Algebraic modular forms - from computation to insights

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What are algebraic modular forms?

~~What are modular forms?~~ Why algebraic modular forms?

- Easy to describe - no analysis required.
- Easy to compute (Lansky-Pollack 2002, Loeffler, 2014, Greenberg-Voight, 2014).
- Transfer to modular forms we care about.
- Yes, we care about modular forms!
- Classical modular forms, Hilbert modular forms, Siegel modular forms,...
- Applications to modularity.

Algebraic modular forms

- Automorphic forms for compact groups G.[1]
- $K \subseteq G(\widehat{\mathbb{Q}}) \implies \operatorname{Cls} K = G(\mathbb{Q}) \backslash G(\widehat{\mathbb{Q}}) / K$ is finite.
- To a weight $\rho : G(\mathbb{Q}) \to \mathrm{GL}(W)$ we associate

$$M_\rho(K) = \left\{ f : G(\widehat{\mathbb{Q}})/K \to W \mid f(\gamma \widehat{g}) = \gamma f(\widehat{g}) \right\}$$

- If $\rho = 1_F$, then $M_1(K) = F^{\operatorname{Cls} K}$.
- Hecke operators

$$K \widehat{g} K = \bigsqcup \widehat{g}_i K \implies T(\widehat{g}) f(\widehat{x}) = \sum f(\widehat{x} \widehat{g}_i)$$

Example (Jacquet-Langlands, 1970)

If $G = B^\times$, $\operatorname{disc} B = D$, $K = \widehat{O}^\times$, $\operatorname{disc} O = N$, then $\operatorname{Cls} K = \operatorname{Cls} O$, and for $\rho_k = \mathrm{Sym}^{k-2}(\mathbb{C}^2)$, then $S_{\rho_k}(\widehat{O}) \hookrightarrow S_k(N)^{D-new}$.

[1] $G^{\mathrm{ad}}(\mathbb{R})$ compact

First applications

- Algorithm for computing Hilbert modular forms (Dembélé-Voight, 2013).
- Applications -
  - Instances of modularity for HMF (Oda's conjecture).
  - Abelian surfaces with good reduction (Dembélé, 2022).
- Advances -
  - Classification of Hilbert modular surfaces (A.-Babei-Breen-Costa-(Duque-Rosero)-Horawa-Kieffer-Kulkarni-Molnar-Schiavone-Voight, 2023).
  - Equations for Hilbert modular varieties (A.-Babei-Breen-Chari-Costa-Dembélé-(Duque-Rosero)-Horawa-Kieffer-Kulkarni-Molnar-Mudigonda-Musty-Schiavone-Sethi-Tripp-Voight*)

Orthogonal modular forms

$G = O(V)$ for a positive definite quadratic space of $\dim_F V = n$, $K = O(\widehat{\Lambda})$ for an integral lattice Λ, then $\mathrm{Cls}\, K = \mathrm{Cls}\, \Lambda$.

- $n = 2$ - Hecke characters (Gauss, 1801, Matchett Wood, 2011).
- $n = 3$ - Hilbert modular forms (Birch 1989, Hein 2016, Hein-Tornaría-Voight*).

$$S_{\rho_k \otimes \theta_d}(O(\widehat{\Lambda})) \simeq S_k(N; w_d)^{D-new},$$

where $\mathrm{disc}\, \Lambda = N$, and for $d \mid D$,

$$\theta_d : O(V) \to \mathbb{Q}^\times / \mathbb{Q}^{\times 2} \to \{\pm 1\}$$

is a spinor character, w_d corr. Atkin-Lehner eigenvalues.

$$\mathfrak{so}_3 \simeq \mathfrak{sl}_2$$

$$\bullet \quad B_1 = A_1 \quad \bullet$$

Rank 4

$$\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$$

$$\bullet\bullet \quad D_2 = A_1 \times A_1 \quad \bullet\bullet$$

Expect $S_\rho(O(\widehat{\Lambda}))$ are Hilbert modular forms over $[K : F] = 2$.

Theorem (A.-Fretwell-Ingalls-Logan-Secord-Voight 2022)

If $\mathrm{Cl}(F) = 1$, $K = F[\sqrt{\operatorname{disc}\Lambda}]$ *and* $\operatorname{disc}\Lambda = D_0 N^2$, *then*

$$S_{\rho_k \otimes \theta_d}(O(\widehat{\Lambda})) \simeq \mathrm{Gal}_{K/F} \backslash S_{k_1,k_2}(N\mathbb{Z}_K; w_d)^{\mathrm{Cl}^+(K)[2], D-new}.$$

Further, if $\phi \mapsto f$, *then* $L(\phi, s) = L_{\mathrm{Asai}}(f, s)$.

Extends the case where $N = \square$ done in (Ponomarev, 1976, Böcherer-(Schulze-Pillot), 1996).

Application to non-vanishing of theta maps.

Rank 5

$$\mathfrak{so}_5 \simeq \mathfrak{sp}_4$$

$$\bullet - \bullet \quad B_2 = C_2 \quad \bullet - \bullet$$

Expect $S_\rho(O(\widehat{\Lambda}))$ are Siegel (para)modular forms.

Theorem (Dummigan-Pacetti-Rama-Tornaría, 2021)

When $F = \mathbb{Q}$,

$$S_{\rho_{k,j} \otimes \theta_d}(O(\widehat{\Lambda}))_{(G)} \simeq S_{k,j}(N; w_d)_{(G)}^{D-new}.$$

Uses Jacquet-Langlands for Sp_4 (Rösner-Weisshauer, 2023).

Theorem (A., Ladd, Rama, Tornaría and Voight,)

There exists an algorithm to compute $S_{k,j}(N)_{(G)}$ for $k \geq 3$, if there exists a prime $p \parallel N$.

Application to modularity of CY 3-folds.

Even higher rank

Natural family of theta maps

$$\theta^{(g)} : S(O(\widehat{\Lambda})) \to S^{(g)}_{n/2}(N, \chi_{D^*}).$$

Theorem (A.-Fretwell-Ingalls-Logan-Secord-Voight 2022)

If n is even, $F = \theta^{(g)}(\phi) \neq 0$ with $2g < n$, and $m = \frac{n}{2} - 1$, then

$$L(\phi, s) = L(\chi_{D^*} \otimes F, \mathrm{std}, s - m) \prod_{i=g-m}^{m-g} \zeta(s + i - m).$$

- Deduce relations between eigensystems.
- Application to proving (generalized) Eisenstein congruences, leading to generalizations of Harder's conjecture. (Atobe-Chida-Ibukiyama-Katsurada-Yamauchi, 2023)
- Case $N = 1$ treated in (Chenevier-Lannes 2019).

Computation $\Longrightarrow$ insights

- Algebraic modular forms are useful for computing Hecke eigensystems.
- Access to L-functions and Galois representations.
- Computations inform us about the theory.
- Future work - exceptional isomorphism in rank 6, algebraic modular forms for exceptional groups.