The Siegel Modular Variety

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Structure of the talk

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1. Symplectic Spaces

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2. Siegel Modular Variety

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Symplectic Spaces

Definition

Symplectic space *is a pair* (V, ψ) *with* V *a* k*-vector space and* $\psi : V \times V \to k$ *a* **symplectic form***.*

- *bilinear.*
- *alternating -* $\psi(v, v) = 0$*.*
- *nondegenerate -* $\psi(u, V) = 0 \Rightarrow u = 0$*.*

Remark

$\dim(V) = 2n$

Definition

$W \subseteq V$ *is* **totally isotropic** *if* $\psi(W, W) = 0$*.*

Definition

Symplectic basis *is a basis* B *such that*

$$[\psi]_B = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

Symplectic Spaces

Lemma (Milne, 6.1)

$W \subseteq V$ totally isotropic, B_W a basis of W. Then there is a symplectic basis B_V of V extending B_W.

Proof.

By induction on n. Identify $W^\vee$ with the complement of $W^\perp$ via $v \mapsto \psi(v, -)$. The dual basis to B_W gives a symplectic basis of $W \oplus W^\vee$. By the induction hypothesis, $(W \oplus W^\vee)^\perp$ also has a symplectic basis. □

Corollary

- *Any two symplectic spaces of the same dimension are isomorphic.*
- *V has a symplectic basis.*

Symplectic Groups

Definition

- *The* **symplectic group** $Sp(\psi)$

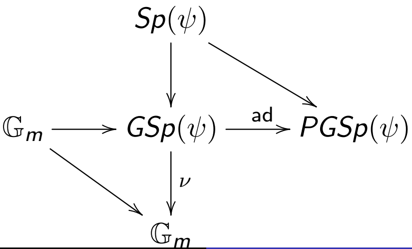
$$Sp(\psi)(k) = \{g \in GL(V) | \psi(gu, gv) = \psi(u, v)\}$$

- *The* **group of symplectic similitudes** $GSp(\psi)$

$$GSp(\psi)(k) = \{g \in GL(V) | \psi(gu, gv) = \nu(g) \cdot \psi(u, v), \nu(g) \in k^{\times}\}$$

- *The* **group of projective symplectic similitudes** $PGSp(\psi)$

$$PGSp(\psi)(k) = GSp(\psi)(k)/k^{\times}$$



Shimura Datum

Reductive group

Let $G(\psi) = GSp(\psi)$. Note that $G(\psi)^{\text{ad}} = PGSp(\psi)$ and $G(\psi)^{\text{der}} = Sp(\psi)$.

$G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$

For J a complex structure on $V(\mathbb{R})$ s.t. $\psi_{\mathbb{R}}(Ju, Jv) = \psi_{\mathbb{R}}(u, v)$, i.e. $J \in Sp(\psi)(\mathbb{R})$ such that $J^2 = -1$, let $\psi_J(u, v) := \psi_{\mathbb{R}}(u, Jv)$. Write
$X(\psi)^+ = \{J \in Sp(\psi)(\mathbb{R}) \mid \psi_J > 0\}$,
$X(\psi)^- = \{J \in Sp(\psi)(\mathbb{R}) \mid \psi_J < 0\}$ and $X(\psi) = X(\psi)^+ \bigsqcup X(\psi)^-$.

Lemma

The map $J \mapsto h_J : X(\psi) \to Hom(\mathbb{C}^\times, G(\mathbb{R}))$ defined by $h_J(a + bi) = a + bJ$ identifies X with a $G(\mathbb{R})$-conjugacy class in $Hom(\mathbb{C}^\times, G(\mathbb{R}))$.

Proof.

For $z = a + bi$, we have

$$\psi(h_J(z)u, h_J(z)v) = \psi(au + bJu, av + bJv) =$$
$$= a^2\psi(u, v) + b^2\psi(Ju, Jv) + ab(\psi(u, Jv) + \psi(Ju, v))$$

But $\psi(Ju, Jv) = \psi(u, v)$ and

$$\psi(Ju, v) = \psi(J^2u, Jv) = \psi(-u, Jv) = -\psi(u, Jv)$$

hence

$$\psi(h_J(z)u, h_J(z)v) = (a^2 + b^2)\psi(u, v) = z\bar{z} \cdot \psi(u, v)$$

In particular, $h_J(z) \in G(\mathbb{R})$. For $g \in G(\mathbb{R})$

$$\psi_{gJg^{-1}}(u, v) = \psi(u, gJg^{-1}v) = \nu(g)\psi(g^{-1}u, J(g^{-1}v)) =$$
$$= \nu(g)\psi_J(g^{-1}u, g^{-1}v)$$

so that for $J \in X(\psi)$, $gJg^{-1} \in X(\psi)$. (Cont...) □

Shimura Datum

Proof.

Also, for $z = a + bi$

$$h_{gJg^{-1}}(z) = a + bgJg^{-1} = g(a + bJ)g^{-1} = gh_J(z)g^{-1}$$

It remains to show transitivity.

For that, let $B(\psi)$ be the set of symplectic bases of (V, ψ). We have $B(\psi) \to X(\psi)^+ : B \mapsto J_B$ defined for $B = (e_i)_{i=1}^{2n}$ by

$$J_B(e_i) = \begin{cases} e_{i+n} & 1 \leqslant i \leqslant n \\ -e_{i-n} & n+1 \leqslant i \leqslant 2n \end{cases}$$

Indeed, $J_B^2 = -1$, and B is orthonormal for ψ_J. This map is surjective (orthonormal basis for ψ_J) and equivariant:

$$J_{gB}(ge_i) = gJ_B(e_i) = gJ_Bg^{-1}(ge_i)$$

$Sp(\psi)(\mathbb{R})$ acts transitively on $B(\psi)$, hence on $X(\psi)^+$.

Finally, the map $g \in G(\mathbb{R})$ swapping e_i with e_{i+n} has $\nu(g) = -1$ and swaps $X(\psi)^+$ with $X(\psi)^-$. □

Shimura Datum

Proposition

The pair $(G(\psi), X(\psi))$ is a Shimura datum. It satisfies SV1-SV6.

(SV1)

For all $h \in X$, the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ defined by $\mathrm{Ad} \circ h$ is of type $\{(-1,1),(0,0),(1,-1)\}$.

Proof.

We have $\mathrm{Lie}(G_{\mathbb{R}}(\psi)) \subseteq \mathrm{Lie}(GL_{\mathbb{R}}(V)) = \mathrm{End}(V)$ and the action defined by $\mathrm{Ad} \circ h$ on $End(V)$ is

$$(zf)(v) = (h(z) \circ f \circ h(z)^{-1})(v)$$

Let $V(\mathbb{C}) = V^+ \oplus V^-$ hence $h(z)v = zv$ for $v \in V^+$ and $h(z)v = \bar{z}v$ for $v \in V^-$. We have

$$\mathrm{End}(V(\mathbb{C})) = \mathrm{End}(V^+) \oplus \mathrm{End}(V^-) \oplus \mathrm{Hom}(V^+, V^-) \oplus \mathrm{Hom}(V^-, V^+)$$

with actions by $1, 1, \bar{z}/z, z/\bar{z}$ respectively. □

Shimura Datum

(SV2)

For all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $G_{\mathbb{R}}^{\mathrm{ad}}$.

Proof.

$J = h(i)$. Let $\psi' : V(\mathbb{C}) \times V(\mathbb{C}) \to \mathbb{C}$ be the sesquilinear form defined by $\psi'(u, v) = \psi_{\mathbb{C}}(u, \bar{v})$. Then for $g \in G(\mathbb{C})$

$$\psi'(gu, J(J^{-1}\bar{g}J)v) = \psi'(gu, \bar{g}Jv) = \psi_{\mathbb{C}}(gu, g\overline{Jv}) = \psi'(u, Jv)$$

so that $\psi'_J(gu, \mathrm{ad}(J)(\bar{g})v) = \psi'_J(u, v)$, and ψ'_J is invariant under $G^{(\mathrm{ad}\,J)}$. Since ψ_J is symmetric and positive(negative)-definite, ψ'_J is Hermitian and positive(negative)-definite. Then $G^{(\mathrm{ad}\,J)} = U(\psi'_J)$ is a definite unitary group, hence compact. □

Shimura Datum

(SV3)

G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Proof.

The root system of $Sp(\psi)$ is irreducible, hence G^{ad} is $\mathbb{Q}$-simple. Finally, $PGSp(\mathbb{R})$ is not compact □

(SV4)

The weight homomorphism $w_X : \mathbb{G}_m \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$.

Proof.

$r \in \mathbb{R}^\times$ acts as r on both V^+ and V^-, so $w_X(r) = r$. □

Shimura Datum

(SV5)

The group $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$.

Proof.

$Z = \mathbb{G}_m$, and $\mathbb{Q}^\times$ is discrete in $\mathbb{A}_f^\times$. □

(SV6)

The torus Z° splits over a CM-field.

Proof.

$Z = \mathbb{G}_m$ is already split over $\mathbb{Q}$. □

The Siegel Modular Variety

Moduli space

$K \subseteq G(\mathbb{A}_f)$ compact open. $\mathcal{H}_K$ - set of triples $((W,h), s, \eta K)$ s.t.

- (W,h) is a rational Hodge structure of type $(-1,0),(0,-1)$.
- s or $-s$ is a polarization for (W,h).
- ηK is a K-orbit in $Hom_{\mathbb{A}_f}(V(\mathbb{A}_f), W(\mathbb{A}_f))$ with $s_{\mathbb{A}_f}(\eta(u),\eta(v)) = \nu(\eta)\cdot\psi_{\mathbb{A}_f}(u,v)$ for some $\nu(\eta) \in \mathbb{A}_f^\times$.

An isomorphism $((W,h), s, \eta K) \to ((W',h'), s', \eta' K)$ is an isomorphism $b : (W,h) \to (W',h')$ s.t.

- $s'(b(u), b(v)) = \mu(b) s(u,v)$ for some $\mu(b) \in \mathbb{Q}^\times$.
- $b \circ \eta K = \eta' K$.

Proposition (Milne, 6.3)

The set $\mathrm{Sh}_K(G(\psi), X(\psi))(\mathbb{C})$ *classifies the elements of* $\mathcal{H}_K$ *modulo isomorphism.*

The Siegel Modular Variety

Proof.

Let $a : W \to V$ be an isomorphism s.t. $\psi(a(u), a(v)) = \mu(a)s(u, v)$ for some $\mu(a) \in \mathbb{Q}^\times$. Then

$$\psi_{\mathbb{R}}(aJa^{-1}u, aJa^{-1}v) = \mu(a)s_{\mathbb{R}}(Ja^{-1}u, Ja^{-1}v) = \\ = \mu(a)s_{\mathbb{R}}(a^{-1}u, a^{-1}v) = \psi_{\mathbb{R}}(u, v)$$

and

$$\psi_{aJa^{-1}}(u, u) = \psi_{\mathbb{R}}(u, aJa^{-1}u) = \mu(a)s_{\mathbb{R}}(a^{-1}u, Ja^{-1}u) = \\ = \mu(a)s_J(a^{-1}u, a^{-1}u)$$

so that $(ah)(z) := a \circ h(z) \circ a^{-1} \in X$. From

$$\psi_{\mathbb{A}_f}((a \circ \eta)(u), (a \circ \eta)(v)) = \mu(a)s_{\mathbb{A}_f}(\eta(u), \eta(v)) = \\ = \mu(a)\nu(\eta)\psi_{\mathbb{A}_f}(u, v)$$

we see that $a \circ \eta \in G(\mathbb{A}_f)$. Define a map

$$\mathcal{H}_K \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K : ((W, h), s, \eta K) \mapsto [ah, a \circ \eta]_K$$

□

Proof.

If $a' : W \to V$ is another such isomorphism, then $a' \circ a^{-1} \in G(\mathbb{Q})$, hence $[ah, a \circ \eta]_K = [a'h, a' \circ \eta]_K$, so the map is independent of a.
If $b : ((W, h), s, \eta K) \to ((W', h'), s', \eta' K)$ is an isomorphism, and $a' : W' \to V$, then $a \circ b : W \to V$, and

$$((a \circ b)h)(z) = a \circ b \circ h(z) \circ b^{-1} \circ a^{-1} = a \circ h'(z) \circ a^{-1} = (ah')(z)$$

so that $[(a \circ b)h, a \circ b \circ \eta]_K = [ah', a \circ \eta']_K$.
Thus, the map factors through the equivalence relation. If $((W, h), s, \eta K)$ and $((W', h'), s', \eta' K)$ map to the same element, take $a : W \to V$ and $a' : W' \to V$. Then $(ah, a \circ \eta) = (a'h', a' \circ \eta' \circ k)$. Take $b = a^{-1} \circ a' : W' \to W$.
Surjectivity - $[h, g]_K$ is the image of $((V, h), \psi, gK)$. □

Abelian Varieties

Definition

An **Abelian variety** *over k is a proper connected group variety.*

Remark

group variety is smooth - translate the smooth locus.
connected group variety is geometrically connected - $1 \in G(k)$.

Lemma (Rigidity Lemma)

$\alpha : X \times Y \to Z$ with X proper and $X \times Y$ geom. irreducible.

$$\alpha(X \times \{y\}) = \{z\} = \alpha(\{x\} \times Y)$$

Then $\alpha(X \times Y) = \{z\}$.

Observations

May assume $k = \bar{k}$. X is connected. $\mathrm{pr} : X \times Y \to Y$ is closed.

Abelian Varieties

Proof.

Let U be an open affine nbd of z, consider $V = \text{pr}(\alpha^{-1}(Z \backslash U)$.
Then $y' \notin V \iff \alpha(X \times \{y'\}) \subseteq U$, so $y \in Y \backslash V$.
X is connected, proper and U is affine, so for $y' \in Y \backslash V$,
$\alpha(X \times \{y'\})$ is a point, so $\alpha(X \times (Y \backslash V)) = \{z\}$.
$X \times (Y \backslash V)$ is nonempty open, hence dense.But Z is separated. □

Corollary

Every regular map $\phi : A \to G$ from an abelian variety to a group variety is the composite of a homomorphism with a translation.

Proof.

May assume $\phi(e_A) = e_G$. Consider $\alpha : A \times A \to G$ given by
$\alpha(a_1, a_2) = \phi(a_1 a_2)\phi(a_2)^{-1}\phi(a_1)^{-1}$. Then
$\alpha(\{e_A\} \times A) = \{e_G\} = \alpha(A \times \{e_A\})$, hence constant, so
$\alpha(A \times A) = \{e_G\}$, hence $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$. □

Abelian Varieties

Corollary

Abelian varieties are abelian.

Proposition

Let X be a proper k-variety, K/k a field extension, $\mathscr{L}$ an invertible sheaf on X. If $\mathscr{L} \times_k K$ is trivial, so is $\mathscr{L}$.

Proof.

X proper, so $k \to H^0(X, \mathscr{O}_X)$ is an isomorphism. Thus $\mathscr{L}$ is trivial iff $H^0(X, \mathscr{L}) \otimes_k \mathscr{O}_X \to \mathscr{L}$ is an isomorphism. □

Theorem (Theorem of the cube)

Let X, Y be proper and Z geometrically connected. If $\mathscr{L}|_{\{x\} \times Y \times Z}$, $\mathscr{L}|_{X \times \{y\} \times Z}$, $\mathscr{L}|_{X \times Y \times \{z\}}$ are trivial, then so is $\mathscr{L}$.

Proof.

Reduce to $k = \bar{k}$. Let Z' be the maximal closed subscheme of Z over which $\mathscr{L}$ is trivial. Enough to show that it contains an open nbd of z. Let $m \subseteq \mathscr{O}_{Z,z}$ be the maximal ideal and I the ideal defining Z' at z. If $I \neq 0$, there is n s.t. $I \subseteq m^n$ but $I \nsubseteq m^{n+1}$. Let $J_1 = m^{n+1} + I$, and $m^{n+1} \subseteq J_2 \subseteq J_1$ s.t. $\dim_k(J_1/J_2) = 1$, hence $J_1 = J_2 + k \cdot a$ for some $a \in J_1$. Let $J_0 = m$, and let $Z_i = \operatorname{Spec}(\mathscr{O}_{Z,z}/J_i)$. Then $I \subseteq J_0, J_1$, so $Z_0, Z_1 \subseteq Z'$.

From $0 \longrightarrow k \xrightarrow{\times a} \mathscr{O}_{Z,z}/J_2 \xrightarrow{\text{res}} \mathscr{O}_{Z,z}/J_1 \longrightarrow 0$, we get

$0 \longrightarrow \mathscr{L}_0 \xrightarrow{\times a} \mathscr{L}_2 \xrightarrow{\text{res}} \mathscr{L}_1 \longrightarrow 0$, where $\mathscr{L}_i = \mathscr{L}|_{X \times Y \times Z_i}$.
Let $\lambda : \mathscr{O}_{X \times Y \times Z_1} \to \mathscr{L}_1$ be a trivialization. It is enough to lift $\lambda(1)$ to $\mathscr{L}_2$. Obstruction is $\xi \in H^1(X \times Y, \mathscr{O}_{X \times Y})$. By assumption the images of ξ in $H^1(X, \mathscr{O}_X)$ and $H^1(Y, \mathscr{O}_Y)$ vanish. Künneth formula then yields $\xi = 0$, contradiction. □

Abelian Varieties

Corollary

$\mathscr{L}$ invertible sheaf on A abelian variety. The sheaf

$$p_{123}^{\star}\mathscr{L} \otimes p_{12}^{\star}\mathscr{L}^{-1} \otimes p_{23}^{\star}\mathscr{L}^{-1} \otimes p_{13}^{\star}\mathscr{L}^{-1} \otimes p_1^{\star}\mathscr{L} \otimes p_2^{\star}\mathscr{L} \otimes p_3^{\star}\mathscr{L}$$

on $A \times A \times A$ is trivial. ($p_{ij} = p_i + p_j$)

Proof.

Restrict to $A \times A \times \{0\}$ to get

$$m^{\star}\mathscr{L} \otimes m^{\star}\mathscr{L}^{-1} \otimes p_2^{\star}\mathscr{L}^{-1} \otimes p_1^{\star}\mathscr{L}^{-1} \otimes p_1^{\star}\mathscr{L} \otimes p_2^{\star}\mathscr{L} \otimes \mathscr{O}_{A\times A}$$

This is trivial, and by symmetry so are the other two. □

Corollary

$f, g, h : X \to A$ regular, A abelian variety, then

$$(f+g+h)^{\star}\mathscr{L}\otimes(f+g)^{\star}\mathscr{L}^{-1}\otimes(g+h)^{\star}\mathscr{L}^{-1}\otimes(f+h)^{\star}\mathscr{L}^{-1}\otimes f^{\star}\mathscr{L}\otimes g^{\star}\mathscr{L}\otimes h^{\star}$$

is trivial.

Proof.

Pullback through $(f, g, h) : X \to A \times A \times A$. □

Theorem (Theorem of the square)

$\mathscr{L}$ invertible sheaf on A. For all $a, b \in A(k)$

$$t_{a+b}^{\star}\mathscr{L} \otimes \mathscr{L} \cong t_a^{\star}\mathscr{L} \otimes t_b^{\star}\mathscr{L}$$

Proof.

Let $f, g, h : A \to A$ be $f(x) = x, g(x) = a, h(x) = b$. □

Remark

Equivalently, the map $a \mapsto t_a^{\star}\mathscr{L} \otimes \mathscr{L}^{-1} : A(k) \to Pic(A)$ is a homomorphism. In terms of divisors, if $D_a = D + a$, then $a \mapsto [D_a - D]$ is a homomorphism, so if $\sum a_i = 0$, $\sum D_{a_i} \sim nD$.

Abelian Varieties

Theorem (Weil)

Abelian varieties are projective.

Proof.

Assume first $k = \bar{k}$. Start by finding prime divisors Z_i such that $\bigcap Z_i = \{0\}$ and $\bigcap T_0(Z_i) = \{0\}$. If $0 \neq P \in A$, let U be an open affine nbd of 0, and $u \in U \cap (U + P)$. Then $U' = U + P - u$ is an open affine nbd of 0 and P. Identify U' with a closed subset of $\mathbb{A}^n$, take a hyperplane H passing through 0 but not through P. Take $Z_1 = \overline{H \cap U'}$ in A. If $0 \neq P' \in Z_1$, find Z_2 passing through 0 but not P'. By d.c.c. this process is finite. Next, let $t \in T_0(P)$ be s.t. $t \in T_0(Z_i)$ for all i. Take an open affine nbd U of P, embed it in $\mathbb{A}^n$ and choose a hyperplane through 0 s.t. $t \notin H$. Add $Z = \overline{H \cap U}$ to the set. Again, this process is finite. □

Abelian Varieties

Proof.

Let $D = \sum Z_i$ Then for any $S = \{a_1, \ldots, a_n; b_1, \ldots, b_n\} \subseteq A$, we have

$$D_S = \sum (Z_{i,a_i} + Z_{i,b_i} + Z_{i,-a_i-b_i}) \sim \sum 3Z_i = 3D$$

Let $a \neq b \in A$. May assume $b - a \notin Z_1$. Set $a_1 = a$. Then Z_{1,a_1} passes through a but not b. The sets

$$\{b_1 \mid b \in Z_{1,b_1}\}, \{b_1 \mid b \in Z_{1,-a_1-b_1}\}$$

are proper closed subsets of A, so can choose b_1 in neither. Similarly, can choose a_i, b_i such that none of $Z_{i,a_i}, Z_{i,b_i}, Z_{i,-a_i-b_i}$ passes through b. Thus, $a \in D_S$ but $b \notin D_S$, so the linear system of $3D$ separates points. Similarly, we see that it separates tangents, and so it is very ample, showing that A is projective. Finally, since A has an ample divisor iff $A_{\bar{k}}$ has an ample divisor, we are done. □

Complex Abelian Varieties

Differential geometry

$A(\mathbb{C})$ has a complex structure as a submanifold of $\mathbb{P}^n(\mathbb{C})$. It is a complex manifold which is compact, connected and has a commutative group structure. We may consider the exponent map.

Proposition

A abelian variety of dimension g over $\mathbb{C}$. Then $\exp : T_0(A(\mathbb{C})) \to A(\mathbb{C})$ *is surjective, and its kernel is a full lattice.*

Proof.

Let $H = \text{Im}(\exp)$. It is a subgroup of $A(\mathbb{C})$. exp is a local isomorphism, hence there is some open nbd of 0, $U \subseteq H$. Then for any $h \in H$, $h + U$ is an open nbd of h in H, so H is open. It is open and closed, and $A(\mathbb{C})$ is connected, showing surjectivity. Now there is some U with $U \cap Ker(exp) = 0$. Therefore the kernel is discrete. It must be a full lattice for the quotient to be compact. □

Corollary

$A(\mathbb{C}) \cong \mathbb{C}^n/\Lambda$ for some full lattice Λ.

Theorem

Let $M = \mathbb{C}^n/\Lambda$. There are canonical isomorphisms

$$\bigwedge^r H^1(M,\mathbb{Z}) \to H^r(M,\mathbb{Z}) \to \mathrm{Hom}\left(\bigwedge^r \Lambda, \mathbb{Z}\right)$$

Proof.

The cup product is the left map. Künneth formula shows that if its an isomorphism for all r for both X and Y, then it is also for $X \times Y$. But it holds for S^1. For the right map, $\mathbb{C}^n$ is s.c. hence a universal covering space, and $\pi_1(M) = \Lambda$, so that $H^1(M,\mathbb{Z}) \cong \mathrm{Hom}(\Lambda,\mathbb{Z})$. Use the perfect pairing $\det(f_i(e_j))$. □

Complex Abelian Varieties

Definition

Riemann form *for* M *is an alternating form* $\psi : \Lambda \times \Lambda \to \mathbb{Z}$ *s.t.* $\psi_{\mathbb{R}}(Ju, Jv) = \psi_{\mathbb{R}}(u, v), \psi_{\mathbb{R}}(u, Ju) > 0$.
M *is* **polarizable** *if it admits a Riemann form.*

Definition

Let $\psi : \Lambda \times \Lambda \to \mathbb{Z}$ *be an alternating form s.t.* $\psi(Ju, Jv) = \psi(u, v)$.
Let $\psi'(u, v) = \psi_{\mathbb{R}}(Ju, v) + i \cdot \psi_{\mathbb{R}}(u, v)$ *be the Hermitian form.*
Let $\alpha : \Lambda \to U_1(\mathbb{R})$ *be s.t.* $\alpha(\lambda_1 + \lambda_2) = e^{i\pi\psi(\lambda_1,\lambda_2)}\alpha(\lambda_1)\alpha(\lambda_2)$.
Let $\mu(\lambda, v) = \alpha(\lambda) \cdot e^{\pi\psi'(v,\lambda)+\frac{1}{2}\pi\psi'(\lambda,\lambda)}$. *Let* $\mathscr{L}(\psi, \alpha)$ *be the quotient of* $\mathbb{C} \times V$ *by* $\lambda \cdot (z, v) = (\mu(\lambda, v) \cdot z, v + \lambda)$.

Theorem (Appell-Humbert)

Any line bundle $\mathscr{L}$ *on the complex torus* M *is isomorphic to* $\mathscr{L}(\psi, \alpha)$ *for a unique pair* (ψ, α).

Proof.

$\pi^\star \mathscr{L} \cong \mathbb{C} \times V$ has a natural action by Λ, lifting the translation.
Then $\lambda \cdot (z, v) = (\mu(\lambda, v) \cdot z, v + \lambda)$ for some $\mu(\lambda, -) \in H^0(V, \mathscr{O}_V^\times)$.
From $\lambda_1(\lambda_2(z, v)) = (\lambda_1 + \lambda_2)(z, v)$, get $\mu(\lambda_1 + \lambda_2, v) = \mu(\lambda_1, v + \lambda_2) \cdot \mu(\lambda_2, v)$.
This induces an isomorphism $H^1(M, \mathcal{O}_M^\times) \to H^1(\Lambda, H^0(V, \mathscr{O}_V^\times))$.
From exactness of the exponential sequence, we may write $\mu(\lambda, v) = e^{2\pi i f_\lambda(v)}$ for some holomorphic f_λ. Then the Chern class is $F \in H^2(\Lambda, \mathbb{Z})$ given by

$$F(\lambda_1, \lambda_2) = f_{\lambda_2}(v + \lambda_1) - f_{\lambda_1+\lambda_2}(v) + f_{\lambda_1}(v)$$

Now, the map $A : H^2(\Lambda, \mathbb{Z}) \to \operatorname{Hom}\left(\bigwedge^2 \Lambda, \mathbb{Z}\right)$ defined by $AF(\lambda_1, \lambda_2) = F(\lambda_1, \lambda_2) - F(\lambda_2, \lambda_1)$ is an isomorphism, and $A(a \cup b) = a \wedge b$. We get the form $\psi(\lambda_1, \lambda_2) = f_{\lambda_2}(v + \lambda_1) + f_{\lambda_1}(v) - f_{\lambda_1}(v + \lambda_2) - f_{\lambda_2}(v)$. □

Complex Abelian Varieties

Proof.

Since ψ is in the image of $H^1(M, \mathscr{O}_M^\times) \to H^2(M, \mathbb{Z})$, its image in $H^2(M, \mathscr{O}_M)$ vanishes. This factors through the $\mathbb{R}$-linear extension $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C}) \to H^2(M, \mathscr{O}_M)$. Let $Hom_{\mathbb{R}}(V, \mathbb{C}) = T \oplus \bar{T}$, then

$$H^2(M, \mathbb{C}) \cong \bigwedge^2 (T \oplus \bar{T}) \cong \left(\bigwedge^2 T \right) \oplus \left(\bigwedge^2 \bar{T} \right) \oplus (T \otimes \bar{T})$$

Write $\psi_{\mathbb{R}} = \psi_1 + \psi_2 + \psi_3$. Since it is real, $\psi_1 = \bar{\psi_2}$, and from vanishing under the projection onto the second factor, we see that $\psi = \psi_3$ which precisely means $\psi(ix, iy) = \psi(x, y)$. Conversely, given such ψ, we can set (and these are all the linear solutions)

$$f_\lambda(v) = \frac{1}{2i} \psi'(v, \lambda) + \beta_\lambda$$

s.t.

$$\frac{1}{2} \psi'(\lambda_1, \lambda_2) + i\beta_{\lambda_1} + i\beta_{\lambda_2} - i\beta_{\lambda_1 + \lambda_2} \in i\mathbb{Z}$$

Proof.

Write $i\beta_\lambda = \gamma_\lambda + \frac{1}{4}\psi'(\lambda,\lambda)$, to reduce to

$$\gamma_{\lambda_1} + \gamma_{\lambda_2} - \gamma_{\lambda_1+\lambda_2} + \frac{1}{2} i\psi(\lambda_1,\lambda_2) \in i\mathbb{Z}$$

Modifying by a coboundary, we may assume that $\gamma \in i\mathbb{R}$. Write $\alpha(\lambda) = e^{2\pi\gamma_\lambda}$. Then $|\alpha(\lambda)| = 1$ and

$$\alpha(\lambda_1+\lambda_2) = e^{i\pi\psi(\lambda_1,\lambda_2)}\alpha(\lambda_1)\alpha(\lambda_2)$$

This establishes that every $\mathscr{L}(\psi,\alpha)$ is a line bundle and conversely, that every line bundle is of this form. It remains to establish uniqueness. It follows from

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom(\Lambda, S^1) & \longrightarrow & \{(\psi,\alpha)\} & \longrightarrow & \{\psi : \psi(\Lambda\times\Lambda) \subseteq \mathbb{Z}\} & \longrightarrow & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(M) & \longrightarrow & \mathrm{Pic}(M) & \longrightarrow & \ker(H^2(M,\mathbb{Z}) \to H^2(M,\mathscr{O}_M)) & \longrightarrow & \end{array}$$

□

Sections of $\mathscr{L}(\psi, \alpha)$

These lift to holomorphic functions $\theta : V \to \mathbb{C}$ s.t.

$$\theta(v + \lambda) = \alpha(\lambda) \cdot e^{\pi\psi'(v,\lambda) + \frac{1}{2}\pi\psi'(\lambda,\lambda)}\theta(v)$$

If ψ is degenerate, for $\lambda \in Rad(\psi) \cap \Lambda$ we have $\theta(v + \lambda) = \alpha(\lambda)\theta(v)$. Recall that $|\alpha(\lambda)| = 1$, so θ is bounded, hence constant on cosets mod $Rad(\psi)$, and $\alpha(Rad(\psi)) = 1$. But then θ factors through $Rad(\psi)$, so $\mathscr{L}(\psi, \alpha)$ can't be ample.
Next, assume $\psi|_W < 0$, then for $w \in W$

$$\Re\psi'(v_0 + w - \lambda, \lambda) + \frac{1}{2}\psi(\lambda, \lambda) = \Re\psi'(v_0 + w - \lambda, w) -$$

$$- \Re\psi'(v_0 + w - \lambda, w - \lambda) + \frac{1}{2}\psi'(w, w) + \frac{1}{2}\psi'(w - \lambda, w - \lambda) -$$

$$- \Re\psi'(w, w - \lambda) = \frac{1}{2}\psi'(w, w) + \Re\psi'(v_0, w) + f(w - \lambda, v_0)$$

Sections of $\mathscr{L}(\psi, \alpha)$

Then this tends to $-\infty$, showing that $\theta = 0$, so $H^0(\mathscr{L}(\psi, \alpha)) = 0$. Thus, we must have ψ positive-definite.

Proposition (Mumford, p.26)

If $\psi > 0$ *then* $\dim H^0(A, \mathscr{L}(\psi, \alpha)) = \sqrt{\det \psi}$.

Proof.

Rough idea - in $\mu(\lambda, v)$, the exponent is linear in v. Multiply by $e^{Q(v)}$ for a suitable quadratic form Q to have periodicity w.r.t. a big sublattice Λ'. The Fourier coefficients then determine the dimension. □

Theorem (Lefschetz)

$\mathscr{L}(\psi, \alpha)$ *is ample if and only if* $\psi > 0$.

Proof.

If $\theta \in H^0(\mathscr{L}(\psi, \alpha))$, then $\phi_{a,b}(v) = \theta(v-a)\theta(v-b)\theta(v+a+b)$ is a section of $\mathscr{L}(3\psi, \alpha^3)$. Thus, for any v, we get a section of $\mathscr{L}(3\psi, \alpha^3)$ not vanishing at v, showing it is base-point free.
For injectivity, taking log derivatives of $\phi_{a,b}(v_1) = c \cdot \phi_{a,b}(v_2)$ we get $\pi\psi(v_2 - v_1, \lambda) - l(\lambda) \in 2\pi i\mathbb{Z}$ for some linear form $l(\lambda)$. This then shows that θ is a theta function for $\Lambda + (v_2 - v_1) \cdot \mathbb{Z}$.
Counting dimensions, we get a contradiction.
Injectivity on the tangent space is shown in a similar manner. □

Corollary (Milne, Theorem 6.7)

The complex torus $\mathbb{C}^n/\Lambda$ is projective iff it is polarizable.
By Chow's theorem, this is iff it is an algebraic variety. Riemann form for M = polarization of the integral Hodge structure Λ.

Complex Abelian Varieties

Theorem (Riemann,..., see Milne, Theorem 6.8)

$A \rightsquigarrow H_1(A, \mathbb{Z})$ is an equivalence of catgeories

$$\underbrace{\left\{ \begin{array}{c} \textit{abelian varieties} \\ \textit{over } \mathbb{C} \end{array} \right\}}_{AV} \leftrightarrow \left\{ \begin{array}{c} \textit{polarizable integral Hodge} \\ \textit{structures of type } (-1,0),(0,-1) \end{array} \right\}$$

Corollary (Milne, Corollary 6.9)

$A \rightsquigarrow H_1(A, \mathbb{Q})$ is an equivalence of categories

$$\underbrace{\left\{ \begin{array}{c} \textit{abelian varieties} \\ \textit{over } \mathbb{C} \textit{ with} \\ Hom_{AV^0}(A,B) = \\ = Hom_{AV}(A,B) \otimes \mathbb{Q} \end{array} \right\}}_{AV^0} \leftrightarrow \left\{ \begin{array}{c} \textit{polarizable rational Hodge} \\ \textit{structures of type} \\ (-1,0),(0,-1) \end{array} \right\}$$

Modular Siegel Variety

Moduli Space

(V, ψ) symplectic space over $\mathbb{Q}$. $\mathcal{M}_K$ is triples $(A, s, \eta K)$ where

- A is an abelian variety over $\mathbb{C}$, as object in AV^0.
- s or $-s$ is a polarization on $H_1(A, \mathbb{Q})$.
- $\eta : V(\mathbb{A}_f) \to T_f(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is s.t. $s(\eta(u), \eta(v)) = \psi(u, v)$.

Theorem (Milne, Theorem 6.11)

The set $\mathrm{Sh}_K(\mathbb{C})$ *classifies the elements of* $\mathcal{M}_K$ *modulo isom.*

Specific Level Example

Assume $\exists$ a $\mathbb{Z}$-lattice $V(\mathbb{Z})$ in V s.t. $\det(\psi|_{V(\mathbb{Z})}) = \pm 1$.

$$K(N) = \left\{ g \in G(\mathbb{A}_f) \mid g(V(\hat{\mathbb{Z}})) = V(\hat{\mathbb{Z}}), g \equiv 1 \bmod NV(\hat{\mathbb{Z}}) \right\}$$

Modular Siegel Variety

Specific Level Example

Let $\Gamma(N) = K(N) \cap Sp(\psi)(\mathbb{Z})$. Then

$$\Gamma(N) = \{g \in Sp(\psi)(\mathbb{Z}) \mid g \equiv 1 \bmod NV(\mathbb{Z})\}$$

The components are

$$\pi_0(\mathrm{Sh}_{K(N)}(G, X)) = T(\mathbb{Q})^{\dagger} \backslash T(\mathbb{A}_f) / \nu(K(N)) \cong (\mathbb{Z}/N\mathbb{Z})^{\times}$$

$Sp(\psi)$ satisfies Hasse principle, hence $\mathrm{Sh}_{K(N)}(\mathbb{C})^{\circ} = \Gamma(N) \backslash X^{+}$.
If $\lambda : A \to A^{\vee}$ is p.p., it induces a perfect alternate pairing

$$e_N^{\lambda} : A(\mathbb{C})[N] \times A(\mathbb{C})[N] \to \mu_N$$

a **level N structure** is

$$\eta_N : V(\mathbb{Z}/N\mathbb{Z}) \to A(\mathbb{C})[N]$$

s.t. ψ_N is a multiple of e_N^{λ}.
Then $\mathrm{Sh}_K(\mathbb{C})$ classifies (A, λ, η_N), and $\mathrm{Sh}_K(\mathbb{C})^{\circ}$ are those for which ψ_N corresponds exactly to e_N^{λ}.