated to the scb-calculus in [Vas21a]. Associated to the leC-calculus is a five-parameter family

\[
\{ H_{m,s,\varsigma,l,\ell}^{s,l}(X) \}_{m,s,\varsigma,l,\ell \in \mathbb{R}}
\] of families

\[
H_{m,s,\varsigma,l,\ell}^{s,l}(X) = \{ H_{m,s,\varsigma,l,\ell}^{s,l}(X)(\sigma) \}_{\sigma \geq 0}
\] (139)
of “leC-based Sobolev spaces,” where

- for each \( \sigma > 0 \), \( H_{m,s,\varsigma,l,\ell}^{s,l}(X)(\sigma) \) is a Hilbertizable Banach space equal to \( H_{scb}^{m,s,l}(X) \subset S'(X) \) at the level of TVSs (i.e. equivalent at the level of Banach spaces),
- \( H_{m,s,\varsigma,l,\ell}^{m,s,l}(X)(0) = H_{scb}^{m,s,l+n/2,l+n/2}(X_{1/2}) \) (at the level of TVSs).

The family \( H_{m,s,\varsigma,l,\ell}^{m,s,l}(X) \) had ought to be thought of as interpolating between the Sobolev spaces \( H_{scb}^{m,s,l}(X) \) and \( H_{scb}^{m,s,l+n/2,l+n/2}(X_{1/2}) \) as \( \sigma \to 0^+ \).

Besides using the leC-Sobolev spaces to relate \( H_{scb}^{m,s,l}(X) \) and \( H_{scb}^{m,s,l}(X_{1/2}) \) we can, more crudely, observe that

\[
H_{m,s,\varsigma,l,\ell}^{m,s,l}(X) = H_b^{m,l}(X) = H_{scb}^{m,s,l+n/2,l+n/2}(X_{1/2}) = H_{scb}^{m,m+l,n/2,2l+n/2}(X).
\] (140)

The spaces \( H_{scb}^{m,s,l}(X) \) are more refined than \( H_b^{m,s,l}(X) \) in that the decay rate of terms like \( \exp(i/x) \) need not be treated the same as the decay rate of the constant function as measured by the former. Similarly, the spaces \( H_{scb}^{m,s,l}(X_{1/2}) \) are more refined than \( H_b^{m,s,l}(X_{1/2}) \) in that the decay rate of terms like \( \exp(i/x^{1/2}) \) need not be the same as the decay rate of the constant function as measured by the former. As shown in [Vas21a], the scb-Sobolev spaces are well-suited for formulating a conjugated version of the Sommerfeld radiation condition, and for similar reasons the leC-Sobolev spaces are well-suited to the study of attractive Coulomb-like Schrödinger operators down to zero energy.

In order to define the \( \sigma \)-dependent norm on \( H_{m,s,\varsigma,l,\ell}^{m,s,l}(X)(\sigma) \), we first note:

**Lemma 2.12.** For any two elliptic \( A, B \in \Psi_{\text{leC}}^{m,s,l,\ell} \) and \( N, M \in \mathbb{Z} \) with \( N, M \geq -\min\{ m, s - l, \varsigma - \ell \} \), and for any \( \Sigma > 0 \), there exist \( c, C > 0 \) such that

\[
c(\| B(\sigma)u \|_{L^2}(X) + \| u \|_{H^{-M,l,\ell}(X)}(\sigma)) \leq \| A(\sigma)u \|_{L^2} + \| u \|_{H^{-M,l,\ell}(X)}(\sigma) \leq C(\| B(\sigma)u \|_{L^2} + \| u \|_{H^{-M,l,\ell}(X)}(\sigma))
\] holds for all \( u \in S'(X) \) and \( \sigma \in [0, \Sigma] \).

**Proof.** It suffices to prove \( \| A(\sigma)u \|_{L^2} + \| u \|_{H^{-M,l,\ell}(X)}(\sigma) \leq C(\| B(\sigma)u \|_{L^2} + \| u \|_{H^{-M,l,\ell}(X)}(\sigma)) \), the other inequality following by symmetry.
By the elliptic parametrix construction, for arbitrary $N_0 \in \mathbb{N}$, we can find $\Lambda \in \Psi_{leC}^{-m_s, r, l, -\ell}$ and $R \in \Psi_{leC}^{-N_0, 0, 0}$ with $\Lambda B = 1 + R$. Now, setting $m_0 = \min\{m, s - l, \varsigma - \ell\}$,

$$
\|u\|_{H^{-N, l, \ell}_{B, leC}(\sigma)} = \|\Lambda(\sigma)B(\sigma) - R(\sigma)u\|_{H^{-N, l, \ell}_{B, leC}(\sigma)} \\
\leq \|\Lambda(\sigma)B(\sigma)u\|_{H^{-N, l, \ell}_{B, leC}(\sigma)} + \|R(\sigma)u\|_{H^{-N, l, \ell}_{B, leC}(\sigma)} \\
\leq \|\Lambda(\sigma)B(\sigma)u\|_{H^{-N, l, \ell}_{B, leC}(\sigma)} + \|u\|_{H^{-M, l, \ell}_{B, leC}(\sigma)} \\
\leq \|B(\sigma)u\|_{H^{-N, m_0, 0, 0}_{B, leC}(\sigma)} + \|u\|_{H^{-M, l, \ell}_{B, leC}(\sigma)} \leq \|B(\sigma)u\|_{L^2} + \|u\|_{H^{-M, l, \ell}_{B, leC}(\sigma)}
$$

(142)

for sufficiently large $N_0$, where we used that $\Lambda \in \Psi_{B, leC}^{-m_0, -l, -\ell}$.

Similarly, for sufficiently large $N_0$,

$$
\|A(\sigma)u\|_{L^2} = \|A(\sigma)(\Lambda(\sigma)B(\sigma) - R(\sigma))u\|_{L^2} \leq \|A(\sigma)\Lambda(\sigma)B(\sigma)u\|_{L^2} + \|A(\sigma)R(\sigma)u\|_{L^2} \\
\leq \|A(\sigma)\Lambda(\sigma)B(\sigma)u\|_{L^2} + \|u\|_{H^{-M, l, \ell}_{B, leC}(\sigma)} \\
\leq \|\Lambda(\sigma)B(\sigma)u\|_{L^2} + \|u\|_{H^{-M, l, \ell}_{B, leC}(\sigma)} \tag{143}
$$

Combining eq. (142) with eq. (143) yields the desired inequality. \qed

We can now define a Hilbertizable norm on $H^{m, s, l, \ell}_{leC}(X)(\sigma)$, for each $\sigma \geq 0$, by writing

$$
\|u\|_{H^{m, s, l, \ell}_{leC}(X)(\sigma)} = \|\Lambda(\sigma)u\|_{L^2} + \|u\|_{H^{-N, l, \ell}_{B, leC}(\sigma)} \tag{144}
$$

for $N \in \mathbb{N}$ with $N \geq -\min\{m, s - l, \varsigma - \ell\}$ and arbitrary elliptic $\Lambda \in \Psi_{leC}^{m, s, l, \ell}(X)$ (Such $\Lambda$ can be constructed by the symbol calculus). The previous lemma suffices to guarantee that the estimates we prove do not depend on the particular choice of $\Lambda$ and $N$ used in defining the norm eq. (144) except with regards to the particular constants involved (which we do not keep track of anyways). However, it will be convenient to fix

$$
\Lambda_{m, s, \varsigma, l, \ell}(X) = (1/2)(\text{Op}(\varphi_{\text{diff}}^{m, s, \varsigma, l, \ell}(X) + \text{Op}(\varphi_{\text{diff}}^{-m, s, \varsigma, l, \ell}(X)), \sigma)
$$

which is certainly an elliptic element of $\Psi_{leC}^{m, s, l, \ell}(X)$. Since $\Lambda_{m, s, \varsigma, l, \ell}(X)$ is an elliptic element of $\Psi_{leC}^{m, s, l}(X)$ for $\sigma > 0$ and $\Psi_{leC}^{m, s, l}(X)(1/2)$ for $\sigma = 0$, we see that $H^{m, s, l, \ell}_{leC}(X)(\sigma)$ is indeed equivalent to

- $H^{m, s, l}_{leC}(X)$ for $\sigma > 0$ and
- $H^{m, s, l}_{leC}(X)(1/2)$ for $\sigma = 0$.

On the other hand:

**Lemma 2.13.** If $m, s, \varsigma, l, \ell, c \in \mathbb{R}$ satisfy $\varsigma = m + \ell$ and $s = m + l$, for each $\Sigma > 0$ there exist constants $c(m, s, \varsigma, l, \ell, \Sigma), C(m, s, \varsigma, l, \ell, \Sigma) > 0$ such that

$$
c\|u\|_{H^{m, s, l, \ell}_{leC}(X)} \leq \|u\|_{H^{m, s, l, \ell}_{leC}(X)} \leq C\|u\|_{H^{m, s, l, \ell}_{leC}(X)} \tag{146}
$$

for all $u \in \mathcal{Sc}(X)$ and $\sigma \in [0, \Sigma]$. In particular, if $\ell = 2l$, $c'\|u\|_{H^{m, s, l, \ell}_{leC}(X)} \leq \|u\|_{H^{m, s, l}_{leC}(X)} \leq C'\|u\|_{H^{m, s, l, \ell}_{leC}(X)}$ for some other constants $c'(m, s, \varsigma, l, \ell, \Sigma), C'(m, s, \varsigma, l, \ell, \Sigma) > 0$. \textbf{□}

**Proof.** We have that $\Lambda_{m, s, \varsigma, l, \ell}(X)$ so

$$
\|u\|_{H^{m, s, l, \ell}_{leC}(X)} = \|\Lambda_{m, s, l, \ell}(X)u\|_{L^2} + \|u\|_{H^{-N, l, \ell}_{B, leC}(\sigma)} \leq \|u\|_{H^{m, s, l, \ell}_{leC}(X)} \tag{147}
$$

for sufficiently large $N$ (by the boundedness of the elements of the resolved family $b$-calculus).

The reverse inequality follows from the ellipticity of $\Lambda_{m, s, l, \ell}$ as an element of $\Psi_{leC}^{m, s, l}(X)$. \textbf{□}
We now check the boundedness of $\text{leC-ΨDOs}$ acting on $\text{leC-Sobolev}$ spaces (from the $L^2$-boundedness of zeroth order $b$-$\text{ΨDOs}$):

**Proposition 2.14.** For any $m, \varsigma, l, \ell, m_0, s_0, q_0, l_0, \xi_0 \in \mathbb{R}$, $A \in \Psi_{\text{leC}}^{m,s,\varsigma,l,\ell}(X)$, and $\Sigma > 0$, there exists some constant $C = C(m, s, \varsigma, l, \ell, m_0, s_0, q_0, l_0, \xi_0, A, \Sigma) > 0$ such that

$$\|A(\sigma)u\|_{H_{\text{leC}}^{m_0,q_0,l_0,\xi_0}(X)(\sigma)} \leq C\|u\|_{H_{\text{leC}}^{m_0,m+s+q_0,l+l_0,\xi+\xi_0}(X)(\sigma)}$$

for all $u \in S'(X)$ and $\sigma \in [0, \Sigma]$.

**Proof.** Pick arbitrary elliptic $A_0 \in \Psi_{\text{leC}}^{m_0,s_0,q_0,l_0,\xi_0}(X)$. Then

$$\|A(\sigma)u\|_{H_{\text{leC}}^{m,q,l,\xi}(X)(\sigma)} \leq \|A_0(\sigma)u\|_{L^2} + \|A(\sigma)u\|_{H_{\text{leC}}^{-N,l,0}(X)(\sigma)}$$

(149)

and

$$\|u\|_{H_{\text{leC}}^{m,m+s+q_0,l+l_0,\xi+\xi_0}(X)(\sigma)} \geq \|A(\sigma)u\|_{L^2} + \|u\|_{H_{\text{leC}}^{-N,l,0}(X)(\sigma)}$$

(150)

for $N, M$ not too negative and arbitrary elliptic $A_1 \in \Psi_{\text{leC}}^{m,s,\varsigma,l,\ell}(X)$.

For $N$ sufficiently large compared to $M$ (dependent on $m, \varsigma, l, \ell, \xi_0$), $\|A(\sigma)u\|_{H_{\text{leC}}^{N,l,0}(X)(\sigma)} \leq \|u\|_{H_{\text{leC}}^{N,l,0}(X)(\sigma)}$.

For each $M' \in \mathbb{R}$, an elliptic parametrix for $A_1$ can be used to construct $A_0 \in \Psi_{\text{leC}}^{0,0,0}(X) \subset L^\infty((0, \infty); \Psi_{\text{leC}}^{0}(X))$ and $R \in \Psi_{\text{leC}}^{-M',l+0,\xi+0}(X)$ such that $A_0A = A_0A_1 + R$. Then, if $M'$ is sufficiently large,

$$\|A_0A_1u\|_{L^2} \leq \|A_0u\|_{L^2} + \|Ru\|_{L^2} \leq \|A_1u\|_{L^2} + \|u\|_{H_{\text{leC}}^{-M',l+0,\xi+0}(X)(\sigma)}$$

(151)

where we have used the uniform $L^2$-boundedness of $A_0$. Combining the estimates above, we deduce eq. (148).

In the rest of the paper, we will abbreviate

$$H_{\text{leC}}^{m,s,\varsigma,l,\ell}(X)(\sigma) = H_{\text{leC}}^{m,s,\varsigma,l,\ell}(X) = H_{\text{leC}}^{m,s,\varsigma,l,\ell},$$

(152)

leaving the dependence on $\sigma$ implicit (and likewise for other $\sigma$-dependent notions). In particular, what we call “estimates” will really be 1-parameter families of estimates with multiplicative constants that – for any fixed $\Sigma > 0$ – are uniform for $\sigma \in [0, \Sigma]$.

The following two lemmas will be used mostly without comment in §5:

**Lemma 2.15.** For any $m, s, \varsigma, l, \ell \in \mathbb{R}$, there exists a constant $C = C(m, s, \varsigma, l, \ell)$ such that, for all $u, v \in S'(X)$ and $\bar{\sigma} > 0$,

$$\langle u, v \rangle_{L^2} \leq C \cdot \bar{\sigma}^{-1}\|u\|_{\text{leC}^{m,s,\varsigma,l,\ell}}^2 + \bar{\sigma}\|v\|_{\text{leC}^{m,s,\varsigma,l,\ell}}^2$$

(153)

for all $\sigma \geq 0$ for which the right-hand side is finite (in the strong sense that if the right-hand side is finite, then the left-hand side makes sense using the duality pairing for sch-Sobolev spaces and obeys the stated inequality).

**Proof.** By the parametrix construction, we can find, for each $N_0 \in \mathbb{N}$, elliptic $V = V_{N_0} \in \Psi_{\text{leC}}^{-m,-s,-\varsigma,-l,-\ell}(X)$ such that $V^*A_{m,s,\varsigma,l,\ell} = 1 + Y$ for $Y = Y_{N_0} \in \Psi_{\text{leC}}^{-N_0,-N_0,-N_0,0,0}(X)$. Then,

$$2\langle u, v \rangle_{L^2} \leq 2\langle A_{m,s,\varsigma,l,\ell}u, Vu \rangle_{L^2} + 2\langle Yu, v \rangle_{L^2}$$

$$\leq \bar{\sigma}^{-1}\|A_{m,s,\varsigma,l,\ell}u\|_{L^2}^2 + \bar{\sigma}\|Vv\|_{L^2}^2 + 2\langle Yu, v \rangle_{L^2}$$

$$\leq \bar{\sigma}^{-1}\|u\|_{\text{leC}^{m,s,\varsigma,l,\ell}}^2 + \bar{\sigma}\|v\|_{\text{leC}^{m,s,\varsigma,l,\ell}}^2 + 2\langle Yu, v \rangle_{L^2}$$

(154)
On the other hand, for any \( N_1 \in \mathbb{N} \),
\[
2| \langle Y u, v \rangle_{L^2} | = 2| \langle x^{-l}(x^2 + Zx)^{-\ell/2+l} Y u, x^{l}(x^2 + Zx)^{\ell/2-l} v \rangle_{L^2} | \\
\leq \delta^{-1} \| x^{-l}(x^2 + Zx)^{-\ell/2+l} Y u \|_{H^{N_1,0}_{\text{leC}}}^2 + \delta \| x^{l}(x^2 + Zx)^{\ell/2-l} v \|_{H^{N_1,0}_{\text{leC}}}^2 \\
\leq \delta^{-1} \| Y u \|_{H^{N_1,1-l,\ell}_{\text{leC}}}^2 + \delta \| v \|_{H^{N_1,1-l,\ell}_{\text{leC}}}^2 \\
\leq \delta^{-1} \| Y u \|_{H^{N_1,N_1+1,1-l,\ell}_{\text{leC}}}^2 + \delta \| v \|_{H^{N_1,N_1+1,1-l,\ell}_{\text{leC}}}^2 + \delta \| v \|_{H^{N_1,N_1+1,1-l,\ell}_{\text{leC}}}^2. \\
\tag{155}
\]

Taking \( N_1 \) sufficiently large, and then taking \( N_0 \) sufficiently large relative to that, we get
\[
| \langle Y u, v \rangle_{L^2} | \leq \delta^{-1} \| u \|_{H^{N_1,N_1+1,1-l,\ell}_{\text{leC}}}^2 + \delta \| v \|_{H^{N_1,N_1+1,1-l,\ell}_{\text{leC}}}^2. \\
\]
Combining this with eq. (154), we get eq. (153).

\[\square\]

**Lemma 2.16.** Let \( m, s, \varsigma, l, \ell, m_0, s_0, \varsigma_0, l_0, \ell_0 \in \mathbb{R} \). Suppose that \( A \in \Psi^{m,s,\varsigma,l,\ell,0}_{\text{leC}}(X) \) and that we have some \( J \in \mathbb{N} \) and \( G_1, \ldots, G_J \in \Psi^{0,0,0,0}_{\text{leC}}(X) \) such that
\[
WF^{m,l,0,0,0}_{\text{leC}}(A) \subseteq \bigcup_{j=1}^{J} \text{Ell}^{0,0,0,0}_{\text{leC}}(G_j). \\
\tag{156}
\]
Then, for each \( \Sigma > 0 \) and \( N \in \mathbb{N} \), there exists some \( C = C(A, G_1, \ldots, G_J, \Sigma, N) > 0 \) such that
\[
\| Au \|_{H^{m_0,0,0,0}_{\text{leC}}} \leq C \left[ \| u \|_{H^{m_0,0,0,0}_{\text{leC}}} + \sum_{j=1}^{J} \| G_j u \|_{H^{m_j,0,0,0,0}_{\text{leC}}} \right] \\
\tag{157}
\]
for all \( \sigma \in [0, \Sigma] \) and \( u \in \mathcal{S}(X) \).

\[\square\]

**Proof.** It suffices to consider the case of \( u \in \mathcal{S}(X) \), the general estimate eq. (157) following from this case via continuity.

Via quantizing some explicit symbols, there exist some \( \tilde{G}_1, \ldots, \tilde{G}_J \in \Psi^{0,0,0,0}_{\text{leC}}(X) \) such that
\[
WF^{0,0,0,0}_{\text{leC}}(G_j) \subseteq \text{Ell}^{0,0,0,0}_{\text{leC}}(G_j), \\
\tag{158}
\]
and
\[
WF^{m,l,0,0,0}_{\text{leC}}(A) \subseteq \bigcup_{j=1}^{J} \text{Ell}^{0,0,0,0}_{\text{leC}}(G_j). \\
\tag{159}
\]

We now apply the leC-analogue of the Gårding’s inequality- type argument. We can choose \( E \in \Psi^{0,0,0,0}_{\text{leC}}(X) \) such that
\[
\text{Ell}^{0,0,0,0}_{\text{leC}}(E) \cup \bigcup_{j=1}^{J} \text{Ell}^{0,0,0,0}_{\text{leC}}(G_j) = \text{df} \cup \text{sf} \cup \text{ff}. \\
\tag{159}
\]

\[
WF^{m,l,0,0,0}_{\text{leC}}(E) \cap WF^{m,l,0,0,0}_{\text{leC}}(A) = \emptyset. \\
\tag{160}
\]
Thus, every representative of \( \sigma^{0,0,0,0}_{\text{leC}}(E^* E + \tilde{G}_1 \tilde{G}_1 + \cdots + \tilde{G}_J \tilde{G}_J) \) is nonvanishing on \( \text{df} \cup \text{sf} \cup \text{ff} \).

We may assume without loss of generality that \( E, \tilde{G}_1, \ldots, \tilde{G}_J \) are constant for \( \sigma \geq 2\Sigma \), in which case (since \( \text{df} \cup \text{sf} \cup \text{ff} \cap \{ \sigma \leq 2\Sigma \} \) is compact) there exists some \( c > 0 \) such that \( \sigma^{0,0,0,0}_{\text{leC}}(E^* E + \tilde{G}_1 \tilde{G}_1 + \cdots + \tilde{G}_J \tilde{G}_J) \geq 2c \) in some neighborhood of \( \text{df} \cup \text{sf} \cup \text{ff} \) (in the sense that every representative
of the principal symbol has this property, for different neighborhoods). Via an iterative symbolic construction: for each $N_0 \in \mathbb{N}$ there exists some elliptic $B = B_{N_0}\epsilon \in \Psi_{\text{loc}}^{0,0,0,0}(X)$ such that

$$E^*E + G^*_1G_1 + \cdots + G^*_JG_J - c - B^*B \in \Psi_{\text{loc}}^{-N_0,-N_0,-0,0}(X).$$

(161)

Then, for $X = H_{\text{loc}}^{m_0,s_0,l_0}(X)$,

$$c\|Au\|_X^2 \leq \|BAu\|_X^2 + \|EAu\|_X^2 + \sum_{j=1}^J \|G_jAu\|_X^2 \leq \|BAu\|_X^2 + \sum_{j=1}^J \|G_jAu\|_X^2$$

(162)

$$\leq \|u\|_{H^{-N,-N,-l_0}}^2 + \sum_{j=1}^J \|G_ju\|_{H^{m_0,m_0}}^2$$

(163)

for $N_0$ sufficiently large, from which eq. (157) follows. □

**Lemma 2.17.** If $m, s, \varsigma, \ell, m_0, s_0, l_0, \ell_0, m_1, s_1, \varsigma_1, \ell_1 \in \mathbb{R}$ satisfy $m_1 > m > m_0, \cdots, \ell_1 > \ell > \ell_0$, then, for each $\epsilon > 0$ and $\Sigma > 0$, there exists a $C(\epsilon) = C(\epsilon, m, \cdots, \ell_1, \Sigma) > 0$ such that

$$\|u\|_{H^{m_0,s_0}} \leq \epsilon \|u\|_{H^{m_1,s_1}} + C(\epsilon)\|u\|_{H^{m_0,s_0}}$$

(164)

for all $u \in S(X)$ and $\sigma \in [0, \Sigma]$. ■

**Proof.** Fix $\Sigma > 0$. Suppose, to the contrary, that there exists some $\epsilon > 0$, $\{\sigma_k\}_{k \in \mathbb{N}} \subset [0, \Sigma]$, and $\{u_k\}_{k \in \mathbb{N}} \subset S(X)$ such that

$$1 = \|u_k\|_{H^{m_0,s_0}} \geq \|u_k\|_{H^{m_1,s_1}} + k\|u_k\|_{H^{m_0,s_0}}$$

(165)

i.e., for sufficiently large $N, M \in \mathbb{N}$,

$$1 = \|u_k\|_{H^{m_0,s_0}} \geq \|x^{-l}(\sigma_k^2 + \Sigma x)^{l-\ell_2/2}u_k\|_{H^{-N}} + k\|u_k\|_{H^{m_0,s_0}}$$

(166)

Passing to a subsequence if necessary, we may assume without loss of generality that $\sigma_k \to \sigma_\infty$ for some $\sigma_\infty \in [0, \Sigma]$.

By the Banach-Alaoglu theorem, by passing to a further subsequence if necessary, we can arrange that $\Lambda_{m_1, s_1, \varsigma_1, \ell_1}(\sigma_k)u_k \to v_i$ weakly in $L^2$ for some $v_0, v_1 \in L^2(X)$ and that

$$x^{-l_1}(\sigma_k^2 + \Sigma x)^{l_1-\ell_2/2}u_k \to w_i$$

(167)

weakly in $H^{-M}(X), H^{-N}(X)$ for some $w_0, w_1 \in H^{-M}(X), H^{-N}(X)$, respectively. Since

$$\Lambda_{m, s, \varsigma, \ell}(X) \in \Psi_{\text{loc}}^{\infty,\infty,\infty,\ell}(X)$$

(168)

is elliptic, Lemma 2.12 yields

$$1 = \|u_k\|_{H^{m_0,s_0}} \leq \|\Lambda_{m_1, s_1, \varsigma_1, \ell_1}(\sigma_k)u_k\|_{L^2} + \|x^{-l}(\sigma_k^2 + \Sigma x)^{l-\ell_2/2}u_k\|_{H^{-N}}.$$
Lemma 2.18. The $L^2_{\text{sc}}(X), L^2_{\text{sc}}(X_{1/2})$-based $b$-Sobolev spaces on $X$ and $X_{1/2}$ are related by $H^m_{b}^{l}(X) = H^m_{b}^{2l+n/2}(X_{1/2})$. □
Proof. For \( f, g \in L^2_\mathcal{g}([0,x] \times \partial X) \),

\[
(f,g)_{L^2_\mathcal{g}([0,x] \times \partial X)} = \int_{\partial X} \left( \int_0^x f^*(x)g(x) \frac{dx}{x^{n+1}} \right) \, d\text{Vol}_{g|X}(y)
= 2 \int_{\partial X} \left( \int_0^{x^{1/2}} f^*(\rho^2)g(\rho^2) \frac{d\rho}{\rho^{2n+1}} \right) \, d\text{Vol}_{g|X}(y)
= 2 \int_{\partial X} \left( \int_0^{x^{1/2}} f^*(\rho^2)g(\rho^2) \frac{d\rho}{\rho^{n+1}} \right) \, d\text{Vol}_{g|X}(y).
\]

Thus, \( L^2_\mathcal{g}([0,x] \times \partial X) \ni f(x) \mapsto \sqrt{2}f(\rho^2) \in \rho^{n/2}L^2_\mathcal{g}([0,x^{1/2}] \times \partial X) \) defines an isomorphism of Hilbert spaces.

This implies that

\[
L^2_\mathcal{g}(X) = x^{n/4}L^2_\mathcal{g}(X_{1/2}).
\]

As follows from the definition [Vas18, Definition 5.15] (see also [Mel93, Definition 4.22] for the case of classicality at the front face of the \( b \)-double space), \( \Psi_{\mathcal{g},m,l}(X) = \Psi_{b,m}^{m,2l}(X_{1/2}) \) for all \( m, l \in \mathbb{R} \). In conjunction with eq. (179), this implies that

\[
H^{m,l}_b(X) = \Psi_{\mathcal{g},m,-l}(X)L^2(X) = \Psi_{b,-m,-2l-n/2}^{m,2l+n/2}(X_{1/2})L^2(X_{1/2}) = H^{m,2l+n/2}_b(X_{1/2}).
\]

For use in §6, we briefly recall the connection between spaces of conormal distributions on \( X^{\text{sp}}_{\mathcal{g}} \), which are defined \( L^\infty \)-based spaces, and the \( b \)-Sobolev spaces, which are defined using \( L^2 \). We have

\[
\mathcal{A}^{\alpha,\beta,0}_{\text{loc}}(X^{\text{sp}}_{\mathcal{g}}) = x^{\alpha}(\beta^2 + Zx)^{-\alpha + \beta / 2}A^{0,0}_{\text{loc}}([0, \infty) \times X),
\]

and

\[
\mathcal{A}^{0,0,0}_{\text{loc}}(X^{\text{sp}}_{\mathcal{g}}) = \{ u \in \mathcal{A}^0(X^{\text{sp}}_{\mathcal{g}}) : [\hat{E} \mapsto u|_{E/E=b}] \in C^\infty([0, \infty)_E; \mathcal{A}^0(X_{1/2})]\},
\]

\[
\mathcal{A}^{\alpha,\beta,0}_{\text{loc}}(X^{\text{sp}}_{\mathcal{g}}) = x^{\alpha}(\beta^2 + Zx)^{-\alpha + \beta / 2}A^{0,0,0}_{\text{loc}}(X^{\text{sp}}_{\mathcal{g}}),
\]

where \( \hat{E} = E/x \) and we are identifying level sets of \( \hat{E} \) with \( zf \cong X_{1/2} \). Note that greater indices \( \alpha, \beta \) means greater decay, the convention opposite of that used for symbols and \( \Psi \)DOs. We also use

\[
\mathcal{A}^{\alpha,\beta,0}_{\text{loc}}(X^{\text{sp}}_{\mathcal{g}}) = \bigcap_{\alpha' < \alpha, \beta' < \beta} \mathcal{A}^{\alpha',\beta',0}_{\text{loc}}(X^{\text{sp}}_{\mathcal{g}})
\]

\[
\mathcal{A}^{\alpha,\beta,0}_{\text{loc}}(X^{\text{sp}}_{\mathcal{g}}) = \bigcap_{\alpha' < \alpha, \beta' < \beta} \mathcal{A}^{\alpha',\beta',0}_{\text{loc}}(X^{\text{sp}}_{\mathcal{g}}).
\]

for \( \alpha, \beta \in \mathbb{R} \). These are all Fréchet spaces of (locally) conormal distributions on \( X^{\text{sp}}_{\mathcal{g}} \) that are smooth at \( zf \) (where, to reiterate, “local” just means that we do not require uniformity as \( \sigma \to \infty \), only as \( \sigma \to 0^+ \)). Via Sobolev embedding,

\[
H^{\infty,l}_b(X) \subseteq A^{l+n/2}(X) \subseteq H^{\infty,l-}_b(X) = \cap_{l' < l} H^{\infty,l'}_b(X),
\]

and each seminorm of \( A^{l+n/2}(X) \) can be controlled using only finitely many of the norms in the family \( \{|-|H^{\infty,l'}_b|\}_{m \in \mathbb{R}, l' < l} \).

Since, for each \( \alpha \geq 0 \), \( \mathcal{A}^{\alpha,\beta,0}_{\text{loc}}([0, \infty); A^{\alpha,0}(X)) \subseteq \mathcal{A}^{\alpha,\beta,0}_{\text{loc}}([0, \infty); A^{\alpha,0}(X)) \), the preceding observation implies that
Proposition 2.19. For any function $m_0 : \mathbb{R} \to \mathbb{R}$,
\[
\bigcap_{l \neq -1/2} \bigcap_{m > m_0(l)} \mathcal{A}^0_{\text{loc}}([0, \infty)_{\sigma}; H^m_{b} + a(X)) \subseteq \mathcal{A}^0_{\text{loc}}((\alpha + (n-1)/2) - \eta, ((2\alpha + (n-1)) - \delta)(X_{\text{res}}^{\text{sp}})) \tag{186}
\]
holds for each $\alpha \in \mathbb{R}$. \[\square\]

Remark. Implicit in the statements above is the identification of elements of $\mathcal{A}^0_{\text{loc}}([0, \infty)_{\sigma}; A^\alpha(-)(X))$ with extendable distributions on $X_{\text{res}}^{\text{sp}}$, which occurs via
\[
\mathcal{A}^0_{\text{loc}}([0, \infty)_{\sigma}; A^\alpha(-)(X)) \ni \{u(-; \sigma)\}_{\sigma > 0} \mapsto \int_0^\infty \langle u(-; \sigma), \chi(-; \sigma) \rangle \, d\sigma \in \mathcal{D}'(X_{\text{res}}^{\text{sp}}). \tag{187}
\]
We refer to [Mel92][Mel93] for more about conormal distributions. See also [Hin21] for the particular case of $X_{\text{res}}, 0$ (where the notation $X_{\text{res}}^+$ is used instead).

Proposition 2.20. Fix $\chi \in C^\infty_c(X_{\text{res}}^{\text{sp}})$ supported away from $bf$ and nonvanishing near $zf$. Then, if $(\chi x\partial E)^k u \in \mathcal{A}^0_{\text{loc}}(-0, -0)(X_{\text{res}}^{\text{sp}})$ for all $k \in \mathbb{N}$, then $u \in \mathcal{A}^0_{\text{loc}}(-0, -0, -0)(X_{\text{res}}^{\text{sp}})$. \[\square\]

Proof. We want to show that $[\hat{E} \mapsto u|_{E = E}] \in C^\infty((0, \infty)_{\hat{E}}; A^0(X_{1/2}))$. Observe that $x\partial E = \partial \hat{E}$ away from $bf$, where the partial derivative on the right-hand side is taken with $x$ held constant.

Consequently, the $k = 0, \ldots, K + 1$ cases of $(\chi x\partial E)^k u \in \mathcal{A}^0_{\text{loc}}(-0, -0, -0)(X_{\text{res}}^{\text{sp}})$ together show that $[\hat{E} \mapsto u|_{E = E}] \in C^k((0, \infty)_{\hat{E}}; A^0(X_{1/2}))$. Here we are using that, given $v \in \mathcal{A}^0_{\text{loc}}(-0, -0, -0)(X_{\text{res}}^{\text{sp}})$,
\[
[\hat{E} \mapsto v|_{E = E}] \in A^0((0, 1)_{\hat{E}}; A^0_-(X_{1/2})) = A^0((0, 1)_{\hat{E}}; A^0_-(X)). \tag{188}
\]
Since $K$ can be taken arbitrarily large, we conclude the claim. \[\square\]

Remark. As the argument shows, each Fréchet seminorm of $u \in \mathcal{A}^0_{\text{loc}}(-0, -0, -0)(X_{\text{res}}^{\text{sp}})$ is controlled by finitely many Fréchet seminorms of $(\chi x\partial E)^k u \in \mathcal{A}^0_{\text{loc}}(-0, -0)(X_{\text{res}}^{\text{sp}})$ for finitely many $k$. In other words, the map
\[
\prod_{k=0}^\infty \mathcal{A}^0_{\text{loc}}(-0, -0)(X_{\text{res}}^{\text{sp}}) \cap \{(\chi x\partial E)^k u\}_{k=0}^\infty : u \in \mathcal{A}^0_{\text{loc}}(-0, -0)(X_{\text{res}}^{\text{sp}}) \ni \{(\chi x\partial E)^k u\}_{k=0}^\infty \mapsto u \in \mathcal{A}^0_{\text{loc}}(-0, -0, -0)(X_{\text{res}}^{\text{sp}}) \tag{189}
\]
is continuous when we endow the domain with the topology of $\prod_{k=0}^\infty \mathcal{A}^0_{\text{loc}}(-0, -0)(X_{\text{res}}^{\text{sp}})$.

3. The Conjugated Perspective

We now construct the “conjugated” operator $\hat{P} = \{\hat{P}(\sigma)\}_{\sigma \geq 0}$. Given some
\[
\{f(-; \sigma)\}_{\sigma \geq 0}, \{g(-; \sigma)\}_{\sigma \geq 0} \subset C^\infty(X^0)
\]
and a family of differential operators $\{D(\sigma)\}_{\sigma \geq 0} \subset \text{Diff}(X^0)$, we use the somewhat abusive notation $fDg = \{f(-; \sigma)D(\sigma)g(-; \sigma)\}_{\sigma \geq 0} \subset \text{Diff}(X^0)$ to denote the family of differential operators
\[
M_f D M_g = \{M_{f(-)} D(\sigma) M_{g(-)} \sigma \geq 0) \}
\tag{190}
\]
where for $h \subset C^\infty(X^0)$, $M_h : C^\infty_c(X^0) \to C^\infty_c(X^0)$ denotes the multiplication operator $C^\infty_c(X^0) \ni \varphi \mapsto h\varphi$. For us, $f, g$ will be of the form $f(-; \sigma) = \exp(-i\Phi(-; \sigma))$ and $g(-; \sigma) = \exp(+i\Phi(-; \sigma))$ for some $\Phi = \{\Phi(-; \sigma)\}_{\sigma \geq 0} \subset C^\infty(X^0)$. The conjugated operator $\hat{P}$ is defined by
\[
\hat{P} = M_{\exp(-i\Phi)} P M_{\exp(+i\Phi)} = e^{-i\Phi} P e^{+i\Phi} \tag{191}
\]
for our eventual choice of $\Phi$.

As discussed in the introduction, $\Phi$ had to be determined to an order or two (including logarithmic terms) on $X_{\text{reg}}^{\text{sp}}$ by the actual asymptotics of solutions of $P(\sigma)u = f$ for $f \in \mathcal{S}(X)$. Rather than determine what $\Phi$ should be in this manner (that is by solving the PDE, or a model thereof, to a sufficient degree of accuracy), it is actually easier to work backwards, meaning to find the asymptotics of solutions to the given PDE by first finding a choice of $\Phi$ for which $\hat{P}$ has a workable form, where “workable” roughly means qualitatively similar to the conjugated operator in [Vas21a]. (We will then have to actually show that this choice describes the asymptotics of solutions to the PDE.) Such a choice has already been stated in the introduction, §1, eq. (54), and the resultant conjugated operator is computed below. We now motivate that choice. Consider the model operator $P_{\text{Model}} \in \text{Diff}^2_{sc}((0, \infty)_x)$ given by

$$P_{\text{Model}}(\sigma) = -(1 + xa_{00})(x^2 \partial_x)^2 + (a(\sigma) + n - 1)x^2 \partial_x - \sigma^2 - Zx,$$  \hspace{1cm} (192)

where $a_{00}, Z \in \mathbb{R}, Z > 0$, and where we require $a_{00} < 0$ for simplicity. This captures the leading and subleading terms of $P(\sigma)$ in $\Psi_{scb}(X)$, modulo the terms involving nonradial derivatives (which can be expected to be unimportant – i.e. under symbolic control – based on considerations similar to those in [Vas21a]). For $\Phi(-; \sigma) \in C^\infty(\mathbb{R}_x^+ \times \mathbb{R}^+_\sigma)$, let

$$\tilde{P}_{\text{Model}} = M \exp(-i\Phi)P_{\text{Model}} \exp(+i\Phi) = \exp(-i\Phi)P_{\text{Model}} \exp(+i\Phi).$$  \hspace{1cm} (193)

This will be qualitatively similar (at least with regards to the sc-calculus) to Vasy’s conjugated operator family (with $3\alpha_\pm = 0$) if $\tilde{P}_{\text{Model}}$ is equal to

$$P_{\text{Goal}} = -(1 + xa_{00}) \left( x^2 \partial_x - \frac{x(n-1)}{2} \right)^2 + 2i(1 + xa_{00}) \sqrt{\sigma^2 + Zx - \sigma^2 a_{00}x} (x^2 \partial_x) + 2i(1 + xa_{00}) \left( - \frac{(n-1)}{2} x \sqrt{\sigma^2 + Zx} + \frac{Zx^2}{4 \sqrt{\sigma^2 + Zx}} \right),$$  \hspace{1cm} (194)

modulo terms which are two sc-decay orders subleading at $\text{sf}$ and $\text{ff}$ (in particular in $x^2 \text{Diff}^0_{sc}(X)$ for $\sigma > 0$ and $x^2 \text{Diff}^0_{sc}(X^{1/2})$ for $\sigma = 0$, where $X = [0, \infty)_x$). The specific form of the lower order terms in eq. (194) bears comment: the operators $x^2 \partial_x - x(n-1)/2$ and

$$(\sigma^2 + Zx)^{1/2} x^2 \partial_x - x(\sigma^2 + Zx)^{1/2} \frac{(n-1)}{2} + \frac{Zx^2}{4 (\sigma^2 + Zx)^{1/2}}$$  \hspace{1cm} (195)

are both formally anti-self-adjoint with respect to the $L^2([0, 1], x^{-(n+1)} \, dx) = L^2_{sc}[0, 1]$ inner product. Indeed, for any $f, g \in \mathcal{S}((0, \infty))$,

$$\int_0^\infty f^*(x) \partial_x g(x) x^{-(n+1)} \, dx = - \int_0^\infty (\partial_x f)^* g(x) x^{-(n+1)} \, dx + (n+1) \int_0^\infty \left( \frac{f(x)}{x} \right)^* g(x) x^{-(n+1)} \, dx,$$  \hspace{1cm} (196)

so, as bilinear forms $\mathcal{S}((0, \infty))^2 \to \mathbb{C}$,

$$\partial_x^* = -\partial_x + (n+1)/x,$$  \hspace{1cm} (197)

$$x^2 \partial_x^* = x^2 \partial_x^* + [\partial_x^*, x^2] = -x^2 \partial_x + x(n-1)$$  \hspace{1cm} (198)

and

$$(\sqrt{\sigma^2 + Zx^2} \partial_x)^* = \sqrt{\sigma^2 + Zx^2} \partial_x^* + [(x^2 \partial_x)^*, \sqrt{\sigma^2 + Zx}] = -\sqrt{\sigma^2 + Zx^2} \partial_x + x(n-1) \sqrt{\sigma^2 + Zx} - \frac{Zx^2}{2 \sqrt{\sigma^2 + Zx}},$$  \hspace{1cm} (199)
which implies the claimed anti-self-adjointness. The \((1 + xa_{00})\) terms in eq. (194), along with the 
\(-\sigma^2 a_{00} x\) under the square root, spoil anti-self-adjointness or self-adjointness, but only negligibly. 
Thus, the terms in eq. (194) have definite adjointness modulo negligible errors.

As a preliminary step towards \(P_{\text{Goal}}\), we can conjugate away the \((a + n - 1)x^2 \partial_x\) term in eq. (192), 
getting
\[
x^{-\frac{a + n - 1}{2}} P_{\text{Model} x^{\frac{a + n - 1}{2}}} = -(1 + xa_{00})(x^2 \partial_x)^2 - \sigma^2 - Z x \mod x^2 \text{Diff}^{1,0}_{\text{sc}}(X).
\] (200)

The \(x(n - 1)/2\) terms in eq. (194) can be conjugated back in at the end (and the last term in eq. (194) will be included automatically for self-adjointness reasons). Conjugating eq. (200) by \(\exp(+i\varphi)\) for a to-be-decided \(\varphi(x; \sigma) \in C^\infty(\mathbb{R}_+^* \times \mathbb{R}_+^*)\), we get
\[
e^{-i\varphi} x^{-\frac{a + n - 1}{2}} P_{\text{Model}} x^{\frac{a + n - 1}{2}} e^{+i\varphi} = -(1 + xa_{00})(x^2 \partial_x + ix^2 \varphi')^2 - \sigma^2 - Z x \mod x^2 \text{Diff}^{1,0}_{\text{sc}}(X),
\] (201)

assuming the contribution from the first order term in the remainder in eq. (200) is negligible. It is 
not unreasonable to expect (perhaps based on the \(Z = 0\) case) that, for our eventual choice of \(\varphi\), 
the leading order new contribution to eq. (201) is \(x^4(\varphi')^2\). Since the Coulomb term in eq. (201) is 
subleading order relative to \(\sigma^2\), we should really be keeping track of the new contributions to 
one subleading order. To this order, the new contribution to the effective potential is \(x^4(1 + xa_{00})(\varphi')^2\) 
(assuming the terms with second derivatives of \(\varphi\) are negligible). Thus, we seek to arrange
\[
x^4(1 + xa_{00})(\varphi')^2 = \sigma^2 + Z x \mod x^2 C^\infty(X).
\] (202)

Multiplying through by \((1 + xa_{00})^{-1} = 1 - xa_{00} \mod x^2 C^\infty(X)\), this suggests setting 
\(x^4(\varphi')^2 = \sigma^2 + Z x - \sigma^2 a_{00} x\), the solution of which (up to an arbitrary additive constant and conventional 
choice of sign) is
\[
\varphi(x; \sigma) = \frac{1}{x} \sqrt{\sigma^2 + Z x - \sigma^2 a_{00} x} + \frac{1}{\sigma} (Z - \sigma^2 a_{00}) \arcsinh \left( \frac{\sigma x^{1/2}}{1 - (\sigma^2 a_{00})^{1/2}} \right).
\] (203)

Recall that \(\arcsinh(z) = \log(z + (1 + z^2)^{1/2})\) for all \(z \geq 0\). Expanding \(\arcsinh(z)\) in Taylor series 
around \(z = 0\), we see that the apparent singularity in Equation (203) at \(\sigma = 0\) is removable (to all 
orders), and hence \(\varphi(x; \sigma)\) defines a smooth function on \(\mathbb{R}_+^* \times \mathbb{R}_+\), and it is even in \(\sigma\). We observe 
that, given eq. (202), the \(x^4 \varphi''\) term in eq. (201) is indeed negligible except at the \(\text{ff}\), where there is 
one non-negligible contribution, and it is precisely the final term in eq. (194) (modulo negligible terms).

Given this definition of \(\varphi\), \(e^{-i\varphi} x^{-\frac{a + n - 1}{2}} P_{\text{Model} x^{\frac{a + n - 1}{2}}} e^{+i\varphi}\) is given, modulo \(x^2 \text{Diff}^{1,0}_{\text{sc}}(X)\), by 
\[
-(1 + xa_{00})(x^2 \partial_x)^2 + 2ix^4 \varphi' \partial_x - x^4 \varphi'' + i(x^2 \partial_x)^2 \varphi - \sigma^2 - Z x = -(1 + xa_{00})(x^2 \partial_x)^2 \\
- 2ix^2 \sqrt{\sigma^2 + Z x - \sigma^2 a_{00} x} \partial_x + i(x^2 \partial_x)^2 \varphi \mod C^\infty(\mathbb{R}_+^*; x^2 C^\infty(X))
\] (204)

(plus the term in \(\text{Diff}_{\text{leC}}^{0,-2,-5,-2,-5}\) that results from applying the first order operator in the 
\(x^2 \text{Diff}^{1,0}_{\text{sc}}(X)\) remainder in eq. (201) to \(\varphi\). We can now add back in the \(x(n - 1)/2\) terms:
\[
e^{-i\varphi} x^{-a/2} P_{\text{Model}} e^{+i\varphi} x^{a/2} = P_{\text{Goal}} \mod \text{Diff}_{\text{leC}}^{1,-2,-4,-2,-4}.
\] (205)

So, at least in this model case, conjugation by \(e^{+i\Phi} = \exp(+i\varphi - (i/2)a \log x)\) has the required properties.
Returning to the full problem, we consider the family $\Phi = \{\Phi(-; \sigma)\}_{\sigma \geq 0}$ of $\Phi(-; \sigma) \in C^\infty(X^0 \times \mathbb{R}^+_x)$ given by
\[
\Phi(x; \sigma) = \frac{1}{x} \sqrt{\sigma^2 + \frac{1}{2} (Z - \sigma^2 a_0 x)} \arcsinh \left( \frac{\sigma}{x^{1/2}} \frac{1}{(Z - \sigma^2 a_0 x)^{1/2}} \right) - i \sigma \log x. \tag{206}
\]
Observe that (after removing the removable singularity at $\sigma = 0$) eq. (58) holds. We then define, for each $\sigma \geq 0$,
\[
\hat{P}(\sigma) = M_{e^{-i\Phi(-; \sigma)}} P(\sigma) M_{e^{i\Phi(-; \sigma)}}, \tag{207}
\]
\[\text{i.e. } \hat{P} = \exp(-i\Phi) P \exp(+i\Phi). \]
Thus, $\hat{P}(\sigma) \in S \text{ Diff}_{sc}(X)$ for each $\sigma \geq 0$, and the coefficients all depend smoothly on $\sigma$ all the way down to $\sigma = 0$ in compact subsets of $X^0$. Of course, this does not mean that $[0, \infty)_\sigma \ni \sigma \mapsto \hat{P}(\sigma) \in S \text{ Diff}_{sc}^{(0)}(X)$ is smooth all the way down to $\sigma = 0$. This map is continuous for $\sigma > 0$ but discontinuous at $\sigma = 0$, as can be verified by computing sc-principal symbols.

Let $\hat{P}_0 = e^{-i\Phi} P_0 e^{i\Phi}$, $\hat{P}_1 = e^{-i\Phi} P_1 e^{i\Phi}$, and $\hat{P}_2 = e^{-i\Phi} P_2 e^{i\Phi}$, where $P_0, P_1, P_2$ are as in §1. We have:

**Proposition 3.1.** For each $\sigma \geq 0$, $\hat{P}_0(\sigma)$ is given with respect to the boundary-collar $\iota$ by
\[
\hat{P}_0(\sigma) = -(1 + x a_0)(x^2 \partial_x)^2 + x^2 \Delta_{\partial X} + (n - 1)x^3 \partial_x + L(\sigma) + V_{\text{eff}}(x; \sigma), \tag{208}
\]
where

\[
L(\sigma) = 2i x (1 + x a_0) \sqrt{\sigma^2 + \frac{1}{2} (Z - \sigma^2 a_0 x)} (x \partial_x - \frac{n - 1}{2} + \frac{Z x}{4 \sigma^2 + Z x}) - a a_0 x^4 \partial_x
\tag{209}
\]
and $V_{\text{eff}} \in x^2 C^\infty(X^{sp})$.

**Proof.** We work on $\hat{X} = [0, \tilde{x}]_x \times \partial X$.

We may write $\hat{P}_0 = P_0 + e^{-i\Phi}[P_0, e^{i\Phi}]$. Observe that
\[
[P_0, e^{i\Phi}] = -(1 + x a_0)(x^2 \partial_x)^2, \quad e^{i\Phi} = x^3(a + n - 1) \partial_x, \quad e^{i\Phi} = x^2[\Delta_{\partial X}, e^{i\Phi}] \tag{210}
\]
\[e^{-i\Phi}(x^2 \partial_x^2, e^{i\Phi}) = 2i x^4 \Phi' \partial_x - x^4 \Phi' \partial_x + i(x^2 \partial_x)^2 \Phi \tag{211}
\]
\[e^{-i\Phi}[\partial_x, e^{i\Phi}] = i \Phi' \tag{212}
\]
\[e^{-i\Phi}[\Delta_{\partial X}, e^{i\Phi}] = 0, \tag{213}
\]
where the primes denote differentiation in $x$. (If $a, a_0$ were nonconstant functions on $\partial X$, then eq. (213) would not hold. This is ultimately the reason for assuming that $a, a_0$ are constant.) Thus, if we set
\[
L(\sigma) = a x^3 \partial_x - 2i x^4 (1 + x a_0) \Phi' \partial_x + V_{SA}, \tag{214}
\]
\[V_{\text{eff}}(x; \sigma) = -(1 + x a_0)(-x^4 \Phi' \partial_x + i(x^2 \partial_x)^2 \Phi) + i x^3(a + n - 1) \Phi' - V_{SA} - \sigma^2 - Z x \tag{215}
\]
for
\[
V_{SA} = -2i x (1 + x a_0) \sqrt{\sigma^2 + \frac{1}{2} (Z - \sigma^2 a_0 x)} \left( \frac{n - 1}{2} - \frac{Z x}{4 \sigma^2 + Z x} \right), \tag{216}
\]
then eq. (208) holds, and it only remains to verify eq. (209) and the fact that $V_{\text{eff}} \in x^2 C^\infty(X^{sp}_{\text{res}})$. 

We compute that

$$+x^4\Phi'\Phi' = x^4 \left( \frac{1}{\sigma^2} \sqrt{\sigma^2 + Zx - \sigma^2 a_{00} x} + \frac{ia}{2x} \right)^2$$

$$= \sigma^2 + Zx - \sigma^2 a_{00} x + iax\sqrt{\sigma^2 + Zx - \sigma^2 a_{00} x} - \frac{x^2 a^2}{4}, \quad (217)$$

$$-i(x^2 \partial_x)\Phi = -\frac{a x^2}{2} + \frac{i}{2} x^2 \frac{(Z - \sigma^2 a_{00})}{(\sigma^2 + Zx - \sigma^2 a_{00} x)^{1/2}}, \quad (218)$$

$$+ix^3\Phi' = \frac{a x^2}{2} - i x \sqrt{\sigma^2 + Zx - \sigma^2 a_{00}}. \quad (219)$$

From eq. (219) and eq. (214), we get eq. (209).

When adding up the various contributions to \(V_{\text{eff}}\), as written in eq. (215), a few key cancellations happen by design:

(I) the first \(\sigma^2\) in eq. (215) (coming from \(x^4\Phi'\Phi'\), eq. (217)) cancels with the last \(-\sigma^2\) in eq. (215),

(II) the first \(Zx\) term, also coming from \(x^4\Phi'\Phi'\), cancels with the \(-Zx\) in eq. (215), so that the original Coulomb-like term has been “conjugated away,”

(III) multiplying eq. (217) by \((1 + x a_{00})\) the terms in \((1 + x a_{00})(\sigma^2 - \sigma^2 a_{00} x)\) linear in \(a_{00}\) cancel per difference-of-squares,

(IV) the \(i a x(\sigma^2 + Zx - \sigma^2 a_{00} x)^{1/2}\) term in eq. (217) cancels with the term in \(i a x^3\Phi'\) coming from the last term in eq. (219).

(V) The \(i x(\sigma^2 + Zx - \sigma^2 a_{00} x)^{1/2}(n - 1)\) term in \(V_{\text{eff}}\) coming from \(V_{\text{eff}}\) cancels with the identical term in \(i(n + 1)x^3\Phi'\) coming from the last term in eq. (219).

All in all, \(V_{\text{eff}}\) is given by

$$a_{00}(Z - \sigma^2 a_{00}) x^2 + i a_{00}(a + n - 1) x^2 \sqrt{\sigma^2 + Zx - \sigma^2 a_{00} x} - \left[ (1 + x a_{00}) \left( \frac{\sigma^2}{4} + \frac{a}{2} \right) - \frac{a}{2} (a + n - 1) \right] x^2$$

$$+ \frac{i x}{2} (1 + x a_{00}) \left[ \frac{Zx - \sigma^2 a_{00} x}{(\sigma^2 + Zx - \sigma^2 a_{00} x)^{1/2}} - \frac{Zx - \sigma^2 a_{00} x}{\sigma^2 + Zx} \right]. \quad (220)$$

It is clear that the first line of eq. (220) defines an element of \(x^2 C^\infty(X_{\text{res}}^{\text{sp}})\). On the other hand, 

$$-(i x/2)(1 + x a_{00})\sigma^2 a_{00} x/(\sigma^2 + Zx - \sigma^2 a_{00} x)^{1/2}$$

is in \(x^2 C^\infty(X_{\text{res}}^{\text{sp}})\) as well. Thus, it remains to verify that

$$\frac{Zx}{(\sigma^2 + Zx)^{1/2}} - \left( \frac{\sigma^2}{\sigma^2 + Zx} \right)^{1/2} - \frac{Zx}{\sigma^2 + Zx} \in x C^\infty(X_{\text{res}}^{\text{sp}}), \quad (221)$$

i.e. that

$$1 - \left( 1 - \frac{\sigma^2}{\sigma^2 + Zx} \right)^{1/2} \in (\sigma^2 + Zx)^{1/2} C^\infty(X_{\text{res}}^{\text{sp}}). \quad (222)$$

This is of course not true for each term on the left-hand side individually, but we can expand

$$\left( 1 - \frac{\sigma^2 a_{00} x}{\sigma^2 + Zx} \right)^{1/2} \equiv 1 \pmod{\frac{\sigma^2 a_{00} x}{\sigma^2 + Zx} C^\infty(X_{\text{res}}^{\text{sp}}) = 1 \pmod{x C^\infty(X_{\text{res}}^{\text{sp}})} \quad (223)$$

(since \(f(\zeta) = \zeta^{-1}((1 - \zeta)^{1/2} - 1) \in C^\infty(\Lambda, 1), f(\sigma^2 a_{00} x/(\sigma^2 + Zx)) \in C^\infty(X_{\text{res}}^{\text{sp}}), \) which implies eq. (223)) so in fact eq. (222) is true with some room to spare. We can then conclude that \(V_{\text{eff}}\) is in \(x^2 C^\infty(X_{\text{res}}^{\text{sp}})\). $$\square$$
Thus, \( L \) is \( L(\sigma) \in \text{Diff}^{1,-2,-1,-3}_{\text{leC}}(X) \) satisfies

\[
L = 2i(1 + x a_{00}) \left( 1 - \frac{\sigma^2 a_{00} x}{2 (\sigma^2 + x z_x)} \right) x \sqrt{\sigma^2 + z_x} \left( x \partial_x - \frac{n - 1}{2} + \frac{z}{4 \sigma^2 + z_x} \right)
\]

\( \mod \text{Diff}^{1,-2,-5,-3,-6}_{\text{leC}}(X) \) (224)

near \( \partial X \).

\textbf{Proof.} It suffices to restrict attention to \( \hat{X} = [0, \tilde{x}] \times \partial X \).

We refer to eq. (209). Expanding \((1 - \sigma^2 a_{00} x / (\sigma^2 + z_x))^{1/2} = 1 - (1/2) \sigma^2 a_{00} x / (\sigma^2 + z_x) + O(\sigma^4 x^2 / (\sigma^2 + z_x)^2)\) in Taylor series, we deduce that

\[
2i x (1 + x a_{00}) \left( \frac{1 - \sigma^2 a_{00} x}{2(\sigma^2 + z_x)} \right) \sqrt{\sigma^2 + z_x} - \sqrt{\sigma^2 + z_x - \sigma^2 a_{00} x} \left( x \partial_x - \frac{n - 1}{2} + \frac{z}{4 \sigma^2 + z_x} \right)
\]

\( \in \text{Diff}^{1,-2,-6,-3,-7}_{\text{leC}}(X) \). (225)

Thus,

\[
L = 2i(1 + x a_{00}) \left( 1 - \frac{\sigma^2 a_{00} x}{2 (\sigma^2 + x z_x)} \right) x \sqrt{\sigma^2 + z_x} \left( x \partial_x - \frac{n - 1}{2} + \frac{z}{4 \sigma^2 + z_x} \right) - a a_{00} x^4 \partial_x
\]

\( \mod \text{Diff}^{1,-2,-6,-3,-7}_{\text{leC}}(X) \). (226)

On the other hand, \( a a_{00} x^4 \partial_x \in \text{Diff}^{1,-2,-5,-3,-6}_{\text{leC}}(X) \). We conclude eq. (224) from the above. \( \square \)

**Proposition 3.3.** \( \hat{P}_0 \in \text{Diff}^{2,0,-2,-1,-3}_{\text{leC}}(X) \), and

\[
\hat{P}_0 = -(x^2 \partial_x)^2 + x^2 \triangle_{\partial X} + 2i x \sqrt{\sigma^2 + z_x} \left( x \partial_x - \frac{n - 1}{2} + \frac{z}{4 \sigma^2 + z_x} \right)
\]

\( \mod \text{Diff}^{2,-1,-3,-2,-4}_{\text{leC}}(X) \). (227)

\textbf{Proof.} We have \( L(\sigma) \in \text{Diff}^{1,-2,-1,-3}_{\text{leC}}(X) \) and

\[
-(x^2 \partial_x)^2 + x^2 \triangle_{\partial X} \in \text{Diff}^{2,0,-2,-2,-4}_{\text{leC}}(X)
\]

\[
x a_{00} (x^2 \partial_x) \in \text{Diff}^{2,-1,-3,-3,-5}_{\text{leC}}(X)
\]

\[
(n - 1) x^3 \partial_x \in \text{Diff}^{1,-1,-3,-2,-4}_{\text{leC}}(X)
\]

\[
V_{\text{eff}} \in \text{Diff}^{0,-2,-4,-2,-4}_{\text{leC}}(X)
\]

Thus, by Proposition 3.1, \( \hat{P}_0 \in \text{Diff}^{2,0,-2,-1,-3}_{\text{leC}}(X) \), and

\[
\hat{P}_0 = -(x^2 \partial_x)^2 + x^2 \triangle_{\partial X} + L \mod \text{Diff}^{2,-1,-3,-2,-4}_{\text{leC}}(X).
\] (229)

Simplifying \( L \) modulo \( \text{Diff}^{2,-1,-3,-2,-4}_{\text{leC}}(X) \) using Proposition 3.2, we get eq. (227). \( \square \)

**Proposition 3.4.** For some \( Y_1, \ldots, Y_J \in S \text{Diff}^{1,-1,-4,-2,-5}_{\text{leC}}(X) \) which near \( \partial X \) are given by \( Y_j = i x^4 \Phi b_j P_{1,j} \), we have

\[
\hat{P}_1(\sigma) = P_1(\sigma) + \sum_{j=1}^J Y_j + R
\] (230)
for some $R \in C^\infty(0, \infty)_{\sigma^2}; \text{Diff}^2(X)$ which is supported outside of some neighborhood $U \subset X$ of $\partial X$. Thus $\tilde{P}_1 = \{\tilde{P}_1(\sigma)\}_{\sigma \geq 0} \in S \text{Diff}_{\text{leC}}^{2,-1,3,-2,-4}(X)$. If $P_1$ is classical to order $\beta_1 > 0$, then

$$\tilde{P}_1 \in \text{Diff}_{\text{leC}}^{2,-1,3,-2,-4}(X) + S \text{Diff}_{\text{leC}}^{2,-1-\beta_1,-3-2\beta_1,-2-\beta_1,-4-2\beta_1}(X). \quad (231)$$

**Proof.** First observe that $\chi\tilde{P}_1 \in C^\infty([0, \infty)_{\sigma^2}; \text{Diff}^2(X^e))$ for any $\chi \in C^\infty(X^e)$. Now let $P_{j,\text{ext}} \in C^\infty([0, \infty)_{\sigma^2}; S \text{Diff}_{\text{scb}}^{2,-1,-3}(X))$ be equal to $x^4P_{\beta_1,2}\partial_x$ near $\partial X$. Now define

$$\Upsilon_j = e^{-i\Phi}[P_{j,\text{ext}}, e^{i\Phi}] = e^{-i\Phi}[1 - \chi]P_{j,\text{ext}}, e^{i\Phi} + e^{-i\Phi}[\chi P_{j,\text{ext}}, e^{i\Phi}], \quad (232)$$

where $\chi$ is identically equal to one in a sufficiently large open set such that $1 - \chi$ is supported in a neighborhood for which eq. (43) applies. Evidently, we have $e^{-i\Phi}[\chi P_{j,\text{ext}}, e^{i\Phi}] \in C^\infty([0, \infty)_{\sigma^2}; \text{Diff}^2(X \setminus U))$ for some neighborhood $U \subset X$ of $\partial X$. On the other hand, $e^{-i\Phi}[(1 - \chi)P_{j,\text{ext}}, e^{i\Phi}] = i(1 - \chi)x^4\partial_x P_{\beta_1,2}$, so $\Upsilon_j = e^{-i\Phi}[P_{j,\text{ext}}, e^{i\Phi}]$ near $\partial X$.

We now write

$$\tilde{P}_1 = P_1 + e^{-i\Phi}[(1 - \chi)P_1, e^{i\Phi}] + e^{-i\Phi}[\chi P_1, e^{i\Phi}] = P_1 + \sum_{j=1}^{\infty} \Upsilon_j + e^{-i\Phi}[\chi P_1, e^{i\Phi}] \quad (233)$$

Set $R = e^{-i\Phi}[\chi P_1, e^{i\Phi}]$. Then eq. (230) holds, and $R \in C^\infty(0, \infty)_{\sigma^2}; \text{Diff}^2(X)$ is supported outside of some neighborhood $U \subset X$ of $\partial X$.

We observe, from eq. (43) (and Proposition 2.10), that

$$P_1 \in S \text{Diff}_{\text{leC}}^{2,-1,3,-2,-4}(X) + S \text{Diff}_{\text{leC}}^{2,-1,3,-2,-4}(X) \subseteq S \text{Diff}_{\text{leC}}^{2,-1,3,-2,-4}(X) \quad (234)$$

Since $\Upsilon_1, \ldots, \Upsilon_j \in S \text{Diff}_{\text{leC}}^{1,-4,-2,-5}(X)$ (and the same holds for $R$, trivially), we conclude that

$$\tilde{P}_1 \in S \text{Diff}_{\text{leC}}^{2,-1,3,-2,-4}(X), \quad (235)$$

as claimed.

If, $b_j, b_j', b_j''$ satisfy eq. (47), then the conclusion is similar, except now we can write $P_1 \in \text{Diff}_{\text{leC}}^{2,-1,3,-2,-4}(X) + x^{\beta_1}S \text{Diff}_{\text{leC}}^{2,-1,3,-2,-4}(X)$ and

$$\Upsilon_1, \ldots, \Upsilon_j \in \text{Diff}_{\text{leC}}^{1,-4,-2,-5}(X) + x^{\beta_1}S \text{Diff}_{\text{leC}}^{1,-4,-2,-5}(X), \quad (236)$$

which leads to the conclusion eq. (233) as a strengthening of eq. (235). \hfill \Box

**Proposition 3.5.** Given $\delta > 0$ such that $P_2 \in C^\infty([0, \infty)_{\sigma^2}; S \text{Diff}_{\text{scb}}^{2,-1,3,-2,-4}(X))$, $\tilde{P}_2 = \{\tilde{P}_2(\sigma)\}_{\sigma \geq 0} \in S \text{Diff}_{\text{leC}}^{2,-1,3,-2,-4}(X)$.

$$\tilde{P}_2 = \{\tilde{P}_2(\sigma)\}_{\sigma \geq 0} \in \text{Diff}_{\text{leC}}^{2,-4,-4,-2,-4}(X) + S \text{Diff}_{\text{leC}}^{2,-1-\beta_2,-3-2\beta_2,-1-\beta_2,-3-2\beta_2}(X) + x^{3/2+\beta_3}S^0(X). \quad (238)$$

**Proof.** We restrict attention to $\hat{X} = [0, \bar{x}]_x \times \partial X$, which suffices by an argument similar to that in the proof of Proposition 3.4. \hfill \Box
Let \( y = (y_1, \ldots, y_n) \) denote local coordinates on \( \partial X \). In terms of these, we can write

\[
P_2(\sigma) = x^5 c \phi^2 + x^4 \sum_{j=1}^{n-1} c_j \partial_{x_j} \phi + x^3 \sum_{j,k=1}^{n-1} c_{j,k} \partial_{y_j} \phi \partial_{y_k} + dx^3 \partial_x + \sum_{j=1}^{n-1} d_j x^2 \partial_{y_j} + x^{3/2} c \]

where \( \{c, d, e\} \cup \{c_{j,k}, d_{j,k}\}_{j,k=1}^{n-1} \subset C^\infty([0, \infty)_\sigma; S^0(X)) \) and \( c_{j,k} = c_{k,j} \). We then have

\[
P_2(\sigma) - \tilde{P}_2(\sigma) = x^5 \left[ 2ix^5 c \phi' \partial_x + ix^4 \sum_{j=1}^{n-1} c_j (\phi' \partial_{y_j} + \partial_{y_j} \phi \partial_x) + 2ix^3 \sum_{j,k=1}^{n-1} c_{j,k} \partial_{y_j} \phi \partial_{y_k} - x^5 c \phi' \phi' \right]
\]

\[+ x^3 \left[ ix^5 \phi'' + x^4 \sum_{j=1}^{n-1} c_j (i \partial_{y_j} \phi' - \phi' \partial_{y_j} \phi) + x^3 \sum_{j,k=1}^{n-1} c_{j,k} (i \partial_{y_j} \partial_{y_k} \phi - \partial_{y_j} \phi \partial_{y_k} \phi) \right]
\]

\[+ x^\delta \left[ idx^3 \phi' + \sum_{j=1}^{n-1} id_j x^2 \partial_{y_j} \phi \right],
\]

i.e. – since \( \phi \) does not depend on tangential coordinates –

\[
P_2(\sigma) - \tilde{P}_2(\sigma) = x^5 \left[ 2ix^5 c \phi' \partial_x + ix^4 \sum_{j=1}^{n-1} c_j \phi' \partial_{y_j} - x^5 c \phi' \phi' + ix^5 c \phi'' + idx^3 \phi' \right].
\]

It follows from eq. (219) that

\[
x^{3+\delta} d\phi' \in \Psi_{leC}^{0,1-\delta,-3-2\delta,-1-\delta,-3-2\delta}(X), \tag{242}
\]

\[
x^{5+\delta} c \phi' \phi' \in \Psi_{leC}^{0,1-\delta,-4-2\delta,-1-\delta,-4-2\delta}(X) \text{ by eq. (217), and } x^{5+\delta} c \phi'' \in \Psi_{leC}^{0,2-\delta,-5-2\delta,-2-\delta,-5-2\delta}(X)
\]

by eq. (219). On the other hand,

\[
x^{5+\delta} \phi' \partial_x, x^4 \phi' \phi' \partial_{y_j} \in \Psi_{leC}^{1,-1-\delta,-4-2\delta,-2-\delta,-5-2\delta}(X)
\]

by eq. (219). Combining the observations above, in particular eq. (242) and eq. (243), we conclude that

\[
P_2 - \tilde{P}_2 \in S \text{ Diff}_{leC}^{1-1-\delta,-3-2\delta,-1-\delta,-3-2\delta}(X).
\]

By eq. (239) and (Proposition 2.10, eq. (131)),

\[
P_2 \in S \text{ Diff}_{leC}^{1-1-\delta,-3-2\delta,-2-\delta,-4-2\delta}(X) + S \text{ Diff}_{leC}^{0,-3/2-\delta,-3-2\delta,-3-2\delta}(X)
\]

\[
\subset S \text{ Diff}_{leC}^{2-1-\delta,-3-2\delta,-3-2\delta}(X), \tag{245}
\]

We conclude eq. (237) from eq. (244) and eq. (245).

If eq. (48) holds, then instead of eq. (244) we conclude

\[
P_2 - \tilde{P}_2 \in S \text{ Diff}_{leC}^{1-1-\beta_2,-3-2\beta_2,-1-\beta_2,-3-2\beta_2}(X), \tag{246}
\]

and instead of eq. (245) we have

\[
P_2 \in S \text{ Diff}_{leC}^{2-1-\beta_2,-3-2\beta_2,-3-2\beta_2,-3-2\beta_2}(X) + x^{3/2+\beta_3} S^0(X). \tag{247}
\]

Equation (238) follows from eq. (246) and eq. (247).
Proposition 3.6. \( \tilde{P} \in \text{Diff}_{b, \text{leC}}^{2,0,\alpha} (X) + S \text{Diff}_{b, \text{leC}}^{2,-1,\beta} (X) \), with \( \tilde{P} = \tilde{P}_0 \mod S \text{Diff}_{b, \text{leC}}^{2,-1,\gamma} (X) \). If \( P_1, P_2 \) are classical to orders \( \beta_1 \) and \( (\beta_2, \beta_3) \) respectively, then \( \tilde{P} \in \text{Diff}_{b, \text{leC}}^{2,0,\alpha} (X) + x^{\beta_1} S \text{Diff}_{b, \text{leC}}^{2,-1,\beta} (X) + x^{\beta_2} S \text{Diff}_{b, \text{leC}}^{2,-1,\gamma} (X) + x^{3/2 + \beta_3} S^0 (X). \) (248)

Moreover,
\[
\tilde{P} = -(x^2 \partial_x)^2 + x^2 \Delta_{\partial X} + 2ix \sqrt{\sigma^2 + ZX} \left( x \partial_x - \frac{n - 1}{2} + \frac{Z}{4 \sigma^2 + Zx} \right) \mod \text{Diff}_{b, \text{leC}}^{2,-1,\alpha} (X) + S \text{Diff}_{b, \text{leC}}^{2,-1,\beta} (X). \tag{249}
\]

Thus,
\[
\tilde{P} = -(x^2 \partial_x)^2 + x^2 \Delta_{\partial X} + 2ix (\sigma^2 + Zx)^{1/2} x \partial_x \mod \text{Diff}_{b, \text{leC}}^{2,-1,\alpha} (X) + S \text{Diff}_{b, \text{leC}}^{2,-1,\beta} (X). \tag{250}
\]

\[\Box\]

Proof. We have \( \tilde{P}(\sigma) = \tilde{P}_0 + \tilde{P}_1 + \tilde{P}_2 \). As seen in Proposition 3.4 and Proposition 3.5, \( \tilde{P}_1 \in S \text{Diff}_{b, \text{leC}}^{2,-1,\alpha} (X) \) and \( \tilde{P}_2 \in S \text{Diff}_{b, \text{leC}}^{2,-1,\beta} (X) \), so
\[
\tilde{P}_1 + \tilde{P}_2 \in S \text{Diff}_{b, \text{leC}}^{2,-1,\alpha} (X). \tag{251}
\]
(where we are using \( \delta < 1/2 \)). Likewise, if \( P_1, P_2 \) are classical to orders \( \beta_1 \) and \( (\beta_2, \beta_3) \) then Proposition 3.4 and Proposition 3.5 yield
\[
\tilde{P}_1 + \tilde{P}_2 \in S \text{Diff}_{b, \text{leC}}^{2,-1,\gamma} (X) + x^{\beta_1} S \text{Diff}_{b, \text{leC}}^{2,-1,\beta} (X) + x^{\beta_2} S \text{Diff}_{b, \text{leC}}^{2,-1,\gamma} (X) + x^{3/2 + \beta_3} S^0 (X). \tag{252}
\]

By Proposition 3.3, \( \tilde{P}_0 \in S \text{Diff}_{b, \text{leC}}^{2,0,\alpha} (X) \), so \( \tilde{P} \) is in the claimed spaces.

Furthermore, by eq. (251), \( \tilde{P} = \tilde{P}_0 \mod S \text{Diff}_{b, \text{leC}}^{2,-1,\alpha} (X) \). Combining with Proposition 3.3, we get eq. (249).

To conclude this discussion, we let
\[
N(\tilde{P}) = \{ N(\tilde{P}(\sigma)) \}_{\sigma \geq 0}, \quad N(\tilde{P}(\sigma)) = 2i \sqrt{\sigma^2 + ZX} \left( x \partial_x - \frac{n - 1}{2} + \frac{Z}{4 \sigma^2 + Zx} \right) \tag{253}
\]
denote the leC-normal operator, defined initially near \( \partial X \). To avoid technicalities, we extend \( N(\tilde{P}(\sigma)) \) to a differential operator on \( X^\circ \) such that, in any compact subset of \( X^\circ \), \( N(\tilde{P}(\sigma)) \) depends smoothly on \( E = \sigma^2 \), all the way down to \( \sigma = 0 \). Thus:

Proposition 3.7. \( N(\tilde{P}) \in \text{Diff}_{b, \text{leC}}^{1,\alpha} (X) \). \[\Box\]

The following proposition justifies the term “leC-normal operator;”

Proposition 3.8. \( N(\tilde{P}) - \tilde{P} \in S \text{Diff}_{b, \text{leC}}^{2,0,\beta} (X) \subseteq S \text{Diff}_{b, \text{leC}}^{2,\alpha - 1,\delta, \gamma - 3, 2\delta} (X) \). If \( P_1, P_2 \) are classical to orders \( \beta_1 \) and \( (\beta_2, \beta_3) \) respectively, then
\[
N(\tilde{P}) - \tilde{P} \in S \text{Diff}_{b, \text{leC}}^{2,0,\beta} (X) + x^{\beta_1} S \text{Diff}_{b, \text{leC}}^{2,\alpha - 1,\gamma - 3, 2\delta} (X) = S \text{Diff}_{b, \text{leC}}^{2,\gamma - 3, 2\delta} (X) + x^{3/2 + \beta_3} S^0 (X). \tag{254}
\]

\[\Box\]
Proof. We have $N(\tilde{P}) - \tilde{P}(\sigma) = (N(\tilde{P}) - \tilde{P}_0) - \tilde{P}_1 - \tilde{P}_2$. We first check that $N(\tilde{P}) - \tilde{P} \in S\text{Diff}^{2,0,-2,-1,\delta,-3\delta}(X)$.

- By eq. (251), $\tilde{P}_1 + \tilde{P}_2 \in S\text{Diff}^{2,0,-2,-1,\delta,-3\delta}(X)$, and by eq. (228) the same holds for $\tilde{P}_0 - L$, so it suffices to check that

$$N(\tilde{P}) - L \in S\text{Diff}^{2,0,-2,-1,\delta,-3\delta}(X).$$

Indeed, by Proposition 3.2,

$$N(\tilde{P}) - L \in S\text{Diff}^{1,2,-5,-3,-6}(X) + x S\text{Diff}^{-2,-1,3}(X) + \sigma^2 x/(\sigma^2 + z) S\text{Diff}^{1,0,-2,-1,3}(X)$$

$$\subseteq S\text{Diff}^{1,2,-5,-3,-6}(X) + S\text{Diff}^{1,-1,4,-2,-5}(X) + S\text{Diff}^{1,-1,4,-2,-5}(X)$$

$$= S\text{Diff}^{2,0,-2,-1,\delta,-3\delta}(X).$$

(256)

If $P_1, P_2$ are classical to orders $\beta_1, (\beta_2, \beta_3)$, then we instead get eq. (254). □

We now consider the $L^2_{\text{sc}}(X) = L^2(X,g_0)$-based adjoint $\tilde{P}^*$, defined such that

$$\int_X f^* \tilde{P}^* g \, d\text{Vol}_{g_0} = \int_X (\tilde{P}^* f) g \, d\text{Vol}_{g_0}$$

(257)

for all $f,g \in S(X)$. This is a family of differential operators and, by Proposition 2.10, an element of $S\text{Diff}^{2,0,-2,-1,3}(X) \subset S\text{Diff}^{2,0,-2,-1,3}(X)$. We form the differential operators

$$\Re \tilde{P} = \frac{1}{2} (\tilde{P} + \tilde{P}^*), \quad \Im \tilde{P} = \frac{1}{2i} (\tilde{P} - \tilde{P}^*)$$

(258)

the self-adjoint and anti-self-adjoint parts of $\tilde{P}$.

Proposition 3.9. For any exactly conic metric $g_0$, there exists a differential operator $R = R_{g_0} \in S\text{Diff}^{2,0,-2,-1,3-2\delta}(X)$ such that

$$\tilde{P}^* = \tilde{P} + (P_1^* - P_1) + \sum_{j=1}^J (Y_j^* - Y_j) + R,$$

(259)

and, near $\partial X$,

$$Y_j^* \in x^2 \partial_{tt} S^0(X_{pp}^{\text{sc}}) \text{Diff}^1(\partial X) \subseteq S\text{Diff}^{1,-1,4,-2,5}(X)$$

(260)

$$P_1^* = \sum_{j=1}^J [ - x^2 P_{zz,j}^* (b_j^* x^2 \partial_x + x b_{*,j}) + x^3 b_j^* P_{\partial X,j}^* + x^2 b_j^* Q_{\partial X,j}^* ] \in S\text{Diff}^{2,-1,3,2-4}(X)$$

(261)

for some $b_{*,1}, \cdots, b_{*,J} \in S^0(X)$.

Proof. It clearly suffices to restrict attention to $\tilde{X} = [0, \bar{\epsilon}) x \times \partial X$, that is to compute the formal adjoint of $\tilde{P}$ with respect to

$$L^2_{\text{sc}}(\tilde{X}) = L^2([0, \bar{\epsilon}) x \times \partial X, x^{-(n+1)} \, dx \, d\text{Vol}_{g_{0x}}(y))$$

(262)

up to the required order.

Begin with $\tilde{P}_0$, which we rewrite as

$$\tilde{P}_0 = -(1 + xa_{00}) \left( x^2 \partial_x - \frac{x(n-1)}{2} \right)^2 + x^2 \Delta_{\partial X} - x^4(n-1)a_{00} \partial_x + L(\sigma) + W$$

(263)
for $W \in x^2 C^\infty (X_{sc})$.

- We see that $x^2 \Delta_{\partial X} = x^2 \Delta_{a_{0x}}$ is formally self-adjoint on $L^2_{sc}([0,\bar{x}) \times \partial X)$, and
- the adoint of $W$ is its complex conjugate $W^* \in x^2 C^\infty (X_{sc})$.
- We also have $a_{0x} x^2 \partial_x \in S \text{Diff}_{leC}^{1,-2,-5,-3,-6} (X)$, thus $(a_{0x} x^2 \partial_x)^* \in S \text{Diff}_{leC}^{1,-2,-5,-3,-6} (X)$.
- On the other hand, $x^2 \partial_x - x(n-1)/2$ is formally anti-self-adjoint on $L^2_{sc}([0,\bar{x}) \times \partial X)$, so

$$
\left[ (1 + x a_{00}) \left( x^2 \partial_x - \frac{x(n-1)}{2} \right) \right]^* = \left( x^2 \partial_x - \frac{x(n-1)}{2} \right)^2 (1 + x a_{00})
= (1 + x a_{00}) \left( x^2 \partial_x - \frac{x(n-1)}{2} \right)^2 + a_{00} \left[ \left( x^2 \partial_x - \frac{x(n-1)}{2} \right)^2, x \right]
= (1 + x a_{00}) \left( x^2 \partial_x - \frac{x(n-1)}{2} \right)^2 + 2 a_{00} x^4 \partial_x + (3-n) a_{00} x^3.
$$

- By the same computation opening this subsection $N(\tilde{P})$, is formally self-adjoint. So, by Proposition 3.2,

$$
L^* = L + [N(\tilde{P}), (1 + x a_{00})(1 - \sigma^2 a_{00} x/(\sigma^2 + Z x)^{-1})] \mod \text{Diff}_{leC}^{1,-2,-5,-3,-6} (X)
= L + 2i x \sigma^2 + Z x \partial_x, (1 + x a_{00})(1 - \sigma^2 a_{00} x/(\sigma^2 + Z x)^{-1})] \mod \text{Diff}_{leC}^{1,-2,-5,-3,-6} (X)
= L \mod \text{Diff}_{leC}^{2,-5,-2,-5}(X) = L \mod \text{Diff}_{leC}^{1,-2,-4,-2,-4}(X).
$$

(264)

So,

$$
\tilde{P}_0^* = \tilde{P}_1 \mod S \text{Diff}_{leC}^{1,-2,-4,-2,-4}(X).
$$

(265)

On the other hand, we trivially have from Proposition 3.5 that

$$
\tilde{P}_2^* \in S \text{Diff}_{leC}^{2,-1-\delta,-3-2\delta,-1-\delta,-3-2\delta}(X).
$$

(266)

Now define $R = \tilde{P}^* - \tilde{P} - (P_1 - P_1) - \sum_{j=1}^J (Y_j - Y_j)$, so that eq. (259) holds by construction. Rearranging this definition,

$$
R = (\tilde{P}_0^* - \tilde{P}_0) + (\tilde{P}_1^* - \tilde{P}_1) + (\tilde{P}_2^* - \tilde{P}_2) - (P_1 - P_1) - \sum_{j=1}^J (Y_j - Y_j)
= (\tilde{P}_0^* - \tilde{P}_0) + (\tilde{P}_2^* - \tilde{P}_2) + \left( \left[ P_1 - P_1 - \sum_{j=1}^J Y_j \right]^* - \left[ P_1 - P_1 - \sum_{j=1}^J Y_j \right] \right).
$$

(267)

Thus, using eq. (265), eq. (266), Proposition 3.5, and Proposition 3.4,

$$
R \in S \text{Diff}_{leC}^{1,-2,-4,-2,-4}(X) + S \text{Diff}_{leC}^{2,-1-\delta,-3-2\delta,-1-\delta,-3-2\delta}(X)
= S \text{Diff}_{leC}^{2,-1-\delta,-3-2\delta,-1-\delta,-3-2\delta}(X),
$$

(268)

as claimed.
Equation (260) follows from the observation that $\mathcal{Y}_j^* = -ix^4 \Phi b_j P_{\perp,j}^*$ near $\partial X$, where $P_{\perp,j}^*$ is computed using the $L^2(\partial X, g_{\partial X})$-inner product. On the other hand,

$$P_1^* = \sum_{j=1}^J \left[ P_{\perp,j}^* (x^2 b_j^* (x^2 \partial_x)^* + (x^2 \partial_x)^* b_j^*) + x^3 b_j^* P_{\partial X}^* + x^2 b_j^* Q_{\partial X,j}^* \right]$$

$$= \sum_{j=1}^J \left[ P_{\perp,j}^* (x^2 b_j^* (-x^2 \partial_x + x(n-1)) - x[x \partial_x, x^2 b_j^*]) + x^3 b_j^* P_{\partial X,j}^* + x^2 b_j^* Q_{\partial X,j}^* \right]$$

$$= \sum_{j=1}^J \left[ P_{\perp,j}^* (x^2 b_j^* (-x^2 \partial_x + x(n-1)) - 2x^3 b_j^* - x^3 (x \partial_x b_j^*)) + x^3 b_j^* P_{\partial X,j}^* + x^2 b_j^* Q_{\partial X,j}^* \right]$$

(269)

near $\partial X$, where $b_j, b_j^*$ are as in eq. (43). Equation (261) follows from this, for some choice of $b_{*,j}$.

We see from the above that

$$\mathfrak{P} \in S \text{Diff}_{leC}[-1, -4, -2, -5](X) + S \text{Diff}_{leC}[-2, -1, -3, -2, -4](X) + S \text{Diff}_{leC}[-1, -\delta, -3, -2\delta, -1, -\delta, -3, -2\delta](X)$$

$$\subset S \text{Diff}_{leC}[-1, -3, -1, -\delta, -3, -2\delta](X).$$

(270)

Thus, $\mathfrak{P}$ is one order lower than $\mathfrak{P}$ at sf and ff and slightly lower order at bf and tf, and we have been entirely explicit about the leading terms of $\mathfrak{P}$ (namely $\mathfrak{P}P_1$ and $\mathfrak{P}\mathcal{Y}_j$) at sf and ff, the remainder $(2i)^{-1} R$ being slightly more than one order lower than $\mathfrak{P}$ at both faces.

4. Situation at Zero Energy

In this section we consider $P(0)$ and $\mathfrak{P}(0)$ in some detail. Specifically, we apply [Vas21a, Theorem 1.1] to study the strong limit

$$R(E = 0; Z \pm i\epsilon) = \lim_{\epsilon \to 0^+} R(E = 0; Z \pm i\epsilon)$$

(271)

used in the statement of Theorem 1.1 to characterize the resolvent output at zero energy. The mapping properties of this operator will be used in §6 in order to prove the smoothness of the output of the conjugated resolvent at positive energy all the way down to zero energy (as used e.g. in Corollary 1.3).

Recall that $x_{1/2} = 2^{-1/2} x^{1/2}$. Then, from the form eq. (41) of $P_0$,

$$P_0(0) = -(1 + 2x_2^2 a_0(0))(x_2^2 \partial_{x_2})^2 + 4x_2^2 \Delta_{\partial X} + 2x_2^3 (a(0) + n - 1) \partial_{x_2} - 2Z x_2^2$$

$$= -x_2^2 (1 + 2x_2^2 a_0(0))(x_2^2 \partial_{x_2})^2 + 4x_2^2 \Delta_{\partial X} + 2x_2^3 \left[a(0) + n - \frac{3}{2} - x_2^2 a_0(0) \right] \partial_{x_2} - 2Z x_2^2$$

(272)

(Recall that we are notationally suppressing the dependence of $a_0, a$ on $y \in \partial X$.) (The extra $-x_{1/2}^{-n/2} P_0(0)x_{1/2}^{-n/2}$ in eq. (272) is the source of the $(\sigma^2 + Z x)^{-1/4}$ term in eq. (10).) Therefore

$$x_{1/2}^{-1-n/2} P_0(0)x_{1/2}^{-n/2} = -(1 + 2x_{1/2}^2 a_0(0))(x_{1/2}^2 \partial_{x_{1/2}})^2 + 4x_{1/2}^2 \Delta_{\partial X} - 2Z$$

$$+ x_2^3 \left[2a(0) + n - 1 - 2(n - 1)x_2^2 a_0(0) \right] \partial_{x_2} + (n - 2) \left[a(0) + \frac{3n}{4} - \frac{3}{2} - nx_2^2 a_0(0) \right] x_2^2.$$
Thus, \( x_{1/2}^{-1-n/2} \tilde{P}_0(0)x_{1/2}^{-1+n/2} \in \text{Diff}_b(X_{1/2}) \) has the same form as the conjugated spectral family at positive energy, except with respect to \( x_{1/2} \) instead of \( x \) (and with an extra short-range potential in eq. (273)).

We generalize this observation:

**Proposition 4.1.** If \( g \) is an asymptotically conic metric on \( X \), and if we set \( g_{1/2} = x_{1/2}^2 g \), then there exists some \( V_{\text{eff}} \in x^2 S^0(X) \) such that

\[
x_{1/2}^{-1-n/2} \Delta_g x_{1/2}^{-1+n/2} = x_{1/2}^{-2} \Delta_{g_{1/2}} + V_{\text{eff}}.
\] (274)

holds.

**Proof.** We first want to show that \( x_{1/2}^{-1-n/2} \Delta_g x_{1/2}^{-1+n/2} - x_{1/2}^{-2} \Delta_{g_{1/2}} \) is zeroth order (and therefore a function on \( X^\circ \)). Indeed, \( \Delta_g = x_{1/2}^2 \Delta_{g_{1/2}} + (n-2)x_{1/2} \nabla_{g_{1/2}} x_{1/2} \), so

\[
x_{1/2}^{-1-n/2} \Delta_g x_{1/2}^{-1+n/2} = x_{1/2}^{-2} \Delta_{g_{1/2}} + x_{1/2}^{-2} \Delta_{g_{1/2}} x_{1/2}^{-1+n/2} + (n-2)x_{1/2} g_{1/2} (dx_{1/2}, dx_{1/2}^{(n-2)/2}) \tag{275}
\]

It is therefore the case that \( x_{1/2}^{-1-n/2} \Delta_g x_{1/2}^{-1+n/2} - x_{1/2}^{-2} \Delta_{g_{1/2}} = V_{\text{eff}} \) for

\[
V_{\text{eff}} = x_{1/2}^{-3-n/2} \Delta_{g_{1/2}} x_{1/2}^{-1+n/2} + (n-2)x_{1/2} g_{1/2} (dx_{1/2}, dx_{1/2}^{(n-2)/2}) \tag{276}
\]

Since \( g_{1/2} \) is an asymptotically conic metric on \( X_{1/2} \) (see below), the fact that \( V_{\text{eff}} \in x^2 S^0(X) \) can be read off eq. (276), but we check in local coordinates.

It suffices to restrict attention to a neighborhood of \( \partial X \). Let \( y_1, \ldots, y_{n-1} \) denote a local system of coordinates on \( \partial X \). Then, using \( g = x_{1/2}^2 g_{1/2} \),

\[
\Delta_g = -\frac{x_{1/2}^2}{|g_{1/2}|^{1/2}} \partial_i (x_{1/2}^{-2} |g_{1/2}|^{1/2} g_{1/2}^{ij} \partial_j)
\] (277)

where \( \partial_0 = \partial x_{1/2} \) and \( \partial_i = \partial y_i \) for \( i = 1, \ldots, n-1 \). Conjugating the right-hand side of eq. (277) by \( x_{1/2}^{-(n-2)/2} \), we see that

\[
x_{1/2}^{-1-n/2} \Delta_g x_{1/2}^{-1+n/2} = -\frac{x_{1/2}^2}{|g_{1/2}|^{1/2}} \partial_i (|g_{1/2}|^{1/2} g_{1/2}^{ij} \partial_j) + 4^{-1} n(n-2) g_{1/2}^{00}
\]

\[
= x_{1/2}^2 \Delta_{g_{1/2}} - \frac{x_{1/2}^2}{|g_{1/2}|^{1/2}} \partial_i (|g_{1/2}|^{1/2} g_{1/2}^{0i}) + 4^{-1} n(n-2) g_{1/2}^{00} \tag{278}
\]

Since \( g_{1/2}^{00} \in x_{1/2}^4 S^0(X_{1/2}) = x^2 S^0(x) \), \( 4^{-1} n(n-2) g_{1/2}^{00} \in x_{1/2}^4 S^0(X_{1/2}) \). Likewise, we see that \( |g_{1/2}|^{-1/2} \partial_0 |g_{1/2}|^{1/2} \in x_{1/2}^{-1} S^0(X_{1/2}) \), \( |g_{1/2}|^{-1/2} \partial_i |g_{1/2}|^{1/2} \in S^0(X_{1/2}) \) for \( i \neq 0 \), and \( \partial_i g^{00} \in x_{1/2}^4 S^0(X_{1/2}) \) for \( i \neq 0 \). So,

\[
V_{\text{eff}} = -\frac{x_{1/2}}{|g_{1/2}|^{1/2}} \frac{n-2}{2} \partial_i (|g_{1/2}|^{1/2} g_{1/2}^{0i}) + 4^{-1} n(n-2) g_{1/2}^{00} \in x_{1/2}^4 S^0(X_{1/2}) = x^2 S^0(X). \tag{279}
\]
Observe that if $g$ is an asymptotically conic metric on $X$, then $g_{1/2}$ is an asymptotically conic metric on $X_{1/2}$. Indeed, $g_{1/2}$ is certainly a Riemannian metric on $X^0 = X^1_{1/2}$, and it is a sum of $x_1^2 g_0$, which is exactly conic on $X_{1/2}$, and terms in

\[
x^2 C^\infty(X; \text{sc} \Sym^2 T^* X) \subset x_1^2 C^\infty(X; \text{sc} \Sym^2 T^* X_{1/2}),
\]

(280)

\[
x^{2+\delta} S^0(X; \text{sc} \Sym^2 T^* X) \subset x_1^{2+\delta} S^0(X; \text{sc} \Sym^2 T^* X_{1/2}).
\]

(281)

Consequently, by Proposition 4.1, if $P$ is the spectral family of an attractive Coulomb-like Schrödinger operator, then

\[
x_{1/2}^{-1-n/2} P(0) x_{1/2}^{-1-n/2} = x_{1/2}^{-2} (x_{1/2}^{-1+n/2})^{-1} P(0) x_{1/2}^{-1+n/2}
\]

(282)

is a member $P_0(2Z) = \Delta_{g_{1/2}} - 2Z - W$ of the spectral family $\{P_0(\zeta) = P_0(0) - \zeta\}_{\zeta \geq 0}$ of a Schrödinger operator

\[
P_0(0) = \Delta_{g_{1/2}} + W
\]

(283)

on $X_{1/2}$, where the potential $W \in x S^0(X) = x_1^2 S^0(X_{1/2})$ is short-range.

Thus, $P_0(2Z)$ satisfies the hypotheses of [Vas21a, §3], with $2^{1/2} Z^{1/2}$ in place of $\sigma$ and $X_{1/2}$ in place of $X$. Moreover, as seen from eq. (58) with $a = 0$, the phase $\Phi(\zeta)$ is just that used by Vasy’s in his conjugation. Thus, $\tilde{P}(0)$ has the form of Vasy’s conjugated operator (with $2^{1/2} Z^{1/2}$ in place of $\sigma$ and $X_{1/2}$ in place of $X$). In order to denote the $Z$ dependence of $P(0)$ and $\tilde{P}(0)$, we write

\[
P(0) = P(0; Z) = x_{1/2}^{1+n/2} P_0(2Z) x_{1/2}^{-1-n/2}
\]

(284)

and $\tilde{P}(0) = \tilde{P}(0; Z)$. For $\epsilon > 0$, let $P(0; Z + i\epsilon)$ denote $P(0)$ with $Z$ replaced by $Z + i\epsilon$. Since $S(X_{1/2}) = S(X)$ and $S'(X_{1/2}) = S'(X)$, the limiting absorption principle (as in [Mel94]) applies in the following form: for $\epsilon > 0$, the resolvent

\[
R_0(2Z + 2i\epsilon) = x_1^{-1-n/2} R(0; Z + i\epsilon) x_1^{(n-2)/2} : S(X) \to S'(X)
\]

(285)

of $P_0(0)$ (evaluated at “energy” $\zeta = 2Z + 2i\epsilon$) — defined e.g. via the functional calculus — admits a strong limit $x_1^{-1-n/2} R(0; Z + i0) x_1^{1+n/2} : S(X) \to S'(X)$. Using [Vas21a, Theorem 1.1], we can construct this resolvent as a map between suitable Sobolev spaces using the conjugated perspective:

**Proposition 4.2.** If $P(\sigma)$ is the spectral family of an attractive Coulomb-like Schrödinger operator on $X$, then, for any $m, \varsigma, \ell \in \mathbb{R}$ satisfying $\ell < -3/2 < \varsigma$,

\[
\tilde{P}(0; Z) : \{u \in H_{\text{scb}}^{m,\varsigma+n/2,\ell+n/2}(X_{1/2}) : \tilde{P}(0) u \in H_{\text{scb}}^{m-2,\varsigma+3+n/2,\ell+3+n/2}(X_{1/2})\}
\]

\[
\to H_{\text{scb}}^{m-2,\varsigma+3+n/2,\ell+3+n/2}(X_{1/2})
\]

(286)

is invertible, and the inverse

\[
\tilde{R}_+(0; Z) : H_{\text{scb}}^{m-2,\varsigma+3+n/2,\ell+3+n/2}(X_{1/2}) \to H_{\text{scb}}^{m,\varsigma+n/2,\ell+n/2}(X_{1/2})
\]

(287)

is related to $R(0; Z + i0)$ by the formula $e^{-i\Phi(\zeta)} \tilde{R}_+(0; Z) e^{-i\Phi(\zeta)} f = R(0; Z + i0) f$, which holds for all $f \in S(X)$. 


Proof. Suppose that we are given $m, \varsigma_1/2, \ell_{1/2} \in \mathbb{R}$ with $\ell_{1/2} < -1/2 < \varsigma_1/2$. Then, combining the observations above and [Vas21a, Theorem 1.1],

$$
e^{-i\Phi(-0)}P_0(2Z)e^{+i\Phi(-0)} = x_{1/2}^{-1-n/2} \tilde{P}(0; Z)x_{1/2}^{-1+n/2} : 
$$

$$
\{ u \in H_{scb}^{m,\varsigma_1/2,\ell_{1/2}}(X_{1/2}) : x_{1/2}^{-1-n/2} \tilde{P}(0; Z)x_{1/2}^{-1+n/2} u \in H_{scb}^{m-2,\varsigma_1/2,1,\ell_{1/2}+1}(X_{1/2}) \} 
\rightarrow H_{scb}^{m-2,\varsigma_1/2,1,\ell_{1/2}+1}(X_{1/2}) \tag{288}
$$
is invertible, defining a continuous linear map

$$
\tilde{R}_0(Z) : H_{scb}^{m-2,\varsigma_1/2,1,\ell_{1/2}+1}(X_{1/2}) \rightarrow H_{scb}^{m,\varsigma_1/2,\ell_{1/2}}(X_{1/2}), 
$$

and this is related to the limiting resolvent $R_0(2Z + i0)$ of the spectral family $P_0 = \{ P_0(0) - \varsigma \}_{\varsigma \geq 0}$ of the Schrödinger operator $P_0(0)$ on $X_{1/2}$ by

$$
e^{+i\Phi(-0)}\tilde{R}_0(0; Z)e^{-i\Phi(-0)} f = R_0(2Z + i0)f = x_{1/2}^{-1-n/2} R(0; Z + i0)x_{1/2}^{-1+n/2} f, \tag{290}
$$

which holds for all $f \in \mathcal{S}(X)$.

Call the ($Z$-dependent) domain and codomain of eq. (288)

$$\begin{align*}
\mathcal{X}_{m,\varsigma_1/2,\ell_{1/2}} &= \{ u \in H_{scb}^{m,\varsigma_1/2,\ell_{1/2}}(X_{1/2}) : x_{1/2}^{-1-n/2} \tilde{P}(0; Z)x_{1/2}^{-1+n/2} u \in H_{scb}^{m-2,\varsigma_1/2,1,\ell_{1/2}+1}(X_{1/2}) \} \\
\mathcal{Y}_{m,\varsigma_1/2,\ell_{1/2}} &= H_{scb}^{m-2,\varsigma_1/2,1,\ell_{1/2}+1}(X_{1/2}).
\end{align*}
$$

Setting

$$\begin{align*}
\mathcal{X}_{m,\varsigma,\ell} &= x_{1/2}^{-1+n/2} \mathcal{X}_{m,\varsigma_0,\ell_0} \\
\mathcal{Y}_{m,\varsigma,\ell} &= H_{scb}^{m-2,\varsigma_0+3+n/2,\ell_0+3+n/2}(X_{1/2})
\end{align*}
$$

for $\varsigma = \varsigma_1/2 - 1$ and $\ell = \ell_{1/2} - 1$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X}_{m,\varsigma,\ell} & \xrightarrow{x_{1/2}^{-1+n/2}} & \mathcal{Y}_{m,\varsigma_0,\ell_0} \\
\mathcal{X}_{m,\varsigma,\ell} & \xrightarrow{\tilde{P}(0)} & \mathcal{Y}_{m,\varsigma,\ell}
\end{array}
$$

in the category of Banach spaces. The vertical arrows are manifestly isomorphisms, and as observed the top horizontal arrow is as well.

Hence, $\tilde{P}(0) = \tilde{P}(0; Z)$ is invertible, and the inverse $\tilde{R}_+(0) = \tilde{R}_+(0; Z)$ has the properties specified in the proposition. \square

And for the b-Sobolev spaces:

**Proposition 4.3.** If $P(\sigma)$ is the spectral family of an attractive Coulomb-like Schrödinger operator on $X$, for any $m, \ell = 2l \in \mathbb{R}$ satisfying $\ell < -3/2 < m + \ell$,

$$
\begin{align*}
\tilde{P}(0; Z) : \{ u \in H_{b}^{m,\ell+n/2}(X_{1/2}) : \tilde{P}(0; Z)u \in H_{b}^{m,\ell+3+n/2}(X_{1/2}) \} &\rightarrow H_{b}^{m,\ell+3+n/2}(X_{1/2}) \\
\{ u \in H_{b}^{m,\ell}(X) : \tilde{P}(0; Z)u \in H_{b}^{m,\ell+3/2}(X) \} &\rightarrow H_{b}^{m,\ell+3/2}(X)
\end{align*}
\tag{294}
$$
is invertible.

Proof. The equality
\[
\{ u \in H^m_{b,\ell,n+2}(X_{1/2}) : \hat{P}(0; Z)u \in H^m_{b,\ell+3+n/2}(X_{1/2}) \} = \{ u \in H^m_{b,\ell}(X) : \hat{P}(0; Z)u \in H^m_{b,\ell+3/2}(X) \}
\]
follows from Lemma 2.18.

First of all, setting \( \varsigma = m + \ell \),
\[
\{ u \in H^m_{b,\ell,n+2}(X_{1/2}) : \hat{P}(0; Z)u \in H^m_{b,\ell+3+n/2}(X_{1/2}) \} \subset X_{m,\varsigma,\ell},
\]
and so eq. (294) is injective (by Proposition 4.2). Conversely,
\[
H^m_{b,\ell+3/2}(X) \subset \hat{Y}_{m,\varsigma,\ell},
\]
so given any \( f \in H^m_{b,\ell+3/2}(X) \) there exists a \( u \in \hat{X}_{m,\varsigma,\ell} \) such that \( \hat{P}(0; Z)u = f \). Thus, \( u \in H^m_{b,\ell+n/2}(X_{1/2}) \), and we already know that \( f = \hat{P}(0; Z)u \in H^m_{b,\ell+3+n/2}(X_{1/2}) \), so \( u \) is actually in the codomain of eq. (294). Thus, eq. (294) is surjective. \( \square \)

5. Symbolic Estimates

We now proceed to establish quantitative control of \( u \in S'(X) \) in terms of \( \hat{P}u \) microlocally in the symbolic region of the leC-phase space \( \text{leC} T^* X \), meaning at \( df \cup sf \cup tf \). Since we make no attempt to be uniform in the \( \sigma \to \infty \) limit, we simply restrict attention to \( \sigma \in [0, \Sigma] \) for some arbitrary \( \Sigma > 0 \), and the estimates will all depend on \( \Sigma \) in some unexamined way. The main result of this section, duplicated below as Proposition 5.14, says:

- for every \( \Sigma > 0 \), \( N \in \mathbb{N} \), and \( m, s, \varsigma, l, \ell, s_0, s_0 \in \mathbb{R} \) satisfying \( l < -1/2 < s_0 < s \) and \( \ell < -3/2 < \varsigma \leq \ell + s - l \), there exists a constant \( C = C(\hat{P}, \Sigma, N, m, s, \varsigma, l, \ell) > 0 \) such that
\[
\| u \|_{H^m_{\text{leC},s,\varsigma,\ell}} \leq C (\| \hat{P}u \|_{H^m_{\text{leC},-2,+,s+3,\varsigma+3,\ell+3,\ell+3}} + \| u \|_{H^{-N,\ell,\ell}}) \tag{298}
\]
holds for all \( u \in S'(X) \) and \( \sigma \in [0, \Sigma] \) such that \( \| u \|_{H^{-N,N,m,s,\varsigma,l,\ell}}(X(\sigma)) < \infty \).

(As mentioned in the introduction, [Vas21a, §5] suffices for the analysis of the \( \sigma \to \infty \) regime.) The contents of this section should be compared to the contents of [Vas21a, §4], as our argument below is very similar to some of the symbolic computations there. Using Lemma 2.16, in order to prove eq. (298) it suffices to establish quantitative control of \( u \) within each member of some finite collection of open subsets of the leC-phase space covering \( df \cup sf \cup tf \) — see Figure 5.

By eq. (250),
\[
\hat{P} = -(x^2 \partial_x)^2 + x^2 \Delta_{\partial X} + 2ix(\sigma^2 + Zx)^{1/2}x \partial_x \mod S \text{Diff}^2_{\text{leC}}(X) \tag{299}
\]
(near \( \partial X \)). Thus, the leC-principal symbol \( \sigma^2_{\text{leC}} = \sigma^2_{\text{leC}}(\hat{P}) \in S^2_{\text{leC}}(X) \) of \( \hat{P} \) has a representative of the form
\[
\tilde{p} = \tilde{p}_0 + \tilde{p}_{1,2}
\]
for \( \tilde{p}_{1,2} \in S^2_{\text{leC}}(X) \), where \( \tilde{p}_0 \) is a representative of \( \sigma^2_{\text{leC}} = \sigma^2_{\text{leC}}(\hat{P}_0) \). Thus,
\[
\tilde{p}_0 = x^2 \xi_b^2 + x^2 \eta_b^2 - 2x(\sigma^2 + Zx)^{1/2} \xi_b \tag{301}
\]
(near \( \partial X \)). (Recall from §2 that \( \xi_b \) is the b-cofiber coordinate dual to \( x \) and \( \eta = \eta_b \in T^* \partial X \).) The ellipticity of \( \hat{P} \in S \text{Diff}^2_{\text{leC}}(X) \) (see Proposition 5.2) at \( \partial \) and near \( df \) makes establishing control there trivial, so we work on \( \text{leC} T^* X = \text{leC} T^* X \setminus df \) and establish control at \( \{ x = 0 \} \subset \text{leC} T^* X \).
Figure 5. Schematic of the proof of eq. (298) when \( \dim \partial X = 1 \). The situation over a point \( p \in (\partial (0, \vec{x})_{\text{res}} \setminus z f^o) \times \partial X \) in the sc,leC-phase space is illustrated, with \( \Sigma = \text{Char}_{\text{sc,leC}}^2(\tilde{P}) \) in red. There are four subsets of interest: the dark gray neighborhood of \( \mathcal{R}_+ \), controlled using a high regularity radial point estimate (§5.2), the dark gray neighborhood of \( \mathcal{R}_0 = o_\partial \), controlled using a low regularity radial point estimate (§5.3) and an elliptic estimate (§5.1), the dashed gray set (which stays away from \( \mathcal{R}_+, \mathcal{R}_0 \)) over which we can propagate regularity (§5.2), and the rest of the cofiber (light gray background), where elliptic estimates apply (§5.1). Cf. [Mel94, Figure 2]. The direction of the Hamiltonian flow of \( \tilde{P} \) is indicated with arrows. (Note that only the “vertical” components of the flow are drawn — the Hamiltonian flow also changes \( p \).) The characteristic set of \( P \) is depicted as a small dotted circle.

Thus, \( \tilde{p}_{1,2} \) is irrelevant for the symbolic considerations below (except those in §5.1), all of which are restricted away from \( df \).

In order to investigate the dynamics away from \( bf \cup tf \), we introduce new coordinates on \( \mathbb{R}^+ \times T^* X^o \) (over some collar neighborhood of \( \partial X \), not including the boundary itself) by

\[
\begin{align*}
\xi_{\text{sc,leC}} &= \xi_\partial \theta_\partial \theta_0 \theta_\partial \theta_0 = \xi_\partial \theta_\partial \theta_0 \theta_\partial \theta_0 \\
\eta_{\text{sc,leC}} &= \eta \theta_\partial \theta_0 \theta_\partial \theta_0 = \eta \theta_\partial \theta_\partial \theta_\partial \theta_0 \\
&\in T^* \partial X, \\
&\text{(302)}
\end{align*}
\]

for \( \xi_\partial \in \mathbb{R} \) and \( \eta = \eta_\partial \in T^* \partial X \). More explicitly, \( \xi_{\text{sc,leC}} = \xi_\partial \theta_\partial (\sigma^2 + Z x)^{-1/2}, \eta_{\text{sc,leC}} = \eta \theta_\partial (\sigma^2 + Z x)^{-1/2} \). These extend to fiber coordinates on the bundle \([0, \vec{x})_{\text{res}} \times \mathbb{R}_{\xi_{\text{sc,leC}}} \times (T^* \partial X)_{\eta_{\text{sc,leC}}} \rightarrow [0, \vec{x})_{\text{res}} \), and we write

\[
\begin{align*}
\text{sc,leC}_{T^*_\partial X} \partial X &= (\partial ([0, \vec{x})_{\text{res}} \setminus z f^o) \times \mathbb{R}_{\xi_{\text{sc,leC}}} \times (T^* \partial X)_{\eta_{\text{sc,leC}}} \\
&= \partial (([0, \vec{x})_{\text{res}} \setminus z f^o) \times \mathbb{R}_{\xi_{\text{sc,leC}}} \times (T^* \partial X)_{\eta_{\text{sc,leC}}}). \\
&\text{(304)}
\end{align*}
\]

(We do not endow \( \text{sc,leC}_{T^*_\partial X} \partial X \) with any more structure than that of a set.) Let \( o \) denote the zero section of \( T^* \partial X \). While \( \text{sc,leC}_{T^*_\partial X} \partial X \) is not a nwc, \( [0, \vec{x})_{\text{res}} \times \mathbb{R}_{\xi_{\text{sc,leC}}} \times T^* \partial X \) is.

The punctured space \( \text{sc,leC}_{T^*_\partial X} \partial X \setminus o_\partial, o_\partial = \{ x = 0, \xi_{\text{sc,leC}} = 0 \text{ and } \eta_{\text{sc,leC}} \in o \} \), can be identified with \( sf \cup ff \setminus (df \cup bf \cup tf) \).
Proposition 5.1. The identity map \( i : \{ 0 \leq x < \bar{x} \} \cap \mathbb{R}^+_\bar{x} \times T^*X^0 \rightarrow \{ 0 \leq x < \bar{x} \} \cap \mathbb{R}^+_\bar{x} \times T^*X^0 \) composed with \((D_i)^* : \{ 0 \leq x < \bar{x} \} \cap (\mathbb{R}^+_\bar{x} \times T^*X^0) \rightarrow \mathbb{R}^+_\bar{x} \times \xi \times T^*\partial X \) (where \( i \) is the boundary collar) extends (uniquely) to a smooth map
\[
\tilde{i} : \{ 0 \leq x < \bar{x} \} \cap \text{leC}T^*X \rightarrow \{ 0, \bar{x} \}\text{res} \times \mathbb{R}_{\xi, \text{leC}} \times (T^*\partial X)_{\eta_{\text{bc,leC}}} \quad (305)
\]
restricting to a diffeomorphism \( \{ 0 \leq x < \bar{x} \} \cap \text{leC}T^*X \setminus (df \cup \text{bf} \cup \text{tf}) \rightarrow \{ 0, \bar{x} \}\text{res} \times \mathbb{R}_{\xi, \text{leC}} \times (T^*\partial X)_{\eta_{\text{bc,leC}}} \setminus \partial \).

Proof. Here we are using the boundary collar \( i \) (really \((D_i)^*\)) to identify \( \{ 0 < x < \bar{x} \} \cap (\mathbb{R}^+_\bar{x} \times T^*X^0) \) with \( \mathbb{R}^+_\bar{x} \times (0, \bar{x}) \times \xi \times T^*\partial X \).

Using \( i \), we see that \( \xi_{\text{bc,leC}} \) is a smooth function on the domain of eq. \((305)\), as is \( \eta_{\text{bc,leC}} \in T^*\partial X \). Likewise, the (smooth) projection \( \text{leC}T^*X \rightarrow X_{\text{res}} \) fits into a composition
\[
\{ 0 \leq x < \bar{x} \} \cap \text{leC}T^*X \rightarrow \{ 0 \leq x < \bar{x} \} \cap X_{\text{res}} \xrightarrow{i^{-1}} X_{\text{res}} = \{ 0, \bar{x} \}\text{res} \times \partial X \rightarrow \{ 0, \bar{x} \}\text{res}, \quad (306)
\]
where \( \tilde{X} = \{ 0, \bar{x} \} \times \partial X \), which shows that the \( \{ 0, \bar{x} \}\text{res} \) component of eq. \((305)\) is smooth. Thus, \( \tilde{i} \) is smooth.

The diffeomorphism clause follows from the inversion formulas
\[
\varrho_{\text{df}} = \left[ 1 + \frac{x}{\sigma^2 + \bar{Z}x} + \left( \frac{x^2}{\sigma^2 + \bar{Z}x} + \xi_{\text{bc,leC}}^2 + \eta_{\text{bc,leC}}^2 \right)^{1/2} \right]^{-1}, \quad (307)
\]
\[
\varrho_{\text{bf}} = \frac{x}{\sigma^2 + \bar{Z}x} \left[ 1 + \left( \frac{x}{\sigma^2 + \bar{Z}x} + \left( \frac{x^2}{\sigma^2 + \bar{Z}x} + \xi_{\text{bc,leC}}^2 + \eta_{\text{bc,leC}}^2 \right)^{1/2} \right)^{-1} \right], \quad (308)
\]
\[
\varrho_{\text{tf}} = (\sigma^2 + \bar{Z}x)^{1/2} \left( \frac{x}{\sigma^2 + \bar{Z}x} + \xi_{\text{bc,leC}}^2 + \eta_{\text{bc,leC}}^2 \right)^{-1/2} + 1, \quad (309)
\]
\[
\varrho_{\text{bf}} = \left[ 1 + \frac{x}{\sigma^2 + \bar{Z}x} \right] \left( \frac{x^2}{\sigma^2 + \bar{Z}x} + \xi_{\text{bc,leC}}^2 + \eta_{\text{bc,leC}}^2 \right)^{-1/2} \right]^{-1}, \quad (310)
\]
and
\[
\varrho_{\text{bf}} = \left[ 1 + \left( \frac{x}{\sigma^2 + \bar{Z}x} \right) \left( \frac{x^2}{\sigma^2 + \bar{Z}x} + \xi_{\text{bc,leC}}^2 + \eta_{\text{bc,leC}}^2 \right)^{-1/2} \right]^{-1} \times \left[ 1 + \frac{x}{\sigma^2 + \bar{Z}x} \right] \left( \frac{x^2}{\sigma^2 + \bar{Z}x} + \xi_{\text{bc,leC}}^2 + \eta_{\text{bc,leC}}^2 \right)^{-1/2}, \quad (311)
\]
(holding on \( \{ 0 < x < \bar{x} \} \cap \mathbb{R}^+_\bar{x} \times T^*X^0 \)), which certainly suffice to define a smooth two-sided inverse
\( \{ 0, \bar{x} \}\text{res} \times \mathbb{R}_{\xi, \text{leC}} \times (T^*\partial X)_{\eta_{\text{bc,leC}}} \setminus \partial \) of \( \tilde{i} \).

In terms of \( \xi_{\text{bc,leC}} \) and \( \eta_{\text{bc,leC}} \), \( \bar{p}_0 \) can be written as
\[
\bar{p}_0 = (\sigma^2 + \bar{Z}x)(\xi_{\text{bc,leC}}^2 + \eta_{\text{bc,leC}}^{-1}\xi_{\text{bc,leC}}; \eta_{\text{bc,leC}}^{-1}\xi_{\text{bc,leC}} - 2\xi_{\text{bc,leC}}) \quad (312)
\]
\[
= (\sigma^2 + \bar{Z}x)(\xi_{\text{bc,leC}}^2 + \eta_{\text{bc,leC}}^{-1}\xi_{\text{bc,leC}} - 2\xi_{\text{bc,leC}}) \quad (313)
\]
in \( T^*X^0 \). Weighting, \( \bar{p}_0^2 \varrho_{\text{df}}^2 \varrho_{\text{bf}}^{-2} \varrho_{\text{tf}}^{-3} = \bar{p}_0^2 \eta_{\text{leC}}^{-2} \eta_{\text{leC}}^{-1} \varrho_{\text{bf}}^{-3} \) induces a well-defined function on \( \text{leC}T^*X \) and thus on \( \text{leC}T^*X_{\partial \xi} \setminus \partial \). Note that this restriction does not depend on \( \bar{p}_{1,2} \). The portion of the \( \text{leC} \)-characteristic set of \( \bar{P} \) disjoint from \( \text{bf} \cup \text{tf} \) can be written as
\[
\text{Char}^{2,0,-2,-1,-3}_{\text{leC}}(\bar{P}) \setminus (\text{bf} \cup \text{tf}) = \left( \bar{p}_0^2, \eta_{\text{leC}}^{-2} \eta_{\text{leC}}^{-1} \right)_{\text{leC}T^*X_{\partial \xi} \setminus \partial}^{-1}(\{0\}). \quad (314)
\]
which is a translated version of \[Mel94, Figure 2\]. The portion of which consists of an off-center sphere over each point of 

\[sc, leC\] coordinates above by 

\[\text{the lift of} \quad \pi \quad \text{over a point} \quad \xi \quad \text{in terms of} \quad \eta \quad \text{of radius one and centered at} \quad \xi_{sc, leC} = 1 \quad \text{and} \quad \eta_{bc, leC} \in o. \quad \text{See Figure 5,} \]

\[\text{which is a translated version of} \quad \text{[Mel94, Figure 2]. The portion of} \quad \text{Char}^{2,0,-2,1,3}_{sc, leC}(P) \quad \text{over a point} \quad p \quad \text{in} \quad \partial(0, \bar{x})_{\text{res}} \times \partial X \quad \text{is depicted in Figure 6. Note that} \quad \text{Char}^{2,0,-2,1,3}_{sc, leC}(P) \quad \text{only contains a codimension one subset of} \quad \eta \quad \text{and} \quad \eta_{bc, leC} = 0, \quad \text{while the lift of} \quad \text{Char}^{2,0,-2,1,3}_{sc, leC}(P) \quad \text{contains the whole sets.} \]

Elliptic estimates control \( u \) in terms of \( Pu \) away from this set — see \[5.1\].

As shown in \[5.2\], the Hamiltonian flow \( H_{\tilde{p}} \in \mathcal{V}(\mathbb{R}^*_+ \times T^* X^0) \) associated to \( \tilde{p} \) is given in terms of the \( sc, leC\)-coordinates above by

\[
H_{\tilde{p}} = x(\sigma^2 + Zx)^{1/2} \left[ 2(x_{sc, leC} - 1)x\partial_x + 2g_{\partial X}^{-1}(\sigma_{bc, leC}, -) 
+ \frac{2\sigma^2 + Zx}{\sigma^2 + Zx} \left( (x_{sc, leC} - 1)\sigma_{bc, leC}\partial_{\sigma_{bc, leC}} + x_{sc}(x_{sc, leC} - 2)\partial_{x_{sc, leC}} \right) - \frac{2\tilde{p}_0}{\sigma^2 + Zx}\partial_{x_{sc, leC}} \right] \]

\[\text{(317)}\]
to leading order at $\partial X$, where $\eta_{bc,leC} \partial_{\eta_{bc,leC}} \in \mathcal{V}(T^* \partial X) \subset \mathcal{V}(\mathbb{R}^+_x \times [0, \tilde{x}]_x \times \mathbb{R}_{\eta_{bc,leC}} \times T^* \partial X)$ denotes the vector field on $T^* \partial X$ given in local coordinates $y_1, \ldots, y_{n-1}$ for $\partial X$ by

$$
\eta_{bc,leC} \partial_{\eta_{bc,leC}} = \sum_{j=1}^{n-1} \eta_{bc,leC,j} \partial_{\eta_{bc,leC,j}},
$$

where $\eta_{bc,leC,j}$ is the cofiber component of $\eta_{bc,leC}$ dual to $y_j$. It will be convenient to work with the weighted Hamiltonian vector fields $H_{\hat{p},0,0,-2} = \partial_{\eta_{bc,leC}} x^{-1}(\sigma^2 + Zx)^{-1/2} H_{\hat{p}}$, $H_{\hat{p},2,0,-2} = \partial_{\eta_{bc,leC}} x^{-1}(\sigma^2 + Zx)^{-1/2} H_{\hat{p}}$.

The vector field $H_{\hat{p}}$, as defined by eq. (317), defines on the $sC,leC$-characteristic set $\text{Char}^{2,0,-2}(\hat{P})$ a source-to-sink flow, with

- $\{\xi_{sc,leC} = \eta_{bc,leC} = 0\} = R_0 \subset \mathcal{R}_{sc,leC} T^* X$, the “selected radial set,” and
- $\{\xi_{sc,leC} = 2, \eta_{bc,leC} = 0\} = R_+ \subset \mathcal{R}_{sc,leC} \mathcal{R}_{bf,tf}$, the “unselected radial set.”

the sink and source (respectively, under our sign conventions) of the flow. Hence, $leC$-analogues of standard propagation and radial point estimates apply away from $\partial \mathcal{R}_{sc,leC}$ (the dependence on the one parameter being unimportant), and the proofs are straightforward modifications of the analogous estimates in [Mel94]. These estimates are proven in §5.2. Note that the $\{\xi_{sc,leC} = 1\} x \partial_x$ term in eq. (317) does not actually vanish on the portion of $\mathcal{R}_0$, $\mathcal{R}_+$ over the interior of $tf$ — rather, it induces a flow from the $bf$ side to the $zf$ side. Thus, it is also possible to prove a propagation result in which regularity at $\mathcal{R}_+ \cap \mathcal{R}_+ \cap \mathcal{R}_+ \cap \mathcal{R}_+ \cap \mathcal{R}_+$ is propagated into $\mathcal{R}_+ \cap \mathcal{R}_+ \cap \mathcal{R}_+ \cap \mathcal{R}_+ \cap \mathcal{R}_+$.

The radial point estimate at the selected radial set is somewhat more nonstandard. Rather than $\mathcal{R}_0$, we work with $\mathcal{R} = \text{Char}^{2,0,-2,1,3}(\hat{P}) \cap \mathcal{R}_{bf,tf}$. See [Vas21a, Figure 3] for the $\sigma > 0$ case. The face $ff$ meets $bf$ at an edge — see Figure 3, Figure 7 — and so $\mathcal{R}_{bf,tf} \cap \partial X$ is not just the boundary of

$$
[(0, \tilde{x})_{sp}^p \times \mathbb{R}_{\eta_{bc,leC}} \times T^* \partial X, [0, \tilde{x})_{sp}^p \times \{0\} \times \{n\}],
$$

which is why Figure 6 depicts the situation only over $p \in [0, \tilde{x})_{sp}^p \times \partial X$ not on the zero face or corner. Away from that edge, our situation looks (even at $ff$) very much like that in [Vas21a]. An argument similar to that used to prove the radial point estimate there suffices to prove an estimate here that is uniform down to $\sigma = 0$, albeit without the desired number of independent orders. In order to prove an estimate with the desired number of independent orders, it will be necessary to take into account the aberrant edge. See §5.3.

The elliptic, propagation, and radial point estimates in §5.1, §5.2, §5.3 are combined in a short epilogue §5.4, which contains Theorem 5.12, as well as the corollary Proposition 5.14 stated above.

5.1. Ellipticity.

**Proposition 5.2.** $\hat{P} \in \text{Diff}^{2,0,-2,1,3}(X)$ is elliptic in some neighborhood of $df$.

**Proof.** By assumption, $\hat{P}$ is elliptic at every point of $df$. By eq. (249), $\hat{P}$ will be elliptic at a point of $\mathcal{R}_{bf,tf}$ (including the points of $\partial df \subset \mathcal{R}_{bf,tf}$) if and only if $\hat{P}_0$ is. $\text{Char}^{2,0,-2,1,3}(\hat{P}_0)$ is contained away from $df$, so the same then holds for $\text{Char}^{2,0,-2,1,3}(\hat{P})$. □

Via the $leC$-calculus analogue of the usual argument via parametrix:
Proposition 5.3. Let \( A, A_0, B \in \Psi^{0,0,0,0}_{\text{leC}}(X) \), with \( \text{WF}^{0,0}_{\text{leC}}(A), \text{WF}^{0,0}_{\text{leC}}(A_0) \subseteq \text{Ell}^{0,0,0,0}_{\text{leC}}(B) \cap \text{Ell}^{2,0,-2,-1,-3}_{\text{leC}}(P) \).

Then, for each \( \Sigma > 0 \), \( m, s, \varsigma, l, \ell \in \mathbb{R} \), and \( N \in \mathbb{N} \), there exists a

\[
C = C(P, A, A_0, B, m, s, \varsigma, l, \ell, N, \Sigma) > 0
\]

such that, for any \( u \in S'(X) \),

\[
\|Au\|_{H^{m,s,\varsigma,l,\ell}_{\text{leC}}} \leq C\left[\|B\hat{P}u\|_{H^{m,-2,s,\varsigma+2,l+1,\ell+3}_{\text{leC}}} + \|u\|_{H^{N,-N,-N,-N}_{\text{leC}}}\right]
\]

(320)

\[
\|A_0u\|_{H^{m,s,\varsigma,l,\ell}_{\text{leC}}} \leq C\left[\|B\hat{P}u\|_{H^{m,-2,s,\varsigma+2,l+1,\ell+3}_{\text{leC}}} + \|u\|_{H^{N,-N,-N,-N}_{\text{leC}}}\right]
\]

(321)

for all \( \sigma \in (0, \Sigma] \) (in the strong sense that if the right-hand side is finite then the left-hand side is as well).

\[\square\]

Proof. Let \( b \in S^{0,0,0,0}_{\text{leC}}(X) = \mathbb{S}^{0,0,0}_{b,\text{leC}}(X) \) denote a representative of \( \sigma^{0,0,0,0}_{\text{leC}}(B) \). Let \( \varphi \in C^\infty_c(\mathbb{R}) \) be identically equal to one in some neighborhood of \([0, \Sigma]\) and supported in \((-\infty, 2\Sigma)\).

The set

\[
K = \text{WF}^{0,0}_{\text{leC}}(A) \cap \{\sigma \leq 2\Sigma\} \subseteq \text{df} \cup \text{sf} \cup \text{ff} \subseteq \partial^{\text{leC}} T^* X
\]

(322)

is a compact subset of \( \text{leC} T^* X \), so we can find some \( \chi \in S^{0,0,0,0}_{\text{leC}}(X) \) such that \( \chi = 1 \) identically in some neighborhood of \( K \) and such that \( \chi = 0 \) identically in some neighborhood of \( \text{Char}^{2,0,-2,-1,-3}_{\text{leC}}(B\hat{P}) = \text{Char}^{0,0,0,0}_{\text{leC}}(B) \cup \text{Char}^{2,0,-2,-1,-3}_{\text{leC}}(\hat{P}) \). We can moreover choose \( \chi \) such that \( \chi = 0 \) identically in some neighborhood of \( b^{-1}(\{0\}) \cup \hat{p}^{-1}(\{0\}) \). Consider \( f = \chi/b\hat{p} \in S^{2,0,2,1,3}_{\text{leC}}(X) \).

Quantizing, we get some \( F = \text{Op}(f) \in \Psi^{-2,0,2,1,3}_{\text{leC}}(X) \) such that – via the leC-principal symbol short exact sequence –

\[
\varphi A(1 - FB\hat{P}) \in \Psi^{-1,-1,-1,0,0}_{\text{leC}}(X).
\]

(323)

Indeed, \( \varphi A(1 - FB\hat{P}) \in \Psi^{0,0,0,0}_{\text{leC}}(X) \) by Proposition 2.6. By Proposition 2.9, for any representative \( a \in S^{0,0,0,0}_{\text{leC}}(X) \) of the principal symbol \( \sigma^{0,0,0,0}_{\text{leC}}(A) \),

\[
\sigma^{0,0,0,0}_{\text{leC}}(\varphi A(1 - FB\hat{P})) = \varphi a(1 - \chi) \mod S^{-1,-1,-1,0,0}_{\text{leC}}(X)
\]

(324)
which implies eq. (323). So, for \( \sigma \in [0, \Sigma] \),
\[
\| Au \|_{H^{m,s,c,\ell,\ell}_{0}} = \| \varphi Au \|_{H^{m,s,c,\ell,\ell}_{0}} \leq \| \varphi AFB \tilde{p} u \|_{H^{m,s,c,\ell,\ell}_{0}} + \| \varphi A(1 - F B \tilde{p}) u \|_{H^{m,s,c,\ell,\ell}_{0}} \\
\leq \| B \tilde{p} u \|_{H^{m-2,s,c,\ell+1,\ell+3}_{0}} + \| \varphi Au \|_{H^{m-1,s,\ell-1,\ell\ell}_{0}} \\
= \| B \tilde{p} u \|_{H^{m-2,s,c,\ell+1,\ell+3}_{0}} + \| Au \|_{H^{m-1,s,\ell-1,\ell\ell}_{0}}
\]  
(325)

Inducting, we conclude the estimate eq. (320).

The second estimate, eq. (321), is proven in a completely analogous manner. \( \square \)

5.2. Propagation. The Hamiltonian vector field \( H_{\tilde{p}} \in \mathcal{V}(\mathbb{R}^+_n \times T^*X^o) \) associated with the symbol \( \tilde{p} \in C^\infty((0, \infty) \sigma \times T^*X^o) \) is given near \( \partial X \) by
\[
H_{\tilde{p}} = (\partial_{\xi_b} \tilde{p}) x \partial_x - (x \partial_x \tilde{p}) \partial_{\xi_b} + \sum_{j=1}^{n-1} ((\partial_{\eta_j} \tilde{p}) \partial_{\eta_j} - (\partial_{\eta_j} \tilde{p}) \partial_{\eta_j}) \in C^\infty((0, \infty) \sigma; \mathcal{V}(T^*X^o))
\]  
(326)

with respect to any set of local coordinates \( y = (y_1, \ldots, y_{n-1}) \) on \( \partial X \). (We will alternatively identify \( H_{\tilde{p}} \) as a smooth family of elements of \( \mathcal{V}(T^*X^o) \) and as a vector field on \( \mathbb{R}^+_n \times T^*X^o \) without \( \partial_\sigma \) component.) Together, \( x, y, \xi_{bc}, \eta_{bc} \) constitute a coordinate chart for \( T^*X^o \), so we can rewrite \( H_{\tilde{p}}(\sigma) \in C^\infty(T^*X^o; T^*X^o) \) in terms of them, and the result (patching together the various coordinate charts for \( \partial X \)) can be interpreted as a weighted \( b \)-vector field on \( \mathcal{V}(\mathbb{R}^+_n \times \mathbb{R}_{x,\xi_{bc},\eta_{bc}} \times (T^*X^o)) \) (after restricting attention to a small collar neighborhood of \( \partial X \)). To perform this rewrite, we need the following substitutions:
\[
x \partial_x \rightarrow x \partial_x + \frac{\partial \xi_{bc}}{\partial x} x \partial_{\xi_{bc},j} + \sum_{j=1}^{n-1} \frac{\partial \eta_{bc,j}}{\partial x} x \partial_{\eta_{bc,j}}
\]
(327)
and
\[
\partial_{\xi_b} \rightarrow \frac{x}{\sqrt{\sigma^2 + Z^2}} \partial_{\xi_{bc}}
\]
(328)

for \( j = 1, \ldots, n-1 \), where the partial derivatives are taken with respect to the coordinate system \( x, y, \xi_b, \eta \). In other words, letting \( (x \partial_x)_{old}, (\partial_{\eta_j})_{old} \) denote the local vector fields defined using the coordinate system \( x, y, \xi_b, \eta \), we have
\[
(\partial_{\eta_j})_{old} = \partial_{\eta_j}
\]
(329)
\[
(x \partial_x)_{old} = x \partial_x + \frac{2\sigma^2 + Z^2}{2(\sigma^2 + Z^2)} \xi_{bc} \partial_{\xi_{bc},j} + \frac{2\sigma^2 + Z^2}{2(\sigma^2 + Z^2)} \sum_{j=1}^{n-1} \eta_{bc,j} \partial_{\eta_{bc,j}}
\]
(330)

where the partial derivatives on the right-hand side are defined using the coordinate system \( x, y, \xi_{bc}, \eta_{bc} \). In terms of this new notation, eq. (326) says
\[
H_{\tilde{p}} = (\partial_{\xi_b} \tilde{p})(x \partial_x)_{old} - ((x \partial_x)_{old} \tilde{p}) \partial_{\xi_b} + \sum_{j=1}^{n-1} ((\partial_{\eta_j} \tilde{p}) (\partial_{\eta_j})_{old} - ((\partial_{\eta_j})_{old} \tilde{p}) \partial_{\eta_j})
\]
(331)
The same holds for \( \tilde{p}_0 \) in place of \( \tilde{p} \). The \( \partial_{y_j} \) component of \( H_{\tilde{p}_0} = H_{\tilde{p}} - H_{\tilde{p}_{1,2}} \) is given by

\[
\partial_{y_j} \tilde{p}_0 = \frac{x}{\sqrt{\sigma^2 + Zx}} \frac{\partial \tilde{p}}{\partial \eta_{he,leC,k}} = 2x \sqrt{\sigma^2 + Zx} \sum_{k=1}^{n-1} g^{kj} \eta_{he,leC,k}.
\]

(332)

On the other hand, the \( \partial_{\eta_{he,leC,j}} \) component of \( H_{\tilde{p}_0} \) is given by

\[
\frac{2\sigma^2 + Zx}{2\sigma^2 + 2Zx} \eta_{he,leC,j} \frac{\partial \tilde{p}_0}{\partial \xi_{he,leC}} - \frac{x}{(\sigma^2 + Zx)^{1/2}} \frac{\partial \tilde{p}_0}{\partial y_j} = \frac{2\sigma^2 + Zx}{2\sigma^2 + 2Zx} \frac{x}{(\sigma^2 + Zx)^{1/2}} \eta_{he,leC,j} \frac{\partial \tilde{p}_0}{\partial \xi_{he,leC}}
\]

\[
= \frac{2\sigma^2 + Zx}{(\sigma^2 + Zx)^{1/2}} \eta_{he,leC,j} \xi_{he,leC} - \frac{2\sigma^2 + Zx}{(\sigma^2 + Zx)^{1/2}} \xi_{he,leC}(\xi_{he,leC} - 1),
\]

(333)

while the \( \partial_{\xi_{he,leC}} \) component is given by

\[
\frac{2\sigma^2 + Zx}{2\sigma^2 + 2Zx} \xi_{he,leC} \frac{\partial \tilde{p}_0}{\partial \xi_{he,leC}} - \frac{x^2}{(\sigma^2 + Zx)^{1/2}} \left( \frac{\partial \tilde{p}_0}{\partial x} \right)_{\text{old}}
\]

\[
= \frac{2\sigma^2 + Zx}{(\sigma^2 + Zx)^{1/2}} \xi_{he,leC}(\xi_{he,leC} - 1) - \frac{x}{(\sigma^2 + Zx)^{1/2}} \left[ 2\tilde{p}_0 + \xi_{he,leC}(2\sigma^2 + Zx) \right]
\]

\[
= \frac{2\sigma^2 + Zx}{(\sigma^2 + Zx)^{1/2}} \xi_{he,leC}(\xi_{he,leC} - 2) - \frac{2x\tilde{p}_0}{(\sigma^2 + Zx)^{1/2}},
\]

(334)

and the \( x\partial_x \) component is \( x(\sigma^2 + Zx)^{-1/2} \partial_{\xi_{he,leC}} \tilde{p}_0 = 2x(\sigma^2 + Zx)^{1/2}(\xi_{he,leC} - 1) \). To summarize:

**Proposition 5.4.** In terms of the coordinates \( x, y, \xi_{he,leC}, \eta_{he,leC} \),

\[
H_{\tilde{p}_0} = x(\sigma^2 + Zx)^{-1/2} \left[ 2(\xi_{he,leC} - 1)x\partial_x + 2g^{-1}_{\partial X}(\eta_{he,leC}, -) \right.
\]

\[
+ \frac{2\sigma^2 + Zx}{\sigma^2 + Zx} \left( \xi_{he,leC}(\xi_{he,leC} - 2)\partial_{\xi_{he,leC}} + \sum_{j=1}^{n-1} \eta_{he,leC,j}(\xi_{he,leC} - 1)\partial_{\eta_{he,leC,j}} \right) - \frac{2\tilde{p}_0}{\sigma^2 + Zx} \partial_{\xi_{he,leC}}
\]

\[
\left. \right] \right) - \frac{2\tilde{p}_0}{\sigma^2 + Zx} \partial_{\xi_{he,leC}}
\]

\[
(335)
\]

(in the relevant neighborhood of \( \text{leC}T^*X \)).

Letting \( H_{\tilde{p}}^{-0,-2} = x^{-1}(\sigma^2 + Zx)^{-1/2}H_{\tilde{p}} \in \mathcal{V}(\mathbb{R}_+^* \times T^*X^0) \), \( H_{\tilde{p}}^{-0,-2} \) defines a b-vector field on \( \text{leC}T^*X = [0, \tilde{x}]_{\text{leC}} \times \mathbb{R}_+ \times \mathbb{R}_+ \times (T^*X)_{y, \eta_{he,leC}} \).

(336)

Likewise for \( \tilde{p}_0 \). Note that \( H_{\tilde{p}}^{-0,-2} \) and \( H_{\tilde{p}_0}^{-0,-2} \) agree at every point of \( \text{sF} \cup \text{ff} \). Restricting to \( \{ x = 0 \} \), \( H_{\tilde{p}}^{-0,-2} \) can be considered as a family

\[
H : \partial([0, \tilde{x}]_{\text{leC}} \times \mathbb{R}_+) \to \mathcal{V}(\mathbb{R}_+ \times (T^*X)_{y, \eta_{he,leC}})
\]

(337)

of vector fields on the fiber \( \mathbb{R} \times T^*X \). In order to understand \( H \), consider \( H_{\tilde{p}_0}^{-0,-2} \) over a neighborhood of a subset of the interior of the transition face \( \text{tf} \) of \( [0, \tilde{x}]_{\text{leC}} \), which we can parametrize in terms of \( x \in [0, \tilde{x}] \) and \( \lambda \in \mathbb{R}_+ \) by writing \( \sigma^2 = 2\lambda x \). In this neighborhood, \( x^{1/2} \) is a bdf for the transition face. Then, \( H \) can be thought of as a family of vector fields on \( \mathbb{R}_+ \times T^*X \) dependent on the parameter \( \lambda \), with explicit formula

\[
H = \frac{2\lambda + 1}{\lambda + 1} \left[ (\xi_{he,leC} - 1)\eta_{he,leC}\partial_{\eta_{he,leC}} + (\xi_{he,leC} - 2)\xi_{he,leC}\partial_{\xi_{he,leC}} \right] + 2g^{-1}_{\partial X}(\eta_{he,leC}, -)
\]

(338)

when \( \tilde{p}_0 = 0 \). The \( \partial_x \) component of \( H \) is independent of \( \lambda \) and comes from the projection of the geodesic flow on \( T^*X \) down to \( \partial X \). The cofiber components of \( H \) (bracketed) depend on \( \lambda \) only
in the form of an overall factor that – crucially – is nonzero for all \( \lambda \in [0, \infty) \). (In fact, eq. (338) makes sense as a family of vector fields on \( \mathbb{R} \times T^*X \) for all \( \lambda > -1 \), and it is non-vanishing for \( \lambda > -1/2 \), but we do not consider this “extended transition face” here The vanishing at \( \lambda = -1/2 \) is one sign that the consideration of negative \( \lambda \) would require solving a b-problem analogous to the b-problem encountered in the low-energy analysis of Coulomb-free Schrödinger operators. Cf. Remark 7.) From eq. (338), we read off the following crucial observation: over the transition face of \( \{0, \bar{x}\}_{\text{res}} \), parametrized as above, \( H \), when restricted to the characteristic set of \( \bar{p} \) and away from \( \text{bf} \cup \text{tf} \), vanishes if and only if \( \eta_{\text{sc,leC}} = 0 \) and \( \xi_{\text{sc,leC}} = 2 \), i.e. on \( \mathcal{R}_+ \). Between this submanifold and the zero section \( \mathcal{R}_0 \), \( H \) has (over each point in \( \partial\{0, \bar{x}\}_{\text{res}} \)) a source-to-sink flow within \( \text{Char}^{2,0,-2}_{\text{sc,leC}}(\bar{P}) \).

In order to see that \( \{\xi_{\text{sc}} = \eta_{\text{sc}} = 0\} \) is a sink of the flow, observe that

\[
H(\xi_{\text{sc,leC}}^2 + \eta_{\text{sc,leC}}^2) = \frac{4\lambda + 2}{\lambda + 1} \left[ \eta_{\text{sc}}^2(\xi_{\text{sc,leC}} - 1) + \xi_{\text{sc}}^2(\xi_{\text{sc,leC}} - 2) \right] - \frac{4\bar{p}_0 \zeta_{\text{sc,leC}}}{\sigma^2 + Zx} \tag{339}
\]

(note that \( \bar{p}_0 / (\sigma^2 + Zx) = \xi_{\text{sc,leC}}^2 + \eta_{\text{sc,leC}}^2 - 2\xi_{\text{sc,leC}} \) is well-defined).

The same computation yields:

**Proposition 5.5.** \( H_{\xi_{\text{sc,leC}}^2 + \eta_{\text{sc,leC}}^2}^{0,0,-2}(\zeta_{\text{sc,leC}}) = \beta_{0,1}(\zeta_{\text{sc,leC}}^2 + \eta_{\text{sc,leC}}^2) + F_{0,2} + F_{0,3} \) for

\[
\beta_{0,1} = \frac{4\sigma^2 + 2Zx}{\sigma^2 + Zx} (\xi_{\text{sc,leC}} - 1), \quad F_{0,2} = -\frac{\xi_{\text{sc,leC}}^2(4\sigma^2 + 2Zx)}{\sigma^2 + Zx}, \quad F_{0,3} = -\frac{4\bar{p}_0 \xi_{\text{sc,leC}}}{\sigma^2 + Zx} \tag{340}
\]

These extend to symbols on the leC-phase space. The first two are nonpositive in the vicinity of \( \text{bf} \cup \text{tf} \), while \( F_{0,3} \) vanishes cubically there.

Note that the same statement holds if we replace \( \bar{p}_0 \) by \( \bar{p}_1 \), if instead of \( F_{0,3} = -4\bar{p}_0 \xi_{\text{sc,leC}}/(\sigma^2 + Zx) \) we use \( H_{\xi_{\text{sc,leC}}^2 + \eta_{\text{sc,leC}}^2}^{0,0,-2}(\zeta_{\text{sc,leC}}) - 4\bar{p}_0 \xi_{\text{sc,leC}}/(\sigma^2 + Zx) \).

Consider, for each pair of \( \Theta_1, \Theta_2 \in (0, \pi) \) with \( \Theta_1 < \Theta_2 \), the set \( \mathcal{P}[\Theta_1, \Theta_2] \subset \text{sf} \cup \text{ff} \) defined by \( \mathcal{P}[\Theta_1, \Theta_2] = \text{Char}^{2,0,-2}_{\text{leC}}(\bar{P}) \cap \{\arccos(\xi_{\text{sc,leC}} - 1) \in [\Theta_1, \Theta_2]\} \). The following proposition is a symbolic version of the statement that the Hamiltonian flow is source-to-sink, \( \mathcal{R}_+ \) to \( \mathcal{R}_0 \).

**Proposition 5.6.** Let \( \Theta \in S_{\text{leC}}^{0,0,0,0}(X) \) satisfy \( \Theta = \arccos(\xi_{\text{sc,leC}} - 1) \) in some neighborhood of \( \mathcal{P}[\Theta_1, \Theta_2] \).

For any pair if \( \Theta_1, \Theta_2 \in (0, \pi) \) with \( \Theta_1 < \Theta_2 \), the symbol \( \alpha \in S_{\text{leC}}^{0,0,0,0}(X) \) defined by \( H_{\xi_{\text{sc,leC}}^2}^{0,0,-2}\Theta = \alpha \) satisfies \( \alpha > 0 \) on \( \mathcal{P}[\Theta_1, \Theta_2] \).

**Proof.** Given such \( \Theta \in S_{\text{leC}}^{0,0,0,0}(X) \), \( \Theta \) is equal to \( \arccos(\xi_{\text{sc,leC}} - 1) \) in some neighborhood \( U \subset \text{leC}T^*X \) of \( \mathcal{P}[\Theta_1, \Theta_2] \). Thus,

\[
H_{\xi_{\text{sc,leC}}^2}^{0,0,-2}\Theta = \frac{2\sigma^2 + Zx}{\sigma^2 + Zx} \xi_{\text{sc,leC}}^{1/2}(2 - \xi_{\text{sc,leC}})^{1/2} + \frac{2\bar{p}_0}{\sigma^2 + Zx} \xi_{\text{sc,leC}}^{1/2}(2 - \xi_{\text{sc,leC}})^{1/2} \tag{341}
\]

in some neighborhood of \( \mathcal{P}[\Theta_1, \Theta_2] \), where we are taking positive square roots. Since \( \bar{p}_0 \) vanishes on \( \mathcal{P}[\Theta_1, \Theta_2] \), the expression on the right-hand side is positive on \( \mathcal{P}[\Theta_1, \Theta_2] \).

Since \( \eta_{\text{ff}} \) is positive on \( \mathcal{P}[\Theta_1, \Theta_2] \), the same statement applies to \( H_{\xi_{\text{sc,leC}}^2}^{2,0,-2}\Theta \). And since \( H_{\xi_{\text{sc,leC}}^2}^{2,0,-2}\Theta \) vanishes at \( \mathcal{P}[\Theta_1, \Theta_2] \), the same statement applies to \( \alpha = H_{\xi_{\text{sc,leC}}^2}^{2,0,-2}\Theta \).

Clearly, there exist \( \Theta = \Theta_{\Theta_1, \Theta_2} \in S_{\text{leC}}^{0,0,0,0}(X) \) satisfying the hypotheses of the previous proposition.
Proposition 5.7. Suppose that $G_1, G_2, G_3 \in \mathcal{P}_{\text{leC}}^{\infty,0,-\infty,-\infty}(X)$ satisfy the following: there exist
\[
\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5 \in (0, \pi)
\] (342)
satisfying $\Theta_1 < \Theta_2 < \Theta_3 < \Theta_4 < \Theta_5$ and
\begin{itemize}
  \item $WF_{\text{leC}}(G_1) \cap \text{Char}_{\text{leC}}^{2,0,-2,-1,-3}(\tilde{P}) \subseteq \mathcal{P}[\Theta_3, \Theta_4],$
  \item $\mathcal{P}[\Theta_1, \Theta_2] \subseteq \text{Ell}_{\text{leC}}^{0,0,0,0}(G_3),$
  \item $\mathcal{P}[\Theta_1, \Theta_5] \subseteq \text{Ell}_{\text{leC}}^{0,0,0,0}(G_2),$
\end{itemize}
and $WF'_{\text{leC}}(G_1) \subseteq \text{Ell}_{\text{leC}}^{0,0,0,0}(G_2)$. Then, for every $\Sigma > 0$, $N \in \mathbb{N}$, $m, s, \varsigma, l, \ell \in \mathbb{R}$, there exists a constant
\[
C = C(\tilde{P}, G_1, G_2, G_3, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Sigma, N, m, s, \varsigma, l, \ell) > 0
\] (343)
such that
\[
\|G_1 u\|_{H^{m,s,l,\ell}_{\text{leC}}} \leq C \left[ \|G_2 \tilde{P} u\|_{H^{N,\varsigma+l+3,-\infty,-\infty}_{\text{leC}}} + \|G_3 u\|_{H^{N,-\varsigma,-\infty,-\infty}_{\text{leC}}}, \|u\|_{H^{N,-\varsigma,-\infty,-\infty,-\infty}_{\text{leC}}}, \right]
\] (344)
holds for all $u \in S'(X)$ and $\sigma \in [0, \Sigma]$ (in the strong sense that if the right-hand side is finite, then the left-hand side is as well, and the stated inequality holds).

Proof. Throughout the argument below, we can take $N$ to be sufficiently large such that any of the finitely many functions of $m, s, \varsigma, l, \ell$ that arise can be bounded below $-N$.

We may assume without loss of generality that $G_2$ is essentially supported away from $\mathcal{R}_+:$
\[
\mathcal{R}_+ \cap WF_{\text{leC}}(G_2) = \emptyset.
\] (345)

Let $\varphi \in C_c^\infty(\mathbb{R})$ satisfy $\text{supp} \varphi \subset [\Theta_1, \Theta_3]$ and
\[
\varphi'(\theta) = \varphi_0(\theta)^2 + \varphi_1(\theta)
\] (346)
for $\varphi_0 \in C^\infty(\mathbb{R})$ nonvanishing on $[\Theta_3, \Theta_4]$ and $\varphi_1 \in C^\infty(\mathbb{R})$ supported within $(\Theta_1, \Theta_2)$. The construction of such $\varphi$ is standard. Moreover, for any closed interval $I \subset ((\Theta_1 + 3\Theta_2)/4, \Theta_3)$ and any $\epsilon > 0$, we can construct $\varphi$ such that $\epsilon \varphi_0^2 \geq \varphi$ in $I$. This construction is also standard — we consider $\varphi_{00} \in C^\infty(\mathbb{R})$ given by
\[
\varphi_{00} = \begin{cases} 
  e^{-F/(\Theta_3 - \Theta)} & (\Theta \leq \Theta_5) \\
  0 & (\Theta > \Theta_5)
\end{cases}
\] (347)
for a parameter $F = F(\sigma) > 0$ and $\varphi_{01} \in C^\infty(\mathbb{R})$ that is identically equal to one in some neighborhood of $[\Theta_2, \infty)$ and identically zero in some neighborhood of $(\infty, \Theta_1]$; setting $\varphi = -\varphi_{00}\varphi_{01}'$ we have
\[
\varphi' = -2\varphi_{00}\varphi_{01}\varphi_0' - \varphi_{00}\varphi_0'^2 = -2\varphi_{00}\varphi_{01}\varphi_0' + F(\Theta_5 - \Theta)^{-1} \varphi_{00}\varphi_{01}^2.
\] (348)
Setting $\varphi_0 = F^{1/2}(\Theta_5 - \Theta)^{-1} \varphi_{01}^{1/2}$ and $\varphi_1 = -2\varphi_{00}\varphi_{01}\varphi_0'$, we see that $\varphi, \varphi'$ have the desired form.

Fix $\Theta \in S_{\text{cl,leC}}^{0,0,0,0}(X)$ that is equal to $\arccos(\xi_{\text{lc,leC}} - 1)$ in some neighborhood of $\mathcal{P}[\Theta_1, \Theta_5]$. Pick a neighborhood $U_0$ of $\mathcal{P}[\Theta_1, \Theta_5]$ on which $\Theta$ is identically $\arccos(\xi_{\text{lc,leC}} - 1)$ and such that $\alpha$ is bounded below on $U_0$ (in compact sets worth of $\sigma$). Let
\[
\varphi(\Theta) \in S_{\text{cl,leC}}^{\infty,0,0,-\infty,-\infty}(X)
\] (349)
denote a symbol equal to $\varphi \circ \Theta$ on some neighborhood $U \in U_0$ of $\mathcal{P}[\Theta_1, \Theta_5]$, and let $\psi \in S_{\text{cl,leC}}^{\infty,0,0,-\infty,-\infty}(X)$ have $\text{supp} \psi \subset U$ and be identically equal to 1 on some neighborhood of $\mathcal{P}[\Theta_1, \Theta_5]$. We can choose $\psi$ such that
\[
\text{supp}(\varphi(\Theta)H_{\tilde{P}}^{2,0,-2} \psi) \cap (\tilde{p}^{2,0,-2,-1,-3})^{-1}\{\{0\}\} = \emptyset
\] (350)
and supp $\psi \cap (df \cup R_+) = \emptyset$. Consider the symbol
\[
a_0 = \varphi(\Theta)\psi^2 \in S^\infty_{\text{cl,leC}}(X).
\]
We then compute
\[
H_p^{2,0,-2}a_0 = (\varphi_0(\Theta)^2 + \varphi_1(\Theta))\psi^2\alpha + 2\psi\varphi(\Theta)H_p^{2,0,-2}\psi,
\]
where $\alpha$ is as in Proposition 5.6.

Now set, for to-be-decided $K > 0$, set $\phi_\varepsilon(x) = (1 + \varepsilon x^{-1})^{-K}$, for each $\varepsilon \geq 0$. Now set $a_0^{(c)} = \phi_\varepsilon^2a_0$.

In §3, we checked that $3P \in S_{\text{Diff}_{\text{leC}}}^{2,-1,-3,-1,-1,-3,-3-2\delta}(X) \subset \Psi^{2,-1,-3,-1,-1,-3-2\delta}(X)$. Let $p_1$ denote a representative of $\sigma^{2,-1,-3,-1,-1,-3-3-2\delta}_{\text{leC}}(-2\delta P)$. Then,
\[
H_p^{2,0,-2}a_0^{(c)} = (\varphi_0(\Theta)^2 + \varphi_1(\Theta))\phi_\varepsilon^2\psi^2\alpha + 2\psi\varphi(\Theta)H_p^{2,0,-2}\psi + 2K\psi^2\phi_\varepsilon^2(\varepsilon x^{-1})(1 + \varepsilon x^{-1})^{-1}\varphi_\beta_1, \tag{535}
\]
where $\beta_1 \in S^{0,0,0,0,0}_{\text{cl,leC}}(X)$ is defined by $H_p^{2,0,-2}x = \beta_1x$.

We now let $a = \phi_\varepsilon^{2a-1}x^{-2a-3} - \varepsilon^{-2a-1}x^{-2a-3}a_0$, $a^{(c)} = \phi_\varepsilon^{2a-1}x^{-2a-3} - \varepsilon^{-2a-1}x^{-2a-3}a_0^{(c)}$. Then, $a^{(c)} \in L^\infty([0,1]_\varepsilon; S^{\infty,\infty,\infty,\infty,\infty}_{\text{leC}}(X))$ and
\[
H_p^{2,0,-2}a^{(c)} = \phi_\varepsilon^{2a-1}x^{-2a-3} - \varepsilon^{-2a-1}x^{-2a-3}a^{(c)} + a^{(c)}p_2, \tag{536}
\]
for some $p_2 \in S^{0,0,0,0,0}_{\text{cl,leC}}(X)$. Thus,
\[
H_p^{2,0,-2}a^{(c)} + \varphi_\beta_1x^{-1}(\sigma^2 + Zx^{-1})^{-1/2},
\]
\[
H_p^{2,0,-2}a^{(c)} + p_1a^{(c)} = \phi_\varepsilon^{2a-1}x^{-2a-3} - \varepsilon^{-2a-1}x^{-2a-3}a^{(c)} + a^{(c)}p_2 + 2\psi\varphi(\Theta)H_p^{2,0,-2}\psi + 2K\psi^2\phi_\varepsilon^2(\varepsilon x^{-1})(1 + \varepsilon x^{-1})^{-1}\varphi_\beta_1 + \phi_\varepsilon^2\varphi(\Theta)\psi^2p_2
\]
\[
\quad + \psi^2x^{-1}(\sigma^2 + Zx^{-1})^{-1/2} \tag{537}
\]
Dividing by $\varepsilon x^{-1}(\sigma^2 + Zx^{-1})^{-1/2}$,
\[
H_p^{2,0,-2}a^{(c)} + p_1a^{(c)} = \phi_\varepsilon^{2a-1}x^{-2a-3} - \varepsilon^{-2a-1}x^{-2a-3}a^{(c)} + a^{(c)}p_2 + 2\psi\varphi(\Theta)H_p^{2,0,-2}\psi + 2K\psi^2\phi_\varepsilon^2(\varepsilon x^{-1})(1 + \varepsilon x^{-1})^{-1}\varphi_\beta_1 + \phi_\varepsilon^2\varphi(\Theta)\psi^2p_2
\]
\[
\quad + \psi^2x^{-1}(\sigma^2 + Zx^{-1})^{-1/2} \tag{538}
\]
For each $K, \xi > 0$, we may choose $\psi = \psi_{K,\xi}$ (perhaps dilating it if necessary) such that its essential support is a subset of $\text{Ell}_{\text{leC}}^{0,0,0,0,0}(G_2)$ and such that, taking $F = F_{K,\xi,\psi} \in C^\infty((0,\infty)_\sigma; \mathbb{R}^+)$ sufficiently large,
\[
b_\varepsilon = \phi_\varepsilon^{1/2}x^{-1/2} - \varepsilon^{-1/2}x^{-1/2} - \Phi_\varepsilon\varphi^{1/2}x^{1/2}/\varphi_{01}\psi \left[ F(\Theta_5 - \Theta)^{-2}\alpha - 2K\frac{\varepsilon x^{-1}}{1 + \varepsilon x^{-1}}\varphi_{\beta_1} - p_2 - p_1\psi_{\beta_1}x^{-1/2} - x^{-1/2} - 2\Phi_\varepsilon\varphi_{\beta_1} / \varphi_{01}\psi \right]^{1/2} \tag{539}
\]
is a well-defined uniform family of $\text{leC}$-symbols, specifically $b_\varepsilon \in L^\infty([0,1]_\varepsilon; S^{\infty,\infty,\infty,\infty,\infty}_{\text{leC}}(X))$. In addition, we set
\[
c_\varepsilon = \phi_\varepsilon^{1/2}x^{-1/2} - \varepsilon^{-1/2}x^{-1/2} - \Phi_\varepsilon\varphi^{1/2}x^{1/2}/\varphi_{01}\psi, \tag{539}
\]
\[ f_\varepsilon = 2\partial_d^{-1} \partial_d^{2s} \partial_d^{2s} \partial_d^{2s} \varphi(\Theta) \tilde{p}^{-1} \psi \mathcal{H}^{2,0,-2}_\varepsilon \]  

(360)

(the division by \( \tilde{p} \) in eq. (360) being unproblematic by eq. (350)). Thus,

\[ e_\bullet \in L^\infty([0, 1]; S^{2s, 2s, -\infty, -\infty}) \]  

(361)

\[ f_\bullet \in L^\infty([0, 1]; S^{2s, 2s, -\infty, -\infty}) \]  

(362)

In terms of these,

\[ H_{\partial d}^{\epsilon} + p_1 a^{\epsilon} = 2\partial_d^{-1} \partial_d^{2s} \partial_d^{2s} \partial_d^{2s} \varphi(\Theta) \tilde{p}^{-1} \psi \mathcal{H}^{2,0,-2}_\varepsilon + b_\varepsilon^2 + c_\varepsilon + f_\varepsilon \tilde{p}. \]  

(363)

Setting \( A_\varepsilon = \frac{1}{2}(\text{Op}(a^{\varepsilon}) + \text{Op}(a^{\varepsilon})^*) \), \( B_\varepsilon = \text{Op}(b_\varepsilon) \), \( E_\varepsilon = \text{Op}(c_\varepsilon) \), \( F_\varepsilon = \text{Op}(f_\varepsilon) \),

\[- i[\Re \tilde{P}, A_\varepsilon] - \{\Im \tilde{P}, A_\varepsilon \} = 2\partial_d A_\varepsilon A_\varepsilon^{\infty, 2s-1, 2s-1, 2s-1} + b_\varepsilon^2 + c_\varepsilon + f_\varepsilon \tilde{p} \]  

(364)

for some \( R_\bullet \in L^\infty([0, 1]; \Psi^{2s, 2s, -\infty, -\infty}_r) \). We have

\[ A_\bullet \in L^\infty([0, 1]; \Psi^{2s, 2s, -\infty, -\infty}_r) \]  

(365)

\[ F_\bullet \in L^\infty([0, 1]; \Psi^{2s, 2s, -\infty, -\infty}_r) \]  

(366)

\[ B_\bullet \in L^\infty([0, 1]; \Psi^{2s, 2s, -\infty, -\infty}_r) \]  

(367)

\[ E_\bullet \in L^\infty([0, 1]; \Psi^{2s, 2s, -\infty, -\infty}_r) \]  

and

\[ \text{WF}^{r, \text{leC}}_{L^\infty,s}(A_\bullet), \text{WF}^{r, \text{leC}}_{L^\infty,s}(B_\bullet), \text{WF}^{r, \text{leC}}_{L^\infty,s}(E_\bullet), \]  

\[ \text{WF}^{r, \text{leC}}_{L^\infty,s}(F_\bullet), \text{WF}^{r, \text{leC}}_{L^\infty,s}(R_\bullet) \subset \text{supp} \psi(\Theta), \]  

where the last of these inclusions (the one for \( R_\bullet \)) follows from the one for \( A_\bullet \) and \( \text{WF}^{r, \text{leC}}_{L^\infty,s}(\tilde{P}, A_\bullet) \subset \text{WF}^{r, \text{leC}}_{L^\infty,s}(A_\bullet) \).

Now, for each \( m_0, s_0, l_0, \ell_0 \in \mathbb{R} \), there exist \( K > 0 \) (dependent on \( m_0, s_0, l_0, \ell_0 \) and \( m, l, \ell \) but nothing else) such that, given \{\( u(-; \sigma) \)}\( \sigma > 0 \) \( \in L^\infty([0, 2\Sigma]; \mathcal{H}^{m_{0,s},s_{0,l},l_0,\ell_0}_{\text{leC}}(X)) \), it is the case that, for any \( \varepsilon > 0 \) (and for each \( \sigma > 0 \), implicit in the notation below),

\[ 2\Sigma(\tilde{P}u, A_\varepsilon u)_{L^2} = -\{\Im \tilde{P}, A_\varepsilon \} u, u\}_{L^2} + i[\Re \tilde{P}(\sigma), A_\varepsilon] u, u\}_{L^2}, \]  

(368)

where the pairings above are well-defined (and where we are using the convention that \( \langle -,-\rangle_{L^2} \) is antilinear in the first slot). Indeed, \( A_\varepsilon \), \( \{\Im \tilde{P}, A_\varepsilon \} \), and \( \Re \tilde{P}(\sigma), A_\varepsilon \) are all smoothing operators – i.e. lying in \( \Psi^{2s, 2s, -\infty}_r(X) \) if \( \sigma > 0 \) and \( \Psi^{2s, 2s, -\infty}_r(X_{1/2}) \) if \( \sigma = 0 \) (in a uniform sense made precise by the leC-calculus, but since we simply need to justify some integration by parts \( \sigma \)-wise the uniformity is not important here) – and by taking \( K \) large they can be made to induce an arbitrarily large amount of decay for each \( \varepsilon > 0 \). Given \( N \), we fix \( m_0, s_0, l_0, \ell_0 \in \mathbb{R} \) such that \( m_0, s_0, l_0, \ell_0 < -N \). Then, we can take \( K \) depending on \( m, l, \ell, N \) and nothing else such that eq. (367) holds for all \{\( u(-; \sigma) \)}\( \sigma > 0 \) \( \in L^\infty([0, 2\Sigma]; \mathcal{H}^{m_{0,s},s_{0,l},l_0,\ell_0}_{\text{leC}}(X)) \).

Applying eq. (367) to \( \{u(-; \sigma)\}_{\sigma > 0} \in \mathcal{H}^{m_{0,s},s_{0,l},l_0,\ell_0}_{\text{leC}}(X) \) and pairing against \( u \) (after taking \( K \) large enough), we have

\[ 2\Sigma(\tilde{P}u, A_\varepsilon u)_{L^2} = \|B_\varepsilon u\|_{L^2}^2 + \langle u, E_\varepsilon u\rangle_{L^2} + \langle \tilde{P}u, F_\varepsilon u\rangle_{L^2} + \langle R_\varepsilon u, u\rangle_{L^2} + 2\delta\|A_\varepsilon u\|_{L^2}^2 \]  

(369)
for $\varepsilon > 0$. So,

$$
\|B_\varepsilon u\|_{L^2}^2 + 2\varepsilon \|A_{1/2,-s-1,-\varsigma-3,-l-1,-\ell-3} u\|_{L^2}^2 \\
\leq 2\|\tilde{P}u, A_\varepsilon u\|_{L^2}^2 + \|\tilde{P}u, F_\varepsilon u\|_{L^2}^2 + \|Ru, u\|_{L^2}^2 + \|u, E_\varepsilon u\|_{L^2}^2.
$$

(369)

Fix self-adjoint $G \in \Psi^{-\infty,0,0,-\infty,-\infty}(X)$ (constructed via $\text{Op}$) such that $\text{WF}_{\text{leC}}(1 - G)$ is disjoint from a neighborhood of the $L^\infty$-essupp of $a, b, f, e$ and such that $\text{WF}_{\text{leC}}(G)$ is disjoint from $\mathcal{R}_+$ and satisfies $\text{WF}_{\text{leC}}(G) \subset \text{Ell}_{\text{leC}}^{0,0,0,0}(C_2)$ (both $a, f$ are supported away from $\mathcal{R}_+$ and $\text{Char}_{\text{leC}}^{0,0,0,0,0}(C_2)$, so such a $G$ exists.) Then (for $K$ large enough):

- Writing $\tilde{P} = (1 - G)\tilde{P} + G\tilde{P}$ and noting that

$$
(1 - G)A_* \in L^\infty([0, 1], \Psi^{-\infty,-\infty,-\infty,-\infty,-\infty}(X)),
$$

we have, for each $N \in \mathbb{N},$

$$
2\|\tilde{P}u, A_\varepsilon u\|_{L^2}^2 \leq 2\|\tilde{P}u, A_\varepsilon u\|_{L^2}^2 + 2\|\tilde{P}u, (1 - G)A_\varepsilon u\|_{L^2}^2
$$

(371)

for any $\overline{\delta} > 0$, where the constant in eq. (371) is independent of $\overline{\delta}$. We have abbreviated

$$
\mathcal{E}_N = H_{\text{leC}}^{-N,N,-N,-N,-N}(X),
$$

$$
\mathcal{Y}_N = H_{\text{leC}}^{-N,N,-(s+1),-\varsigma-3,-l-1,-\ell-3}(X).
$$

(372)

We also set $\mathcal{Y}_{s,N} = H_{\text{leC}}^{-N,-(s+1),-\varsigma-3,-(l+1),-\ell-3}(X)$ (so dual in all orders except that at $df$).

- Similarly, we can choose self-adjoint $\tilde{G}_3 \in \Psi^{-\infty,0,0,-\infty,-\infty}(X)$ with $\text{WF}_{\text{leC}}(1 - \tilde{G}_3) \cap \text{WF}_{\text{leC}}(\tilde{G}_3) \cap \text{Char}_{\text{leC}}^{2,0,-2,-1,-3}(\tilde{P}) \subset \text{Ell}_{\text{leC}}^{0,0,0,0,0}(G_3)$. Then

$$
\|u, E_\varepsilon u\|_{L^2}^2 \leq \|\tilde{G}_3 u, E_\varepsilon u\|_{L^2}^2 + \|u, (1 - \tilde{G}_3)E_\varepsilon u\|_{L^2}^2
$$

(373)

$$
\leq \|\tilde{G}_3 u\|_{X_N}^2 + \|E_\varepsilon u\|_{X_N}^2 + \|u\|_{X_N}^2,
$$

$$
\leq \|\tilde{G}_3 u\|_{X_N}^2 + \|E_\varepsilon u\|_{X_{s,N}}^2 + \|u\|_{X_N}^2
$$

where $X_N = H_{\text{leC}}^{-N,N,-s,-l,\ell}(X)$, $X_{s,N} = H_{\text{leC}}^{-N,N,-s,-s,-l-1,-\ell}(X)$.


The bound $\|E_\varepsilon u\|_{X_N}^2 \leq \|u\|_{X_N}^2 + \|u\|_{X_N}^2$ follows (using eq. (365)) from the construction via $\text{Op}$ of $H_1 \in \Psi^{-1,0,0,0,0}(X)$ and $H_2 \in \Psi^{0,0,0,0,0}(X)$ such that $\text{WF}_{\text{leC}}(H_2) = \mathcal{O}$ and $1 = H_1 + H_2$. Then, we can compute

$$
\|E_\varepsilon u\|_{X_N}^2 \leq \|H_1 E_\varepsilon u\|_{X_N}^2 + \|H_2 E_\varepsilon u\|_{X_N}^2
$$

(374)

$$
\leq \|E_\varepsilon u\|_{H_{N-1,-s,-l,\ell}}^2 + \|H_2 E_\varepsilon u\|_{X_N}^2
$$

$$
\leq \|E_\varepsilon u\|_{H_{N-1,-s,-l,\ell}}^2 + \|u\|_{X_N}^2.
$$

Proceeding inductively, we deduce $\|E_\varepsilon u\|_{X_N}^2 \leq \|E_\varepsilon u\|_{X_{s,N}}^2 + \|u\|_{X_N}^2$. This argument will be used below without further comment.

- Writing $\tilde{P}u, F_\varepsilon u\|_{L^2}^2 \leq \|\tilde{P}u, (1 - G)F_\varepsilon u\|_{L^2}^2$

$$
\leq \|G\tilde{P}u\|_{L^2}^2 + \|F_\varepsilon u\|_{X_{s,N}}^2 + \|u\|_{X_N}^2.
$$

(375)
We have $WF_{L^{\infty, \leq C}(F_{\bullet})} \cap \text{Char}^{2,0,-2,-1,-3}(G\bar{P}) = \emptyset$ (since by eq. (350), $f_{\bullet}$ is supported away from the characteristic set), so we can deduce via the elliptic parametrix construction that
\[
\|F_{\varepsilon}u\|_{Y_{\varepsilon}} \leq \|G\bar{P}u\|_{H_{\leq C}^{0, N, s+1, -N, -N}} + \|u\|_{E_{\varepsilon}}
\leq \|G\bar{P}u\|_{Y_{\varepsilon}}^2 + \|u\|_{E_{\varepsilon}}^2
\tag{376}
\]
for $N$ sufficiently large, so
\[
|\langle \bar{P}u, F_{\varepsilon}u \rangle_{L^2} | \leq \|G\bar{P}u\|_{Y_{\varepsilon}}^2 + \|u\|_{E_{\varepsilon}}^2.
\tag{377}
\]
\bullet Writing $R_{\varepsilon} = (1 - G^2)R_{\varepsilon} + G^2R_{\varepsilon}$, since $1 - G^2 = (1 - G)(1 + G)$ implies $WF_{L^{\infty, \leq C}(1 - G^2)} \subset WF_{L^{\infty, \leq C}(1 - G)}$, for each $N \in \mathbb{N}$ we have
\[
|\langle R_{\varepsilon}u, u \rangle_{L^2} | \leq \|GR_{\varepsilon}u\|_{Z_{\varepsilon}, N} \|Gu\|_{Z_{\varepsilon}, N} + \|u\|_{E_{\varepsilon}}^2 \leq (\|GR_{\varepsilon}u\|_{Z_{\varepsilon}, N}^2 + \|Gu\|_{Z_{\varepsilon}, N}^2 + \|u\|_{E_{\varepsilon}}^2)
\leq (\|\tilde{R}_{\varepsilon}G\bar{u}\|_{Z_{\varepsilon}, N}^2 + \|Gu\|_{Z_{\varepsilon}, N}^2 + \|u\|_{E_{\varepsilon}}^2)
\leq (\|\tilde{G}u\|_{Z_{\varepsilon}, N}^2 + \|u\|_{E_{\varepsilon}}^2)
\tag{378}
\]
for some $\tilde{R}_{\varepsilon} \in L^\infty([0, 1], \Psi_{\leq C}^{0, 0, 0, 0, 0}(X))$, where
\[
\begin{align*}
Z_{\varepsilon} &= H_{\leq C}^{-N, (2s-1)/2, (2s-1)/2, l, \ell} (X), \\
Z_{\varepsilon, N} &= H_{\leq C}^{-N, -(2s-1)/2, -(2s-1)/2, -l, -\ell} (X).
\end{align*}
\tag{379}
\tag{380}
\]
Combining eq. (369), eq. (371) with $\delta$ sufficiently small, eq. (377), eq. (378), we have proven that
\[
\|B_{\varepsilon}u\|_{L^2}^2 \leq \|G\bar{P}u\|_{Y_{\varepsilon}}^2 + \|Gu\|_{Z_{\varepsilon}, N}^2 + \|\tilde{G}u\|_{\bar{X}_{\varepsilon}, N}^2 + \|u\|_{E_{\varepsilon}}^2,
\tag{381}
\]
i.e.
\[
\begin{align*}
\|\tilde{B}_{\varepsilon}u\|_{\bar{Y}_{\varepsilon}}^2 &\leq \|G\bar{P}u\|_{\bar{Y}_{\varepsilon}}^2 + \|Gu\|_{\bar{Z}_{\varepsilon}, N}^2 + \|\tilde{G}u\|_{\bar{X}_{\varepsilon}, N}^2 + \|u\|_{E_{\varepsilon}}^2, \\
\|\hat{B}_{\varepsilon}u\|_{\bar{X}_{\varepsilon}} &\leq \|G\bar{P}u\|_{\bar{Y}_{\varepsilon}}^2 + \|Gu\|_{\bar{Z}_{\varepsilon}, N}^2 + \|\tilde{G}u\|_{\bar{X}_{\varepsilon}, N}^2 + \|u\|_{E_{\varepsilon}}^2,
\end{align*}
\tag{382}
\tag{383}
\]
where $\tilde{B}_{\varepsilon} = \Lambda_{0, -s, -c, 0, 0} B_{\varepsilon}$ and $\hat{E}_{\varepsilon} = \Lambda_{-2s, -2c, 0, 0} E_{\varepsilon}$. Hence, $\tilde{B}_{\varepsilon}, \hat{E}_{\varepsilon} \in \Psi_{\leq C}^{0, 0, 0, 0, 0}(X)$. By the choice $\psi$, $WF_{L^{\infty, \leq C}(B_{\bullet})} \cap \text{Char}^{2,0,-2,-1,-3}(\bar{P}) \subset \mathcal{P}[\Theta_1, \Theta_2]$, $WF_{L^{\infty, \leq C}(E_{\bullet})} \subset \mathcal{E}_{\leq C}^{0, 0, 0, 0, 0}(G_2)$. Since $\tilde{E}_{\bullet} \in L^\infty([0, 1], \Psi_{\leq C}^{0, 0, 0, 0, 0}(X))$, \[
\|\tilde{E}_{\varepsilon}u\|_{\bar{X}_{\varepsilon}} \leq \|\hat{E}_{\varepsilon}u\|_{\bar{X}_{\varepsilon}},
\tag{384}
\]
for $\bar{X}_{\varepsilon} = H_{\leq C}^{0, s+1, -1, -N, -N}(X)$. Since $WF_{L^{\infty, \leq C}(E_{\bullet})} \cap \text{Char}^{2,0,-2,-1,-3}(\bar{P}) \subset \mathcal{P}[\Theta_1, \Theta_2]$, $WF_{L^{\infty, \leq C}(E_{\bullet})} \subset \mathcal{E}_{\leq C}^{0, 0, 0, 0, 0}(G_3)$. Consequently, $\|\tilde{F}_{\varepsilon}u\|_{\bar{X}_{\varepsilon}} \leq \|G\bar{P}u\|_{\bar{Y}_{\varepsilon}} + \|G_3u\|_{\bar{X}_{\varepsilon}}^2 + \|u\|_{E_{\varepsilon}}$ for sufficiently large $N$, where $\bar{Y}_{\varepsilon} = H_{\leq C}^{0, s+1, -1, -N, -N}(X)$.

Likewise, $\|\tilde{G}u\|_{\bar{X}_{\varepsilon}} \leq \|G\bar{P}u\|_{\bar{Y}_{\varepsilon}} + \|G_3u\|_{\bar{X}_{\varepsilon}}^2 + \|u\|_{E_{\varepsilon}}.$
\[
\tag{386}
\]
Letting $\bar{Z}_{\varepsilon} = H_{\leq C}^{0, s+1/2, -1/2, -N, -N}(X)$, since $G \in \Psi_{\leq C}^{0, 0, 0, 0, 0}(X)$, $\|Gu\|_{\bar{Z}_{\varepsilon}} \leq \|Gu\|_{\bar{Z}_{\varepsilon}}$.

Because $WF_{\leq C}(G) \subset \mathcal{E}_{\leq C}^{0, 0, 0, 0, 0}(G_3)$, $\|G\bar{P}u\|_{\bar{Y}_{\varepsilon}} \leq \|G_2\bar{P}u\|_{\bar{Y}_{\varepsilon}} + \|u\|_{E_{\varepsilon}}$ and $\|Gu\|_{\bar{Z}_{\varepsilon}} \leq \|G_2u\|_{\bar{Z}_{\varepsilon}} + \|u\|_{E_{\varepsilon}}$ for sufficiently large $N$. We have therefore shown that
\[
\|\tilde{B}_{\varepsilon}u\|_{\bar{X}_{\varepsilon}} \leq \|G_2\bar{P}u\|_{\bar{Y}_{\varepsilon}} + \|G_2u\|_{\bar{Z}_{\varepsilon}} + \|G_3u\|_{\bar{X}_{\varepsilon}}^2 + \|u\|_{E_{\varepsilon}}.
\tag{387}
\]
Thus, for each $\sigma \geq 0$, $\tilde{B}_{\varepsilon}u(-; \sigma)$ is uniformly bounded in $\bar{X}_{\varepsilon}(\sigma)$ as $\varepsilon \to 0^+$. 

For each $\sigma \geq 0$, given any sequence $\{\varepsilon_k\}_{k \in \mathbb{N}} \subset (0, 1]$ with $\varepsilon_k \to 0$ as $k \to \infty$, there exists – by the Banach-Alaoglu theorem – a subsequence $\varepsilon_{k_n}$ thereof such that
\[
\{\tilde{B}_{\varepsilon_{k_n}}(\sigma)u(-; \sigma)\}_{k \in \mathbb{N}} \subset \mathcal{X}_N(\sigma)
\] (388)
is weakly convergent in the scb-Sobolev space $\mathcal{X}_N(\sigma)$ and in fact in any closed ball in $\mathcal{X}_N(\sigma)$ in which $\tilde{B}_{\varepsilon_k}(\sigma)u(-; \sigma)$ is eventually contained. Call the weak limit $v = v(N, \sigma, \{\varepsilon_{k_n}\}_{k \in \mathbb{N}}) \in \mathcal{X}_N(\sigma)$. The preceding clause means that
\[
\|v\|_{\mathcal{X}_N(\sigma)} \leq \limsup_{k \to \infty} \|\tilde{B}_{\varepsilon_{k_n}}(\sigma)u(-; \sigma)\|_{\mathcal{X}_N(\sigma)}.
\] (389)
The family $\{B_\varepsilon(\sigma)\}_{\varepsilon \in [0, 1]}$ was constructed so that it is continuous down to $\varepsilon = 0$ with respect to the topology of some space of high order PDOs. (This follows from the analogous observation for $b_\bullet(\sigma)$ and the continuity of the quantization map.) Consequently,
\[
\tilde{B}_{\varepsilon_k}(\sigma)u(-; \sigma) \to \tilde{B}_0(\sigma)u(-; \sigma)
\] (390)
in the topology of $\mathcal{S}'(X)$ as $k \to \infty$ But $\tilde{B}_{\varepsilon_k}(\sigma)u(-; \sigma) \to v$ in $\mathcal{S}'(X)$, so $v = \tilde{B}_0(\sigma)u(-; \sigma)$. The sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ was arbitrary, so we can actually conclude from eq. (389) that
\[
\|\tilde{B}_0(\sigma)u(-; \sigma)\|_{\mathcal{X}_N(\sigma)} \leq \liminf_{\varepsilon \to 0^+} \|\tilde{B}_\varepsilon(\sigma)u(-; \sigma)\|_{\mathcal{X}_N(\sigma)}.
\] (391)
This applies for each $\sigma \geq 0$, so
\[
\|\tilde{B}_0u\|_{\mathcal{X}_N} \leq \|G_2\tilde{P}u\|_{\mathcal{Y}_N} + \|G_2u\|_{\mathcal{Z}_N} + \|G_3u\|_{\mathcal{X}_N} + \|u\|_{\mathcal{X}_N}.
\] (392)
Since $\varphi$ is nonvanishing on $[\Theta_3, \Theta_4]$, we have $\text{Ell}^{0, 0, 0, 0, 0}_{\text{leC}}(\tilde{P}) \supseteq \mathcal{P}[\Theta_3, \Theta_4] \supseteq \text{Char}^{2, 0, -2, -1, -3}_{\text{leC}}(\tilde{P}) \cap \text{WF}^{\infty}_{\text{leC}}(G_1)$, so (via elliptic regularity) $\|G_1u\|_{\mathcal{X}_N} \leq \|\tilde{B}_0u\|_{\mathcal{X}_N} + \|G_2\tilde{P}u\|_{\mathcal{Y}_N} + \|u\|_{\mathcal{X}_N}$, where $\mathcal{X}_N = H^{m,s,c,l,t}_{\text{leC}}(X)$. This yields
\[
\|G_1u\|_{\mathcal{X}_N} \leq \|G_2\tilde{P}u\|_{\mathcal{Y}_N} + \|G_2u\|_{\mathcal{Z}_N} + \|G_3u\|_{\mathcal{X}_N} + \|u\|_{\mathcal{X}_N}.
\] (393)
Observe that the leC-Sobolev space $Z_N$ is lower order than $X$ at sf and ff. Since the leC- Sobolev spaces $\mathcal{X}_N, \mathcal{Y}_N, \mathcal{Z}_N$ get bigger as $s, c$ decrease, an inductive argument (which we can carry out because eq. (345)) upgrades eq. (393) to eq. (344).

Since $H^{-2, 0, -2}_\rho$, viewed as a vector field on $\text{sc, leC} T^*X$, vanishes at the two radial sets, in order to carry out a positive commutator argument we must take into account the previously negligible radial component
\[
(H^{-2, 0, -2}_\rho x)\partial_x = 2(\xi_{\text{sc, leC}} - 1)x\partial_x
\] (394)
of the rescaled flow. For $\xi_{\text{sc, leC}} = 0, 2$, this is $\pm 2x\partial_x$, which (projecting down to $X$) is the ur-example of a nondegenerate radial b-vector field on $X$. Moreover, $\mathcal{R}_+$ is a source in the radial direction (as well as the other directions, as seen earlier), so $\mathcal{R}_+$ is a source for the Hamiltonian flow in all directions. (Similarly, $\mathcal{R}_0$ is a sink in all directions.) It is therefore straightforward to prove a radial point estimate at $\mathcal{R}_+$.

To begin:

**Proposition 5.8.** There exist $\beta_1, \beta_2 \in \mathcal{S}^{0, 0, 0, 0, 0}_{\text{leC}}(X)$ such that
\[
H^{2, 0, -2}_\rho x = \beta_1 x
\] (395)
\[
H^{2, 0, -2}_\rho (\sigma^2 + Zx)^{1/2} = \beta_2 (\sigma^2 + Zx)^{1/2},
\] (396)
with $\beta_1, \beta_2 > 0$ on $\mathcal{R}_+$. \hfill \qed
Then, for every \( u \in \mathbb{S}(\Sigma) \),
\[
H_{\rho_0}^{-0,-2} x = (\xi_{\text{sc,leC}} - 1) x,
\]
and
\[
H_{\rho_0}^{-0,-2} (\sigma^2 + Z x)^{1/2} = (\xi_{\text{sc,leC}} - 1) Z x (\sigma^2 + Z x)^{-1/2}.
\]
So, defining \( \beta_1, \beta_2 \) by eq. (395) and eq. (396), \( \beta_1, \beta_2 \in \mathcal{S}_{\text{cl,leC}}^{0,0,0,0,0}(X) \) and are given at \( \{ x = 0 \} \) by \( \beta_1 = 2 \xi u (\xi_{\text{sc,leC}} - 1) \) and \( \beta_2 = \xi u (\xi_{\text{sc,leC}} - 1) Z x (\sigma^2 + Z x)^{-1} \). Thus, \( \beta_1, \beta_2 > 0 \) on \( \mathcal{R}_+ \).

**Proposition 5.9.** Suppose that \( G_1, G_2, G_3 \in \Psi_{\text{leC}}^{-\infty,0,-\infty,-\infty}(X) \) satisfy

1. \( \mathcal{W}^f_{\text{leC}}(G_1) \subseteq \mathcal{E}_{\text{leC}}^{0,0,0,0,0}(G_2) \),
2. \( \mathcal{R}_+ \subseteq \mathcal{E}_{\text{leC}}^{0,0,0,0,0}(G_3) \) \( \cap \mathcal{E}_{\text{leC}}^{0,0,0,0,0}(G_1) \),
3. there exists some \( \Theta \in (0, \pi) \) such that
\[
\mathcal{W}^f_{\text{leC}}(G_1) \cap \mathcal{C}^{2,0,0,-1,-3}(\mathcal{P}) \subseteq \mathcal{R}_+ \cup \bigcup_{0 < \Theta' < \Theta} \mathcal{P}[\Theta', \Theta] \subseteq \mathcal{E}_{\text{leC}}^{0,0,0,0,0}(G_2).
\]

Then, for every \( \Sigma > 0 \) and \( N \in \mathbb{N}, m, s, \varsigma, l, \ell, s_0, \varsigma_0 \in \mathbb{R} \) such that \( s > s_0 > -1/2 \) and \( \varsigma > \varsigma_0 > -3/2 \), there exists some constant \( C = C(\mathcal{P}, G_1, G_2, \Sigma, m, s, \varsigma, l, \ell, s_0, \varsigma_0, N) > 0 \)
\[
\| G_1 u \|_{H^{m,s,\varsigma,l,\ell} \mathcal{R}} \leq C \left[ \| G_2 \mathcal{P} u \|_{H^{-N,+1,-1,-N} \mathcal{R}} + \| G_3 u \|_{H^{N,0,0,-N,-N} \mathcal{R}} + \| u \|_{H^{N,0,0,-N,-N} \mathcal{R}} \right]
\]
holds (in the usual strong sense, i.e. the left-hand side is finite if the right-hand side is) for all \( \sigma \in [0, \Sigma] \).

**Proof.** The symbolic constructions will only be specified for \( \sigma \in [0, \Sigma] \), which is evidently unproblematic. Also, it suffices to take \( N \) to be sufficiently large such that any of the finitely many functions of \( m, s, \varsigma, l, \ell \) that arise below can be bounded below \( -N \).

We can assume without loss of generality that \( \mathcal{W}^f_{\text{leC}}(G_3) \subseteq \mathcal{E}_{\text{leC}}^{0,0,0,0,0}(G_2) \) and
\[
\mathcal{W}^f_{\text{leC}}(G_3) \cap \mathcal{C}^{2,0,0,-1,-3}(\mathcal{P}) \subseteq \mathcal{R}_+ \cup \bigcup_{0 < \Theta' < \Theta} \mathcal{P}[\Theta', \Theta] \subseteq \mathcal{E}_{\text{leC}}^{0,0,0,0,0}(G_2).
\]

We read off Proposition 5.8 that \( \sup_{\mathcal{R}_+} \beta_1, \sup_{\mathcal{R}_+} \beta_2 > 0 \).

Now choose nonnegative \( \rho \in \mathcal{S}_{\text{cl,leC}}^{0,0,0,0,0}(X) \) equal to \( (\xi_{\text{sc,leC}} - 2)^2 + \eta_{\text{sc,leC}}^2 \) in some neighborhood of \( \mathcal{R}_+ = \{ \xi_{\text{sc}} = 2, \eta_{\text{leC}} = 0 \} \subset \mathfrak{S} \cup \mathfrak{F} \).

There exist some symbols, which we call \( \tilde{\beta}_0, \tilde{F}_2, \tilde{F}_3 \in \mathcal{S}_{\text{cl,leC}}^{0,0,0,0,0}(X) \) such that
\[
\inf_{\mathcal{R}_+} \beta_0 |_{\mathcal{R}_+} > 0,
\]
\( \beta_0, \tilde{F}_2, \tilde{F}_3 \geq 0 \) everywhere, and \( F_3 \) vanishes cubically at \( \mathcal{R}_+ \) (uniformly in \( \sigma \)). This computation is completely analogous to the one in the proof of Proposition 5.5. We now consider the weight (for to-be-decided \( l', l'' \in \mathbb{R} \))
\[
a_0 = x^{-l'} (\sigma^2 + Z x)^{l'/2 + l'} \in \mathcal{S}_{\text{cl,leC}}^{0,l',l',l'}(X).
\]
Then, the symbol \( \beta \in \mathcal{S}_{\text{cl,leC}}^{0,0,0,0,0}(X) \) defined by
\[
\beta = -l'' \beta_1 + (2 l'' - l') \beta_2 = a_0^{-1} H_{\tilde{\beta}}^{2,0,-2} a_0
\]
has a definite sign near $\mathcal{R}_+$ if $l', -l' + l'/2 \neq 0$ and have the same sign. Using the explicit formula for $\beta_1, \beta_2$ in the proof of Proposition 5.8,

$$
\beta = -\partial_{\mathtt{df}} \left( 2l' + (l' - 2l) \frac{Z_x}{\sigma^2 + Z_x} \right) (\xi_{\mathtt{sc,leC}} - 1)
$$

(406)
at $\{x = 0\}$. Negativity on $\mathcal{R}_+ \cap \mathtt{sf}$ requires that $l' > 0$, negativity on $\mathcal{R}_+ \cap \mathtt{ff}$ requires that $l' > 0$. And this suffices; for $l', l' > 0$, $\beta < 0$ in some neighborhood of $\mathcal{R}_+$.

There exists $\chi \in C^\infty_c(\mathbb{R})$ such that $-\text{sgn}(t)\chi'(t)\chi(t) = \chi_0^2(t)$ for some $\chi_0 \in C^\infty_c(\mathbb{R})$ and such that $\chi = 1$ identically in some neighborhood of the origin. (The construction is standard and uses a translate of $\exp(-1/t)$. ) Replacing $\chi$ with $\chi \circ \text{dil}_\lambda = \chi(\lambda \bullet)$ for sufficiently large $\lambda = \lambda(l', l', \chi)$ if necessary, choose $\chi$ such that $\beta|_{\text{supp} \chi(\rho)} < 0$

and such that $\text{supp} \chi(\tilde{p}^{2,0,-2}) \chi(\rho)$ is disjoint from $\mathtt{df} \cup \mathtt{bf} \cup \mathtt{tf}$, where $\tilde{p}^{2,0,-2} = (\sigma^2 + Zx)^{-1}\tilde{p}$.

Choose $\psi \in S^{0,0,0,0}_{\mathtt{cl,leC}}(X)$ such that $\psi$ is identically equal to one in some neighborhood of $\{x = 0\}$ and such that the formula $(H_{\tilde{p}}^{2,0,-2}\rho)^{1/2} \psi \chi(\rho)$ defines a symbol:

$$
\chi(\rho)\psi \sqrt{H_{\tilde{p}}^{2,0,-2}\rho} \in S^{0,0,0,0}_{\mathtt{cl,leC}}(X).
$$

(408)

(The existence of such a $\psi$ follows from eq. (403).) Now set

$$
a = a_0 \psi^2 \chi(\tilde{p}^{2,0,-2}) \chi(\rho)^2 \in S^{0,0,0,0}_{\mathtt{cl,leC}}(X).
$$

(409)

We compute

$$
H_{\tilde{p}}^{2,0,-2} a = \chi(\tilde{p}^{2,0,-2}) \chi(\rho)^2 H_{\tilde{p}}^{2,0,-2} a_0 - 2a_0 \chi(\rho)^2 \chi(\tilde{p}^{2,0,-2}) H_{\tilde{p}}^{2,0,-2} \rho
$$

$$
+ 2a_0 \chi(\tilde{p}^{2,0,-2}) \chi'(\tilde{p}^{2,0,-2}) \chi(\rho)^2 \tilde{p}^{2,0,-2} \tilde{q} + 2a_0 \chi(\tilde{p}^{2,0,-2}) \chi(\rho)^2 \psi H_{\tilde{p}}^{2,0,-2} \psi.
$$

(410)

Here $\tilde{q} \in S^{0,0,0,0}_{\mathtt{cl,leC}}(X)$ is defined by $\tilde{q} = (\sigma^2 + Zx) H_{\tilde{p}}^{2,0,-2} (\sigma^2 + Zx)^{-1}$, so that $H_{\tilde{p}}^{2,0,-2} \tilde{p}^{2,0,-2} = \tilde{q}^2 (\sigma^2 + Zx)^{-1}$.

(411)

In terms of $\beta$, eq. (410) says

$$
H_{\tilde{p}}^{2,0,-2} a = \chi(\tilde{p}^{2,0,-2}) \chi(\rho)^2 a_0 - 2a_0 \chi(\rho)^2 \chi(\tilde{p}^{2,0,-2}) H_{\tilde{p}}^{2,0,-2} \rho
$$

$$
+ 2a_0 \chi(\tilde{p}^{2,0,-2}) \chi'(\tilde{p}^{2,0,-2}) \chi(\rho)^2 \tilde{p}^{2,0,-2} \tilde{q} + 2a_0 \chi(\tilde{p}^{2,0,-2}) \chi(\rho)^2 \psi H_{\tilde{p}}^{2,0,-2} \psi.
$$

(412)

The first two terms have definite sign (the same sign for $l', l' > 0$), while the third and fourth terms are unproblematic (being controllable by elliptic or propagation estimates).

Set $\phi_e = (1 + \varepsilon x^{-1})^{-K_1} (1 + \varepsilon (\sigma^2 + Zx)^{-1/2})^{-K_2}$ and $\alpha^{(e)} = \phi_e^2 a \in L^\infty([0,1]; S^{0,0,0,0}_{\mathtt{leC}}(X))$ for to-be-decided $K_1, K_2 \in \mathbb{R}$. The replacement for eq. (412) is

$$
H_{\tilde{p}}^{2,0,-2} \alpha^{(e)} = \phi_e^2 \left[ -\chi(\tilde{p}^{2,0,-2}) \chi(\rho)^2 a_0 \psi^2 \left( \beta_1 \left( l' - \frac{K_1 \varepsilon x^{-1}}{1 + \varepsilon x^{-1}} \right) + \beta_2 \left( l' - 2l' - \frac{K_2 \varepsilon (\sigma^2 + Zx)^{-1/2}}{1 + \varepsilon (\sigma^2 + Zx)^{-1/2}} \right) \right)ight.
$$

$$
- 2a_0 \chi(\rho)^2 \chi(\tilde{p}^{2,0,-2}) \psi^2 H_{\tilde{p}}^{2,0,-2} \rho + 2a_0 \chi'(\tilde{p}^{2,0,-2}) \chi(\rho)^2 \psi^2 p^{2,0,-2} \tilde{q}
$$

$$
+ 2a_0 \chi(\tilde{p}^{2,0,-2}) \chi(\rho)^2 \psi H_{\tilde{p}}^{2,0,-2} \psi.
$$

(413)
Rewriting the first parenthetical,
\[
\beta_1 \left( \ell' - \frac{K_1 \varepsilon x^{-1}}{1 + \varepsilon x^{-1}} \right) + \beta_2 \left( \ell' - 2\ell' - \frac{K_2 \varepsilon (\sigma^2 + Zx)^{-1/2}}{1 + \varepsilon (\sigma^2 + Zx)^{-1/2}} \right)
= \varrho_{df} \left( \frac{2(\ell' - K_1 \varepsilon x^{-1})}{1 + \varepsilon x^{-1}} + \frac{Zx}{\sigma^2 + Zx} \left( \ell' - 2\ell' - \frac{K_2 \varepsilon (\sigma^2 + Zx)^{-1/2}}{1 + \varepsilon (\sigma^2 + Zx)^{-1/2}} \right) \right) (\xi_{sc} - 1).
\] (414)

So, we will require that
\[
K_1 < \ell', \quad K_2 < \ell',
\] (415)

and then the quantity in eq. (414) is positive in some neighborhood of \( R_+ \). Thus, only a limited amount of “regularization” can be performed. This is a standard technicality, and we can deal with it via citing the standard arguments used to handle it elsewhere — see [Vas18]. We do not even need to worry about uniformity: we can justify the formal integrations-by-parts below \( \sigma \)-wise, by citing essentially verbatim the arguments in [Vas18] for the \( \sigma > 0 \) and applying the argument with \( X_{1/2} \) in place of \( X \) to handle the \( \sigma = 0 \) case. (In fact, since the \( \sigma = 0 \) case of the proposition follows from the estimates in [Vas21a] applied on \( X_{1/2} \), to prove the proposition here it suffices to prove estimates that are uniform as \( \sigma \to 0^+ \), and thus to restrict attention to the \( \sigma > 0 \) case, for which we can take \( K_2 = 0 \) and apply [Vas18] essentially verbatim.)

Given that the inequalities eq. (415) are satisfied, we can (perhaps dilating \( \chi \) or shrinking the support of \( \psi \) if necessary) find
\[
\delta = \delta(K_1, K_2, \ell', \ell, \chi) > 0
\] (416)
sufficiently small such that there exist uniform families of leC-symbols
\[
b_\bullet \in L^\infty([0, 1]; S^\infty_{leC}((\ell' - 1)/2, (\ell' - 3)/2, -\infty, -\infty) (X)),
\]
\[
e_\bullet \in L^\infty([0, 1]; S^\infty_{leC}((\ell' - 1)/2, (\ell' - 3)/2, -\infty, -\infty) (X)),
\]
\[
f_\bullet \in L^\infty([0, 1]; S^\infty_{leC}((\ell' - 1)/2, (\ell' - 3)/2, -\infty, -\infty) (X)),
\]
\[
r_\bullet \in L^\infty([0, 1]; S^\infty_{leC}((\ell' - 1)/2, (\ell' - 3)/2, -\infty, -\infty) (X)),
\] (417)
given by
\[
b_\varepsilon = \varrho_{df}^{-1/2} \varrho_{df}^{3/2} \varrho_{df}^{-3/2} \varrho_{df}^{-3} \varrho_{df}^{-3} \varphi(x(\rho^2)^{-1}) \varphi(x(\rho^2)^{-1}) \left[ - \varrho_{df}^{-1} \varrho_{df}^{-3} \varrho_{df}^{-1} \varrho_{df}^{-3} p_1 + \beta_1 \left( \ell' - \frac{K_1 \varepsilon x^{-1}}{1 + \varepsilon x^{-1}} \right) \right]^{1/2}
\]
\[
e_\varepsilon = \varrho_{df}^{-1/2} \varrho_{df}^{3/2} \varrho_{df}^{-3/2} \varrho_{df}^{-3/2} \varrho_{df}^{-3} \varphi(x(\rho^2)^{-1}) \varphi(x(\rho^2)^{-1}) \left[ - \varrho_{df}^{-1} \varrho_{df}^{-3} \varrho_{df}^{-1} \varrho_{df}^{-3} p_1 + \beta_1 \left( \ell' - \frac{K_1 \varepsilon x^{-1}}{1 + \varepsilon x^{-1}} \right) \right]^{1/2}
\]
\[
f_\varepsilon = 2\varrho_{df} \varrho_{df} \varrho_{df} \varrho_{df} \chi(\rho(\rho^2)^{-1}) \chi(\rho(\rho^2)^{-1}) \left[ - \varrho_{df}^{-1} \varrho_{df}^{-3} \varrho_{df}^{-1} \varrho_{df}^{-3} p_1 + \beta_1 \left( \ell' - \frac{K_1 \varepsilon x^{-1}}{1 + \varepsilon x^{-1}} \right) \right]^{1/2}
\]
\[
r_\varepsilon = 2\varrho_{df} \varrho_{df} \varrho_{df} \varrho_{df} \chi(\rho(\rho^2)^{-1}) \chi(\rho(\rho^2)^{-1}) \left[ - \varrho_{df}^{-1} \varrho_{df}^{-3} \varrho_{df}^{-1} \varrho_{df}^{-3} p_1 + \beta_1 \left( \ell' - \frac{K_1 \varepsilon x^{-1}}{1 + \varepsilon x^{-1}} \right) \right]^{1/2}
\] (418)

Here \( p_1 \in S^2_{leC}((\ell' - 1)/2, (\ell' - 3)/2, -\infty, -\infty) (X) \) is as in the proof of the propagation estimate, and we are using Proposition 3.9, which shows that \( \varrho_{df}^{3/2} \varrho_{df}^{-3} \varrho_{df}^{-3} \varrho_{df}^{-3} p_1 \) vanishes to some fractional order at \( R_+ \) and therefore does not spoil the sign of the quantity under the first square root in eq. (418) for an appropriate choice of \( \chi, \psi \).
In terms of these new symbols, we can write
\begin{equation}
H_\rho a^{(e)} + p_1 a^{(e)} = -2\delta^{-1} \partial_{\alpha} \partial_{\beta} \partial_{\alpha} \partial_{\beta} a^{(e)} \chi(p^{2,0,2})^2 \chi(\rho)^2 - b_\varepsilon^2 - \varepsilon^2 + f_\varepsilon \bar{p} + r_\varepsilon \tag{419}
\end{equation}
We apply the quantization map $\text{Op}$. Setting $A_\varepsilon = (1/2)(\text{Op}(a^{(e)}) + \text{Op}(a^{(e)})^*)$, $B_\varepsilon = \text{Op}(b_\varepsilon)$, $E_\varepsilon = \text{Op}(e_\varepsilon)$, $F_\varepsilon = \text{Op}(f_\varepsilon)$,
\begin{align*}
A_\varepsilon &\in L^\infty([0,1]; \Psi_{\text{leC}}^{-\infty,l',-\infty,-\infty}(X)) \\
B_\varepsilon, E_\varepsilon &\in L^\infty([0,1]; \Psi_{\text{leC}}^{-\infty,2,2,l'-3/2,-\infty,-\infty}) \\
F_\varepsilon &\in L^\infty([0,1]; \Psi_{\text{leC}}^{-\infty,l'-1,l'-1,-\infty,-\infty})
\end{align*}
and
\begin{equation}
-i[\Re \tilde{P}, A_\varepsilon] - \{\Im \tilde{P}, A_\varepsilon\} = -2\delta A_\varepsilon A_\varepsilon^* - \frac{1}{2} - \frac{1}{2} - (t' + 1)/2, -(t' + 3)/2, -(t' + 1)/2, -(t' + 3)/2 A_\varepsilon - B_\varepsilon^* B_\varepsilon - E_\varepsilon^* E_\varepsilon + F_\varepsilon^* \tilde{P} + r_\varepsilon \tag{421}
\end{equation}
for some $R_\varepsilon \in L^\infty([0,1]; \Psi_{\text{leC}}^{-\infty,l',-2,l'-4,-\infty,-\infty}(X))$. We have
\begin{equation}
\text{WF}_{L^\infty,\text{leC}}(A_\varepsilon), \text{WF}_{L^\infty,\text{leC}}(B_\varepsilon), \text{WF}_{L^\infty,\text{leC}}(F_\varepsilon), \text{WF}_{L^\infty,\text{leC}}(E_\varepsilon), \text{WF}_{L^\infty,\text{leC}}(R_\varepsilon) \subset \supp \chi(p^{2,0,2})\chi(\rho)^\varepsilon. \tag{422}
\end{equation}
We now set the parameters $l', \ell'$ in the definition eq. (404) of $a_0$ to $l' = 2s + 1 > 0$ and $\ell' = 2s + 3 > 0$. Fix $K_1 \in (0, l')$, $K_2 \in (0, \ell')$ such that $-1/2 < s - K_1 < s_0$ and $-3/2 < s - K_2 < s_0$. Suppose now that $u \in S'(X)$, $\sigma \in [0, \Sigma]$ such that
\begin{equation}
\|G_3 u\|_{L^2_{\text{leC}}(N, \varepsilon, -\varepsilon, -N)} < \infty. \tag{423}
\end{equation}
The argument in [Vas18, §4.7, above Proposition 5.27] justifies the computation
\begin{align}
2\Im \langle \tilde{P} u, A_\varepsilon u \rangle_{L^2} &= -\{\Im \tilde{P}, A_\varepsilon\} u, u \rangle_{L^2} + i\langle [\Re \tilde{P}, A_\varepsilon] u, u \rangle_{L^2} \\
&= -\|B_\varepsilon u\|^2_{L^2} - \|E_\varepsilon u\|^2_{L^2} + \langle \tilde{P} u, F_\varepsilon u \rangle_{L^2} + \langle R_\varepsilon u, u \rangle_{L^2} \\
&\quad - 2\delta \|A_{1/2, -s-1, -s-1, -s-1, -s-3} u, u \|^2_{L^2} \tag{424}
\end{align}
where the individual terms above are all well-defined distributional pairings (in the sense of Hörmander) or (finite) norms. Thus,
\begin{equation}
\|B_\varepsilon u\|^2_{L^2} + \|E_\varepsilon u\|^2_{L^2} + 2\delta \|A_{1/2, -s-1, -s-1, -s-1, -s-3} A_\varepsilon u\|^2_{L^2} \leq 2\|\tilde{P} u, A_\varepsilon u \|^2_{L^2} + \|\tilde{P} u, F_\varepsilon u \|^2_{L^2} + \|R_\varepsilon u, u \|^2_{L^2}. \tag{425}
\end{equation}
We estimate each of the terms on the right-hand side as in the proof of the propagation estimate: for self-adjoint $G \in \Psi_{\text{leC}}^{-\infty,0,0,0,0,0,0}(X)$ with $1 - G$ essentially supported away from the $L^\infty$-essential support of $a, f, e, b$ and with $\text{WF}_{L^\infty,\text{leC}}(G) \subset \text{Ell}_{0,0,0,0,0,0,0}(G)$,
\begin{equation}
\|B_\varepsilon u\|_{X_N} \leq \|B_\varepsilon u\|_{X_N} + \|\tilde{E}_\varepsilon u\|_{X_N} \leq \|G \tilde{P} u\|_{Y_N} + \|G u\|_{Z_N} + \|u\|_{E_N}, \tag{427}
\end{equation}
where $E_N = H_{\text{leC}}^{-N, -N, -N, -N}(X)$, $Z_N = H_{\text{leC}}^{-N, (2s-1)/2, (2s-1)/2, -N, -N}(X)$, $X_N = H_{\text{leC}}^{-N, s, s, -N, -N}(X)$, $Y_N = H_{\text{leC}}^{-N, s, s, -N, -N}(X)$, and $\tilde{B}_\varepsilon$ and $\tilde{E}_\varepsilon$ are given by $\tilde{B}_\varepsilon = A_{0, -s-1, -s-1, -s-1, 0} B_\varepsilon$ and $\tilde{E}_\varepsilon = A_{0, -s-1, -s, -s-1, 0} E_\varepsilon$. Via elliptic regularity, we can estimate
\begin{equation}
\|G \tilde{P} u\|_{Y_N} \leq \|G_2 \tilde{P} u\|_{Y_N} + \|u\|_{E_N}. \tag{428}
\end{equation}
By shrinking the support of $\chi, \psi$ if necessary, we can arrange that the $L^\infty$-esssupp of $a, b, f, e$ is a subset $\text{Eff}_{\le C}^{0,0,0,0}(G_3)$, and then we can choose $G$ such that $\text{WF}_{\le C}^{0,0,0,0}(G_3) \subset \text{Eff}_{\le C}^{0,0,0,0}(G_3)$, so that the estimate eq. (427) implies
\[
\|\hat{B}_z u\|_{\mathcal{X}_N} \leq \|G_2 \hat{P} u\|_{\mathcal{Y}_N} + \|G_3 u\|_{\mathcal{Z}_N} + \|u\|_{\mathcal{E}_N}.
\] (429)

Using the Banach-Alaoglu theorem, applied as during the proof of the propagation estimate, we can take $\varepsilon \to 0^+$ to conclude
\[
\|\hat{B}_0 u\|_{\mathcal{X}_N} \leq \|G_2 \hat{P} u\|_{\mathcal{Y}_N} + \|G_3 u\|_{\mathcal{Z}_N} + \|u\|_{\mathcal{E}_N}.
\] (430)

Let $\mathcal{X} = H^{\text{leC},s,k,1,\ell}_{\le C}(X)$. Since $\text{Eff}_{\le C}^{0,0,0,0}(\hat{B}_0) \supset \mathcal{R}_+$, eq. (430) implies
\[
\|G_1 u\|_{\mathcal{X}} \leq \|\hat{B}_0 u\|_{\mathcal{X}_N} + \|G_2 \hat{P} u\|_{\mathcal{Y}_N} + \|u\|_{\mathcal{E}_N},
\] (431)

where we used the propagation estimate to control $\hat{G}_1 u$ on $\text{Char}^{2,0,1,-3,-3}_{\text{leC}}(\hat{P})$ away from $\mathcal{R}_+$. So,
\[
\|G_1 u\|_{\mathcal{X}} \leq \|\hat{B}_0 u\|_{\mathcal{X}_N} + \|G_2 \hat{P} u\|_{\mathcal{Y}_N} + \|u\|_{\mathcal{E}_N} \leq \|G_2 \hat{P} u\|_{\mathcal{Y}_N} + \|G_3 u\|_{\mathcal{Z}_N} + \|u\|_{\mathcal{E}_N}
\]
\[
\leq \|G_2 \hat{P} u\|_{\mathcal{Y}_N} + \|G_3 u\|_{H^{\text{leC},s,-1,1,1/2,-1/2,-N,-N}_{\le C}} + \|u\|_{\mathcal{E}_N}.
\] (432)

An inductive argument (using eq. (402)) estimating $\|G_3 u\|_{H^{\text{leC},s,-1,1,1/2,-1/2,-N,-N}_{\le C}}$ finishes the proof. \(\square\)

5.3. The Radial “Point” $\mathcal{R}$. In order to analyze matters uniformly near the corners of $\text{leC}$-phase space (in particular the highlighted edge $bf \cup ff$ in Figure 7), we work with the bdfs $\theta_{df}, \theta_{dt}, \theta_{ff}$ defined in eq. (78) rather than the $\text{leC}$-adapted momentum coordinates $\xi_{sc, \text{leC}}, \eta_{sc, \text{leC}}$ used in the previous section.

In terms of the bdfs,
\[
\tilde{p}_0 = \theta_{df}^2 \theta_{dt}^2 \theta_{ff}^2 (1 - \theta_{df}^2 \theta_{dt}^2 \theta_{ff}^2) - 2 \left( \frac{\xi_b}{(1 + \xi_b^2 + g_{\text{df}}(\eta_b, \eta_b))^{1/2}} \right) \theta_{df}^{-1} \theta_{dt} \theta_{ff}^3
\] (433)
\[
= \theta_{df}^2 \theta_{dt}^2 \theta_{ff}^2 (1 - \theta_{df}^2 \theta_{dt}^2 \theta_{ff}^2) - 2 \tilde{\mathfrak{B}} \theta_{df}^{-1} \theta_{dt}^2 \theta_{ff}^3
\] (434)

near $\{x = 0\} \subset \text{leC} T^\star X$, where $\tilde{\mathfrak{B}} \in S^{0,0,0,0}_{\text{leC}}(X)$ is given by $\xi_b (1 + \xi_b^2 + g_{\text{df}}(\eta_b, \eta_b))^{-1/2}$ near $\{x = 0\}$. Factoring out common powers of bdfs from eq. (434), we are left with
\[
\tilde{p}_0^{2,0,-2,-1,-3} = \theta_{df} \theta_{dt} (1 - \theta_{df}^2 \theta_{dt}^2 \theta_{ff}^2) - 2 \tilde{\mathfrak{B}} \theta_{df},
\] (435)
\[
= \theta_{df} \theta_{dt} - 2 \tilde{\mathfrak{B}} \theta_{df} \text{mod} S^{1,-1,1,-1,-1}_{\text{leC}}(X).
\] (436)

On $sf \cup ff$, $\tilde{p}_0^{2,0,-2,-1,-3}$ and thus $\tilde{p}_0^{2,0,-2,-1,-3}$ vanishes if and only if $2 \tilde{\mathfrak{B}} = \theta_{df} \theta_{dt} / \theta_{df}$. Therefore
\[
\text{Char}_{\text{leC}}^{2,0,-2,-1,-3}(\hat{P}) = \{2 \mathfrak{B} = \theta_{df} \theta_{dt} / \theta_{df}\} \subset \text{leC} T^\star X.
\] (437)

In particular, the portion of the characteristic set that is on the boundary of $bf \cup tf$ is precisely $\{2 \mathfrak{B} = 0\} \cap (bf \cup tf) \cap (sf \cup ff)$.

**Proposition 5.10.** We have $H_{\tilde{p}_0}^{2,0,-2} \theta_{df} = F_{0,1} \theta_{df}$ for $F_{0,1} \in S_{\text{leC}}^{0,0,0,0}(X)$ given by
\[
F_{0,1} = 2 \theta_{df} \theta_{dt} (1 - \theta_{df}^2 \theta_{dt}^2 \theta_{ff}^2) - 2 \theta_{df} \theta_{dt} \mathfrak{B}^2 - \mathcal{Z} \theta_{dt} \theta_{df} \theta_{df} \mathfrak{B}^2
\] (438)

near $\{x = 0\} \subset \text{leC} T^\star X$. \(\blacksquare\)
Proof. Applying eq. (326) to $\vartheta_{\text{df}}$,

$$H_{p_0} \vartheta_{\text{df}} = -(x \partial_x \tilde{p}_0) \partial_{\xi_b} \vartheta_{\text{df}}$$

(439)

near $\{x = 0\}$. Writing $\tilde{p}_0 = x^2 \vartheta_{\text{df}}^2 (1 - \vartheta_{\text{df}}^2) - 2x \xi_b (\sigma^2 + Zx)^{1/2}$,

$$\partial_x \tilde{p}_0 = 2x \vartheta_{\text{df}}^2 (1 - \vartheta_{\text{df}}^2) - 2\xi_b (\sigma^2 + Zx)^{1/2} - Zx \xi_b (\sigma^2 + Zx)^{-1/2}$$

(440)

Thus,

$$H_{p_0} \vartheta_{\text{df}} = 2x^2 \xi_b \vartheta_{\text{df}} (1 - \vartheta_{\text{df}}^2) - 2x \xi_b^2 \vartheta_{\text{df}}^3 (\sigma^2 + Zx)^{1/2} - Z \xi_b^2 \vartheta_{\text{df}}^3 x^2 (\sigma^2 + Zx)^{-1/2}.$$ 

(441)

In terms of bdfs $\mathcal{C}^{+} X$, $x^2 \xi_b \vartheta_{\text{df}} = \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^4 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3$, and $x \xi_b^2 \vartheta_{\text{df}}^3 (\sigma^2 + Zx)^{1/2} = \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^3$,

and $\xi_b \vartheta_{\text{df}} x^2 (\sigma^2 + Zx)^{-1/2} = \vartheta_{\text{df}} \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3$. Adding everything together, we find

$$H_{p_0} \vartheta_{\text{df}} = 2 \vartheta_{\text{df}}^4 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^3 - \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^3$$

(442)

Dividing by $\vartheta_{\text{df}}^{-1} x(\sigma^2 + Zx)^{-1/2} = \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3$, we obtain

$$H_{p_0}^{2,0,-2} \vartheta_{\text{df}} = (2 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3 - \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3 \vartheta_{\text{df}}^3) \vartheta_{\text{df}}$$

(443)

as claimed. $\square$

Letting $\beta_1, \beta_2$ be as in Proposition 5.8, $\beta_1, \beta_2 < 0$ on $\mathcal{R}_0$ and thus on $\mathcal{R}$. By Proposition 5.10, there exists an $F_1 \in S_{\text{cl}, \text{leC}}^{0,0,0,0,0}(X)$ such that

$$H_{p_0}^{2,0,-2} \vartheta_{\text{df}} = F_1 \vartheta_{\text{df}}$$

(444)

with $F_1$ vanishing on $\mathcal{R}$. By Proposition 5.5, it is the case that for any fixed $\rho \in S_{\text{cl}, \text{leC}}^{0,0,0,0,0}(X)$ equal to $\xi_{\text{sc}, \text{leC}}^2 + \eta_{\text{sc}, \text{leC}}^2$ in some neighborhood of $bf \cup tf$, there exist some symbols

$$\beta_0, F_2, F_3 \in S_{\text{cl}, \text{leC}}^{0,0,0,0,0}(X)$$

(445)

such that

$$H_{p_0}^{2,0,-2} \rho = \beta_0 \rho + F_2 + F_3$$

(446)

$$\beta_0 |_{\mathcal{R}_0} < 0,$$

(447)

$\beta_0, F_2 \leq 0$ everywhere, and $F_3$ vanishes cubically at $\mathcal{R}$ uniformly in $[0, \Sigma]$. We may choose $\rho$ such that it is nonnegative everywhere.

It is necessary to have another weight with semidefinite sign under the Hamiltonian flow. We may use

$$\vartheta_\rho = \vartheta_{\text{df}} + \vartheta_{\text{tf}} \in S_{\text{cl}, \text{leC}}^{0,0,0,0,0}(X).$$

(448)

We have $H_{p_0}^{2,0,-2} \vartheta_\rho = \beta_3 \vartheta_\rho$ for $\beta_3 = (F_1 \vartheta_{\text{df}} + \beta_2 \vartheta_{\text{tf}})(\vartheta_{\text{df}} + \vartheta_{\text{tf}})^{-1} \in S_{\text{cl}, \text{leC}}^{0,0,0,0,0}(X)$. We can write this as $\beta_3 = F_1 \vartheta_{\text{df}} + \beta_2 \vartheta_{\text{tf}}$. Thus, $\beta_3 |_{\mathcal{R}} \leq 0$. Note that $\beta_3$ vanishes at $\mathcal{R} \cap tf$. Normally, this would be problematic as far as the radial point estimate is concerned, but this fourth weight is used only to give us an extra independent order and not to manufacture positivity, so a semidefinite sign is actually acceptable.

We now have enough basic weights to construct our commutants: the basic weights are $x = \vartheta_{\text{df}} \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2, (\sigma^2 + Zx)^{1/2} = \vartheta_{\text{df}} \vartheta_{\text{df}}^2$, and $\vartheta_{\text{df}}^2 = \vartheta_{\text{df}} \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^2$. For any $s, l, \ell \in \mathbb{R}$,

$$x^s (\sigma^2 + Zx)^{l/2-1} \vartheta_{\text{df}}^s \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^{s-l} = \vartheta_{\text{df}}^s \vartheta_{\text{df}}^2 \vartheta_{\text{df}}^{s-l} \vartheta_{\text{df}}^{l}.$$ 

(449)
We do not care about the order at df, so the weights of the form eq. (449) (which give four independent orders) suffice. We now consider the weight (dependent on parameters $s, \varsigma, l, \ell \in \mathbb{R}$)

$$a_0 = x^{-l}(\alpha^2 + Zx)^{-\ell/2+l}e^{-x+s-l}e^{-\varsigma+s-l} \in S^{s-l, s, \varsigma, l, \ell}_c(X).$$

(450)

Per the above and eq. (414),

$$\beta = -((s - l)F_1 + (\varsigma - \ell - s + l)\beta_3 + l\beta_1 + (\ell - 2l)\beta_2) = a_0^{-1}H_{\tilde{p}}^{2,0,-2}a_0 \in S^{0,0,0,0}_c(X)$$

(451)

will have a definite sign near $R$ if $l, \ell$ have the same definite sign and $\varsigma - \ell - s + l$ has the same sign semidefinite.

Hence, multiplying $a_0$ by an appropriate microlocal cutoff, we can arrange for $a$ to be everywhere monotonic under the Hamiltonian flow, strictly so near $R$. Specifically, fix $\chi \in C_\infty^0(R; [0, 1])$ as in the proof of Proposition 5.9. Dilating $\chi$ if necessary, we can find $\psi \in S^{0,0,0,0}_c(X)$ that is identically equal to one in some neighborhood of $sf \cup ff$ such that

$$\beta < 0$$

(452)

on supp $\chi(\tilde{p}^{2,0,-2,1-3})\chi(\rho)\psi$ and such that

$$\chi(\tilde{p}^{2,0,-2,1-3})\chi(\rho)\psi\sqrt{H_{\tilde{p}}^{2,0,-2}} \in S^{0,0,0,0}_c(X).$$

(453)

and such that supp $\chi(\tilde{p}^{2,0,-2,1-3})\chi(\rho)\psi \cap (R_+ \cup df) = \emptyset$. Now set

$$a = a_0\chi(\tilde{p}^{2,0,-2,1-3})\chi(\rho)\psi^2 \in S^{s-l, s, \varsigma, \ell}_c(X).$$

(454)

The three factors $\chi(\tilde{p}^{2,0,-2,1-3}), \chi(\rho), \psi \in S^{0,0,0,0}_c(X)$ together microlocalize near $R$.

We can write $H_{\tilde{p}}^{2,0,-2} \tilde{p}^{2,0,-2,1-3} = \tilde{q}^{2,0,-2,1-3}$ for $\tilde{q} \in S^{0,0,0,0}_c(X)$ defined by

$$\tilde{q} = \tilde{q}_{df}^2 \tilde{q}_H^2 \tilde{q}_h^3 \tilde{q}_s^2 \tilde{q}_t^2 \tilde{q}_x^2 \tilde{q}_\tilde{t} H_{\tilde{p}}^{2,0,-2}.$$

(455)

Then,

$$H_{\tilde{p}}^{2,0,-2}a = \psi^2 \chi(\tilde{p}^{2,0,-2,1-3})\chi(\rho)^2 H_{\tilde{p}}^{2,0,-2}a_0 - 2a_0 \chi(\rho)^2 \chi(\tilde{p}^{2,0,-2,1-3})\chi(\rho)^2 \psi^2 H_{\tilde{p}}^{2,0,-2}$$

$$+ 2a_0 \chi(\tilde{p}^{2,0,-2,1-3})\chi(\rho)^2 \psi H_{\tilde{p}}^{2,0,-2} \psi$$

(456)

Observe:

1. by eq. (452), the first term,

$$\psi^2 \chi(\tilde{p}^{2,0,-2,1-3})\chi(\rho)^2 H_{\tilde{p}}^{2,0,-2}a_0 = \psi^2 \chi(\tilde{p}^{2,0,-2,1-3})\chi(\rho)^2 \beta a_0$$

(457)

will have a definite sign (the same as $\beta$, negative if $l, \ell < 0$) for appropriate $\chi, \psi$,

2. the second term,

$$-a_0 \chi(\rho)^2 \chi(\tilde{p}^{2,0,-2,1-3})\psi^2 H_{\tilde{p}}^{2,0,-2} \rho = -a_0 \chi(\rho)^2 \chi(\tilde{p}^{2,0,-2,1-3})\psi^2 (\beta_0 \rho + F_2 + F_3)$$

(458)

also has a definite sign (positive, since $\beta_0, F_2, F_3 \leq 0$) for appropriate $\chi, \psi$ and is supported in an annulus around $R_+$, which should intersect $\text{Char}_{c}^{2,0,-2,1-3}(\tilde{P}) \backslash (R \cup R_+)$,

3. the third sand fourth terms are supported away from $\text{Char}_{c}^{2,0,-2,1-3}(\tilde{P})$ and are therefore unproblematic (as that region of phase space is controlled via elliptic estimates or, in the case of $bf^u \cup tf^u$ where the fourth term might have some support, cannot be controlled by symbolic considerations anyways).
We will prove a low order radial point estimate. This means that
\[ l < 0, \quad \ell < 0 \quad \varsigma \leq \ell + s - l, \]
so that \( \beta > 0 \) near \( \mathcal{R} \). Thus, the first and second terms in eq. (456) have the opposite sign near \( \mathcal{R} \). The second term thus contributes to the right-hand side of the radial point estimate, but this term can itself be controlled using a radial point estimate at \( \mathcal{R}_+ \) in conjunction with a propagation estimate and will therefore be unproblematic as well.

In order to “regularize,” set \( \phi_e = (1 + \varepsilon x^{-1})^{-K_1(1 + \varepsilon \theta_{df,0}^{-1})^{-K_3}}, \)
\[ a^{(e)} = \phi^2_e a \in L^\infty([0, 1]; S^{-\infty, s, \varsigma, l, \ell}_c(X)), \]
for to-be-decided \( K_1, K_3 \in \mathbb{R} \). We then compute that
\[ H^{2,0, -2}_p a^{(e)} = \phi^2_e \left[ -\chi(p^{2,0, -2, -1, -3})^2 \chi(\rho)^2 \psi^2 a_0 \right] \left( \varsigma - \ell - s + l \right) \beta_3 
+ F_1 \left( m - \frac{K_3 \varepsilon \theta_{df,0}^{-1}}{1 + \varepsilon \theta_{df,0}^{-1}} \right) + \beta_1 \left( l - \frac{K_1 \varepsilon x^{-1}}{1 + \varepsilon x^{-1}} \right) + \beta_2 (\ell - 2l) \right) 
- 2a_0 \chi_0(\rho)^2 \psi^2 \chi(p^{2,0, -2, -1, -3})^2 \psi^2 H^{2,0, -2}_p 
+ 2a_0 \chi(p^{2,0, -2, -1, -3}) \psi^2 \chi(p^{2,0, -2, -1, -3})^2 \psi \psi H^{2,0, -2}_p. \] (461)

In contrast to the previous radial point estimate, we can make \( K_1, K_3 \) arbitrarily large without affecting the sign of the parenthetical term, although we might need to choose \( \psi \) with smaller support and replace \( \chi \) with \( \chi \circ \text{dil}_\lambda \) for
\[ \lambda = \lambda(m, l, \ell, K_1, K_3) > 0 \] (462)
sufficiently large to ensure that the term proportional to \( F_1 \) in eq. (461) does not spoil that sign. So, for some choice of \( \psi, \chi \), we can choose \( \delta = \delta(K_1, K_3, m, l, \ell, \chi, \psi) > 0 \) sufficiently small such that there exist well-defined uniform families of leC-degrees
\[ b_* \in L^\infty([0, 1]; S^{-\infty, (s-1)/2, (l-1)/2, (-l-1)/2, (-\varsigma-1)/2}_c(X)), \]
\[ e_* \in L^\infty([0, 1]; S^{-\infty, (s-1)/2, (l-1)/2, (-\varsigma-1)/2, -\infty, -\infty}_c(X)), \]
\[ f_* \in L^\infty([0, 1]; S^{-\infty, -\infty, -\infty, -\infty}_c(X)), \]
\[ r_* \in L^\infty([0, 1]; S^{-\infty, -\infty, -\infty, -\infty}_c(X)) \] (463)
such that, in some neighborhood of \( \{ x = 0 \} \subset \text{leC} T^* X, \)
\[ b_x = -\theta_{dt}^{1/2} \theta_{df}^{1/2} \theta_{dt}^{3/2} \theta_{dt}^{1/2} a_0 \chi(p^{2,0, -2, -1, -3}) \psi^2 \phi_0 \phi_e \left[ F_1 \left( m - \frac{K_3 \varepsilon \theta_{df,0}^{-1}}{1 + \varepsilon \theta_{df,0}^{-1}} \right) + \beta_1 \left( l - \frac{K_1 \varepsilon x^{-1}}{1 + \varepsilon x^{-1}} \right) \right]^1/2 
+ \beta_2 (\ell - 2l) \right) + \beta_3 (\varsigma - \ell - s + l) \right) - 2\delta \phi^2 \chi(p^{2,0, -2, -1, -3}) \psi^2 \hat{\phi}_0 \hat{\phi}_{dt}^{3/2} \theta_{dt}^{1/2} \theta_{dt}^{3/2} \theta_{dt}^{1/2} [a_0 \chi(p^{2,0, -2, -1, -3}) \psi \psi H^{2,0, -2}_p]^{1/2} 
+ \delta \phi^2 \chi(p^{2,0, -2, -1, -3}) \psi \psi H^{2,0, -2}_p. \] (464)
In terms of these new symbols, we can write

\[ H_p a^{(e)} + p_1 a^{(e)} = -2 \delta_e a^{(-1)} \psi_{bf} \psi_{bf}^3 \psi_{t1}^3 a^{(-1)} - b^2_e + c^2_e + f_e \bar{p} + r_e. \]  

(465)

We now apply \( \text{Op} \). Setting \( A_\varepsilon = (1/2)(\text{Op}(a^{(e)}) + \text{Op}(a^{(e)^*})) \), \( B_\varepsilon = \text{Op}(b_\varepsilon) \), \( E_\varepsilon = \text{Op}(e_\varepsilon) \), \( F_\varepsilon = \text{Op}(f_\varepsilon) \), we have

\[
\begin{align*}
A_* &\in L^\infty \left( \{0, 1\}; \Psi^{-\infty, s, \zeta, l, t}_\text{leC}(X) \right), \\
B_* &\in L^\infty \left( \{0, 1\}; \Psi^{-\infty, (s-1)/2, (c-3)/2, (l-1)/2, (\ell-3)/2}_\text{leC}(X) \right), \\
E_* &\in L^\infty \left( \{0, 1\}; \Psi^{-\infty, (s-1)/2, (c-3)/2, (l-1)/2, (\ell-3)/2}_\text{leC}(X) \right), \\
F_* &\in L^\infty \left( \{0, 1\}; \Psi^{-\infty, s-1, \zeta-1, l, t}_\text{leC}(X) \right),
\end{align*}
\]

(466)

and

\[
- i [\mathbb{R} \tilde{P}, A_\varepsilon] - \{3 \tilde{P}, A_\varepsilon \} = -2 A_\varepsilon A_1/2, -(s+1)/2, -(c+3)/2, -(l+1)/2, -(\ell+3)/2 A_\varepsilon - B^*_e B_e + E^*_e E_e + \bar{F}^*_e \tilde{P} + R_\varepsilon
\]

(467)

for some \( R_* \in L^\infty \left( \{0, 1\}; \Psi^{-\infty, s-2, c-4, l-1, \ell-3}_\text{leC}(X) \right) \). Moreover, we necessarily have

\[
\begin{align*}
\text{WF}'_{L^\infty, \text{leC}}(A_*), \text{WF}'_{L^\infty, \text{leC}}(B_*), \text{WF}'_{L^\infty, \text{leC}}(F_*), \\
\text{WF}'_{L^\infty, \text{leC}}(E_*), \text{WF}'_{L^\infty, \text{leC}}(R*) \subset \text{supp} \chi(p^{2, 0, -2, -1, -3}_\text{leC}) \chi(\rho) \psi,
\end{align*}
\]

(468)

where the last of these inclusions (the one for \( R_* \)) follows from the one for \( A_* \) and \( \text{WF}'_{L^\infty, \text{leC}}([\tilde{P}, A_*]) \subset \text{WF}'_{L^\infty, \text{leC}}(A_*) \).

For each \( m_0, s_0, l_0, \ell_0 \in \mathbb{R} \), there exist some \( K_{1,0}, K_{3,0} > 0 \) (dependent on \( m_0, s_0, l_0, \ell_0 \) and \( m, s, \zeta, l, \ell \)) such that, given \( \{u(-\sigma)\}_{\sigma>0} \subset \mathcal{S}'(X) \) with \( u(-0) \in H^{m_0, s, l}_\text{leC}(X_{1/2}) \) and \( u(-\sigma) \in H^{m_0, s, l}_\text{leC}(X) \) for all \( \sigma > 0 \), if we take \( K_1 > K_{1,0}, K_3 > K_{3,0} \) in the construction above then it is the case that (for any \( \varepsilon > 0 \), and for each \( \sigma > 0 \), implicit in the notation),

\[
2 \mathfrak{H} \{ \tilde{P} u, A_\varepsilon u \} = - \{3 \tilde{P}, A_\varepsilon \} u, u \}_{L^2} + i \{ [\mathbb{R} \tilde{P}(\sigma), A_\varepsilon] u, u \}_{L^2},
\]

(469)

where the pairings above are well-defined distributional pairings (with the left argument of each inner product in the dual Sobolev space to a Sobolev space in which the right argument lies). Applying eq. (467) to \( \{u(-\sigma)\}_{\sigma>0} \) as above and pairing against \( u \) (and taking \( K_1, K_3 \) large enough), we have

\[
2 \mathfrak{H} \{ \tilde{P} u, A_\varepsilon u \} = - \|B_\varepsilon u\|_{L^2}^2 + \|E_\varepsilon u\|_{L^2}^2 + \langle \tilde{P} u, F_\varepsilon u \rangle_{L^2} + \langle R_\varepsilon u, u \rangle_{L^2}
- 2 \delta \|x A_1/2, -(s+1)/2, -(c+3)/2, -(l+1)/2, -(\ell+3)/2 A_\varepsilon u\|_{L^2}^2.
\]

(470)

Equation (470) implies

\[
\|B_\varepsilon u\|_{L^2}^2 + 2 \delta \|A_1/2, -(s+1)/2, -(c+3)/2, -(l+1)/2, -(\ell+3)/2 A_\varepsilon u\|_{L^2}^2
\leq 2 \|\tilde{P} u, A_\varepsilon u \|_{L^2} + \|\tilde{P} u, F_\varepsilon u \|_{L^2} + \|R_\varepsilon u, u \|_{L^2} + \|E_\varepsilon u\|_{L^2}^2.
\]

(471)

Suppose \( G \in \Psi^{-\infty, 0, 0, 0, 0}_\text{leC}(X) \) is self-adjoint and such that \( \text{WF}'_{\text{leC}}(1 - G) \) is disjoint from \( \text{supp} \chi(p^{2, 0, -2, -1, -3}_\text{leC}) \chi(\rho) \psi \). We now estimate the terms in eq. (471) as in the propagation estimate, except we now must keep track of orders at \( b, t \),

- Writing \( \tilde{P} = (1 - G) \tilde{P} + G \tilde{P} \) we have, for each \( N \in \mathbb{N} \) (and \( K_1, K_3 \) large enough),

\[
\|\tilde{P} u, A_\varepsilon u \|_{L^2} \leq \|G \tilde{P} u, A_\varepsilon u \|_{L^2} + \|\tilde{P} u, (1 - G) A_\varepsilon u \|_{L^2}.
\]

(472)
Thus, taking \( \delta \) sufficiently small (relative to \( \delta \)),

\[
2|\langle \hat{P}u, A_\varepsilon u \rangle_{L^2^2}| - 2\delta \|\Lambda_{1/2,-(s+1)/2,-(c+3)/2,-(l+1)/2,-(l+3)/2}A_\varepsilon u\|_{L^2^2}^2 \leq \|G\hat{P}u\|_{Y_N}^2 + \|u\|_{\tilde{E}_N}^2.
\] (476)

\[
\|F_\varepsilon u\|_{Y_N}^2 + \|u\|_{\tilde{E}_N}^2.
\] (477)

for \( N_0 \) sufficiently large (relative to \( N_0 \)). Thus,

\[
|\langle \hat{P}u, F_\varepsilon u \rangle_{L^2^2}| \leq \|G\hat{P}u\|_{Y_N}^2 + \|u\|_{\tilde{E}_N}^2.
\] (478)

\[
\|B_\varepsilon u\|_{L^2^2} \leq \|G\hat{P}u\|_{Y_N}^2 + \|u\|_{\tilde{E}_N}^2 + \|E_\varepsilon u\|_{L^2^2}^2 + \|u\|_{\tilde{E}_N}^2.
\] (480)

I.e., letting \( X_N = H_{l_{\varepsilon}}^{-N,(s-2)/2,(c-4)/2,(l-1)/2,(l-3)/2}(X) \) and \( \bar{X}_N = H_{l_{\varepsilon}}^{-N,(s-1)/2,(c-3)/2,(l-2)/2,(l-3)/2}(X) \),

\[
|\langle \hat{B}_\varepsilon u, X_N \rangle| \leq \|G\hat{P}u\|_{Y_N}^2 + \|u\|_{\tilde{E}_N}^2 + \|\bar{E}_\varepsilon u\|_{\bar{X}_N}^2 + \|u\|_{\bar{E}_N}^2,
\] (481)

\[
|\langle \hat{B}_\varepsilon u, \bar{X}_N \rangle| \leq \|G\hat{P}u\|_{Y_N}^2 + \|u\|_{\tilde{E}_N}^2 + \|\bar{E}_\varepsilon u\|_{\bar{X}_N}^2 + \|u\|_{\bar{E}_N}^2,
\] (482)
where
\[
\hat{B}_\varepsilon = \Lambda_{0,-(s-1)/2,-(\varsigma-3)/2,-(\ell-1)/2,-(\ell-3)/2B_\varepsilon}
\]
and
\[
\hat{E}_\varepsilon = \Lambda_{0,-(s-1)/2,-(\varsigma-3)/2,0,0,E_\varepsilon},
\]
so that \(\hat{B}_\bullet, \hat{E}_\bullet \in \Psi_{k\in C}^{0,0,0,0,0}(X)\).

**Proposition 5.11.** Suppose that \(G_0, G_1 \in \Psi_{k\in C}^{-\infty,0,0,0,0}(X)\), \(G_2 \in \Psi_{k\in C}^{-\infty,0,0,-\infty,-\infty}(X)\) satisfy

1. \(\text{WF}^0_{\text{leC}}(G_2) \cap \mathcal{R}_+ = \emptyset\) and

\[
\text{WF}^{0,0}_{\text{leC}}(G_1) \cap \mathcal{R}_+ = \emptyset,
\]

2. \(\mathcal{R}, \text{WF}^0_{\text{leC}}(G_1) \subset \text{Ell}_{\text{leC}}^{0,0,0,0,0}(G_0)\),

3. there exist \(\Theta_1, \Theta_2 \in (0, \pi)\) with \(\Theta_1 < \Theta_2\) such that \(\text{Ell}_{\text{leC}}^{0,0,0,0,0}(G_2) \supset \mathcal{P}[\Theta_1, \Theta_2]\) and

\[
\text{Ell}_{\text{leC}}^{0,0,0,0,0}(G_0) \supset \mathcal{R} \cup \bigcup_{\Theta_2 \in (0, \pi)} \mathcal{P}[\Theta_1, \Theta_2]
\]

Then, for any \(\Sigma > 0\), \(N \in \mathbb{N}\), and \(s, \varsigma, \ell, l \in \mathbb{R}\) with \(l < -1/2\), \(\ell < -3/2\), and \(\varsigma \leq \ell + s - l\), there exists a constant

\[
C = C(\dot{P}, G_0, G_1, G_2, G_3, \Sigma, N, m, s, \varsigma, l, \ell) > 0
\]

such that

\[
\|G_1 u\|_{H_{k\in C}^{m,s,\varsigma,l,\ell}} \leq C \|G_0 \dot{P} u\|_{H_{k\in C}^{-N,s,\varsigma,-N,-N}(X)} + \|G_2 u\|_{H_{k\in C}^{-N,s,\varsigma,-N,-N}} + \|u\|_{H_{k\in C}^{-N,-N,-N,1,\ell}}
\]

holds for all \(u \in \mathcal{S}'(X)\) and \(\sigma \in [0, \Sigma]\).

**Proof.** We can assume without loss of generality that

\[
\mathcal{R} \subseteq \text{Ell}_{\text{leC}}^{0,0,0,0,0}(G_1)
\]

and

\[
\text{Ell}_{\text{leC}}^{0,0,0,0,0}(G_0) \supset \text{WF}^0_{\text{leC}}(G_2).
\]

Let \(s_0 = 2s + 1, l_0 = 2l + 1, l_0 = 2\ell + 3,\) and \(s_0 = 2\varsigma + 3\). The condition \(l < -1/2\) is equivalent to \(l_0 < 0\), \(\ell < -3/2\) is equivalent to \(\ell_0 < 0\), and \(\varsigma \leq \ell + s - l\) is equivalent to \(s_0 \leq \ell_0 + s_0 - l_0\). We can therefore apply eq. (482) with \(s_0, s_0, l_0, l_0\) in place of what we called \(s, \varsigma, l, l_0\) there. Thus, for each \(\varepsilon > 0\),

\[
\|\hat{B}_\varepsilon u\|_{X_N} \leq \|G_0 \dot{P} u\|_{Y_N} + \|G_0 u\|_{Z_N} + \|\hat{E}_\varepsilon u\|_{X_N} + \|u\|_{E_N},
\]

where \(G, \hat{B}_\bullet, \hat{E}_\bullet\) are as above and now

\[
X_N = H_{k\in C}^{-N,s,\varsigma,l,\ell}(X),
\]

\[
Z_N = H_{k\in C}^{-N,s+1/2,\varsigma+1/2,l+1,\ell+1}(X),
\]

\[
Y_N = H_{k\in C}^{-N,s,\varsigma+1,l+1,\ell+1}(X),
\]

\[
E_N = H_{k\in C}^{-N,-N,-N,l,\ell}(X).
\]

If necessary, we can retroactively replace \(\chi, \psi\) such that

\[
\text{WF}^0_{L^\infty, k\in C}(E_\bullet), \text{WF}^0_{L^\infty, k\in C}(B_\bullet) \subset \text{Ell}_{k\in C}^{0,0,0,0,0}(G_0).
\]

We can apply the propagation estimate Proposition 5.7 to bound

\[
\|\hat{E}_\varepsilon u\|_{X_N} \leq \|G_0 \dot{P} u\|_{Y_N} + \|G_2 u\|_{X_N} + \|u\|_{E_N}.
\]
By eq. (489), we can now retroactively choose $G$ (for appropriate $\chi, \psi$) such that $WF_{leC}^{0,0,0,0}(G_0) \cap Ell_{leC}^{0,0,0,0}(G_1)$, so that

$$\|G\tilde{P}u\|_{Y_N} \leq \|G_0\tilde{P}u\|_{Y_N} + \|u\|_{\mathcal{E}_N}$$

and $\|G_0u\|_{Z_N} \leq \|G_1u\|_{Z_N} + \|u\|_{\mathcal{E}_N}$.

The estimate eq. (491) therefore implies $\|\hat{B}_\varepsilon u\|_{X_N} \leq \|G_0\tilde{P}u\|_{Y_N} + \|G_1u\|_{Z_N} + \|G_2u\|_{X_N} + \|u\|_{\mathcal{E}_N}$.

Via the compactness argument utilized in the proof of the propagation estimate and previous radial point estimate, $\|\hat{B}_0u(-;\sigma)\|_{X_N} \leq \liminf_{\varepsilon \to 0} \|\hat{B}_\varepsilon u(-;\sigma)\|_{X_N}$ for each $\sigma \geq 0$, so

$$\|\hat{B}_0u\|_{X} \leq \|G_0\tilde{P}u\|_{Y_N} + \|G_1u\|_{Z_N} + \|G_2u\|_{X_N} + \|u\|_{\mathcal{E}_N}.$$ (497)

Unlike $\hat{B}_\varepsilon$ for $\varepsilon > 0$, $Ell_{leC}^{0,0,0,0}(\hat{B}_0) \supset \mathcal{R}$. Thus, (for $\chi, \psi$ with sufficiently small support) $\|\hat{B}_0u\|_{X_N} \leq \|G_1u\|_{X_N} + \|u\|_{\mathcal{E}_N}$. Substituting this into the estimate above, we get

$$\|G_1u\|_{X_N} \leq \|G_0\tilde{P}u\|_{Y_N} + \|G_1u\|_{Z_N} + \|G_2u\|_{X_N} + \|u\|_{\mathcal{E}_N}.$$ (498)

Since the leC- Sobolev spaces $X_N, Y, Z_N$ in eq. (498) get weaker as $s, \varsigma, l, \ell$ decrease, we can inductively use the family of estimates eq. (498) to bound the $\|G_1u\|_{Z_N}$ term on the right-hand side: for all $N \in \mathbb{N}$,

$$\|G_1u\|_{X_N} \leq \|G_0\tilde{P}u\|_{Y_N} + \|G_1u\|_{Z_N} + \|G_2u\|_{X_N} + \|u\|_{\mathcal{E}_N}$$

(499)

$$\|G_1u\|_{X_N} \leq \|G_0\tilde{P}u\|_{Y_N} + \|G_2u\|_{X_N} + \|u\|_{\mathcal{E}_N}.$$ (500)

This implies eq. (488). □

5.4. Upshot. Combining the estimates above (e.g. using Lemma 2.16), we get:

**Theorem 5.12.** Suppose that $G_1, G_2, G_3 \in \Psi_{leC}^{0,0,0,0}(X)$ satisfy

- $WF_{leC}^{0,0,0,0}(G_1), \text{Char}_{leC}^{2,0,-2,-1,-3}(\check{P}) \subseteq Ell_{leC}^{0,0,0,0}(G_2)$,
- $\mathcal{R}_+ \subseteq Ell_{leC}^{0,0,0,0}(G_3)$.

For every $m, s, \varsigma, l, \ell, s_0, s_0 \in \mathbb{R}$ satisfying

$$l < -1/2, \quad \ell < -3/2, \quad -1/2 < s_0 < s, \quad -3/2 < s_0 < \varsigma < l + s - l,$$

(501)

there exists, for each $\Sigma > 0$ and $N \in \mathbb{N}$, a constant $C = C(\check{P}, G_1, G_2, G_3, \Sigma, N, m, s, \varsigma, l, \ell, s_0, s_0) > 0$ such that

$$\|G_1u\|_{H_{leC}^{m,s,\varsigma,l,\ell}} \leq C\left(\|G_2\tilde{P}u\|_{H_{leC}^{m,-2,1,\varsigma+3,1,\ell+3}} + \|G_3u\|_{H_{leC}^{-N,s,0,-N,-N}} + \|u\|_{H^{-N,l,\ell}}\right)$$

(502)

for all $u \in \mathcal{S}'(X)$ and $\sigma \in [0, \Sigma]$ (in the usual strong sense that if the right-hand side is finite, then the left-hand side as well, and the stated inequality holds).

It will also be useful to have the following refinement of Theorem 5.12:

**Proposition 5.13.** Given $G_1, G_2, m, s, \varsigma, l, \ell, s_0, s_0$ as in the setup of Theorem 5.12, if in addition $Ell_{leC}^{0,0,0,0}(G_1) \supset \mathcal{R}_+$, then there exists a constant $c = c(\check{P}, G_1, G_2, \Sigma, N, m, s, \varsigma, l, \ell, s_0) > 0$ such that

$$\|G_1u\|_{H_{leC}^{m,s,\varsigma,l,\ell}} \leq c\left(\|G_2\tilde{P}u\|_{H_{leC}^{m,-2,1,\varsigma+3,1,\ell+3}} + \|u\|_{H^{-N,l,\ell}}\right)$$

(504)

for all $\sigma \in [0, \Sigma]$ such that $\|G_1(\sigma)u\|_{H_{leC}^{-N,s,0,-N,-N}(X) (\sigma)} < \infty$. □
Proof. It suffices to consider the case \(-N < m, l, \ell\).

Applying Theorem 5.12 with \(G_3 = G_1\),
\[
\|G_1 u\|_{H^{m,r,s,l,\ell}_\text{leC}} \leq C' \left( \|G_2 \tilde{P} u\|_{H^{m-2,s+1,c+3,l+1,\ell+3}_{\text{leC}}} + \|G_1 u\|_{H^{m-2,s_0,c+3,l+1,\ell+3}_{\text{leC}}} \right) + \|u\|_{H^{-N,r,s,l,\ell}_{h,b,\text{leC}}} \quad (505)
\]
for some constant \(C' > 0\). We now use the interpolation inequality Lemma 2.17, which, for each \(\epsilon > 0\), allows us to bound
\[
\|G_1 u\|_{H^{-N,r,s_0,c+3,l+1,\ell+3}_{h,b,\text{leC}}} \leq \epsilon \|G_1 u\|_{H^{m,r,s,l,\ell}_\text{leC}} + C''(\epsilon) \|u\|_{H^{-N,r,s,l,\ell}_\text{leC}} \quad (506)
\]
for some \(C''(\epsilon) = C(\tilde{P}, G_1, G_2, \Sigma, N, m, s, c, l, \ell, s_0, s_0, \epsilon) > 0\). Taking \(\epsilon < 1/2C''\), plugging eq. (506) into eq. (505) yields
\[
\|G_1 u\|_{H^{m,r,s,l,\ell}_\text{leC}} \leq C' \left( \|G_2 \tilde{P} u\|_{H^{m-2,s+1,c+3,l+1,\ell+3}_{\text{leC}}} + (1 + C'') \|u\|_{H^{-N,r,s,l,\ell}_\text{leC}} \right) + (1/2) \|G_1 u\|_{H^{m,r,s,l,\ell}_\text{leC}} \quad (507)
\]
If \(\sigma \in [0, \Sigma]\) is such that
\[
\|G_1(\sigma)u(\cdot; \sigma)\|_{H^{m,r,s,l,\ell}_\text{leC}} < \infty, \quad (508)
\]
they can subtract the last term on the right-hand side from both sides, getting eq. (504),
\[
\|G_1 u\|_{H^{m,r,s,l,\ell}_\text{leC}} \leq 2C'(1 + C'') \|G_2 \tilde{P} u\|_{H^{m-2,s+1,c+3,l+1,\ell+3}_{\text{leC}}} + \|u\|_{H^{-N,r,s,l,\ell}_\text{leC}}. \quad (509)
\]
If \(\|G_1(\sigma)u(\cdot; \sigma)\|_{H^{-N,r,s_0,c+3,l+1,\ell+3}_{h,b,\text{leC}}} < \infty\), then we break into two cases:

1. If one of \(\|G_2 \tilde{P} u\|_{H^{m-2,s+1,c+3,l+1,\ell+3}_{\text{leC}}} \) or \(\|u\|_{H^{-N,r,s,l,\ell}_\text{leC}} \) is infinite, then eq. (504) holds trivially;
2. If both are finite, then Theorem 5.12 implies that \(\|G_1 u\|_{H^{m,r,s,l,\ell}_\text{leC}} < \infty\), and therefore eq. (509) holds.

We conclude that eq. (504) holds if we take \(C = 2C'(1 + C'')\).

Taking \(G_1 = G_2 = 1\) in Proposition 5.13, we get the main claim eq. (298) at the beginning of this section:

**Proposition 5.14.** For every \(\Sigma > 0\), \(N \in \mathbb{N}\), and \(m, s, c, l, \ell, s_0, s_0 \in \mathbb{R}\) satisfying \(l < -1/2 < s_0 < s\) and \(\ell < -3/2 < s_0 < c \leq \ell + s - l\), there exists a constant \(C = C(P, \Sigma, N, m, s, c, l, \ell) > 0\) such that
\[
\|u\|_{H^{m,r,s,l,\ell}_\text{leC}} \leq C \left[ \|\tilde{P} u\|_{H^{m-2,s+1,c+3,l+1,\ell+3}_{\text{leC}}} + \|u\|_{H^{-N,r,s,l,\ell}_\text{leC}} \right] \quad (510)
\]
holds for all \(u \in \mathcal{S}'(X)\) and \(\sigma \in [0, \Sigma]\) such that \(\|u\|_{H^{-N,r_0,s_0,c+3,l+1,\ell+3}_{h,b,\text{leC}}} < \infty\). □

6. Proof of Main Theorem

Most of the results in this section are split among three subsections:

1. §6.1, where estimates regarding the “leC-normal operator” \(N(\tilde{P})\) (see §3) are proven,
2. §6.2, where – using the estimates from the previous subsection – the conormality of the output of the conjugated resolvent family on \(X^\text{sp}_{\text{res}}\) is established, along with smoothness (in terms of \(E = \sigma^2\)) at \(zf\) (with the terms in the Taylor series being conormal distributions on \(zf\)) – see Proposition 6.15 — and
3. §6.3, where the proposition, Proposition 6.16, needed to upgrade the conormality established in the previous subsection to smoothness is proven.
Central to this section is the analysis of the model problem
\[ 2i \left[ \hat{x} \partial_x + \frac{\hat{x}}{1 + \hat{x}} \left( k + \frac{1}{4} \right) + l + \frac{1}{2} \right] u = f, \quad (511) \]
for some \( c \in \mathbb{C} \). The model problem above arises rewriting
\[ \tilde{N}(\tilde{P}) = x^{-l-n/2}(\sigma^2 + Zx)^{-k} N(\tilde{P})x^{l+n/2}(\sigma^2 + Zx)^k \quad (513) \]
in terms of \( \hat{x} = Zx/\sigma^2 \). Indeed,
\[ x^{-1}(\sigma^2 + Zx)^{-1/2} \tilde{N}(\tilde{P}) = 2i \left[ x\partial_x + l + \frac{1}{2} \right] + \frac{2iZx}{\sigma^2 + Zx} \left( k + \frac{1}{4} \right) \]
\[ = 2i \left[ \hat{x} \partial_x + \frac{\hat{x}}{1 + \hat{x}} \left( k + \frac{1}{4} \right) + l + \frac{1}{2} \right]. \quad (514) \]

If \( N(\tilde{P})u = f \) then \( \tilde{N}(\tilde{P})u_0 = f_0 \) for \( u_0 = x^{-l-n/2}(\sigma^2 + Zx)^{-k} u \) and \( f_0 = x^{-l-n/2}(\sigma^2 + Zx)^{-k} f \).

We now spell out the deduction of Theorem 1.1 (in the form of the more general Proposition 6.3) from the results in §6.2, §6.3.

**Proposition 6.1.** Suppose that \( g = g_0 + xg_1 + x^{\alpha_1}g_2 \) for \( g_1 \in C^\infty(X; \text{csc} \Sym^2 T^* X) \) and \( g_2 \in S^0(X; \text{csc} \Sym^2 T^* X) \) for \( \alpha_1 > 1 \). Then the Laplace-Beltrami operator \( \triangle_g \) has the form
\[ \triangle_g = \triangle_{g_0} + x \text{Diff}_{\text{csc}}^{2,0,-2}(X) + x^{\alpha_1} \text{Diff}_{\text{csc}}^{2,0,-2}(X), \quad (515) \]
where \( g_0 \) is an exactly conic metric. \( \square \)

**Proof.** Let \( h \mapsto [h] \) denote the natural bundle monomorphism \( \text{csc} \Sym^2 T^* X \to \text{End}(\text{csc} \Sym^2 T^* X) \).

The matrix identity \([g]^{-1} = [g_0]^{-1} - [g]^{-1}(x[g_1] + x^{\alpha_1}[g_2])[g_0]^{-1} \), applied inductively, yields
\[ [g]^{-1} - [g_0]^{-1} = [g_0]^{-1} \sum_{k=1}^{K} (-1)^k ((x[g_1] + x^{\alpha_1}[g_2])[g_0]^{-1})^k + (-1)^{K+1}[g]^{-1}((x[g_1] + x^{\alpha_1}[g_2])[g_0]^{-1})^{K+1} \quad (516) \]
for each \( K \in \mathbb{N} \). Noting that \( g^{-1} \in S^0(X; \text{csc} \Sym^2 T^* X) \), taking \( K > \alpha_1 \) leads to \( g^{-1} - g_0^{-1} \in xC^\infty(X; \text{csc} \Sym^2 T^* X) + x^{\alpha_1}S^0(X; \text{csc} \Sym^2 T^* X) \). Also, from \([g] = [g_0][1 + x[g_0]^{-1}[g_1] + x^{\alpha_1}[g_0]^{-1}[g_2]] \),
det \( g \in S^0(X; \text{csc} \Omega X) \) satisfies
\[ \det g = (\det g_0)(1 + x \text{tr}([g_0]^{-1}[g_1])) + x^2 C^\infty(X) + x^{\alpha_1}S^0(X). \quad (517) \]

Writing
\[ \triangle_g = \sum_{i,j=1}^{n} (g^{ij}\partial_i \partial_j + \partial_i g^{ij} \partial_j + (1/2)(\det g)^{-1} (\partial_i \det g) g^{ij} \partial_j) \quad (518) \]
in local coordinates, we conclude that eq. (515) holds locally, which suffices to show that the decomposition eq. (515) can be done globally. \( \square \)

Thus:
Proposition 6.2. Suppose that \( g = g_0 + xg_1 + x^{\alpha_1}g_2 \) for \( g_1 \in C^\infty(X, \ast \text{Sym}^2 T^*X) \) and \( g_2 \in S^0(X, \ast \text{Sym}^2 T^*X) \) for \( \alpha_1 > 1 \). If \( P(\sigma) = \Delta_g - \sigma^2 - \nabla x + V \) is the spectral family of an attractive Coulomb-like Schrödinger operator with \( V \in x^2 C^\infty(X) + x^{\alpha_2} S^0(X) \) for \( \alpha_2 > 3/2 \), then we can decompose

\[
P(\sigma) = P_0(\sigma) + P_1 + P_2
\]

for \( P_0, P_1, P_2 \) having the form specified in the introduction (with \( a = 0 \)), except that

\[
a_{00} = -x^4(g_1)_{00}|_{\partial X} = -\lim_{x \to 0^+} x^4 g_1(\partial x, \partial_x)
\]

is not necessarily constant (and the attractivity condition eq. (42) might only be satisfied for small \( \sigma \)). We can arrange that \( P_1 \) is fully classical and that \( P_2 \) is classical to order \( \beta_2, \beta_3 \) with \( \beta_2 = \alpha_1 - 1 \) and \( \beta_3 = \alpha_2 - 3/2 \).

Proof. Let \( V_0 \in C^\infty(X), V_1 \in S^0(X) \) satisfy \( V = x^2 V_0 + x^{\alpha_1} V_1 \). It suffices to restrict attention to the boundary-collar. There, we define \( P_0 \) by

\[
P_0(\sigma) = \Delta_{g_0} - x a_{00}(x^2 \partial_x)^2 - \sigma^2 - \nabla x.
\]

This has the form specified in the introduction, in the sense that eq. (41) holds with \( a = 0 \).

By the proof of Proposition 6.1, there exists some \( P_{2,0} \in \text{Diff}^{2,-\alpha_1}_\text{scb} (X) + x^{\alpha_1} S \text{Diff}^{2, -\alpha_2}_\text{scb} (X) \) such that, in any local coordinate patch,

\[
\Delta_g = \Delta_{g_0} + \sum_{i,j,k} x g^{0k}_0 (g_1)_{i\ell} g_0^{\ell j} \partial_i \partial_j + P_{2,0}.
\]

Now set \( P_1 = \sum_{i,j,k} x g^{0k}_0 (g_1)_{i\ell} g_0^{\ell j} \partial_i \partial_j - x g^{0k}_0 (g_1)_{i\ell} g_0^{\ell j} \partial_i \partial_j - x^9 (g_1)_{00} \partial_x^2 \partial_x \).

Then, \( P_1 \) has the form specified in eq. (43) and is even fully classical.

Defining \( a_{00} \) by eq. (520), \((g_1)_{00} + x^{-4} a_{00} \in x^{-3} C^\infty(X) \), so the operator \( P_{2,1} = x^9 (g_1)_{00} \partial_x^2 + x a_{00}(x^2 \partial_x)^2 \) is in \( \text{Diff}^{2, -\alpha_1}_\text{scb} (X) \). We therefore set

\[
P_2 = P_{2,0} + P_{2,1} + x^2 V_0 + x^{\alpha_1} V_1 \in \text{Diff}^{2, -\alpha_1}_\text{scb} (X) + S \text{Diff}^{2, -\alpha_1, -2 - \alpha_1}_\text{scb} (X) + x^{\alpha_2} S^0(X).
\]

Then eq. (48) applies, for \( \beta_2 = \alpha_1 - 1 \) and \( \beta_3 = \alpha_2 - 3/2 \).

Proposition 6.3. Given an asymptotically conic manifold \((X, g)\) of dimension \( \dim X = n \geq 2 \) such that \( g \) satisfies eq. (38), with \( a_{00} \in \mathbb{R} \), and given \( Z > 0 \) and \( V \in x^2 C^\infty(X) + x^{\alpha_2} S^0(X) \) for some \( \alpha_2 > 3/2 \), consider the Schrödinger operator

\[
P = \Delta_g - Z x + V : S'(X) \to S'(X).
\]

Set

\[
\Phi(x; \sigma) = \frac{1}{x} \sqrt{\sigma^2 + Z x - \sigma^2 a_{00} x} + \frac{1}{\sigma} (Z - \sigma^2 a_{00}) \arcsinh \left( \frac{\sigma}{x^{1/2} (Z - \sigma^2 a_{00})^{1/2}} \right)
\]

for all \( \sigma > 0 \) such that \( Z > \sigma^2 a_{00} \). Suppose that \( g \) is classical to \( \alpha_1 \)th order, \( \alpha_1 > 1 \). Set \( \delta_1 = \min \{ \alpha_1 - 1, \alpha_2 - 1 \} \) and \( \delta_0 = \min \{ \alpha_1 - 1, \alpha_2 - 3/2 \} \).

Then, for any \( f \in S(X) \):

(1) there exist some \( u_{0,\pm} \in C^\infty(X_{\text{res}} \cap \{ Z > E a_{00} \}) + A^{(0,0), \delta_0, (0,0)}_{\text{loc}}(X_{\text{res}} \cap \{ Z > E a_{00} \}) \)

such that, for \( E > 0 \) satisfying \( Z > E a_{00} \), \( u_{\pm}(-; E^{1/2}) = R(E \pm i0) f \in S'(X) \) can be written as

\[
u_{\pm} = e^{\pm i \Phi(x; E^{1/2})} x^{(n-1)/2} (E + Z x)^{-1/4} u_{0,\pm}.
\]
(II) \( u_{\pm}(-;0) = R(E = 0; Z \pm i0)f \) as \( u_{\pm}(-;0) = e^{\pm i\Phi(x;0)2(n-1)/2(Zx)^{-1/4}u_{0,\pm}(-;0)} \), where 
\( u_{0,\pm}(-;0) \in C^\infty(X_{1/2}) + A_{loc}^{2k}(X) \) is the restriction of \( u_{0,\pm} \) to \( zf = cl\{\sigma = 0, x > 0\} \subset X_{res}^{sp} \).

Moreover, the map 
\[ S(X) \ni f \mapsto u_{0,\pm} \in C^\infty(X_{res}^{sp} \cap \{Z > E_{a00}\}) + A_{loc}^{((0,0),\delta_1),2\delta_0,(0,0)}(X_{res}^{sp} \cap \{Z > E_{a00}\}) \] (528)
is continuous.

Proof. It suffices to prove only the ‘+’ case of the proposition, since the ‘−’ case is similar (and follows via complex conjugation). Moreover, it suffices to construct \( u_{0,\pm} = u_{0,\pm,E_0} \) over every interval of the form \([0,E_0]\) for \( E_0 > 0 \) satisfying \( Z > E_{a00} \), since then for \( E \leq E_0 \) the function \( u_{0,\pm,E_0}(-;E^{1/2}) \) does not depend on \( E_0 \) in the sense that for any \( E_0, E_0' \) satisfying \( Z > E_{a00}, E_{a00}' \) it is the case that \( u_{0,\pm,E_0} \) and \( u_{0,\pm,E_0'} \) agree on \([0,\min\{E_0, E_0'\}]\).

Given the setup of the proposition, Proposition 6.2 applies, so we can define a family \( P = \{P(\sigma)\}_{\sigma \geq 0} \) satisfying the assumptions listed in §1 such that \( P(\sigma) = P(0) - \sigma^2 \leq E_0 \), with \( P_1 \) fully classical and \( P_2 \) classical to order \((\beta_2, \beta_3)\) for \( \beta_2 = \alpha_1 - 1 \) and \( \beta_3 = -2/3 \).

Defining \( \tilde{P} = \exp(-i\Phi)P \exp(+i\Phi) \), since \( Pu_+ = f, \tilde{P}u_{0,+,+} = \tilde{f} \) for \( u_{0,+,+} = \exp(-i\Phi)u_+ \in S'(X) \) and \( \tilde{f} = \exp(-i\Phi)f \) \( f \in A^{\infty,\infty,(0,0)}(X_{res}^{sp}) \). Referring §6.2 for the definition of \( \tilde{R}_+(\sigma) \), by [Vas21a, Theorem 1.1] it is the case that 
\[ u_{0,+,+}(-;\sigma) = \tilde{R}_+(\sigma)\tilde{f}(-;\sigma) \] (529)
for each \( \sigma > 0 \) (since, for each \( \sigma > 0, \Phi \) differs from Vasy’s phase by a remainder the exponential of which acts as a bounded multiplication operator on b-Sobolev spaces — cf. eq. (55), with \( a = 0 \)).

By Proposition 6.15, \( u_{0,+,+} = x^{(n-1)/2}(\sigma^2 + 2k)u_{0,+,+} \) for some \( u_{0,+,+} \in A_{loc}^{0,-0,(0,0)}(X_{res}^{sp}) \), depending continuously on \( f \). Then, by Proposition 6.16, we conclude that 
\[ u_{0,+,+} \in A_{loc}^{((0,0),\delta_1),2\delta_0,(0,0)}(X_{res}^{sp}), \] (530)
depending continuously on \( f \in S(X) \), with \( \delta_1 = \min\{\beta_2/2 + \beta_3\} = \{\alpha_1 - 1, \alpha_2 - 1\} \) and \( \delta_0 = \min\{\beta_2, \beta_3\} = \min\{\alpha_1 - 1, \alpha_2 - 3/2\} \). This yields the first half of the proposition (as well as the continuity clause).

The second clause of this proposition then follows from the second clause of Proposition 6.15 and Proposition 4.2. Indeed, by the latter,
\[ e^{-i\Phi(-;0)}u_+ = e^{-i\Phi(-;0)}R(0;Z + i0)f = \tilde{R}_+(0)\tilde{f}(-;0). \] (531)
By the second half of Proposition 6.15, the right-hand side is the restriction to zf of \( u_{+,00} \).
Proof. We write the proof for \([0, \bar{x})\), and the proof for \(\hat{X}\) is verbatim.

Via the Mellin transform, \(\hat{N}_l,N(k,0) = 2i(x\partial_x + k + 3/4) : H^{-N,0}_b[0,\bar{x}) \rightarrow H^{-N-l,0}_b[0,\bar{x})\) is invertible for \(l, k\) as in the lemma statement, so it suffices to restrict attention to the case \(\sigma > 0\), i.e. to prove that

\[
\|u\|_{H^{-N,0}_b[0,\bar{x})} \leq \|x^{-l}N_l,N(k,\sigma)x^{l}u\|_{H^{-N-l,0}_b[0,\bar{x})} \tag{534}
\]

for all \(u \in H^{-N,0}_b[0,\bar{x})\) and \(\sigma > 0\) (the estimate required to be uniform in \(\sigma\)). Let \(\hat{x} = x/\sigma^2\bar{x} \).

Via the dilation invariance of the b-Sobolev spaces, the estimate eq. (534) is equivalent to the following:

\[
\|u\|_{H^{-N,0}_b[0,\bar{x})} \leq \|\hat{N}_{l,k}u\|_{H^{-N-l,0}_b[0,\bar{x})} \tag{535}
\]

for all \(u \in H^{-N,0}_b[0,\bar{x})\) and \(\sigma > 0\), where

\[
\hat{N}_{l,k} = 2i\left[\hat{x}\partial_{\hat{x}} + l + \frac{1}{2}\right] + \frac{2i\hat{x}}{1 + \hat{x}} \left(k + \frac{1}{4}\right).
\]

We now “radially” compactify the nonnegative real axis \([0, \infty)\) \((\hat{x})\) so that \(1/(1 + \hat{x})\) becomes a bdf for the new boundary face, and we call the result \([0, \infty]\). Observe that \(\hat{N}_{l,k}\) is a b-differential operator on \([0, \infty]\), with b-decay rate zero at both ends. In order to prove the estimate eq. (535), it suffices to prove that

\[
\|u\|_{H^{-N,0}_b[0,\infty]} \leq \|\hat{N}_{l,k}u\|_{H^{-N-l,0}_b[0,\infty]} \tag{537}
\]

for all \(u \in H^{-N,0}_b[0,\infty]\), where the third index is the b-decay order at \(\hat{x} = \infty\).

For \(l \neq -1/2\) and \(l + k \neq -3/4\), \(\hat{N}_{l,k}\) is (via b-ellipticity and the Taylor series expansion of \(\hat{x}/(1 + \hat{x})\) around \(\hat{x} = [0, \infty]\)) Fredholm as an operator \(H^{-N,0}_b[0, \infty] \rightarrow H^{-N-l,0}_b[0, \infty]\), and for any \(N_0 \in \mathbb{N}\) we have the estimate

\[
\|u\|_{H^{-N,0}_b[0,\infty]} \leq \|\hat{N}_{l,k}u\|_{H^{-N-l,0}_b[0,\infty]} + \|u\|_{H^{-N_0,-1,-1}_b[0,\infty]} \tag{538}
\]

for \(u \in H^{-N,0}_b[0,\infty]\). Once \(\hat{N}_{l,k}\) is known to be injective, a standard argument allows us to remove the last term of eq. (538), yielding the desired estimate eq. (537). It suffices to consider the case \(N_0 > N\). The standard argument is as follows:

- If we could not remove the last term of eq. (538), then we would be able to find a sequence \(\{u_j\}_{j \in \mathbb{N}} \subset H^{-N,0}_b[0,\infty]\) with \(\|u_j\|_{H^{-N,0}_b[0,\infty]} = 1\) for all \(j\) and

\[
\|\hat{N}_{l,k}u_j\|_{H^{-N-l,0}_b[0,\infty]} \to 0 \tag{539}
\]

as \(j \to \infty\). By the Banach-Alaoglu theorem, we may assume without loss of generality (by passing from \(\{u_j\}_{j \in \mathbb{N}}\) to a subsequence if necessary) that there exists some \(u_\infty \in H^{-N-1,0,0}_b[0,\infty]\) such that \(u_j \to u_\infty\) weakly as \(j \to \infty\).

Via the compactness of the inclusion \(H^{-N,0}_b[0,\infty] \hookrightarrow H^{-N_0,-1,-1}_b[0,\infty]\) for \(N_0 > N\), \(u_j \to u_\infty\) strongly in the latter space. From eq. (538), we deduce that

\[
\|u_\infty\|_{H^{-N_0,-1,-1}_b[0,\infty]} = \lim_{j \to \infty} \|u_j\|_{H^{-N_0,-1,-1}_b[0,\infty]} \geq 1. \tag{540}
\]

In particular, \((I)\) \(u_\infty\) is nonzero.

Also, from the strong convergence of \(u_j \to u_\infty\) in \(H^{-N_0,-1,-1}_b[0,\infty]\),

\[
\hat{N}_{l,k}u_j \to \hat{N}_{l,k}u_\infty \tag{541}
\]
distributionally. But \( \widehat{N}_{l,k}u_j \to 0 \) strongly in \( H_b^{-N-1,0,0}[0,\infty] \), by eq. (539). Thus, (II) 
\( \widehat{N}_{l,k}u_\infty = 0 \).

From (I) and (II), we conclude that \( \widehat{N}_{l,k} \) is not injective.

In order to show that

\[
\ker_{H_b^{-N,0,0}[0,\infty]} \widehat{N}_{l,k} = \{ u \in H_b^{-N,0,0}[0,\infty] : \widehat{N}_{l,k}u = 0 \}
\]

is trivial, we simply appeal to the solution eq. (512) of the ODE (although it is slightly simpler to integrate in the other direction). Indeed, any element \( u \) of the kernel eq. (542) must be given by 
\( u(\hat{x}) = c\hat{x}^{-l+k+3/4} \) for some \( c \in \mathbb{C} \). If this is nonzero, then it is \( \Omega(\hat{x}^{-l+k+3/4}) \) as \( \hat{x} \to \infty \). If \( l + k + 3/4 \leq 0 \), then \( u \) fails to lie in \( L_b^2[0,\infty] \). By eq. (538) (applied with 0 in place of \( N \) and \( N \) in place of \( N_0 \)),

\[
\|u\|_{L_b^2[0,\infty]} \leq \|\widehat{N}_{l,k}u\|_{H_b^{-N,0,0}[0,\infty]} + \|u\|_{H_b^{-N,-1,1}[0,\infty]} \leq \|\widehat{N}_{l,k}u\|_{H_b^{-N,0,0}[0,\infty]} + \|u\|_{H_b^{-N,0,0}[0,\infty]}.
\]

Since \( \|u\|_{L_b^2[0,\infty]} = \infty \) and \( u \in \ker_{H_b^{-N,0,0}[0,\infty]} \widehat{N}_{l,k} \Rightarrow \|\widehat{N}_{l,k}u\|_{H_b^{-N,0,0}[0,\infty]} = 0 \), eq. (543) implies

\[
u \notin H_b^{-N,0,0}[0,\infty],
\]
which contradicts \( u \in \ker_{H_b^{-N,0,0}[0,\infty]} \widehat{N}_{l,k} \). Thus, \( u \in \ker_{H_b^{-N,0,0}[0,\infty]} \widehat{N}_{l,k} \Rightarrow u = 0 \).

This completes the proof of the lemma.

\[
\square
\]

**Proposition 6.5.** For any \( l < -1/2 \), \( k + l \leq -3/4 \), and \( N \in \mathbb{N} \), there exists a constant \( C = C(\bar{P},N,l,\ell,k) > 0 \) such that, for all \( \sigma \geq 0 \),

\[
\|v\|_{H_b^{-N,l}(X)} \leq C \cdot \|((\sigma^2 + \bar{Z}x)^{-k} N(\bar{P}(\sigma))((\sigma^2 + \bar{Z}x)^k v)\|_{H_b^{-N,1,2l+3}(X)}
\]

\[
\|v\|_{H_b^{-N,l,2k+2}(X)} \leq C \cdot \|N(\bar{P}(\sigma))v\|_{H_b^{-N,1,2k+3}(X)}
\]

for all \( v \in S'(X) \) supported in \( \{ x < \bar{x} \} \subset X \).

**Proof.** Letting \( \tilde{N}(\bar{P}(\sigma)) = (\sigma^2 + \bar{Z}x)^{-k} N(\bar{P}(\sigma))((\sigma^2 + \bar{Z}x)^k) \), we can choose the \( b,\ell, C \)-Sobolev and \( b \)-Sobolev norms such that

\[
\|\tilde{N}(\bar{P}(\sigma))v\|_{H_b^{-N,l,1,2l+3}} = \|((2i\sigma x\partial_x - (n - 1)/2 + 2\sigma^2 x + Zx))v\|_{H_b^{-N,1,2l+3}}.
\]

We can work on \( \hat{X} = [0,1]_x \times \mathbb{R}X \), as

\[
\|x^{-n/2} u\|_{\tilde{H}_b^{-N,l}(\hat{X})} \leq \|u\|_{H_b^{-N,l}(X)} \leq \|x^{-n/2} u\|_{\tilde{H}_b^{-N,l}(\hat{X})}
\]

for all \( \frac{1}{2} \leq \sigma \leq 1 \), \( \frac{1}{2} \leq k \leq 2 \), and \( w \in S'(X) \) supported in \( \{ x < \bar{x} \} \). Equation (545) is therefore equivalent to

\[
\|x^{-n/2} v\|_{\tilde{H}_b^{-N,l}(\hat{X})} \leq \|2i(x\partial_x - (n - 1)/2 + 2\bar{Z}ix^2/(\sigma^2 + \bar{Z}x))v\|_{\tilde{H}_b^{-N,1,2l+3}}
\]

for \( v \in H_b^{-N,l}(X) \) supported in \( \{ x < \bar{x} \} \), which follows if

\[
\|w\|_{\tilde{H}_b^{-N,l}(\hat{X})} \leq \|(2i\sigma x\partial_x - 1/2 + 2\bar{Z}ix/(\sigma^2 + \bar{Z}x))w\|_{\tilde{H}_b^{-N,l}(\hat{X})}
\]

holds for \( w \in \tilde{H}_b^{-N,l}(\hat{X}) \), which was the conclusion of Lemma 6.4. \( \square \)
Proposition 6.6. For each $\Sigma > 0$, $N \in \mathbb{N}$, $m, s, \varsigma, l, \ell \in \mathbb{R}$ satisfying $l < -1/2$, $\ell < -3/2$, $s > s_0 > -1/2$, $-3/2 < \varsigma < \ell + s - l$, there exists a constant $C = C(P, \Sigma, N, m, s, \varsigma, l, \ell) > 0$ such that, for any $u \in S'(X)$
\[
\|u\|_{H_{\text{loc}}^{m, s, \varsigma, l, \ell}} \leq C \cdot (\|\hat{P}u\|_{H_{\text{loc}}^{m-2, s+1, \varsigma+3, l+1, \ell+3}} + \|u\|_{H_{\text{loc}}^{-N, l-\delta, \ell-2\delta}}) \tag{551}
\]
holds for any $\sigma \in [0, \Sigma]$ such that $u(-\sigma) \in H_{\text{loc}}^{-N, s_0, \varsigma_0}(X)$.

Proof. Consider $u \in S'(X)$ and $\sigma \in [0, \Sigma]$ as in the proposition statement. By Proposition 5.13, we have
\[
\|u\|_{H_{\text{loc}}^{m, s, \varsigma, l, \ell}} \leq \|\hat{P}u\|_{H_{\text{loc}}^{m-2, s+1, \varsigma+3, l+1, \ell+3}} + \|u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}}, \tag{552}
\]
where $N_0 \in \mathbb{N}$ is arbitrary. We now apply Proposition 6.5 to estimate the remainder term. Let $\chi \in C_c^\infty(X)$ be supported in $x \leq \bar{x}$ and identically equal to one in some neighborhood of $x = 0$. First of all,
\[
\|u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}} \leq \|\chi u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}} + \|u\|_{H_{\text{loc}}^{-N_0, -N_0, -N_0}}. \tag{553}
\]
Set $k = (\ell - 2l)/2$. Then, $k + l \leq -3/4$. We now apply the previous proposition with $v = \chi u$, the result being
\[
\|\chi u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}} \leq \|N(\hat{P})\chi u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}} \leq \|\hat{P}\chi u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}} + \|E\chi u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}} \tag{554}
\]
for $E = N(\hat{P}) - \hat{P}$. By Proposition 3.8, $E \in \Psi^{2, -1, -\delta, -3-2\delta}_b(X)$ for some $\delta \in (0, 1/2)$, so
\[
\|E\chi u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}} \leq \|u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}}. \tag{555}
\]
On the other hand, since $\chi$ is identically one in some neighborhood of $\partial X$,
\[
\|\hat{P}\chi u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}} \leq \|\hat{P}u\|_{H_{\text{loc}}^{-N_0, l-\delta, \ell-2\delta}} + \|u\|_{H_{\text{loc}}^{-N_0, -N_0, -N_0}} \leq \|\hat{P}u\|_{H_{\text{loc}}^{-m, s+1, \varsigma+3, l+1, \ell+3}} + \|u\|_{H_{\text{loc}}^{-N_0, -N_0, -N_0}} \tag{556}
\]
for sufficiently large $N_0$. Combining the estimates above (for sufficiently large $N_0$), we get Equation (552).

\[\square\]

For each $m, s, \varsigma, l, \ell \in \mathbb{R}$, we consider the families $\mathcal{X} = \mathcal{X}_{m, s, \varsigma, l, \ell} = \{\mathcal{X}_{m, s, \varsigma, l, \ell}(\sigma)\}_{\sigma \geq 0}$ and $\mathcal{Y} = \mathcal{Y}_{m, s, \varsigma, l, \ell} = \{\mathcal{Y}_{m, s, \varsigma, l, \ell}(\sigma)\}_{\sigma \geq 0}$ given by
\[
\mathcal{X}_{m, s, \varsigma, l, \ell}(\sigma) = \{u \in H_{\text{loc}}^{m, s, \varsigma, l, \ell}(X) : \hat{P}u \in H_{\text{loc}}^{m-2, s+1, \varsigma+3, l+1, \ell+3}(X)\} \tag{557}
\]
\[
\mathcal{Y}_{m, s, \varsigma, l, \ell}(\sigma) = H_{\text{loc}}^{m-2, s+1, \varsigma+3, l+1, \ell+3}(X), \tag{558}
\]
considered as families of Banach spaces in the usual way,
\[
\|u\|_{\mathcal{X}} = \|u\|_{H_{\text{loc}}^{m, s, \varsigma, l, \ell}} + \|\hat{P}u\|_{H_{\text{loc}}^{m-2, s+1, \varsigma+3, l+1, \ell+3}}. \tag{559}
\]
Note that $\mathcal{X}_{m, s, \varsigma, l, \ell}(0) = \mathcal{X}_{m, s, \varsigma, l}$ and $\mathcal{Y}_{m, s, \varsigma, l, \ell}(0) = \mathcal{Y}_{m, \varsigma, l}$, where the right-hand sides are defined eq. (291) and eq. (292). Tautologically, $\hat{P} : \mathcal{X} \to \mathcal{Y}$ is bounded, uniformly in $\sigma \geq 0$, as $\|\hat{P}u\|_{\mathcal{Y}} \leq \|u\|_{\mathcal{X}}$.

Proposition 6.7. Given $m, s, \varsigma, l, \ell$ satisfying the inequalities $l < -1/2$, $\ell < -3/2$, $-1/2 < s$, $-3/2 < \varsigma < \ell + s - l$, one of the following two alternatives holds:

- there exists some $\sigma \geq 0$ and nonzero $u \in \mathcal{X}_{m, s, \varsigma, l, \ell}(\sigma)$ such that $\hat{P}(\sigma)u = 0$,
there exists, for each \( \Sigma > 0 \), a constant \( C_0 = C_0(\tilde{P}, m, s, \varsigma, \ell, \Sigma) > 0 \) such that the estimate

\[
\|u\|_{H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma)} \leq C_0 \|\tilde{P} u\|_{H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma)}
\]

holds for all \( \sigma \in [0, \Sigma] \) and all \( u \in X_{m_s, \varsigma, l, \ell}(\sigma) \).

\[\blacksquare\]

\textbf{Proof.} The following is a variant of the proof of [Hör07, Theorem 26.1.7], also used in the proof of the main theorem in [Vas21a].

Suppose that the second of the two alternatives does not hold, so that there exist \( \Sigma > 0 \) and sequences \( \{\sigma_k\}_{k=0}^{\infty} \subset [0, \Sigma] \) and \( \{u_k\}_{k=0}^{\infty} \subset S'(X) \) with \( \|\tilde{P} u_k\|_{H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma_k)} = 1 \) and

\[
\|\tilde{P}(\sigma_k) u_k\|_{H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma_k)} < 1/k
\]

for all \( k \in \mathbb{N} \). By passing to a subsequence if necessary (and noting that eq. (561) continues to hold upon doing so), we can arrange that \( \sigma_k \to \sigma_\infty \) for some \( \sigma_\infty \in [0, \Sigma] \).

Even though eq. (560) might not hold, by Proposition 6.6 we at least have the bound

\[1 = \|u_k\|_{H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma_k)} \leq C \left( \|\tilde{P} u_k\|_{H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma_k)} + \|u_k\|_{H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma_k)} \right),\]

for any \( N \in \mathbb{N} \), where \( C = C(\tilde{P}, \Sigma, N, m, s, \varsigma, \ell \), is some constant. On the other hand, for sufficiently large \( N_0 > 0 \), we can bound

\[
\|u\|_{H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma_k)} \geq \|\Lambda_{m_s, \varsigma, l, \ell}(\sigma_k) u\|_{L^2(X)} + \|x^{-l}(\sigma^2 + Z x)^{-l/2} u\|_{L^2(X)}
\]

this holding for all \( u \in S'(X) \) and \( \sigma \in [0, \Sigma] \). Consequently, \( \{\Lambda_{m_s, \varsigma, l, \ell}(\sigma_k) u_k\}_{k=0}^{\infty} \) is bounded in \( L^2(X) \) and \( \{x^{-l}(\sigma_k^2 + Z x)^{-l/2} u_k\}_{k=0}^{\infty} \) is bounded in \( H_{b}^{1-N_0, 0}(X) \). By Banach-Alaoglu – passing to a subsequence if necessary (and once again noting that eq. (561) continues to hold upon doing so) – we can assume that there exist some \( v \in L^2(X) \) and \( w \in H_{b}^{1-N_0, 0}(X) \) such that

\[
\Lambda_{m_s, \varsigma, l, \ell}(\sigma_k) u_k \to v
\]

as \( k \to \infty \) weakly in \( L^2(X) \) and \( x^{-l}(\sigma_k^2 + Z x)^{-l/2} u_k \to w \) as \( k \to \infty \) weakly in \( H_{b}^{1-N_0, 0}(X) \).

It follows from the latter that \( u_k \to x^l(\sigma_k^2 + Z x)^{l/2-l} w \) strongly in some \( b \)-Sobolev space. This has two consequences:

- First,

\[
\tilde{P}(\sigma_k) u_k \to \tilde{P}(\sigma_\infty)(x^l(\sigma_\infty^2 + Z x)^{l/2-l} w)
\]

in \( S'(X) \) as \( k \to \infty \) (e.g. using Proposition 2.4 and Proposition 3.6). But, the assumption

\[
\|\tilde{P}(\sigma_k) u_k\|_{H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma_k)} < 1/k
\]

implies that \( \tilde{P}(\sigma_k) u_k \to 0 \) in \( S'(X) \). Therefore

\[
\tilde{P}(\sigma_\infty)(x^l(\sigma_\infty^2 + Z x)^{l/2-l} w) = 0.
\]

- Second (using Proposition 2.4), \( \Lambda_{m_s, \varsigma, l, \ell}(\sigma_k) u_k \to \Lambda_{m_s, \varsigma, l, \ell}(\sigma_\infty)(x^l(\sigma_\infty^2 + Z x)^{l/2-l} w) \) in \( S'(X) \). Since \( S'(X) \) is Hausdorff, this implies that

\[
\Lambda_{m_s, \varsigma, l, \ell}(\sigma_\infty)(x^l(\sigma_\infty^2 + Z x)^{l/2-l} w) = v \in L^2(X),
\]

which in turn implies that \( x^l(\sigma_\infty^2 + Z x)^{l/2-l} w \in H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma_\infty) \) by elliptic regularity (and the fact that \( w \in H_{b}^{1-N_0, 0} \)).

Let \( u = x^l(\sigma_\infty^2 + Z x)^{l/2-l} w \). What we proved above is \( \tilde{P}(\sigma_\infty) u = 0 \) and \( u \in H_{\text{loc}}^{m_s, \varsigma, l, \ell}(\sigma_\infty) \). Thus,

\[
u \in \Lambda_{m_s, \varsigma, l, \ell}(\sigma_\infty).
\]
Since \( N > N_0 \), via the compactness of the inclusion \( H^{-N_0,0}_b(X) \hookrightarrow H^{-N,-\delta}_b(X) \) it is the case that
\[
\| w \|_{H^{-N,-\delta}_b(X)} = \lim_{k \to \infty} \| x^{-\ell}(\sigma_k^2 + Z x)^{1/2} u_k \|_{H^{-N,-\delta}_b(X)}.
\]
(568)
On the other hand, we can bound
\[
\| u_k \|_{H^{-N,-\delta}_b(X)} \leq \| x^{-\ell}(\sigma_k^2 + Z x)^{1/2} u_k \|_{H^{-N,-\delta}_b(X)}. \]
So, eq. (562) yields
\[
1 \leq \| w \|_{H^{-N,-\delta}_b(X)} + \lim_{k \to \infty} \| \bar{P} u_k \|_{H^{-m_2-s+1,\ell+3}_b(X)}.
\]
(569)
Therefore \( w \neq 0 \). It follows that \( u \neq 0 \). We have therefore succeeded in showing that the second of the two alternatives listed in the proposition holds.

**Proposition 6.8.** Suppose that \( P \) is the spectral family of an attractive Coulomb-like Schrödinger operator for \( \sigma \leq \Sigma \). Then, given \( m, s, \zeta, l, \ell \in \mathbb{R} \) satisfying \( l < -1/2, \ell < -3/2, -1/2 < s, -3/2 < \zeta \), it is the case that, for each \( \sigma \geq \Sigma \),
\[
\bar{P}(\sigma) : \mathcal{X}_{m,s,\zeta,l,\ell}(\sigma) \to \mathcal{Y}_{m,s,\zeta,l,\ell}(\sigma)
\]
is invertible.

**Proof.** We already observed the \( \sigma = 0 \) case in §3. The \( \sigma > 0 \) case is essentially proven in [Vas21a, §4]. In order to see this, note that, for each \( \sigma > 0 \),
\[
\mathcal{X}_{m,s,\zeta,l,\ell}(\sigma) = \{ u \in H^{m,s,l}_{scb}(X) : \bar{P}(\sigma) u \in H^{-m_2-s+1,\ell+1}_{scb}(X) \}
\]
(571)
\[
\mathcal{Y}_{m,s,\zeta,l,\ell}(\sigma) = H^{m_2-s+1,\ell+1}_{scb}(X)
\]
(572)
at the level of sets, where
\[
\bar{P}(\sigma) = e^{-i\Phi_0 + i\Phi} \bar{P} e^{i\Phi_0 - i\Phi}
\]
(573)
is Vasy’s conjugated operator, \( \Phi_0 \) simply being defined by \( \Phi_0 = \sigma x^{-1} \). Since the leC-Sobolev spaces are just \( scb \)-Sobolev spaces for \( \sigma > 0 \), the crux of the previous claim is that the \( \bar{P} \) on the right-hand side of eq. (557) can be replaced by Vasy’s eq. (573). Indeed, \( \Phi_0(x;\sigma) - \Phi(x;\sigma) \in \log x C^\infty((0,\bar{x})_x) + C^\infty((0,\bar{x})_x) \) for each \( \sigma > 0 \), so
\[
\bar{P}(\sigma) = \bar{P}(\sigma) + T(\sigma)
\]
(574)
for
\[
T(\sigma) = e^{-i\Phi_0(\cdot;\sigma) + i\Phi(\cdot;\sigma)}[\bar{P}(\sigma), e^{i\Phi_0(\cdot;\sigma) - i\Phi(\cdot;\sigma)}] \in \text{Diff}_{scb}^{1,1}(X). \text{ Thus,}
\]
\[
T(\sigma) : H^{m,s,l}_{scb}(X) \to H^{m_2-s+1,\ell+1}_{scb}(X).
\]
(575)
So, for \( u \in H^{m,s,l}_{scb}(X) \), \( \bar{P}(\sigma) u \in H^{-m_2-s+1,\ell+1}_{scb}(X) \) if and only if \( \tilde{P}(\sigma) u \in H^{m_2-s+1,\ell+1}_{scb}(X) \). The operators we consider have slightly more general \( \sigma \)-dependence than the ones in Vasy (as some additional assumptions are needed for the \( \sigma \notin \mathbb{R} \) case of [Vas21a, Theorem 1.1]), but since we are only considering real \( \sigma \) his proof of the real case of [Vas21a, Theorem 1.1] goes through in this slightly greater generality mutatis mutandis.

Alternatively, the \( \sigma > 0 \) case of Proposition 6.6 suffices as a replacement for [Vas21a, Prop. 4.16] in his proof of [Vas21a, Theorem 1.1], the rest of which is identical. (For the purpose of the proof above, we do not need to know that the estimate in Proposition 6.6 is uniform down to \( \sigma = 0 \), so the \( \zeta \leq \ell + s - \ell \) hypothesis there is not relevant here.)
6.2. Smoothness at \(\sigma\), Conormality elsewhere. For this subsection, we suppose that \(P(\sigma)\) is the spectral family of an attractive Coulomb-like Schrödinger operator for \(\sigma\) in some neighborhood of \([0, \Sigma]\), \(\Sigma > 0\). By analogy with the terminology in [Mc94], we might say that \(u \in S'(X)\) “satisfies the conjugated Sommerfeld radiation condition” for some given \(\sigma \geq 0\) if \(u \in X_{m, s, c, l, \ell}(X)(\sigma)\) for some \(m, s, c, l, \ell \in \mathbb{R}\) satisfying \(l < -1/2, \ell < -3/2, s > -1/2, \ell + s - l \geq \varsigma > -3/2\). One of the main tasks of this subsection is to show that the limiting resolvent output converges as \(\sigma \to 0^+\) to something satisfying the zero energy version of the Sommerfeld radiation condition.

For each \(\sigma \in [0, \Sigma]\), let \(\tilde{R}_+(\sigma) : \mathcal{X}_{m, s, c, l, \ell}(\sigma) \to \mathcal{X}_{m, s, c, l, \ell}(\sigma)\) denote the set-theoretic inverse to eq. (570) (which, of course, must actually be an isomorphism of Banach spaces e.g. by the closed graph theorem). (The ‘+’ subscript of \(\tilde{R}_+ = \{\tilde{R}_+(\sigma)\}_{\sigma \geq 0}\) refers to the choice of sign in defining the conjugation.) This extends the definition of the operator \(\tilde{R}_+(0)\) introduced at the end of §3 to the \(\sigma > 0\) case. For each \(\sigma \in (0, \Sigma]\),

\[
\tilde{R}_+(\sigma) : H^{m-2, s+1, l+1}_{\mathrm{scb}}(X) \to H^{m, s, l}_{\mathrm{scb}}(X)
\]  

(576)

is bounded (but not uniformly in \(\sigma\)). Considering the case \(s = m + l\), \(\tilde{R}_+(\sigma) : H^{m, l+1}_{\mathrm{b}}(X) \to H^{m, l}_{\mathrm{b}}(X)\) if \(l < -1/2 < m + l\). Note that the mapping properties of the resolvent with respect to the b-Sobolev spaces are slightly lossy, in the sense that we can no longer keep track of the fact that the map eq. (576) smooths by two orders. As the notation in eq. (576) indicates, the operator \(\tilde{R}_+(\sigma)\) makes sense as a map

\[
\bigcup_{m, s, l \in \mathbb{R}} H^{m-2, s+1, l+1}_{\mathrm{scb}}(X) \to \bigcup_{m, s, l \in \mathbb{R}} H^{m, s, l}_{\mathrm{scb}}(X),
\]  

(577)

hence we can just write “\(\tilde{R}_+(\sigma)\)” without specifying \(m, s, l\). A similar statement holds for \(\sigma = 0\).

Proposition 6.9. Given \(m, s, c, l, \ell \in \mathbb{R}\) satisfying \(l < -1/2, \ell < -3/2, s > -1/2, \ell + s - l \geq \varsigma > -3/2\), there exists some constant \(C = C(m, s, c, l, \ell, \Sigma) > 0\) such that

\[
\|\tilde{R}_+(\sigma)\|_{H^{m-2, s+1, l+1}_{\mathrm{scb}}(X)(\sigma)} \leq C\|\tilde{f}\|_{H^{m-2, s+1, c+3, l+1, l+3}_{\mathrm{scb}}(X)(\sigma)}
\]  

(578)

for all \(\sigma \in [0, \Sigma]\) and \(\tilde{f} \in H^{m-2, s+1, c+3, l+1, l+3}_{\mathrm{scb}}(X)(\sigma)\). Moreover, for \(m, l \in \mathbb{R}\) with \(l < -1/2\) and \(-1 < m + 2l\):

(I) for any \(\tilde{f} \in H^{m, l+5/4}_{\mathrm{b}}(X)\), we have \(\|\sigma^2 + Z\tilde{x}\|^{1/4}\tilde{R}_+(\sigma)\tilde{f}\|_{H^{m, l}_{\mathrm{b}}} \leq C_0\|\tilde{f}\|_{H^{m, l+5/4}_{\mathrm{b}}}\) for some constant \(C_0 = C_0(\tilde{P}, m, l, \Sigma) > 0\);

(II) for any \(\tilde{f} \in H^{m, l+1}_{\mathrm{b}}(X)\) we have \(\|\sigma^2 + Z\tilde{x}\|^{1/4}\tilde{R}_+(\sigma)\sigma^2 + Z\tilde{x}\|^{1/4}\tilde{f}\|_{H^{m, l}_{\mathrm{b}}} \leq C_1\|\tilde{f}\|_{H^{m, l+1}_{\mathrm{b}}}\) for some constant \(C_1 = C_1(\tilde{P}, m, l, \Sigma) > 0\)

for all \(\sigma \in [0, \Sigma]\).

\(\blacksquare\)

Proof. As a corollary of Proposition 6.7 and Proposition 6.8, we get that for \(m, s, c, l, \ell\) as above and \(P\) the spectral family of an attractive Coulomb-like Schrödinger operator,

\[
\|u\|_{H^{m, s, c, l, \ell}_{\mathrm{scb}}(X)(\sigma)} \leq C(m, s, c, l, \ell, \Sigma)\|P\|_{H^{m, s, c, l, \ell}_{\mathrm{scb}}(X)(\sigma)}
\]  

(579)

holds for all \(\sigma \in [0, \Sigma]\) and all \(u \in \mathcal{X}_{m, s, c, l, \ell}(\sigma)\). We also have

\[
\tilde{R}_+(\sigma) : H^{m-2, s+1, c+3, l+1, l+3}_{\mathrm{scb}}(X)(\sigma) \to \mathcal{X}_{m, s, c, l, \ell}(\sigma) \subseteq H^{m, s, c, l, \ell}_{\mathrm{scb}}(X)(\sigma).
\]  

(580)

Taking \(u = \tilde{R}_+(\sigma)\tilde{f}\) in eq. (579) yields Equation (578).
Suppose now that \( m, l \in \mathbb{R} \) satisfy \( l < -1/2 \) and \(-1 < m + 2l \) (in which case \(-1/2 < m + l \) holds as well). First suppose that \( \tilde{f} \in H^m_{b,l+5/4}(X) \). Applying Equation (578) (observing that the required inequalities \( l < -1/2 < m + l \) and \(-3/2 < m + 2l - 1/2 \) hold),

\[
\|(\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma) \|_{H^m_{b,l}} \leq \|(\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma) f\|_{H^{m+1,m+2l-1/2}^{0,l}} \leq \||\tilde{R}_+(\sigma) f\|_{H^{m+1,m+2l-1/2}^{0,l}} \]  

(581)

Now supposing that \( \tilde{f} \in H^{m,l+1}_{b,l}(X) \),

\[
\|(\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma)(\sigma^2 + Zx)^{1/4} \tilde{f}\|_{H^{m}_{b,l}} \leq \|(\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma)(\sigma^2 + Zx)^{1/4} \tilde{f}\|_{H^{m+1,m+2l-1/2}^{0,l}} \leq \||\tilde{R}_+(\sigma) f\|_{H^{m+1,m+2l-1/2}^{0,l}} \leq \|\tilde{f}\|_{H^{m,l+1}_{b,l}} \]  

(582)

**Proposition 6.10.** For any \( m, l \in \mathbb{R} \) with \( l < -1/2 \) and \(-1 < m + 2l \), for any \( f \in H^{m,l+5/4}_{b,l}(X) \),

\[
(\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma) f \rightarrow Z^{1/4}x^{1/4} \tilde{R}_+(0) f
\]  

(583)

weakly in \( H^{m,l}_{b,l}(X) \) as \( \sigma \rightarrow 0^+ \). In fact, the map \((\sigma, f) \mapsto (\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma) f\) defines a jointly continuous map

\[
[0, \Sigma) \times H^{m,l+5/4}_{b,l}(X) \rightarrow H^{m,-\epsilon,l-\epsilon}_{b,l}(X),
\]  

(584)

for any \( \epsilon > 0 \), where we are using the strong topologies on \( H^{m,l+5/4}_{b,l}(X) \) and \( H^{m,-\epsilon,l-\epsilon}_{b,l}(X) \).

This (applied for slightly smaller \( m, l \)) implies that

\[
\{(\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma)\}_{\sigma \geq 0} \subset \mathcal{L}(H^{m,l+5/4}_{b,l}(X), H^{m,-\epsilon,l-\epsilon}_{b,l}(X))
\]  

(585)

is continuous with respect to the uniform operator topology.

**Proof.** First consider the claim of joint continuity. By the metrizability of \([0, \Sigma) \times H^{m,l+5/4}_{b,l}(X) \) and \( H^{m,-\epsilon,l-\epsilon}_{b,l}(X) \), joint continuity follows from the claim that

\[
\bullet \text{ given } f \in H^{m,l+5/4}_{b,l}(X) \text{ and } \sigma_\infty \in [0, \Sigma), \text{ for any and sequences } \{\sigma_k\}_{k \in \mathbb{N}} \subset [0, \Sigma) \text{ and } \{f_k\}_{k \in \mathbb{N}} \subset H^{m,l+5/4}_{b,l}(X) \text{ with } \sigma_k \rightarrow \sigma_\infty \text{ and } f_k \rightarrow f \text{ as } k \rightarrow \infty,
\]

\[
(\sigma_k^2 + Zx)^{1/4} \tilde{R}_+(\sigma_k) f_k \rightarrow (\sigma_\infty^2 + Zx)^{1/4} \tilde{R}_+(\sigma_\infty) f
\]  

(586)

strongly in \( H^{m,-\epsilon,l-\epsilon}_{b,l}(X) \).

Since a sequence of elements of a metric space converges to some element if and only if every subsequence thereof contains a further subsequence converging to that same element, it suffices to show that
• given any \( \{\sigma_k\}_{k \in \mathbb{N}} \subset (0, \Sigma) \), \( \{f_k\}_{k \in \mathbb{N}} \subset H^{m, l+5/4}_b(X) \) with \( \sigma_k \to \sigma_\infty \) and \( f_k \to f \) as \( k \to \infty \), there exists a subsequence \( \{k_n\}_{n \in \mathbb{N}} \subset \{k\}_{k \in \mathbb{N}} \) such that
\[
(\sigma_{k_n}^2 + Zx)^{1/4} \hat{R}_+(\sigma_{k_n})f_{k_n} \to (\sigma_\infty^2 + Zx)^{1/4} \hat{R}_+(\sigma_\infty)f
\] (587)
strongly in \( H^{m-l, l-\varepsilon}_b(X) \).

We now handle the case of \( \sigma_\infty = 0 \). The case \( \sigma_\infty > 0 \) follows by a similar but even easier argument.

By Proposition 6.9 and Banach-Alaoglu, we can find a subsequence \( \{k_n\}_{n \in \mathbb{N}} \subset \{k\}_{k \in \mathbb{N}} \) such that
\[
(\sigma_{k_n}^2 + Zx)^{1/4} \hat{R}_+(\sigma_{k_n})f_{k_n}
\]
converges weakly in \( H^{m, l}_b(X) \). Let \( w \in H^{m, l}_b(X) \) denote the weak limit. We first want to show that \( w = Z^{1/4}x^{1/4} \hat{R}_+(0)f \).

• We first check that \( w \) solves the PDE \( \hat{P}(0)(Z^{-1/4}x^{-1/4}w) = f \). Indeed, for any \( m_0, l_0 \),
\[
[0, \infty)_\sigma \times H^{m_0, l_0}_b(X) \ni (\sigma, u) \mapsto \hat{P}(\sigma)u \in S'(X)
\] (588)
is jointly continuous with respect to the strong topology on \( H^{m_0, l_0}_b(X) \). (Besides being clear from the explicit formulas for \( \hat{P} \) in §3, this follows from Proposition 2.4 and Proposition 3.6.)

Since
\[
Z^{-1/4}x^{-1/4}(\sigma_{k_n}^2 + Zx)^{1/4} \hat{R}_+(\sigma_{k_n})f_{k_n} \to Z^{-1/4}x^{-1/4}w
\] (589)
weakly in \( H^{m-l-1/4}_b(X) \), this convergence occurs strongly in \( H^{m-l-1/4-\varepsilon}_b(X) \) for any \( \varepsilon > 0 \), so (by the aforementioned joint continuity)
\[
\hat{P}(\sigma_{k_n})(Z^{-1/4}x^{-1/4}(\sigma_{k_n}^2 + Zx)^{1/4} \hat{R}_+(\sigma_{k_n})f_{k_n}) \to \hat{P}(0)(Z^{-1/4}x^{-1/4}w)
\] (590)
in \( S'(X) \).

Moreover, it is not difficult to see that \( \hat{P}(\sigma_{k_n})(Z^{-1/4}x^{-1/4}(\sigma_{k_n}^2 + Zx)^{1/4} \hat{R}_+(\sigma_{k_n})f_{k_n}) \to f \)
in \( S'(X) \): indeed
\[
\hat{P}(\sigma_{k_n})(Z^{-1/4}x^{-1/4}(\sigma_{k_n}^2 + Zx)^{1/4} \hat{R}_+(\sigma_{k_n})f_{k_n})
\]
\[= [\hat{P}(\sigma_{k_n}), Z^{-1/4}x^{-1/4}(\sigma_{k_n}^2 + Zx)^{1/4}] \hat{R}_+(\sigma_{k_n})f_{k_n} + f_{k_n}, \] (591)
and \( [\hat{P}(\sigma), Z^{-1/4}x^{-1/4}(\sigma^2 + Zx)^{1/4}] \in S\text{Diff}^{1-5/4, -3-5/4, -3}_b(X) \) satisfies
\[
[\hat{P}(\sigma), Z^{-1/4}x^{-1/4}(\sigma^2 + Zx)^{1/4}]_{|\sigma=0} = [\hat{P}(0), 1] = 0,
\] (592)
so the boundedness of \( \hat{R}_+(\sigma_{k_n})f_{k_n} \) in \( H^{m}_b(X) \) in some \( \text{b-Sobolev} \) space (as given by Proposition 6.9) and Proposition 2.4 show that \( [\hat{P}(\sigma_{k_n}), Z^{-1/4}x^{-1/4}(\sigma_{k_n}^2 + Zx)^{1/4}] \hat{R}_+(\sigma_{k_n})f_{k_n} \to 0 \)
in \( S'(X) \).

Since \( S'(X) \) is Hausdorff, it follows that \( \hat{P}(0)(Z^{-1/4}x^{-1/4}w) = f \).

• Thus, we have
\[
\hat{P}(0)(Z^{-1/4}x^{-1/4}w) = f.
\] (593)
But, also,
\[
\hat{P}(0)(\hat{R}_+(0)f) = f.
\] (594)
Set \( l_0 = l - 1/4 \), so that \( f \in H^{m, l+5/4}_b(X) = H^{l+3/2}_b(X) \). In terms of \( \ell_0 = 2l_0 \), the inequalities \( l < -1/2 \) and \(-1 < m + 2l \) become \( \ell_0 < -3/2 \) and \(-3/2 < m + \ell \), so Proposition 4.3 applies —
\[
\hat{P}(0) : \{u \in H^{m-l-1/4}_b(X) : \hat{P}(0)u \in H^{m+5/4}_b(X)\} \to H^{m, l+5/4}_b(X)
\] (595)
is invertible, and the inverse is $\tilde{R}_+(0)$. Thus, $u = \tilde{R}_+(0)f$ is the unique solution to $\tilde{P}(0)u = f$ in the domain of eq. (595). But $Z^{-1/4}x^{-1/4}w$ is in the codomain, and as we saw in eq. (594) solves this PDE. We conclude that

$$Z^{-1/4}x^{-1/4}w = \tilde{R}_+(0)f.$$  
(596)

Via the compactness of the inclusion $H^{m,l}_b \hookrightarrow H^{m-\varepsilon,l-\varepsilon}_b$, we conclude that

$$(\sigma_{k_n}^2 + Zx)^{1/4}\tilde{R}_+(\sigma_{k_n})f_{k_n} \rightarrow Z^{1/4}x^{1/4}\tilde{R}_+(0)f$$  
(597)

strongly in $H^{m-\varepsilon,l-\varepsilon}_b$. This completes the proof of joint continuity.

By Proposition 6.9, for each $\Sigma > 0$ the set $\{(\sigma^2 + Zx)^{1/4}\tilde{R}_+(\sigma)f\}_{\sigma \in [0,\Sigma]}$ is bounded in $H^{m,l}_b(X)$, and the result above shows that eq. (583) holds in the topology generated Schwartz test functions (and, in fact, even the strong topologies $H^{m-\varepsilon,l-\varepsilon}_b$). It follows from the conjunction of these observations that eq. (583) holds with respect to the weak topology of $H^{m,l}_b(X)$ (since $S(X)$ is dense in $H^{m,l}_b(X)$).

We now deduce the uniform continuity statement from the joint continuity statement. Suppose, to the contrary, that there are some $m, l \in \mathbb{R}$ with $l < -1/2$ and $-1 < m + 2l$ and some $\varepsilon > 0$ such that $\{(\sigma^2 + Zx)^{1/4}\tilde{R}_+(\sigma)\}_{\sigma \geq 0}$ is not continuous with respect to the uniform operator topology, generated by the norm

$$\|\cdot\|_{L(H^{m,l+5/4}_b(X),H^{m-\varepsilon,l-\varepsilon}_b(X))}.$$  
(598)

We handle the case of a discontinuity at $\sigma = 0$, and the case of $\sigma > 0$ follows via a similar, easier argument. The discontinuity statement means that there exists some $\varepsilon > 0$ such that there exist sequences $\{\sigma_k\}_{k \in \mathbb{N}}$ with $\sigma_k \rightarrow 0^+$ and $\{f_k\}_{k \in \mathbb{N}} \subset H^{m,l+5/4}_b$ with $\|f_k\|_{H^{m,l+5/4}_b} \leq 1$ such that

$$\|\sigma_k^2 + Zx\|^{1/4}\tilde{R}_+(\sigma_k)f_k - (Zx)^{1/4}\tilde{R}_+(0)f_k\|_{H^{m-\varepsilon,l-\varepsilon}_b} \geq \varepsilon$$  
(599)

for all $k$. By Banach-Alaoglu, we can choose these sequences such that there exists some $f_\infty \in H^{m,l+5/4}_b$ such that $f_k \rightharpoonup f_\infty$ weakly, which implies strong convergence in $H^{m-\varepsilon',l+5/4-\varepsilon'}_b$ for any $\varepsilon' > 0$. But then, by the joint continuity statement already proven, as long as $\varepsilon'$ is sufficiently small such that $-1 < m + 2l - 3\varepsilon'$, we have

$$\|\sigma_k^2 + Zx\|^{1/4}\tilde{R}_+(\sigma_k)f_k - (Zx)^{1/4}\tilde{R}_+(0)f_k\|_{H^{m-\varepsilon',l-\varepsilon'-\varepsilon'}_b} \rightarrow 0$$  
(600)

as $k \rightarrow \infty$, for any $\varepsilon' > 0$. But we can take $\varepsilon', \varepsilon''$ sufficiently small such that $\varepsilon' + \varepsilon'' < \varepsilon$, in which case eq. (600) contradicts eq. (599).

Via Sobolev embedding, the previous proposition already yields the following corollary on the continuity of the resolvent output at zero energy:

- for any $f \in H^{m,l+5/4}_b(X)$, $l < -1/2$, and for any $\chi \in C^\infty_c(X^\circ)$, $\chi\tilde{R}_+(\sigma)f \rightarrow \chi\tilde{R}_+(0)f$ in $C^\infty_c(X^\circ)$ as $\sigma \rightarrow 0^+$.

We will need to strengthen this result to apply to $\partial_\sigma$ derivatives of the resolvent output. In order to handle the compositions that arise, we will use the following variant of the preceding proposition.

**Proposition 6.11.** For any $m, l \in \mathbb{R}$ with $l < -1/2$ and $-1 < m + 2l$, the map $(\sigma, f) \mapsto (\sigma^2 + Zx)^{1/4}\tilde{R}_+(\sigma)f(\sigma^2 + Zx)^{1/4}$ defines a jointly continuous map

$$[0, \Sigma) \times H^{m,l+1}_b(X) \rightarrow H^{m-\varepsilon,l-\varepsilon}_b(X),$$  
(601)
for any $\epsilon > 0$ (with respect to the strong topologies on $H^{m,l+1}_0(X)$ and $H^{m-\epsilon,l-\epsilon}_0(X)$). Consequently, 
\{(\sigma^2 + Zx)^{1/4} \tilde{R}_+ (\sigma)(\sigma^2 + Zx)^{1/4}\}_{\sigma \geq 0} \subset \mathcal{L}(H^{m,l+1}_0(X), H^{m-\epsilon,l-\epsilon}_0(X))$ is continuous with respect to the uniform operator topology. \hfill \blacksquare

**Proof.** We mimic the proof of Proposition 6.10. By the metrizability of $[0, \Sigma] \times H^{m,l+1}_0(X)$ and $H^{m-\epsilon,l-\epsilon}_0(X)$, joint continuity follows from the claim that

- given $f \in H^{m,l+1}_0(X)$ and $\sigma_\infty \in [0, \Sigma)$, for any and sequences $\{\sigma_k\}_{k \in \mathbb{N}}$, $\{f_k\}_{k \in \mathbb{N}} \subset H^{m,l+1}_0(X)$ with $\sigma_k \rightarrow \sigma_\infty$ and $f_k \rightarrow f$ as $k \rightarrow \infty$,
  \[(\sigma_k^2 + Zx)^{1/4} \tilde{R}_+ (\sigma_k)(\sigma_k^2 + Zx)^{1/4} f_k \rightarrow (\sigma_\infty^2 + Zx)^{1/4} \tilde{R}_+ (\sigma_\infty)(\sigma_\infty^2 + Zx)^{1/4} f \tag{602} \]

  strongly in $H^{m-\epsilon,l-\epsilon}_0(X)$. It suffices to show that

- given any $\{\sigma_k\}_{k \in \mathbb{N}} \subset [0, \Sigma)$, $\{f_k\}_{k \in \mathbb{N}} \subset H^{m,l+1}_0(X)$ with $\sigma_k \rightarrow \sigma_\infty$ and $f_k \rightarrow f$ as $k \rightarrow \infty$, there exists a subsequence $\{k_n\}_{n \in \mathbb{N}} \subset \{k\}_{k \in \mathbb{N}}$ such that
  \[(\sigma_{k_n}^2 + Zx)^{1/4} \tilde{R}_+ (\sigma_{k_n})(\sigma_{k_n}^2 + Zx)^{1/4} f_{k_n} \rightarrow (\sigma_\infty^2 + Zx)^{1/4} \tilde{R}_+ (\sigma_\infty)(\sigma_\infty^2 + Zx)^{1/4} f \tag{603} \]

  strongly in $H^{m-\epsilon,l-\epsilon}_0(X)$. As before, we only consider the case of $\sigma_\infty = 0$, since the case $\sigma_\infty > 0$ follows by a similar but even easier argument.

By Proposition 6.9 and Banach-Alaoglu, we can find a subsequence $\{k_n\}_{n \in \mathbb{N}} \subset \{k\}_{k \in \mathbb{N}}$ such that $(\sigma_{k_n}^2 + Zx)^{1/4} \tilde{R}_+ (\sigma_{k_n})(\sigma_{k_n}^2 + Zx)^{1/4} f_{k_n}$ converges weakly in $H^{m,l}_0(X)$. Let $w \in H^{m,l}_0(X)$ denote the weak limit. We now show that $w = Z^{1/2} x^{1/4} \tilde{R}_+(0) x^{1/4} f$.

- We first check that $w$ solves the PDE $\tilde{P}(0)(Z^{-1/4} x^{-1/4} w) = Z^{1/4} x^{1/4} f$. As above, we use that, for any $m_0, l_0$,
  \[ [0, \infty)_\sigma \times H^{m_0,l_0}_0(X) \ni (\sigma, u) \mapsto \tilde{P}(\sigma) u \in S'(X) \tag{604} \]

  is jointly continuous with respect to the strong topology on $H^{m_0,l_0}_0(X)$. Since

  \[ Z^{-1/4} x^{-1/4} (\sigma_{k_n}^2 + Zx)^{1/4} \tilde{R}_+ (\sigma_{k_n})(\sigma_{k_n}^2 + Zx)^{1/4} f_{k_n} \rightarrow Z^{-1/4} x^{-1/4} w \tag{605} \]

  weakly in $H^{m-1/2}_0(X)$,

  \[ \tilde{P}(\sigma_{k_n})(Z^{-1/4} x^{-1/4} (\sigma_{k_n}^2 + Zx)^{1/4} \tilde{R}_+ (\sigma_{k_n})(\sigma_{k_n}^2 + Zx)^{1/4} f_{k_n}) \rightarrow \tilde{P}(0)(Z^{-1/4} x^{-1/4} w) \tag{606} \]

  $S'(X)$. Moreover, it is not difficult to see that, as before, \( \tilde{P}(\sigma_{k_n})(Z^{-1/4} x^{-1/4} (\sigma_{k_n}^2 + Zx)^{1/4} \tilde{R}_+ f_{k_n}) \rightarrow Z^{1/4} x^{1/4} f \) in $S'(X)$.

  Since $S'(X)$ is Hausdorff, it follows that $\tilde{P}(0)(Z^{-1/4} x^{-1/4} w) = f$.

- Thus, we have

  \[ \tilde{P}(0)(Z^{-1/4} x^{-1/4} w) = Z^{1/4} x^{1/4} f \tag{607} \]

  \[ \tilde{P}(0)(\tilde{R}_+(0)(Z^{1/4} x^{1/4} f)) = Z^{1/4} x^{1/4} f. \]

  Set $t_0 = l - 1/4$, so that $x^{1/4} f \in H^{m,t_0+5/4}_0(X) = H^{l_0+3/2}_0(X)$. By Proposition 4.3,

  \[ \tilde{P}(0) : \{ u \in H^{m,l}_0(X) : \tilde{P}(0) u \in H^{m,l_0+3/2}_0(X) \} \rightarrow H^{m,l_0+3/2}_0(X) \]

  $: \{ u \in H^{m,l}_{1/4}(X) : \tilde{P}(0) u \in H^{m,l+5/4}_0(X) \} \rightarrow H^{m,l+5/4}_0(X)$.
is invertible, and the inverse is $\tilde{R}_+(0)$. Thus, $u = \tilde{R}_+(0)Z^{1/4}x^{1/4}f$ is the unique solution to $\tilde{P}(0)u = Z^{1/4}x^{1/4}f$ in the domain of eq. (608). But $Z^{-1/4}x^{-1/4}w$ is in the codomain, and as we saw in eq. (607) solves this PDE. We conclude that
\begin{equation}
Z^{-1/4}x^{-1/4}w = \tilde{R}_+(0)Z^{1/4}x^{1/4}f.
\end{equation}
Via the compactness of the inclusion $H^m_l \hookrightarrow H^{m-\epsilon,l-\epsilon}_b$, we conclude that
\begin{equation}
(\sigma^2_k + Zx)^{1/4}\tilde{R}_+(\sigma_k)(\sigma^2_k + Zx)^{1/4}f_k \rightarrow Z^{1/4}x^{1/4}\tilde{R}_+(0)Z^{1/4}x^{1/4}f
\end{equation}
strongly in $H^{m-\epsilon,l-\epsilon}_b$.

Uniform continuity follows as in the proof of Proposition 6.10. □

**Proposition 6.12.** Fix $\psi \in C^\infty_c(\mathbb{R})$ supported sufficiently close to $(-\infty, \Sigma^2]$ and $\chi \in C^\infty_c(X^p_\text{res})$ supported away from $bf$. Then, for each $k, K \in \mathbb{N}$ with $k + K > 0$,
\begin{equation}
\{\psi(E)(E\partial_E)^k(\chi x\partial_E)^K\tilde{P}(E^{1/2})\}_{E \geq 0} \subseteq S \text{Diff}_{k,\text{leC}}^{1,0,-2,-1,-3}(X) \subseteq S \text{Diff}_{k,\text{leC}}^{1,0,-2,-1,-3}(X).
\end{equation}

**Proof.** For $\sigma \in (0, \Sigma)$, $\tilde{P}(\sigma) = K(\sigma) + C$ for $K(\sigma) \in S \text{Diff}_{k,\text{leC}}^{1,0,-2,-1,-3}(X)$ containing the $\sigma$-dependent part of $\tilde{P}(\sigma)$ and $C \in S \text{Diff}_{k,\text{leC}}^{1,0,-2,-1,-3}(X)$ constant in $\sigma$. Thus, for $k, K \in \mathbb{N}$ which are not both zero,
\begin{equation}
\{\psi(E)(E\partial_E)^k(\chi x\partial_E)^K\tilde{P}(\sigma)\}_{\sigma \geq 0} \subseteq (E\partial_E)^k(\chi x\partial_E)^K S \text{Diff}_{k,\text{leC}}^{1,0,-2,-1,-3}(X).
\end{equation}
Since $E\partial_E$ lifts to a conormal vector field on $X^p_\text{res}$ and $\chi x\partial_E$ lifts to a smooth vector field which is conormal at $tf$ (and identically zero near $bf$),
\begin{equation}
(\sigma\partial_\sigma)^k(\chi x\partial_E)^K S \text{Diff}_{k,\text{leC}}^{1,0,-2,-1,-3}(X) \subseteq S \text{Diff}_{k,\text{leC}}^{1,0,-2,-1,-3}(X),
\end{equation}
so eq. (611) follows. □

**Proposition 6.13.** For each $m, l \in \mathbb{R}$ with $m + l > -1/2 > l$ and for each $f \in H^{m,l+1}_b(X)$, the map
\begin{equation}
\tilde{R}_+(\bullet) : (0, \Sigma) \ni \sigma \mapsto \tilde{R}_+(\sigma)f \in S'(X)
\end{equation}
is smooth as a map $(0, \Sigma) \to S'(X)$. Thus, $\partial^k_\sigma(\tilde{R}_+(\sigma)f(-\sigma)) : \mathbb{R}_x^+ \to S'(X)$ is well-defined for $k \in \mathbb{N}$ and $f \in C^\infty((0, \Sigma); H^{m,l+1}_b(X))$ and is given by
\begin{equation}
\left[\sum_{K=0}^k \begin{pmatrix} k \\ K \end{pmatrix} \partial^{K}_\sigma(\tilde{R}_+(\sigma)(\partial^{-\sigma-K}_\sigma f(-\sigma_0)))\right]|_{\sigma = 0}.
\end{equation}

Moreover, for $k \in \mathbb{N}$, it is the case that, for each $\sigma > 0$, the map
\begin{equation}
\partial^k_\sigma \tilde{R}_+(\sigma) = \partial^k_\sigma \tilde{R}_+(\sigma)|_{\sigma_0 = \sigma} : \bigcup_{m,l \in \mathbb{R}} H^{m,l+1}_b(X) \to S'(X)
\end{equation}
is smooth as a map $(0, \Sigma) \to S'(X)$. Thus, $\partial^k_\sigma(\tilde{R}_+(\sigma)f) : \mathbb{R}_x^+ \to S'(X)$ is well-defined for $k \in \mathbb{N}$ and $f \in C^\infty((0, \Sigma); H^{m,l+1}_b(X))$ and is given by
\begin{equation}
\partial^k_\sigma(\tilde{R}_+(\sigma)f) = \partial^k_\sigma(\tilde{R}_+(\sigma)f)|_{\sigma_0 = \sigma} : \bigcup_{m,l \in \mathbb{R}} H^{m,l+1}_b(X) \to S'(X)
\end{equation}
satisfies $\partial^k_\sigma \tilde{R}_+(\sigma) \in L(H^{m,l+1}_b(X), H^{m-k,l}_b(X))$. For each $k \in \mathbb{N}$, the identity
\begin{equation}
\partial^k_\sigma \tilde{R}_+(\sigma) = \left[\sum_{\{k_i\}_{i=1}^l \in \mathcal{I}_k} c_{\{k_i\}_{i=1}^l} \prod_{i=1}^l \left(\tilde{R}_+(\sigma)\partial^{k_i}_\sigma \tilde{P}(\sigma)\right)\right] \tilde{R}_+(\sigma)
\end{equation}
holds for some $c_{\{k_i\}_{i=1}^l} \in \mathbb{Z}$, where $\mathcal{I}_k$ is the set of finite sequences of positive integers summing to $k$. □
Remark 8. By the conjunction of

1. \( \tilde{R}_+(\sigma) : H^{m,l+1}_b(X) \to H^{m,l}_b(X) \) holding for all \( m, l \in \mathbb{R} \) with \( l < -1/2 < m + l \) and \( \sigma > 0 \), and

2. \( \partial_x^k \tilde{P}(\sigma) : H^{m,l}_b(X) \to H^{m-1,l+1}_b(X) \) holding for all \( m, l \in \mathbb{R} \) and \( \sigma > 0 \),

it is the case that

\[
\tilde{R}_+(\sigma) \partial_x^k \tilde{P}(\sigma) : H^{m,l}_b(X) \to H^{m-1,l}_b(X)
\]

for each \( \sigma > 0, k \in \mathbb{N} \), and \( m, l \in \mathbb{R} \) satisfying \( l < -1/2 < m + 1 + l \).

Consequently, for \( \{ i_i \}_{i=1}^l \in \mathcal{I}_k \),

\[
\prod_{i=1}^l \tilde{R}_+(\sigma) \partial_x^k \tilde{P}(\sigma) : H^{m,l}_b(X) \to H^{m-1,l}_b(X)
\]

is a well-defined composition whenever \( l < -1/2 < m + l - I \). The right-hand side of eq. (618) is therefore a well-defined map \( H^{m,l+1}_b(X) \to H^{m-k,l}_b(X) \) whenever \( l < -1/2 < m + l - k \).

The identity eq. (618) should therefore be read as stating that both sides agree as maps \( H^{m,l+1}_b(X) \to H^{m-k,l}_b(X) \).

Proposition 6.14. For each \( k, \sigma \in \mathbb{N} \) and \( l < -1/2 \), there exists an \( m_0(l, k + \sigma) \in \mathbb{R} \) such that for all \( m > m_0 \) and \( \chi \in C_c^\infty(X_{t_0}^{sp}) \) supported away from \( \Sigma \),

\[
\{(E \partial_x^k(\chi x \partial_x^k((E + \Sigma x^{1/4})\tilde{R}_+(E^{1/2})))\}_{E \in (0, 2)} \in L^\infty \cap C^0((0, \Sigma); \mathcal{L}(H^{m,l+5/4}_b(X), H^{m-k-\sigma,l-\epsilon}_b(X)))
\]

for all \( \epsilon > 0 \).

Proof. We explicitly consider the \( K = 0 \) case, and the \( K \in \mathbb{N}^+ \) case is similar (if a bit messier).

So, we want to prove that

\[
\{(\sigma \partial_\sigma)^k((\sigma^2 + \Sigma x)^{1/4}\tilde{R}_+(\sigma))\}_{\sigma \in (0, \Sigma)} \in L^\infty \cap C^0((0, \Sigma); \mathcal{L}(H^{m,l+5/4}_b(X), H^{m-k-\sigma,l-\epsilon}_b(X))).
\]

By eq. (618),

\[
(\sigma \partial_\sigma)^k((\sigma^2 + \Sigma x)^{1/4}\tilde{R}_+(\sigma)) = \sum_{j=0}^k \binom{k}{j} (\sigma^2 + \Sigma x)^{-1/4}(\sigma \partial_\sigma)^j (\sigma^2 + \Sigma x)^{1/4}
\]

\[
\left[ \sum_{\{k_i\}_{i=1}^l \in \mathcal{I}_{k-\sigma}} c_{\{k_i\}}_{l=1} \prod_{i=1}^l (\sigma^2 + \Sigma x)^{1/4}\tilde{R}_+(\sigma)(\sigma \partial_\sigma)^{k_i} \tilde{P}(\sigma)(\sigma^2 + \Sigma x)^{-1/4} \right] (\sigma^2 + \Sigma x)^{1/4}\tilde{R}_+(\sigma)
\]

(624)
for all $\sigma > 0$. By Proposition 6.12, for any $k \in \mathbb{N}^+$ we can write
\[
(\sigma^2 + Zx)^{-1/4}((\sigma \partial_\sigma)^k \tilde{P}(\sigma))(\sigma^2 + Zx)^{-1/4} \in L^\infty \cap C^0([0, \Sigma); S \text{Diff}^{1-1+l}_b(X)) \tag{625}
\]
for any $\epsilon > 0$. On the other hand, by Proposition 6.11, for all $\epsilon > 0$,
\[
(\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma)(\sigma^2 + Zx)^{1/4} \in C^0([0, \Sigma); \mathcal{L}(H^{m-1-1+4/\epsilon}_b, H^{m-1-\epsilon, l-\epsilon}_b(X))) \tag{626}
\]
for any $m, l \in \mathbb{R}$ satisfying $l < -1/2$ and $m > -2l$. Combining these two observations, for any $k \in \mathbb{N}^+$ and $\epsilon > 0$ we have that
\[
(\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma)((\sigma \partial_\sigma)^k \tilde{P}(\sigma))(\sigma^2 + Zx)^{-1/4} \tag{627}
\]
lies in
\[
C^0([0, \Sigma); \mathcal{L}(H^{m-1, l+1, \epsilon}_b, H^{m-1, l-\epsilon}_b) \text{Diff}^{1-1+l}_b(X)) \subseteq C^0([0, \Sigma); \mathcal{L}(H^{m, l}_b, H^{m-1, l-\epsilon}_b)) \tag{628}
\]
for any $m, l \in \mathbb{R}$ satisfying $l < -1/2$ and $m > -2l$. Therefore, by Proposition 6.10,
\[
\left[ \sum_{\{k_i\}_{i=1}^l \subseteq \mathbb{Z}_{<0}} c_{\{k_i\}_{i=1}^l} \prod_{i=1}^l (\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma)(\sigma \partial_\sigma)^k \tilde{P}(\sigma)(\sigma^2 + Zx)^{-1/4} \right]
\]
\[
(\sigma^2 + Zx)^{1/4} \tilde{R}_+(\sigma)
\]
\[
\in C^0([0, \Sigma); \mathcal{L}(H^{m, l+4/\epsilon}_b, H^{m-k, l-\epsilon}_b)) \tag{629}
\]
for any $\epsilon > 0$ and for any $m, l \in \mathbb{R}$ satisfying $l < -1/2$ and $-2l + k - 1 < m$.

Since $(\sigma^2 + Zx)^{-1/4}(\sigma \partial_\sigma)^j(\sigma^2 + Zx)^{1/4} \in C^0([0, \Sigma); S^1_b(X))$ (as checked in Lemma 2.1), we conclude eq. (623), eq. (622).

The preceding results amount to:

**Proposition 6.15.** Given $\tilde{f} \in C^\infty(X^\text{sp}_{\text{res}})$ vanishing rapidly at tf, (i.e. $\tilde{f} \in A^{\infty, \infty, 0, 0}(X^\text{sp}_{\text{res}})$), set $u_{00, +}(\sigma) = \tilde{R}_+(\sigma)\tilde{f}(-\sigma)$ for all $\sigma \in [0, \Sigma)$. Defining $u_{0, +}(\sigma) = x^{-(n-1)/2}(\sigma^2 + Zx)^{1/4}u_{00, +}(\sigma)$, $u_{0, +} = \{u_{00, +}(- \sigma)\}_{\sigma \in [0, \Sigma)} \in A^{0, 0, 0}(X^\text{sp}_{\text{res}} \cap \{\sigma < \Sigma\})$.

Moreover, the mapping $A^{\infty, \infty, 0, 0}(X^\text{sp}_{\text{res}}) \ni \tilde{f} \mapsto u_{0, +} \in A^{0, 0, 0}(X^\text{sp}_{\text{res}} \cap \{\sigma < \Sigma\})$ is continuous.

**Proof.** Fix $\chi \in C^\infty_c(X^\text{sp}_{\text{res}})$ supported away from tf and nonvanishing near zf.

- We first show that $u_{00, +} = \{u_{00, +}(- \sigma)\}_{\sigma \in [0, \Sigma)} \in A^{0, 0, 0}(X^\text{sp}_{\text{res}} \cap \{\sigma < \Sigma\})$.

  For any $k, K \in \mathbb{N}$, $(\chi x \partial_E)^K(\chi x \partial_E)^K \tilde{f} \in A^{\infty, \infty, 0, 0}(X^\text{sp}_{\text{res}})$. Thus, Proposition 6.13 and Proposition 6.14 imply that, for all $m \in \mathbb{R}$, $l < -1/2$, $K \in \mathbb{N}$,
\[
\{(\chi x \partial_E)^K u_{00, +}(- \cdot^{-1/2})\}_{E \in (0, \Sigma^2)} = \left\{ \left( \chi x \frac{\partial}{\partial E} \right)^K \left[ x^{-(n-1)/2}(E + Zx)^{1/4} \tilde{R}_+(E^{1/2}) \tilde{f}(- E^{1/2}) \right] \right\}_{E \in (0, \Sigma^2)} \tag{630}
\]
is in $A^{0, 0, 0}(X^\text{sp}_{\text{res}} \cap \{\sigma < \Sigma\})$. By Proposition 6.19, we deduce that $(\chi x \partial_E)^K u_{00, +} \in A^{0, 0, 0}(X^\text{sp}_{\text{res}} \cap \{\sigma < \Sigma\})$. Moreover, Proposition 6.14 shows that the map
\[
A^{\infty, \infty, 0, 0}(X) \ni \tilde{f} \mapsto (\chi x \partial_E)^K u_{00, +}(- \cdot^{-1/2}) \in A^{0, 0, 0}(X^\text{sp}_{\text{res}} \cap \{\sigma < \Sigma\}) \tag{631}
\]
is continuous, for each $K \in \mathbb{N}$, and the inclusion in Proposition 2.19 is continuous, so the map $\mathcal{A}^{\infty,\infty,(0,0)}(X) \ni \tilde{f} \mapsto (\chi x \partial_x^K u_{0,+}(-;E^{1/2}) \in \mathcal{A}^{0,-0,0}_{loc}(X^\text{sp})$ is continuous as well.

- It now follows from Proposition 2.20 that $\{u_{0,+}(-;\sigma)\}_{\sigma > 0}$ defines an element of the space $\mathcal{A}^{0,-0,0}_{loc}(X^\text{sp} \cap \{\sigma < \Sigma\})$. By the remark following Proposition 2.20,

$$\mathcal{A}^{\infty,\infty,(0,0)}(X^\text{sp}) \ni \tilde{f} \mapsto u_{0,+} \in \mathcal{A}^{0,-0,0}_{loc}(X^\text{sp} \cap \{\sigma < \Sigma\})$$

(632)

is continuous.

- Finally, we observe from Proposition 6.12 that $\{u_{0,+}(-;\sigma)\}_{\sigma > 0}$, considered as an element of $\mathcal{A}^{0,-0,0}_{loc}(X^\text{sp})$, restricts to $x^{-(n-1)/2}(Zx)^{1/4}\hat{R}_+(0)\tilde{f}(-0)$ at $zf$.

\[\square\]

6.3. Smoothness at $tf$, $bf$. For this subsection, we do not assume that $P$ is the spectral family of an attractive Coulomb-like Schrödinger operator, only that $P$ satisfies the hypotheses in §1.

Suppose that $P_1$ is classical to order $\beta_1 > 0$ and $P_2$ is classical to order $(\beta_2, \beta_3)$. Then, by Proposition 3.8,

$$N(\tilde{P}) - \tilde{P} \in \text{Diff}^{2,2,-4}_{b,leC}(X) + S \text{Diff}^{2,-1,3-2\delta_0}_{b,leC}(X)$$

(633)

holds for $\delta_1 = \min\{1 + \beta_1, \beta_2, 1/2 + \beta_3\}$ and $\delta_0 = \min\{1 + \beta_1, \beta_2, \beta_3\}$.

**Proposition 6.16.** Suppose that $P_1$ is classical to $\beta_1$th order and $P_2$ is classical to order $(\beta_2, \beta_3)$. If $u_0 \in \mathcal{A}^{0,-0,0}_{loc}(X^\text{sp}_{\text{res}})$ and $\tilde{f} \in \mathcal{A}^{\infty,\infty,(0,0)}(X^\text{sp}_{\text{res}})$, then, setting

$$u_{00}(-;\sigma) = x^{(n-1)/2}(\sigma^2 + Zx)^{-1/4}u_0(-;\sigma),$$

(634)

if $\tilde{P}u_{00} = \tilde{f}$, then

$$u_0 \in \mathcal{A}^{((0,0)\delta_1),((0,0)2\delta_0),(0,0)}_{loc}(X^\text{sp}_{\text{res}}),$$

(635)

and $\mathcal{A}^{0,-0,0}_{loc}(X^\text{sp}_{\text{res}}) \times \mathcal{A}^{\infty,\infty,(0,0)}_{loc}(X^\text{sp}_{\text{res}})|_{P_0u_{00} = \tilde{f} \mapsto (u_0, \tilde{f}) \mapsto u_0 \in \mathcal{A}^{((0,0)\delta_1),((0,0)2\delta_0),(0,0)}_{loc}(X^\text{sp}_{\text{res}})$ is continuous.

In particular, if $P$ is fully classical, then $u_0 \in \cap_{\alpha > 0} \mathcal{A}^{((0,0)\alpha),((0,0)2\alpha),(0,0)}_{loc}(X^\text{sp}_{\text{res}}) = C^{\infty}(X^\text{sp}_{\text{res}})$. \[\blacksquare\]

**Proof.** We will prove via induction on $\alpha, \beta$ that, for all $\alpha \in (-\infty, \delta_1]$ and $\beta \in (-\infty, \delta_0]$,

$$u_0 \in C^{\infty}(X^\text{sp}_{\text{res}}) + \mathcal{A}^{((0,0)\alpha),2\beta,(0,0)}_{loc}(X^\text{sp}_{\text{res}}) \subseteq \mathcal{A}^{((0,0)\alpha),((0,0)2\beta),(0,0)}_{loc}(X^\text{sp}_{\text{res}}).$$

(636)

The $\alpha, \beta < 0$ case of eq. (636) is of course the hypothesis of the proposition. We now write $\tilde{P} = N(\tilde{P}) + (\tilde{P} - N(\tilde{P}))$.

Consider now $\alpha$ such that eq. (636) holds. Thus, from $\tilde{P}u_{00} = \tilde{f}$ we get

$$N(\tilde{P})u_{00} = \tilde{f} - (\tilde{P} - N(\tilde{P}))u_{00} \in \text{Diff}^{2,2,-4}_{b,leC}(X) + S \text{Diff}^{2,-1,3-2\delta_0}_{b,leC}(X) x^{(n-1)/2}\mathcal{A}^{((0,0)\alpha),((0,0)2\alpha)2\beta,(0,0)}_{loc}(X^\text{sp}_{\text{res}}).$$

(637)

The set on the right-hand side is

$$x^{(n-1)/2} \sum_{\beta=00}^{7/2} C^{\infty}(X^\text{sp}_{\text{res}}) + x^{(n-1)/2}\mathcal{A}^{((2,0)\alpha),2\beta+7/2,(0,0)}_{loc}(X^\text{sp}_{\text{res}}) + x^{(n-1)/2}\mathcal{A}^{(1+\delta_1,1+\alpha+\delta_1)\min\{5/2+2\delta_0,5/2+2\delta_0+2\alpha\},(0,0)}_{loc}(X^\text{sp}_{\text{res}}).$$

(638)
For $\alpha$ sufficiently close to zero or positive, we can apply the results of Proposition 6.17 below to conclude that

$$u_0 \in C^\infty(X_{res}^{sp}) + A_{\text{loc}}^{((0,0),1+\alpha,1+2\beta,(0,0))}(X_{res}^{sp}) + A_{\text{loc}}^{((0,0),\min\{\delta_1,\alpha+\delta_1\},\min\{2\delta_0,2\delta_0+2\beta\},(0,0))}(X_{res}^{sp}) \subseteq C^\infty(X_{res}^{sp}) + A_{\text{loc}}^{((0,0),\min\{1+\alpha,\delta_1,\alpha+\delta_1\},\min\{1+2\beta,2\delta_0,2\delta_0+2\beta\},(0,0))}(X_{res}^{sp}),$$

(639)

If $\delta_1 \leq 1, \alpha + \delta_1$ and $\delta_0 \leq 2\beta + 1, 2\beta + 2\delta_0$, then

$$A_{\text{loc}}^{((0,0),\min\{1+\alpha,\delta_1,\alpha+\delta_1\},\min\{1+2\beta,2\delta_0,2\delta_0+2\beta\},(0,0))}(X_{res}^{sp}) = A_{\text{loc}}^{((0,0),(0,0),\delta_0,(0,0))}(X_{res}^{sp}),$$

(640)

so we have concluded that eq. (635) holds. Otherwise,

$$u_0 \in C^\infty(X_{res}^{sp}) + A_{\text{loc}}^{((0,0),\alpha+\varepsilon),(2\beta+2\varepsilon\delta_0),(0,0)}(X_{res}^{sp}),$$

(641)

for $\varepsilon_1 = \min\{1, \delta_1\}$ and $\varepsilon_0 = \min\{1/2, \delta_0\}$. The claim therefore follows by induction.

The continuity clause of the proposition can be proven using the same argument, keeping track of topologies. □

**Proposition 6.17.** Suppose that $P$ satisfies the minimal hypotheses of §1.

Suppose further that $u \in A_{\text{loc}}^{(-\infty,-\infty),(0,0)}(X_{res}^{sp}) = A_{\text{loc}}^{(-\infty),(0,0)}([0, \infty)_E \times X)$ satisfies $N(\hat{P})u = f$ for some

$$f \in x^{-(n+1)/2}(\sigma^2 + Zx)^{-1/4}C^\infty(X_{res}^{sp}) + x^{-(n+1)/2}A_{\text{loc}}^{((1,0),\alpha+1,\beta+5/2,(0,0))}(X_{res}^{sp}),$$

(642)

for $\alpha, \beta \in \mathbb{R}^+$. Then, setting $u_0 = x^{-(n+1)/2}(\sigma^2 + Zx)^{1/4}u$,

$$u_0 \in C^\infty(X_{res}^{sp}) + A_{\text{loc}}^{((0,0),\alpha,\beta,(0,0))}(X_{res}^{sp})$$

(643)

holds.

□

**Proof.** Now letting $\tilde{N}(\hat{P}) = x^{-n(n-1)/2}(\sigma^2 + Zx)^{1/4}N(\hat{P})x^{(n+1)/2}(\sigma^2 + Zx)^{-1/4}$, $\tilde{N}(\hat{P})u_0 = f_0$ for $u_0 = x^{-(n+1)/2}(\sigma^2 + Zx)^{1/4}u$ and $f_0 = x^{-(n+1)/2}(\sigma^2 + Zx)^{1/4}f$. Equation (642) yields

$$f_1 \in x^{(\sigma^2 + Zx)^{-1/2}}C^\infty(X_{res}^{sp}) + A_{\text{loc}}^{((0,0),\alpha,\beta,(0,0))}(X_{res}^{sp}),$$

(644)

where $f_1 = x^{-1}(\sigma^2 + Zx)^{-1/2}f_0$.

In order to prove the proposition, it suffices to restrict attention to $\hat{X} = [0, \bar{x})_x \times \partial X_y$. By eq. (514) (with $k = -1/4$ and $l = -1/2$), we have

$$x^{-1}(\sigma^2 + Zx)^{-1/2}\tilde{N}(\hat{P}) = 2ix\partial_x.$$  

(645)

Thus, integrating $\tilde{N}(\hat{P})u_0 = f_0$, we get

$$u_0(x, y; \sigma) = c(y; \sigma) + \frac{i}{2} \int_x^{\bar{x}} x_0^{-1}f_1(x_0, y; \sigma) \, dx_0$$

(646)

for some $c(y; \sigma) \in \mathbb{C}$, for each $\sigma \geq 0$. Since $u_0 \in A_{\text{loc}}^{(-\infty,-\infty),(0,0)}([0, \infty)_E \times \hat{X})$, and since the same applies to the integral in eq. (646) (cut-off near $\partial X$), we deduce that $c \in A_{\text{loc}}^{(-\infty,-\infty),(0,0)}([0, \infty)_E \times \hat{X})$. Since $c(y, \sigma)$ does not depend on $x$, this implies that $c \in C^\infty([0, \infty)_E \times \partial X)$.

Equation (643) then follows from the mapping properties of the integral in eq. (646), which we record in Proposition 6.18 and Proposition 6.19 below. □

For the following proposition, we use $\hat{X}^{sp}_{res}$, defined as $X^{sp}_{res}$ with $\hat{X} = [0, \bar{x}) \times \partial X$ in place of $X$.  

**Proposition 6.18.** If $g \in x^{(\sigma^2 + Zx)^{-1/2}}C^\infty(\hat{X}^{sp}_{res})$, then $I = \int_x^{\bar{x}} x_0^{-1}g(x_0, y; \sigma) \, dx_0 \in C^\infty(\hat{X}^{sp}_{res})$. □
Proof. It suffices to consider the case when $g$ is supported in $[0, \bar{r}/2]$. Let us write $G(x, y; E) = x^{-1}(\sigma^2 + \mathbf{x})^{1/2}g$, so

$$I = \int_{x}^{\bar{x}} (E + \mathbf{x})^{1/2}G(x, y; E) \, dx_0 = 2 \int_{x}^{\bar{x}/2} \left( \frac{\rho_0^2}{E + \mathbf{x}^0} \right)^{1/2} G(\rho_0^2, y; E) \, d\rho_0. \quad (647)$$

This is evidently smooth away from $tf$.

By the smoothness of $G$ (in particular at $zf \cap tf$), there exists a $G_0 \in C^\infty([0, \infty) \times [0, \infty) \times \partial X)$ such that $G_0(\rho, y; \hat{E}) = G(x, y; E)$ when $x = \rho^2$ and $E = \hat{E}$.

$$I = 2 \int_{\rho}^{\bar{x}/2} \left( \frac{\rho_0^2}{E \rho^2 + \mathbf{x}^0} \right)^{1/2} G_0(\rho_0, y; \hat{E} \left( \frac{\rho}{\rho_0} \right)^2) \, d\rho_0 = \int_{\rho}^{\bar{x}/2} G_1(\rho_0, y; \hat{E} \left( \frac{\rho}{\rho_0} \right)^2) \, d\rho_0 \quad (648)$$

for $G_1 \in C^\infty([0, \infty) \times [0, \infty) \times \partial X)$ defined by $G_1(\rho, y; \hat{E}) = 2(\hat{E} + \mathbf{x})^{-1/2}G_0(\rho, y; \hat{E})$. By the smoothness at $G_1$, there exist $G_1^{(j,k)} \in C^\infty(\partial X)$, $F_1^{(j,k)} \in L^\infty \cap C^\infty([0, \infty) \times [0, \infty) \times \partial X)$ for $j, k \in \mathbb{N}$ such that, for all $J, K \in \mathbb{N}$,

$$G_1(\rho, y; \hat{E}) = \sum_{j=0}^{J} \sum_{k=0}^{K} \rho^j \hat{E}^k G_1^{(j,k)}(y) + \rho^{J+1} \hat{E}^{K+1} F_1^{(J+1,K+1)}(\rho, y; \hat{E}). \quad (649)$$

Thus,

$$G_1(\rho_0, y; \hat{E} \left( \frac{\rho}{\rho_0} \right)^2) = \sum_{j=0}^{J} \sum_{k=0}^{K} \rho^{j+2k} \rho_0^{-2k} \hat{E}^k G_1^{(j,k)}(y) \quad (650)$$

Integrating,

$$I = \sum_{j=0}^{J} \sum_{k=0}^{K} \rho^{j+2k} \rho_0^{-2k} \hat{E}^k \int_{\rho}^{\bar{x}/2} G_1^{(j,k)}(y) \, d\rho_0 \quad (651)$$

The big sum lies in $C^\infty([0, \infty) \times [0, \infty) \times \partial X)$. The remainder term satisfies

$$\left| \rho^{j+2K+3} \hat{E}^K \int_{\rho}^{\bar{x}/2} \rho^{-2K-2} F_1^{(J+1,K+1)}(\rho_0, y; \hat{E} \left( \frac{\rho}{\rho_0} \right)^2) \, d\rho_0 \right| \leq \rho^{j+1} \hat{E}^{K+1} |x^{1/2} - \rho| ||F_1^{(J+1,K+1)}||_{L^\infty}. \quad (652)$$

Thus, $I$ defines a smooth function on $[0, \infty) \times [0, \infty) \times \partial X$. This shows – in conjunction with the smoothness of $I$ away from $tf$ – that $I$ is smooth on $X_{\partial \mathbf{x}}^{sp}$ everywhere except the corner $bf \cap tf$.

Now suppose that $G$ is supported in some set of the form $\{x/E > C\}$, then, near $bf \cap tf$,

$$I(x, y, E) = I(\rho \sigma^2, y, \sigma^2) = I(C \sigma^2, y, \sigma^2) = I(\rho^2, y, \hat{E} \rho^2) \quad (653)$$

for $\rho = C^{1/2} \sigma$ and $\hat{E} = C^{-1/2}$. Since we already know that $I$ depends smoothly on $\rho, \hat{E}, y$, we conclude that $I$ depends smoothly on $\sigma, y$ alone near $bf \cap tf$. 
On the other extreme, suppose that $G$ is supported in some set of the form $\{x/E < c\}$, for $c > 0$. In order to study $I$ near $bf \cap tf$, we work with the coordinate $\varrho = x/E$. By the smoothness of $G$, there exists a $G_2 \in C^\infty([0, \infty)_\varrho \times [0, \infty)_E \times \partial X)$ such that $G_2(\varrho, y; \sigma) = G(x, y; E)$ whenever $\varrho = x/E$ and $\sigma^2 = E$. We can then write

$$I = \sigma \int_{x/\sigma^2 \in \mathbb{R}} (1 + Z_{\varrho})^{-1/2} G_2(\varrho, y; \sigma) d\varrho = \sigma \int_{x/\sigma^2 \in \mathbb{R}} G_3(\varrho, y; \sigma) d\varrho$$

for $G_3 \in C^\infty([0, \infty)_\varrho \times [0, \infty)_E \times \partial X)$ given by $G_3(\varrho, y; \sigma) = (1 + Z_{\varrho})^{-1/2} G_2(\varrho, y; \sigma)$. The integral eq. (654) is in $\sigma C^\infty([0, \infty)_\varrho \times (0, \infty)_E \times \partial X)$ for $\sigma < \bar{x}^{1/2}/c^{1/2}$.

Since any $G$ can be decomposed $G = G_1 + G_2$ into smooth parts $G_1, G_2 \in C^\infty(\mathcal{X}_{res}^\beta)$, with $G_1$ supported on $\{x/E > C\}$ and $G_2$ supported on $\{x/E < c\}$ for some $c, C > 0$, and since $I$ depends linearly on $G$, we can conclude that $I$ is smooth. \hfill \Box

**Proposition 6.19.** If $g \in \mathcal{A}_{loc}^{(\alpha_0, 0), \beta, (0, 0)}(\mathcal{X}_{res}^\varrho)$ for $\alpha_0 \in \mathbb{N}^+$, $\alpha, \beta > 0$, then

$$\int_x \varrho \varrho^{-1} g(x_0, y; \sigma) d\varrho \in C^\infty(\mathcal{X}_{res}^\varrho) + \mathcal{A}_{loc}^{(0, 0), \beta, (0, 0)}(\mathcal{X}_{res}^\varrho)$$

holds.

**Proof.** Let $I = \int_x \varrho \varrho^{-1} g(x_0, y; \sigma) d\varrho$. It suffices to prove the following two claims:

1. If $g$ is supported in $\{x/E > C\}$, then $I \in C^\infty(\mathcal{X}_{res}^\varrho) + \mathcal{A}_{loc}^{(0, 0), \beta, (0, 0)}(\mathcal{X}_{res}^\varrho)$.
2. If $g$ is supported in $\{x/E < c\}$, then $I \in \mathcal{A}_{loc}^{(0, 0), \alpha, \beta, (0, 0)}(\mathcal{X}_{res}^\varrho)$.

In order to prove (II), we write

$$I = \int_{x/\sigma^2 \in \mathbb{R}} \varrho^{-1} G(\varrho, y; \sigma) d\varrho,$$

where $G \in \mathcal{A}_{loc}^{(\alpha_0, 0), \beta, (0, 0)}([0, \infty)_\varrho \times [0, \infty)_E \times \partial X)$. Note that $\min\{\varrho/\sigma^2, c\} = c$ for $\sigma$ sufficiently small. Thus, it is also the case that

$$I \in \mathcal{A}_{loc}^{(0, 0), \alpha, \beta, (0, 0)}([0, \infty)_\varrho \times [0, \infty)_E \times \partial X)$$

locally. Since $I$ is identically zero in the set $\{E < x\}$ and therefore smooth in some neighborhood of $x$, this suffices to show that, globally speaking, $I \in \mathcal{A}_{loc}^{(0, 0), \alpha, \beta, (0, 0)}(\mathcal{X}_{res}^\varrho)$.

On the other hand, to prove (I), we write

$$I = 2 \int_{\varrho} \varrho^{1/2} \rho^{-1} G(\rho, y; \hat{E}\left(\frac{\rho}{\rho_0}\right)^2) d\rho,$$

where $G(\rho, y; \hat{E}) \in C^\infty([0, \infty)_E; \mathcal{A}_{loc}^{\varrho}(0, \infty)_\varrho \times \partial X)$ is supported in $\hat{E} < C^{-1}$. We expand $G(\rho, y; \hat{E})$ in Taylor series around $\hat{E} = 0$: there exist $G^{(k)} \in \mathcal{A}_{loc}^{\varrho}(0, \infty)_\varrho \times \partial X)$ and $F^{(k)} \in C^\infty([0, \infty)_E; \mathcal{A}_{loc}^{\varrho}(0, \infty)_\varrho \times \partial X))$ such that

$$G(\rho, y; \hat{E}) = \sum_{k=0}^K \hat{E}^k G^{(k)}(\rho, y) + \hat{E}^{K+1} F^{(K+1)}(\rho, y; \hat{E}).$$
Substituting this into eq. (658),
\[ I = \sum_{k=0}^{K} \hat{E}^k \rho^{2k} \int_{\rho}^{1/2} \rho_0^{-1-2k} G(k)(\rho_0, y) \, d\rho_0 + \hat{E}^{K+1} \rho^{2K+2} \int_{\rho}^{1/2} \rho_0^{-3-2K} F(K+1)(\rho, y; \hat{E}(\rho^2/\rho_0^2)) \, d\rho_0 \]
for each \( K \in \mathbb{N} \). We take \( K = \lfloor \beta/2 \rfloor \), in which case we can write
\[ \int_{\rho}^{1/2} \rho_0^{-1-2k} G(k)(\rho_0, y) \, d\rho_0 = \int_{\rho}^{1/2} \rho_0^{-1-2k} G(k)(\rho_0, y) \, d\rho_0 - \int_{\rho}^{1/2} \rho_0^{-1-2k} G(k)(\rho_0, y) \, d\rho_0. \]  
We split \( I = I_0 + J \), where
\[ I_0 = \sum_{k=0}^{K} \hat{E}^k \rho^{2k} \int_{\rho}^{1/2} \rho_0^{-1-2k} G(k)(\rho_0, y) \, d\rho_0 \in C^\infty([0, \infty)_E \times [0, \rho] \times \partial X) \]
\[ J = \sum_{k=0}^{K} \hat{E}^k \rho^{2k} \int_{\rho}^{1/2} \rho_0^{-1-2k} G(k)(\rho_0, y) \, d\rho_0 + \hat{E}^{K+1} \rho^{2K+2} \int_{\rho}^{1/2} \rho_0^{-3-2K} F(K+1)(\rho_0, y; \hat{E}(\rho^2/\rho_0^2)) \, d\rho_0. \]
Observe that \( J \in C^\infty([0, \infty)_E; A^\beta_{loc}([0, \infty)_\rho \times \partial X]) \) — for example,
\[ |\rho^{2K+2} \int_{\rho}^{1/2} \rho_0^{-3-2K} F(K+1)(\rho_0, y; \hat{E}(\rho^2/\rho_0^2)) \, d\rho_0| \leq \rho^{2K+2} \int_{\rho}^{1/2} \rho_0^{-3-2K+\beta} \, d\rho_0 \leq \rho^{2K+2} \left[(\rho^{1/2})^{-2-2K+\beta} + \rho^{-2-2K+\beta}\right] \leq \rho^{2[\beta/2]+2} + \rho^\beta = O(\rho^\beta) \]
as \( \rho \to 0^+ \). Thus, \( I \in C^\infty([0, \infty)_E \times [0, \rho] \times \partial X) + C^\infty([0, \infty)_E; A^\beta_{loc}([0, \infty)_\rho \times \partial X]). \)

Using eq. (653) as in the proof of the previous proposition, this suffices to show that \( I \in C^\infty(\hat{X}_{res}^n) + A^0_{loc}([0, \infty)_\rho \times \partial X)). \)

**Appendix A. The Model ODE**

We record in this appendix some computations regarding the model ODE, eq. (50): now allowing nonzero \( a_{00} > 0 \) and \( f \neq 0 \),
\[ -(1 + x a_{00}) \frac{a^2}{\partial x} u - a^2 u - Z x u = f, \]  
\( f \in C^\infty(\mathbb{R}_+^x), u \in D'[0, \infty)_x = \hat{C}^\infty([0, \infty)'). \) Since the results in this section mainly serve to illustrate the general features of the problem observed in Theorem 1.1, proofs are either sketched (or omitted entirely when elementary). References for many of the elementary statements can be found in [Bat53][Sla60][AS64][Olv97]. We will mostly cite [AS64]. The \( a_{00} > 0 \) case can be reduced to the \( a_{00} = 0 \) case via a simple change of variables:
\• let \( x_0 = (a_{00} + x^{-1})^{-1} = x/(1 + a_{00} x) \), i.e. \( r_0 = a_{00} + r \). Then \( \partial_r = \partial_{r_0} \), and the ODE eq. (665) is equivalent to
\[ -(x_0^2 \frac{\partial}{\partial x_0})^2 u - a^2 u - (Z - a^2 a_{00}) x_0 u = f_0 \]  
(666)
for \( f_0(x_0) = (1 - a_{00}x)f(x) \). The interval \([0, \infty)_x\) becomes \([0, 1/a_{00})_x\), but we can analyze eq. (666) on the larger region \([0, \infty)_x\).

The case of \( a_{00} < 0 \) is computationally similar to the case of \( a_{00} \geq 0 \), but in the former it is necessary to restrict \( x < 1/a_{00} \). We will therefore only state results in the \( a_{00} = 0 \) case.

Fix \( Z > 0 \). Consider the family \( P = \{P(\sigma)\}_{\sigma \geq 0} \) of ordinary differential operators \( P(\sigma) \) on \([0, \infty)_x\) given by

\[
P(\sigma) = -\left(x^2 \frac{\partial}{\partial x}\right)^2 - Zx - \sigma^2.
\]

In terms of \( r = 1/x \), the homogeneous ODE \( P(\sigma)u = 0 \) is

\[
\frac{\partial^2 u}{\partial r^2} + \left(\sigma^2 + \frac{Z}{r}\right)u = 0
\]

on \( \mathbb{R}^+_r = (0, \infty)_r \), with the \( x \to 0^+ \) limit corresponding to \( r \to \infty \). The (nonzero) solutions to eq. (668) cannot be written in terms of elementary functions for any value of \( \sigma \geq 0 \). In fact, for \( \sigma \neq 0 \), eq. (668) is essentially a special case of Whittaker’s ODE

\[
d\frac{d^2 W}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2}\right) W = 0,
\]

where \( \kappa \in \mathbb{C}, \mu \in \mathbb{C} \setminus (-\mathbb{N}^+/2) \) are parameters and \( W \in \mathcal{D}'(\mathbb{R}^+_r) \). There exist two named solutions to eq. (669), which we will denote WhittM\(_{\kappa,\mu}\)\(_{|\mathbb{R}^+_r}\) and WhittW\(_{\kappa,\mu}\)\(_{|\mathbb{R}^+_r}\), and these extend to analytic functions

\[
\text{WhittM}_{\kappa,\mu}, \text{WhittW}_{\kappa,\mu} : \mathbb{C}_z \setminus \{0\} \to \mathbb{C}.
\]

These are Whittaker’s M- and W- functions [AS64][Sla60]. (They can be written in terms of Kummer’s confluent hypergeometric function and Tricomi’s confluent hypergeometric function [AS64, pp. 13.1.32, 13.1.33].) It follows that Whittaker’s functions solve eq. (669) in the complex analytic sense.

**Proposition A.1.** For \( \sigma > 0 \), the set of \( u \in \mathcal{D}'(\mathbb{R}^+_r) \) solving eq. (668) is given by

\[
\mathcal{W}_\sigma = \{c_1 \text{WhittM}_{\kappa,1/2}(2i\sigma r) + c_2 \text{WhittW}_{\kappa,1/2}(2i\sigma r) : c_1, c_2 \in \mathbb{C}\}
\]

for \( \kappa = -iZ/2\sigma \). \( \square \)

The asymptotic expansions of the Whittaker M- and W-functions at large imaginary argument are due originally to Whittaker. For fixed \( \kappa, \sigma > 0 \), we have

\[
\text{WhittM}_{\kappa,1/2}(2i\sigma r) = \left[ \frac{(-2i\sigma)^{-\frac{\kappa}{2}}}{\Gamma(1 + \kappa)} e^{-i\sigma r - \frac{\kappa}{2} \log r} + \frac{(-2i\sigma)^{\frac{\kappa}{2}}}{\Gamma(1 - \kappa)} e^{i\sigma r + \frac{\kappa}{2} \log r} \right] \left(1 + O\left(\frac{1}{r}\right)\right)
\]

\[
\text{WhittW}_{\kappa,1/2}(2i\sigma r) = (2i\sigma)^{-\frac{\kappa}{2}} e^{-i\sigma r - \frac{\kappa}{2} \log r} (1 + O\left(\frac{1}{r}\right))
\]

as \( r \to \infty \). We are using the principal branch of the logarithm in making sense of \((2i\sigma)^{-iZ/2}\sigma\). As we are concerned with \( r \to \infty \) behavior, the Whittaker M- and W- functions (which are singled out of the space of all solutions to Whittaker’s ODE by their behavior at small argument) are not the solutions to the radial ODE of direct interest. For \( \sigma > 0 \),

\[
\mathcal{W}_{\sigma,-} = \text{span}_\mathbb{C}\{\text{WhittW}_{\kappa,1/2}(2i\sigma r)\}
\]

\[
\mathcal{W}_{\sigma,+} = \text{span}_\mathbb{C}\left\{\left[2i\sigma)^{-\frac{\kappa}{2}} \text{WhittM}_{\kappa,1/2}(2i\sigma r) + (-2i\sigma)^{-\frac{\kappa}{2}} \frac{1}{\Gamma(1 + \kappa)} \text{WhittW}_{\kappa,1/2}(2i\sigma r)\right]\right\}
\]
are the spaces of “incoming” or “outgoing” solutions to the ODE, $\mathcal{W}_\sigma = \mathcal{W}_{\sigma,-} \oplus \mathcal{W}_{\sigma,+}$. It is the spaces $\mathcal{W}_{\sigma,-}, \mathcal{W}_{\sigma,+}$ that concern us. Set

$$u_{-0}(r; \sigma) = (2i\sigma)^{\frac{\sigma}{2}} \text{Whitt}_\kappa(e^{i(r + 2\pi R)})$$
$$u_{+0}(r; \sigma) = (2i\sigma)^{\frac{-\sigma}{2}} \Gamma(1 - \kappa) \text{WhittM}_{\kappa,1/2}(2i\sigma r) + (-2i\sigma)^{\frac{-\sigma}{2}} \Gamma(1 - \kappa) \Gamma(1 + \kappa) \text{WhittW}_{\kappa,1/2}(2i\sigma r).$$

These have normalized oscillatory behavior $\exp(\pm i\sigma r)$ as $r \to \infty$,

$$u_{\pm,00}(r, \sigma) = e^{\pm i\sigma r + \frac{\sigma}{2} \log r} (1 + O_\sigma(1/r)).$$

**Proposition A.2.** For each $\sigma > 0$, $u_{\pm,00}$ are the unique solutions to the ODE eq. (668) satisfying eq. (678). □

It follows from the previous proposition and the fact that the ODE eq. (668) has real coefficients that $u_{-00}(r, \sigma) = u_{+00}(r, \sigma)^\ast$. Expanding to higher order [AS64, §13.5]:

**Proposition A.3.** For each $\sigma > 0$ and $K \in \mathbb{N}^+$,

$$u_{\pm,00}(r; \sigma) = e^{\pm i\sigma r + \frac{\sigma}{2} r} \left[ 1 + Z \sum_{k=1}^{K-1} \frac{(-1)^k}{8^k k! (2k)!} (Z \pm 2k \sigma)^2 \prod_{j=1}^{k-1} (Z \pm 2j \sigma)^2 + O_\sigma \left( \frac{1}{r^K} \right) \right]$$

as $r \to \infty$. □

**Proposition A.4.** $u_{\pm,00}(r; \sigma) \in C^\infty(\mathbb{R}_+^+ \times \mathbb{R}_+^+)$.

Turning to the $\sigma = 0$ case of eq. (668):

**Proposition A.5.** For $\sigma = 0$, the set of $u \in \mathcal{D}'(\mathbb{R}_+^+)$ solving eq. (668) is

$$\{ c_1 r^{1/2} J_1(2Z^{1/2} r^{1/2}) + c_2 r^{1/2} Y_1(2Z^{1/2} r^{1/2}) : c_1, c_2 \in \mathbb{C} \},$$

where $J_1, Y_1$ denote the Bessel $J$- and $Y$- functions of order one. □

As $r \to \infty$,

$$r^{1/2} J_1(2Z^{1/2} r^{1/2}) = \frac{r^{1/4}}{2^{1/2}} \left[ -\cos \left( 2\sqrt{Z} r + \frac{\pi}{4} \right) + \frac{3}{16} \frac{1}{\sqrt{Z} r} \sin \left( 2\sqrt{Z} r + \frac{\pi}{4} \right) \right] \left( 1 + O \left( \frac{1}{r^{1/2}} \right) \right)$$

$$r^{1/2} Y_1(2Z^{1/2} r^{1/2}) = \frac{r^{1/4}}{2^{1/2}} \left[ -\sin \left( 2\sqrt{Z} r + \frac{\pi}{4} \right) - \frac{3}{16} \frac{1}{\sqrt{Z} r} \cos \left( 2\sqrt{Z} r + \frac{\pi}{4} \right) \right] \left( 1 + O \left( \frac{1}{r^{1/2}} \right) \right),$$

and we have a full expansion in powers of $r^{-1/2}$. For $\sigma = 0$, set

$$u_{-0}(r; 0) = r^{1/2} J_1(2Z^{1/2} r^{1/2}) - i r^{1/2} Y_1(2Z^{1/2} r^{1/2}) = r^{1/2} H_1^{(2)}(2Z^{1/2} r^{1/2})$$
$$u_{+0}(r; 0) = r^{1/2} J_1(2Z^{1/2} r^{1/2}) + i r^{1/2} Y_1(2Z^{1/2} r^{1/2}) = r^{1/2} H_1^{(1)}(2Z^{1/2} r^{1/2}),$$

where $H_1^{(2)}$ and $H_1^{(1)}$ are the Hankel functions of the second kind.
where $H^{(1)}_1, H^{(2)}_1$ denote the Hankel functions of order one. These have the asymptotics

$$u_{\pm,0}(r; \sigma) = e^{\pm 2iZ^{1/2}r^{1/2} \frac{\pi}{2\sigma}} \frac{r^{1/4}}{\pi^{1/2}Z^{1/4}} \left( 1 + O_Z \left( \frac{1}{r^{1/2}} \right) \right)$$

in the $r \to \infty$ limit.

**Proposition A.6.** $u_{\pm,0}$ are the unique solutions to the $\sigma = 0$ case of eq. (668) satisfying the asymptotic eq. (685).

Expanding eq. (685) to higher order [AS64, §9.2]:

**Proposition A.7.** For each $K \in \mathbb{N}^+$,

$$u_{\pm,0}(r; 0) = e^{\pm 2iZ^{1/2}r^{1/2} \frac{\pi}{2\sigma}} \frac{r^{1/4}}{\pi^{1/2}Z^{1/4}} \sum_{k=0}^{K-1} (\pm i)^k Z^{-k/2} \frac{(2k + 1)!(2k)!}{64^k(k!)^4} \frac{1}{r^{k/2}} + O\left( \frac{1}{r^{K/2-1/4}} \right)$$

as $r \to \infty$.

One can prove:

**Proposition A.8.** Setting

$$C_{\pm}(\sigma) = \frac{Z^{1/2}}{2\pi\sigma} (\mp 2i\sigma)^{\mp 3/2} \Gamma \left( \mp \frac{1}{2}iZ \right)$$

and $u_{\pm,0}(r; \sigma) = C_{\pm}(\sigma) u_{\pm,00}(r; \sigma)$, the functions $u_{\pm,0}(r; E^{1/2}) : [0, \infty)_E \times (0, \infty)_r \to \mathbb{C}$ are both smooth all the way down to $E = 0$. Consequently, if

$$C_{\pm,0}(\sigma) = e^{\pm \pi i/4} \pi^{-1/2} \sigma^{-1/2} \exp \left[ \pm \frac{Z}{\sigma} \left( \log \left( \frac{2\sigma}{Z^{1/2}} \right) + \frac{1}{2} \right) \right],$$

then $C_{\pm,0}(\sigma) u_{\pm,00}(r; \sigma) : [0, \infty)_\sigma \times (0, \infty)_r \to \mathbb{C}$ are smooth all the way down to $\sigma = 0$, with restriction $u_{\pm,0}$ (as defined by eq. (683), eq. (684)) to $\sigma = 0$.

Remark. The $C^0$ case of this proposition is similar to [AS64, 13.3.4 and 13.3.5], except that our $\kappa$ is purely imaginary rather than purely real. See also [Bat53, §6.13.3, Eq. (21) - (24)]. Of course, if we were to multiply $C_{\pm}(\sigma)$ by any element of $C^\infty[0, \infty)_E$, the resultant functions would also satisfy the proposition above, and likewise with $C_{\pm,0}(\sigma)$ in place of $C_{\pm}(\sigma)$ and $C^\infty[0, \infty)_\sigma$ in place of $C^\infty[0, \infty)_E$.

Remark. Observe, using [AS64, p. 6.1.40], that the ratio $C_\pm/C_{\pm,0} : [0, \infty)_\sigma \to \mathbb{C}$ is a smooth function of $\sigma$, not $E = \sigma^2$. This is related to the fact that, for $u_{0,\pm}$ as in Theorem 1.1, $u_{0,\pm}\chi_{1} : [0, \infty)_\sigma \times \partial X \to \mathbb{C}$ is only smooth with respect to $\sigma$, in contrast to $u_{0,\pm}|_{x=\varepsilon} : [0, \infty)_\sigma \times \partial X \to \mathbb{C}$ for $\varepsilon \in (0, \varepsilon)$, which is smooth with respect to $E$.

Up to the $\exp(\pm \pi i/4)\pi^{-1/2}$ in eq. (688), eq. (684) is suggested by Theorem 1.1, as, for each $\sigma > 0$,

$$\Phi(x; \sigma) = \frac{\sigma}{x} - \frac{Z}{2\sigma} \log x + \frac{Z}{2\sigma^2} \log \left( \frac{2\sigma}{Z^{1/2}} \right) + O_\sigma(x)$$

as $x \to 0^+$, while the phase of the bracketed term in eq. (677) consists only of the first two terms of eq. (69). The $\sigma^{-1/2}$ term in eq. (688) is needed to match the $r^{1/2}$ in eq. (683), eq. (684). This heuristic should be taken with a grain of salt, as the $O_\sigma(x)$ term blows up as $\sigma \to 0^+$. 

Figure 8. The real and imaginary parts of the function $C_-(\sigma)$ defined by eq. (687), rescaled so that the oscillations have constant amplitude.

Assuming that there exist some $C_\pm$ satisfying the conclusion of Proposition A.8, the sufficiency of the formula eq. (688) can be seen from the $r \to 0^+$ asymptotics of the Whittaker functions. Indeed, the ODE eq. (668) can be written in the form

$$\left(r \frac{\partial}{\partial r}\right)^2 u - \left(r \frac{\partial}{\partial r}\right)u + (\sigma^2 r^2 + Zr)u = 0,$$

(690)

which has a regular singular point at $r = 0$. The normal operator of $(r \frac{\partial}{\partial r})^2 - (r \frac{\partial}{\partial r}) + \sigma^2 r^2 + Zr$ at $r = 0$ is $(r \frac{\partial}{\partial r})^2 - r \frac{\partial}{\partial r}$, the indicial roots of which are 0 and 1. We can deduce that a family $\{u(\cdot; \sigma)\}_{\sigma \geq 0} \subset D'(\mathbb{R}^+)$ of solutions to eq. (690) can be smooth at $\{r > 0, \sigma = 0\}$ only if it extends continuously to $[0, 1) \times [0, 1)$, in which case $\lim_{\sigma \to 0^+} \lim_{r \to 0^+} u = \lim_{r \to 0^+} \lim_{\sigma \to 0^+} u$ exists. We now observe that

$$u_{\pm,00}(r; \sigma) = \frac{2i}{Z} (\mp 2i\sigma)^{\mp \frac{Z}{2}} \frac{\sigma}{\Gamma(\mp iZ/2\sigma)} + O(r \log r),$$

(691)

$$\lim_{r \to 0^+} u_{\pm,00}(r; \sigma) = \frac{2i}{Z} (\mp 2i\sigma)^{\mp \frac{Z}{2}} \frac{\sigma}{\Gamma(\mp iZ/2\sigma)}$$

(692)

for each $\sigma > 0$. We now compare eq. (692) with

$$\lim_{r \to 0^+} u_{\pm,0}(r; 0) = \mp \frac{i}{\pi Z^{1/2}} + O(r \log r)$$

(693)

[AS64, p. 9.1.9]. This yields $C_\pm(\sigma) = -(2\pi)^{-1} Z^{1/2}(\mp 2i\sigma)^{\pm \frac{Z}{2}} \Gamma(\mp iZ/2\sigma)$, which is Equation (687). A more thorough analysis of the ODE near $r = 0$ suffices to prove Proposition A.8 properly. Once we know that $C_\pm$ satisfies the first clause of the conclusion of Proposition A.8, the second clause of the proposition follows from the large argument asymptotics of $\log \Gamma$, which can be found in [AS64, p. 6.1.40].

Letting $u_{\pm,0}(r; \sigma)$ be as in Proposition A.8. Then, letting $\chi \in C_c^\infty[0, \infty)$ be identically one in some neighborhood of the origin, $w = \chi(1/r) u_{\pm,0}(r; \sigma)$ satisfies

$$\frac{\partial^2 w}{\partial r^2} + (\sigma^2 + \frac{Z}{r}) w = f$$

(694)

for some $f \in \cap_{E_0 > 0} C_c^\infty([0, E_0] \times \mathbb{R}^+)$. We can deduce (e.g. by appealing to Theorem 1.1 plus Remark 1 in the case of spherical symmetry) that $w_\pm$ has the form $w_\pm = \exp(\pm i\Phi(r^{-1}; E^{1/2}))(E + \ldots)$.
As a corollary of Proposition A.9, analogous to Corollary 1.3. The Whittaker W-function \( \text{WhittW}_{\kappa, 1/2} \) satisfies

\[
\text{WhittW}_{-iZ/2\kappa, 1/2}(2i\kappa r) \in C^\infty(X_{\text{res}}^\kappa),
\]

where \( X = [0, \infty) \times [0, \infty) \times \{0\} \times \{0\} \times \{0\} \). Thus, \( u_{+0}(r, \sigma) \) defined by eq. (676), eq. (677) satisfy

\[
\begin{aligned}
\Gamma(\rho Z^2/2\kappa) &\in r^{-1/4} e^{\rho Z^2/4\kappa} e^{-i\rho Z^2/2\kappa} (Z_{\rho}^{1/2}/2\kappa) Z^{1/2} C^\infty([0, \infty) \times [0, \infty) \times [0, \infty) \times [0, \infty) \times [0, \infty)).
\end{aligned}
\]
Figure 10. The real (in orange) and imaginary (in blue) parts of the function \( \exp(-\pi Z^{1/2}/4\varsigma) \text{Whitt}_{n,1/2}(2i\varsigma r^{1/2}) \) as a function of \( r^{1/2} \), for \( \varsigma = 1/2 \) and \( Z = 3 \) fixed, where \( \kappa = -iZr^{1/2}/2\varsigma \). We can see from this plot that \( \text{Whitt}_{n,1/2}(2i\varsigma r^{1/2}) \) is oscillating as \( r^{1/2} \to \infty \), with exponentially growing amplitude.

Figure 11. The real and imaginary parts of the function \( \varsigma^{-1/2} \exp(-\pi Z^{1/2}/4\varsigma) \exp(-ir^{1/2}\varphi) \text{Whitt}_{n,1/2}(2i\varsigma r^{1/2}) \) as a function of \( r^{1/2} \), for \( \varsigma \in \{.05,.25,5,1,2\} \) and \( Z = 3 \). The real parts are converging to \( \approx .7, .8 \), with the \( \varsigma \)-dependent limit converging as \( \varsigma \to 0^+ \), and the imaginary parts are converging to what appears to be zero (or at least something small).

Thus, if we let

\[
\varphi = -\left(\varsigma^2 + Z\right)^{1/2} + \frac{Z}{2\varsigma} - \frac{Z}{\varsigma} \frac{1}{2} \log \left(\frac{Zr^{1/2}}{2\varsigma}\right) + \log \left(\frac{\varsigma}{Z^{1/2}} + \left(1 + \frac{\varsigma^2}{Z}\right)^{1/2}\right),
\]

(699)

then, for each \( \varsigma > 0 \), \( e^{-\pi Z^{1/2}/4\varsigma} e^{-ir^{1/2}\varphi} \text{Whitt}_{n,1/2}(2i\varsigma r^{1/2}) \) is a smooth function of \( r^{-1/2} \), all the way down to \( r^{-1/2} = 0 \). The convergence aspect of this result, in particular the fact that the multiplication by \( \exp(-ir^{1/2}\varphi) \) kills off the oscillations of \( \exp(-\pi Z^{1/2}/4\varsigma) \text{Whitt}_{n,1/2}(2i\varsigma r^{1/2}) \), is depicted in Figure 11 (and the contrast with Figure 10, where the \( \exp(-ir^{1/2}\varphi) \) factor is missing).
The $C^0$ case of this is similar to [Bat53, §6.13.3, Eq. (21) - (24)] (though we did not compute out the leading order term in the asymptotic expansion).

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