THE NEWTON STRATIFICATION ON THE FLAG VARIETY

1. Where From, Where to

1.1. **What we have so far.** Recall that we have proved the following, in Ben's talk (May 29th)

Theorem 1.1. *([Caraiani and Scholze, 2017], Theorem 1.10)*

Let (G, X) be a Shimura datum of Hodge type with reflex field E. Let p be a prime, and $\mathfrak{p} \mid p$ a place of E. Then for any sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$, there is a perfectoid space $\mathcal{S}_{K^p}$ over $E_{\mathfrak{p}}$, such that

$$\mathcal{S}_{K^p} \sim \varprojlim_{K_p} \mathcal{S}_{K_p K^p}$$

where $\mathcal{S}_K$ is the adic space corresponding to $S_K \otimes_E E_{\mathfrak{p}}$. Moreover, there is a Hodge-Tate period map

$$\pi_{HT} : \mathcal{S}_{K^p} \to \mathscr{F}l_{G,\mu}$$

which is functorial in the Shimura datum.

Why is that helpful?

We can now analyze the cohomology of $\mathcal{S}_{K^p}$ using a Leray spectral sequence.

Since on the left we have a moduli space of abelian varieties, and on the right a moduli space of p-divisible groups, (Dan's talk), the fibres of π_{HT} should be thought of as a moduli space of abelian varieties with trivialization of their p-divisible group, i.e. Igusa varieties (to be discussed next week - Pol and Raffael). Therefore, we can compute the fibres of $R(\pi_{HT})_*\mathbb{Z}_l$ in terms of cohomology of Igusa varieties.

1.2. **What we want to do next.** We would like to identify the fibres of π_{HT} with Igusa varieties. This will be done as follows:

(1) We will recall the definition of the Newton stratification on the special fibre of a Shimura variety, parametrized by the finite subset $B(G, \mu) \subset B(G)$ of Kottowitz's set of isocrystals with G-structure (to be defined).
(2) Define a natural stratification on $\mathscr{F}l_{G,\mu}$, which corresponds under π_{HT} to the Newton stratification (pulled back from the special fibre).
(3) For any $b \in B(G, \mu^*)$, we find a p-divisible group $\mathbb{X}_b$ over $\overline{\mathbb{F}}_p$ equipped with a certain extra endomorphism and polarization structures.
(4) We will be able to find a perfect scheme Ig^b over $\overline{\mathbb{F}}_p$ which parametrizes abelian varieties A with extra structures, equipped with an isomorphism $\rho : A[p^\infty] \cong \mathbb{X}_b$.

In this talk we will discuss the first two steps, in particular prove the following:

Theorem 1.2. *([Caraiani and Scholze, 2017], Theorem 1.11)*

Let G be a reductive group over $\mathbb{Q}_p$, and μ a conjugacy class of miniscule cocharacters. There is a natural decomposition $\mathscr{F}l_{G,\mu} = \coprod_{b \in B(G,\mu^*)} \mathscr{F}l_{G,\mu}^b$ into locally closed subsets $\mathscr{F}l_{G,\mu}^b$. The union

$$\coprod_{b \preceq b'} \mathscr{F}l_{G,\mu}^{b'}$$

is closed for all $b \in B(G, \mu^*)$; In particular, $\mathscr{F}l_{G,\mu}^b$ is open when b is the basic element of $B(G, \mu^*)$.

The other two will be discussed next week.

Remark 1.3. This is opposite to the closure relations on the Shimura variety. However, we do not prove that the closure of a stratum is a union of strata. This is related to a subtle behaviour of π_{HT} on certain higher rank points of the adic space (**).

2. Motivation - Modular Curve Case

2.1. **The Modular Curve.** We start by reviewing what we know for the case of the modular curve.

Recall that the special fibre of the modular curve $X_0(p)$ (draw a picture) has a supersingular locus, which is closed, yielding a stratification into two strata (locally closed subsets) - the supersingular stratum and the ordinary stratum, where the closure of the open ordinary stratum is their union (i.e. everything), and the supersingular stratum is closed.

In this case, $\mathscr{F}l_{G,\mu}$ is just the adic projective line $\mathbb{P}^{1,ad}$.

Recall from Chris's talk that we know the following fact:

Fact 2.1. *(*[Scholze, 2015]*, Lemma III.3.19)*
The preimage of $\mathscr{F}l_{G,\mu}(\mathbb{Q}_p) \subset \mathscr{F}l_{G,\mu}$ under π_{HT} is given by the closure of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)$.

In the case of the infinite level modular curve, this is just the ordinary locus (essentially by definition - where the Hasse invariant is invertible), so $\pi_{HT}^{-1}(\mathbb{P}^1(\mathbb{Q}_p)) = \mathcal{X}^{ord}_{\Gamma(p^\infty)}$, and for points $(E/C, \alpha : \mathbb{Z}_p^2 \to T_pE)$ in this locus, there exists a unique canonical subgroup, and the Hodge-Tate period map

$$\pi_{HT}(E, \alpha) = 0 \to (\alpha \otimes 1)^{-1}(\mathrm{Lie}E) \to C^2$$

just measures its position.

The rest, i.e. the supersingular locus is mapped onto Drinfeld's upper half plane $\Omega^2 = \mathbb{P}^1 \backslash \mathbb{P}^1(\mathbb{Q}_p)$.

Note that, in contrast to the classical stratification on the modular curve, in the stratification on the perfectoid modular curve the supersingular locus is open, while the ordinary locus is closed.

3. The structure of $B(G)$

Let G be a connected reductive group over $\mathbb{Q}_p$. Let k be an algebraically closed field of characteristic p. Let $L = W(k)\left[\frac{1}{p}\right]$, with Frobenius σ. Let $\Gamma = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Recall the following definition from Ashwin's talk.

Definition 3.1. We call $B(G) = G(L)/\sim_\sigma$, where $x \sim_\sigma y \iff x = gy\sigma(g)^{-1}, \quad g \in G(L)$ the *Kottwitz Set* of G. (This classifies isocrystals with G-structure)

Remark 3.2. Note that k does not appear in the notation. This is justified by the following Lemma.

Lemma 3.3. *(*[Rapoport and Richartz, 1996]*, Lemma 1.3) Let $k' \subset k$ be an algebraically closed subfield. Let $L^{'} = W(k')\left[\frac{1}{p}\right]$, σ' its Frobenius, and $B^{'}(G) = G(L')/\sim_{\sigma'}$. Then the natural map $B'(G) \to B(G)$ is a bijection.*

Example 3.4. If $G = GL_n$, then these are just isocrystals of height n, and by Dieudonne-Manin classification they can be classified by their slopes. That is, if $(L^n, b\sigma)$ admits a slope decomposition as $\bigoplus_i V_{\lambda_i}$, with $\lambda_i = r_i/s_i$ and $\lambda_1 \geq \lambda_2 \geq \dots$ then we can consider the vector

$$(\underbrace{\lambda_1, \dots, \lambda_1}_{s_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{s_2}, \dots) \in \mathbb{Q}^n_{dom} = X_*(GL_n) \otimes \mathbb{Q}/S_n$$

Since $X_*(GL_n)$ is an ordered set, this induces an ordering on $B(GL_n)$, which will allow us later to define strata by mapping to $B(G)$. Basically, this will be called the Newton map.

However, we would like to define this map for arbitrary G. Although it is possible to just use a faithful representation and pullback, we would like some functorial map with nice properties.

For that we will need the following. Let $W(\overline{L}/\mathbb{Q}_p)$ be the Weil group, i.e. the group of continuous automorphisms which fix $\mathbb{Q}_p$ and induce on the redisue field k an integral power of Frobenius.

Lemma 3.5. *There is a natural injective map $H^1(\mathbb{Q}_p, G) \to B(G)$*

Proof. There is an exact sequence of topological groups

$$1 \to Gal(\overline{L}/L) \to W(\overline{L}/\mathbb{Q}_p) \to \langle\sigma\rangle \to 1$$

where $\langle\sigma\rangle$ denotes the infinite cyclic (discrete) group generated by σ. The Inflation-Restriction exact sequence gives us the induced map

$$0 \to B(G) = H^1(\langle\sigma\rangle, G(L)) \to H^1(W(\overline{L}/\mathbb{Q}_p), G(\overline{L})) \to H^1(Gal(\overline{L}/L), G(\overline{L}))^{\langle\sigma\rangle}$$

But by Steinberg's induced theorem, since G is connected and linear, and the cohomological dimension of L is at most 1, the term on the right vanishes, so we get a bijection.
The restriction homomorphism $W(\overline{L}/\mathbb{Q}_p) \to \Gamma$ and the inclusion $G(\overline{\mathbb{Q}}_p) \subseteq G(\overline{L})$ give us a map

$$H^1(\mathbb{Q}_p, G) \to H^1(W(\overline{L}/\mathbb{Q}_p), G(\overline{L}))$$

However, if F is a finite Galois extension of $\mathbb{Q}_p$ in $\overline{L}$, we have an exact sequence

$$1 \to W\left(\overline{L}/F\right) \to W(\overline{L}/\mathbb{Q}_p) \to Gal(F/\mathbb{Q}_p) \to 1$$

yielding the inflation-restriction exact sequence

$$0 \to H^1(Gal(F/\mathbb{Q}_p), G(F)) \to H^1(W(\overline{L}/\mathbb{Q}_p), G(\overline{L})) \to H^1(W(\overline{L}/F), G(\overline{L}))^{\langle\sigma\rangle}$$

hence injectivity of the above map. Combining the two we get the injective map as claimed. Note that one could realize this map by taking a cocycle $c : \Gamma \to G(\overline{\mathbb{Q}}_p) \hookrightarrow G(\overline{L})$, finding an equivalent cycle $c' : \Gamma \to G(L)$, and restricting it to a lift of σ. □

Let $\mathbb{D}$ be the pro-algebraic torus with character group $\mathbb{Q}$. The following theorem will establish the existence of such a Newton map, as we wanted.

Theorem 3.6. *([Kottwitz, 1985], Section 4) Let $b \in G(L)$. There exists a unique element $\nu_b \in Hom_L(\mathbb{D}, G)$ such that for any $\rho : G \to GL(V)$, $\rho \circ \nu_b$ is the slope decomposition of $(V_L, \Phi) := (V \otimes_{\mathbb{Q}_p} L, \rho(b) \circ (id_V \otimes \sigma))$. The element ν_b is called the slope homomorphism associated to b. It induces a map*

$$B(G) \to \mathcal{N}(G) := (Int(G(L))\backslash Hom_L(\mathbb{D}, G))^{\langle\sigma\rangle}$$

Furthermore, ν_b is trivial if and only if b is in the image of the map $H^1(\mathbb{Q}_p, G) \to B(G)$.

Proof. (Sketch) Let $b \in B(G)$. Then for any finite-dimensional $\mathbb{Q}_p$-representation $\rho : G \to GL(V)$, the pair $(V_L, \Phi) := (V \otimes_{\mathbb{Q}_p} L, \rho(b) \circ (id_V \otimes \sigma))$ is an isocrystal. Its slope decomposition gives us an element $\nu_\rho \in Hom_L(\mathbb{D}, GL(V))$. Let R be an L-algebra, and let $x \in \mathbb{D}(R)$. Let $\mathrm{Rep}_{\mathbb{Q}_p} G$ be the category of finite-dimensional representations $\rho : G \to GL(V)$. Then the elements $\nu_\rho(x)$ give an automorphism of the standard fiber functor of $\mathrm{Rep}_{\mathbb{Q}_p} G$. By Tannaka reconstruction ($End(RepG \to Vect) \cong k[G]$), there exists a unique element $y \in G(R)$ suc h that $\rho(y) = \nu_\rho(x)$ for all ρ. Moreover, the homomorphism $x \mapsto y$ is functorial in R and thus defines an element $\nu = \nu_b \in Hom_L(\mathbb{D}, G)$ such that $\rho \circ \nu_b = \nu_\rho$ for all ρ. This type of argumetnis called Tannakaian formalism, and will be used frequently. Kottwitz goes on to show that ν_b could be defined intrinsically, and this intrinsic description shows that σ-conjugation leads to conjugate weights, and that these are σ-invariant. □

Note that being invariant under $Gal(\overline{\mathbb{Q}}_p/L)$ and under σ is the same as being Γ-invariant.
Thus, for example, if $G = T$ is a torus, conjugation is trivial and

$$\mathcal{N}(T) = (Hom_L(\mathbb{D}, T))^{\langle\sigma\rangle} = Hom(\mathbb{D}, T)^\Gamma = X_*(T)^\Gamma \otimes \mathbb{Q}$$

More generally, if $T \subset G$ is a maximal torus, with Weyl group W, then any morphism $\mathbb{D} \to G$ factors through one of the maximal tori, hence through $N_G(T)$, and the conjugation factors through $W = N_G(T)/T$, so that

$$\mathcal{N}(G) = ((X_*(T) \otimes \mathbb{Q})/W)^\Gamma$$

Thus, we obtain a map $\nu_G : B(G) \to \mathcal{N}(G)$, which we call the Newton map.
It follows that the map $b \mapsto \nu_b$ induces a natural transformation of set-valued functors on the category of connected reductive algebraic groups

$$\nu : B(\cdot) \to \mathcal{N}(\cdot)$$

3.1. **A partial ordering on the set of Newton points.** Let T be a maximal torus, with $\Phi = \Phi(G, T)$ its set of roots, and fix a basis Δ for Φ. Let $\Delta^\vee$ be the corresponding basis for the set of coroots. Let

$$(X_*(T) \otimes \mathbb{Q})_{\mathrm{dom}} = \{x \in X_*(T) \otimes \mathbb{Q} \mid \langle x, \alpha \rangle \geq 0, \quad \forall \alpha \in \Delta\}$$

and

$$(X_*(T) \otimes \mathbb{Q})^\vee = \left\{ x \in X_*(T) \otimes \mathbb{Q} \mid x = \sum_{\alpha^\vee \in \Delta^\vee} n_{\alpha^\vee} \cdot \alpha^\vee, \quad n_{\alpha^\vee} \in \mathbb{Q}_{\geq 0} \right\}$$

be the corresponding *closed* Weyl chamber, and *obtuse* Weyl chamber.
Then $(X_*(T) \otimes \mathbb{Q})_{\mathrm{dom}}$ is a fundamental domain for the action of W on $X_*(T) \otimes \mathbb{Q}$, and we can be identify

$$\mathcal{N}(G) = (X_*(T) \otimes \mathbb{Q})_{\mathrm{dom}}^\Gamma$$

Let $W = N_G(T)/T$ be the Weyl group of G, T. We then have the following Lemma.

Lemma 3.7. *(*[Rapoport and Richartz, 1996]*, Lemma 2.2, rational version) Let $\nu, \nu' \in X_*(T) \otimes \mathbb{Q}$. The following conditions are equivalent.*

(i) ν lies in the convex hull of the finite set $\{w\nu' \mid w \in W\}$.

(ii) Let $\tilde{\nu}$, $\tilde{\nu}'$ be representatives in $(X_*(T) \otimes \mathbb{Q})_{dom}$ of ν, ν' respectively for the action of W. Then $\tilde{\nu}' - \tilde{\nu} \in (X_*(T) \otimes \mathbb{Q})^\vee$.

(iii) Let $\tilde{\nu}'$ be the representative of ν' in $(X_*(T) \otimes \mathbb{Q})_{dom}$. Then $\tilde{\nu}' - w\nu \in (X_*(T) \otimes \mathbb{Q})^\vee$ for all $w \in W$.

Definition 3.8. If the above equivalent conditions are satisfied, we say that $\nu \preceq \nu'$. This defines a partial ordering on $\mathcal{N}(G)$.

Lemma 3.9. *(*[Rapoport and Richartz, 1996]*, Lemma 2.2 (d)) $\nu \preceq \nu'$ iff for any representation $\rho : G \to GL(V)$, if $T' \subset GL(V)$ is a maximal torus containing $\rho(T)$, then $\rho \circ \nu \preceq \rho \circ \nu'$.*

At this point, we could have defined a partial ordering on $B(G)$, by pulling back through the Newton map. However, there are two problems with it:

1. Not every two elements are comparable - indeed, even for GL_n, we must have $\sum \nu_i = \sum \nu_i'$, e.g. $(0,0)$ and $(1,1)$ are incomparable.

You could argue: OK, but we can still order them by the sum.

2. The following example shows that it might not be enough.

Example 3.10. (The Newton map is not injective in general) Let $F/\mathbb{Q}_p$ be a finite extension. Let

$$G := U_1 = \ker(\mathrm{Nm}_{F/\mathbb{Q}_p} : \mathrm{Res}_{F/\mathbb{Q}_p}\mathbb{G}_m \to \mathbb{G}_m)$$

be the norm one elements. Then we have a SES $1 \to U_1 \to \mathrm{Res}_{F/\mathbb{Q}_p}\mathbb{G}_m \to \mathbb{G}_m \to 1$, giving rise to the long exact sequence of cohomology

$$1 \to U_1(\mathbb{Q}_p) \to F^\times \overset{Nm_{F/\mathbb{Q}_p}}{\to} \mathbb{Q}_p^\times \to H^1(\mathbb{Q}_p, U_1) \to H^1(\mathbb{Q}_p, \mathrm{Res}_{F/\mathbb{Q}_p}\mathbb{G}_m) = 0$$

where the last equality follows from Hilbert's Theorem 90. Therefore $H^1(\mathbb{Q}_p, U_1) = \mathbb{Q}_p^\times / Nm_{F/\mathbb{Q}_p}(F^\times)$ is nontrivial, showing that the Newton map ν_{U_1} has a nontrivial kernel, by Theorem 3.6. This could also be seen directly by looking at the slopes, as the weights of U_1 are the same as $[F : \mathbb{Q}_p]$-tuples corresponding to the different embedings $F \hookrightarrow \overline{\mathbb{Q}}_p$, restricted to elements of norm 1. Therefore, all tuples which sum to zero give a trivial weight for U_1.

We see that although in the GL_n case, the Newton map suffices to determine the original element (by Dieudonne-Manin), in general this map is not injective, and one needs another invariant, namely the Kottwitz map.

3.2. **The Kottwitz map.** Next, we would like to define the Kottwitz map.

For that we will need to define the algebraic fundamental group of G.

Definition 3.11. Let $T \subset G_{\overline{\mathbb{Q}}_p}$ be a maximal torus defined over $\overline{\mathbb{Q}}_p$. Let $\Phi(G,T)$ be the set of roots of T, and for $\alpha \in \Phi(G,T)$ we denote by $\alpha^\vee$ by the corresponding root. We denote by

$$\pi_1(G,T) := X_*(T) / \sum_{\alpha \in \Phi(G,T)} \mathbb{Z}\alpha^\vee$$

the *algebraic fundamental group of G with respect to T.*

Lemma 3.12. *$\pi_1(G,T)$ has an action of Γ. For any other maximal torus T', $\pi_1(G,T) \cong \pi_1(G,T')$ as Γ-modules.*

Remark 3.13. If we let $\rho : T^{sc} \to T$ be the simply connected cover of T, then one can write $\pi_1(G) = X_*(T)/\rho_* X_*(T^{sc})$, which gives a more conceptual definition. Ths functor is also exact.

Example 3.14. Let $G = T$ be a torus. Then $\pi_1(G) = X_*(T)$. We will describe in this case a natural map $X_*(T)_\Gamma \to B(T)$. Let F be such that T splits over F. Let F_0 be the largest unramified subfield of F. Consider $\mu \in X_*(T)$. We map μ to the σ -conjugacy class containing $\mathrm{Nm}_{F/F_0}(\mu(\pi_F))$, where π_F is a uniformizing element for F and Nm_{F/F_0} is the norm homomorphism $T(F) \to T(F_0)$. Note that $F_0 \subset L$. If $\gamma \in \Gamma$ and $\mu \in X_*(T)$, then $\gamma\mu$ and μ map to the same value, hence this map factors through the Γ-coinvariants.

This is in fact an isomorphism, and the inverse map generalizes well to a functorial construction. We first extend it to groups G such that G_{der} is simply connected via composition with $B(G) \to B(G/G_{der})$, since $\pi_1(G) = \pi_1(G/G_{der})$. Next, for arbitrary G, take a simply connected central extension $H \to G$, with kernel C. Then the diagram

$$\begin{array}{ccc} B(C) & \longrightarrow & \pi_1(C)_\Gamma \\ \downarrow & & \downarrow \\ B(H) & \longrightarrow & \pi_1(H)_\Gamma \end{array}$$

is commutative. Hence we get a map $B(G) \to \pi_1(G)_\Gamma$ (SES $1 \to C \to H \to G \to 1$).

Example 3.15. Let $G = GL_n$. Then $G_{der} = SL_n$ is simply connected, and $G/G_{der} \cong \mathbb{G}_m$ via the det map. The isomorphism $X_*(\mathbb{G}_m)_\Gamma \to B(\mathbb{G}_m)$ is given by mapping $\mu = t \mapsto t^m$ to $\mu(p) = p^m$. Therefore the map $B(G) \to \pi_1(G)_\Gamma \cong \mathbb{Z}$ is given by

$$[b] \mapsto [\det(b)] = [p^m] \mapsto m$$

i.e. $b \mapsto \mathrm{val}_p(\det(b))$.

The Kottwitz map is so useful since we have the following theorem.

Theorem 3.16. *([Kottwitz, 1997], 4.13) The map $(\nu, \kappa) : B(G) \to (X_*(G) \otimes \mathbb{Q})^\Gamma_{dom} \times \pi_1(G)_\Gamma$ is injective.*

(maybe just draw the diagram with exact rows that commutes, and induces bijections)

3.3. **The set $B(G, \mu)$.** Now, we can define a partial ordering on $B(G)$ by setting

$$b \preceq b' \iff \nu_b \preceq \nu_{b'}, \quad \kappa(b) = \mu^\flat$$

Note that on fibres of the Kottwitz map, we can compare elements, by definition!

Now, given a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$, we can identify it with an element of $X_*(T)_{dom}$. There is a natural map $X_*(T)_{dom} \to (X_*(T) \otimes \mathbb{Q})^\Gamma_{dom}$ given by

$$\mu \mapsto \overline{\mu} := \frac{1}{[F : \mathbb{Q}_p]} \sum_{\gamma \in \mathrm{Gal}(F/\mathbb{Q}_p)} \gamma(\mu)$$

where F is a field where T splits. There is also a natural quotient map $X_*(T) \to \pi_1(G)_\Gamma$, which we denote by $\mu \mapsto \mu^\flat$. (Why the notation - is there a goos reason!?)

Definition 3.17. The subset $B(G, \mu) \subset B(G)$ of *μ-admissible* elements is

$$B(G, \mu) := \left\{ b \in B(G) \mid \nu_b \preceq \overline{\mu}, \quad \kappa(b) = \mu^\flat \right\}$$

Example 3.18. Let $G = GL_n$. Then

$$\pi_1(G) = X_*(T) / \sum \mathbb{Z} \cdot \alpha^\vee \cong \mathbb{Z}^n / \sum_{i=1}^{n-1} \mathbb{Z} \cdot (e_i - e_{i+1}) \cong \mathbb{Z}$$

where the rightmost isomorphism is the map $(\mu_1, \mu_2, \ldots, \mu_n) \mapsto \mu_1 + \mu_2 + \ldots + \mu_n$. Therefore, we may identify $\pi_1(G) = \pi_1(G)_\Gamma$ with $\mathbb{Z}$, and set $\mu^\flat = \mu_1 + \mu_2 + \ldots + \mu_n$.

Also, $\nu \preceq \mu$ means that there exist $a_1, \ldots, a_{n-1} \geq 0$ such that

$$\mu - \nu = \sum_{i=1}^{n-1} a_i (e_i - e_{i+1}) = (a_1, a_2 - a_1, a_3 - a_2, \ldots, a_{n-1} - a_{n-2}, -a_{n-1})$$

so that

$$\begin{aligned} a_1 &= \mu_1 - \nu_1 \\ a_2 &= \mu_1 + \mu_2 - \nu_1 - \nu_2 \\ &\vdots \\ a_{n-1} &= \mu_1 + \ldots + \mu_{n-1} - \nu_1 - \ldots - \nu_{n-1} \end{aligned}$$

Therefore, $b \in B(G,\mu)$ if and only if "the Newton polygon lies above the Hodge polygon": (draw!)

$$\begin{gathered}\nu_1 \le \mu_1 \\ \nu_1+\nu_2 \le \mu_1+\mu_2 \\ \vdots \\ \nu_1+\ldots+\nu_{n-1} \le \mu_1+\ldots+\mu_{n-1} \\ \nu_1+\ldots+\nu_n = \mu_1+\ldots+\mu_n\end{gathered}$$

4. Newton Stratification on (special fibers) of Shimura Varieties

4.1. **Short Review of Shimura Varieties.** The stratification we have seen on the special fibre of the modular curve is only a special case of the much more general Newton stratification on special fibres of Shimura varieties.

Let (G,X) be a Shimura datum. Let $K^p \subset G(\mathbb{A}_f^p)$ a sufficiently small open compact subgroup, and let K_p be maximal. Let $K = K^pK_p \subset G(\mathbb{A}_f)$, and $Sh_K = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ the corresponding Shimura variety. Let E be the reflex field, and let v be a prime of E above p.

Let S_K be the canonical integral model of $Sh_K \otimes_E E_v$ over $\mathcal{O}_{E_v}$, and let $\overline{S}_K$ its special fiber, over $\kappa(v)$.

Recall that $S_K(R)$ can be viewed as classifying (semi-)abelian schemes over R with some additional structure.

Let $\mathcal{A}_{univ}$ be the universal abelian scheme over $\overline{S}_K$, and let $H = \mathcal{A}_{univ}[p^\infty]$ its p-divisible group. Then H has a $G_{\mathbb{Q}_p}$-structure.

After choosing a local embedding at p of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$, $\nu : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, any Hodge cocharacter $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$ determines a local version $\nu \circ \mu_h : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$, which respects the conjugation action.

Therefore X induces a minuscule conjugacy class of cocharacters $\mu_{\overline{\mathbb{Q}}_p} : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$.

Let $L = W(\overline{\mathbb{F}}_p)\left[\frac{1}{p}\right]$, and assume for simplicity that G is connected reductive, quasi-split and μ is unramified (so that it will be defined over L).

The group H is compatible with $\mu_{\overline{\mathbb{Q}}_p}$.

To each geometric closed point $x \in \overline{S}_K$, we associate the the fibre H_x, and the Dieudonne module $\mathbb{D}(H_x)$, $N_x = \mathbb{D}H_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the isocrystal of H_x.

Therefore, the isocrystal N_x determines a unique element $b_x \in B(G_{\mathbb{Q}_p})$. This induces a map $b : \overline{S}_K \to B(G_{\mathbb{Q}_p})$.

4.2. **What is the upshot of all of this?** We can now use the map $b : \overline{S}_K \to B(G_{\mathbb{Q}_p})$, and the partial ordering on $B(G_{\mathbb{Q}_p})$ to define

$$\overline{S}_K(\preceq b) = \left\{x \in \overline{S}_K \mid b_x \preceq b\right\}$$

Theorem 4.1. *(Grothendieck's Specialization Theorem,* [Rapoport and Richartz, 1996]*, Theorem 3.6) Let $b \in B(G)$. Then $\overline{S}_K(\preceq b)$ is a (Zariski-)closed subscheme of $\overline{S}_K$ (endowed with the induced reduced structure).*

Remark 4.2. This is true in general for any isocrystal with G-structure over a scheme.

Definition 4.3. The closed subscheme $\overline{S}_K(\preceq b)$ is the *closed Newton stratum* associated with b. The *Newton Stratum associated with b is the open subscheme of $\overline{S}_K(\preceq b)$ defined by*

$$\overline{S}_K(b) = \left\{x \in \overline{S}_K \mid b_x = b\right\}$$

Moreover, the compatibilty of H with $\mu_{\overline{\mathbb{Q}}_p}$ shows that the strata $\overline{S}_K(b)$ are empty for $b \notin B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}}_p})$. Thus, the strata are parameterized by the set $B(G,\mu)$. However, even though it is conjectures, it is unknown in general whether the strata with $b \in B(G,\mu)$ are nonempty. Viehmann and Wedhorn has proved it for PEL type Shimura varieties, and there are some other results.

5. Newton Stratification on the Adic Flag Variety

The rough idea of the stratification (as in the classical case) is to consider a point in the flag variety as a point in the affine Grassmannian, which is a G-bundle with a trivialization over the generic fiber.

These will correspond, by Fargue's theorem, to isocrystals with G-structure, which are classified by their Newton polygons. (Basically, Dieudonne-Manin)

5.1. **The mixed characteristic affine Grassmannian.** Let (R, R^+) be a perfectoid affine algebra over $\mathbb{Q}_p$. (Here it is defined as a uniform adic Banach algebra with a surjective Frobenius on $R^+/(p)$, and the existence of $x \in R^+$ such that $x^p - p \in p^2R^+$. This only depends on R, and when R contains a perfectoid field, it agrees with the usual definition - take a topologically nilpotent element$a$, $b^p \equiv a \mod p$, $c^p \equiv p/a \mod p$, take the units, and find $d^p \equiv u_1^{-1}u_2^{-1}$, so that $x = bcd$ is the element)

Recall we have a surjective map $\theta : W(R^{\flat+}) \to R^+$ (namely, $\sum [r_n]p^n \mapsto \sum r_n^\sharp p^n$), whose kernel is generated by a non-zero divisor $\xi \in W(R^+)$ (e.g. SW Lemma 6.2.8 - look for $\xi = p + [\pi]\alpha$).

Then $\mathbb{B}^+_{dR,R}$ is defined as the ξ-adic completion of $W(R^{\flat+})[1/p]$, and $\mathbb{B}_{dR.R} = \mathbb{B}^+_{dR,R}[\xi^{-1}]$. Since $W(R^{\flat+})[1/p]/\ker\theta \cong R^+[1/p] = R$, we see that these rings are independent of R^+.

Definition 5.1. Let $Gr_G^{B^+_{dR}}$ be the functor associating to any perfectoid affinoid $\mathbb{Q}_p$-algebra (R, R^+) the set of G-torsors over $\mathrm{Spec}\mathbb{B}^+_{dR,R}$ trivialized over $\mathrm{Spec}\mathbb{B}_{dR,R}$, up to isomorphism.

Example 5.2. Let $G = GL_n$. Then G-torsors are vector bundles of rank n, which in turn are just finite projective $\mathbb{B}^+_{dR,R}$-modules of rank n, Λ, such that $\Lambda \otimes_{\mathbb{B}^+_{dR,R}} \mathbb{B}_{dR,R} \cong \mathbb{B}^n_{dR,R}$ (trivialization), i.e. $\mathbb{B}^+_{dR,R}$-lattices $\Lambda \subset \mathbb{B}^n_{dR,R}$ with $\Lambda[\xi^{-1}] = \mathbb{B}^n_{dR,R}$.

If $(R, R^+) = (K, K^+)$ where K is a perfectoid field, then $\mathbb{B}^+_{dR,K} \cong K[[\xi]]$ (as abstract rings) is a complete discrete valuation ring. In this case, this is our familiar affine Grassmanian, which admits a loop space interpretation:

$$\mathrm{Gr}_G^{B^+_{dR}}(K, K^+) = G(\mathbb{B}_{dR,K})/G(\mathbb{B}^+_{dR,K})$$

In particular, choose $K = C$ algebraically closed, and fix an embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. Then using Cartan's decomposition

$$G(\mathbb{B}_{dR,C}) = \coprod_{\mu \in X_*(T)_{dom}} G(\mathbb{B}^+_{dR,C})\mu(\xi)^{-1}G(\mathbb{B}^+_{dR,C})$$

Here $\mu(\xi)$ is defined for $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ via $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{B}^+_{dR,C}$. Thus, we can map each element $x \in \mathrm{Gr}_G^{B^+_{dR}}(C, \mathcal{O}_C)$ to its (inverse) Schubert cell $\mu(x) \in X_*(T)_{dom}$.

We would like now to begin with a given conjugacy class of Hodge cocharacters $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ and identify a corresponding flag variety with the Schubert cell in the affine Grassmannian. That way, points on the flag variety will give rise to G-bundles.

Definition 5.3. Let E be the field of definition of μ. μ (its conjugacy class) determines an ascending filtration $\mathrm{Fil}_\bullet(\mu)$ on $\mathrm{Rep}_{\overline{\mathbb{Q}}_p} G$ by setting

$$\mathrm{Fil}_m(\mu)(V) = \bigoplus_{m' \geq -m} V^{\mu = m'}$$

Let $\mathscr{F}l_{G,\mu}/E$ be the rigid-analytic flag variety parametrizing such filtrations. The choice of μ identifies $\mathscr{F}l_{G,\mu} = G/P_\mu$, where $P_\mu \subset G$ is the stabilizer of $\mathrm{Fil}_\bullet(\mu)$.

Example 5.4. (maybe replace with GL_n and arbitrary μ)˙Let $G = GL_2$, $\mu = z \mapsto \begin{pmatrix} z & \\ & 1 \end{pmatrix}$. Letting $V = \overline{\mathbb{Q}}_p^2$ be the standard representation of G, we see that $V^{\mu=1} = \overline{\mathbb{Q}}_p \cdot e_1$, $V^{\mu=0} = \overline{\mathbb{Q}}_p \cdot e_2$. Therefore

$$\mathrm{Fil}_m(\mu)(V) = \begin{cases} 0 & m \leq -2 \\ \overline{\mathbb{Q}}_p \cdot e_1 & m = -1 \\ V & m \geq 0 \end{cases}$$

Thus $P_\mu = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ is the standard Borel of upper triangular matrices, fixing e_1.

Definition 5.5. Let $\mathrm{Gr}_{G,\mu}^{B^+_{dR}} \subset \mathrm{Gr}_G^{B^+_{dR}} \times_{\mathbb{Q}_p} E$ be the subfunctor on perfectoid affinoid E-algebras

$$\mathrm{Gr}_{G,\mu}^{B^+_{dR}}(R, R^+) = \left\{ x \in \mathrm{Gr}_{G,E}^{B^+_{dR}}(R, R^+) \mid \mu(x) = \mu \right\}$$

(Note a subtlety - want it here for all the points in the adic spectrum of R, R^+, because we only defined it for the case $(R, R^+) = (C, \mathcal{O}_C)$ an algebraically closed field.)

Proposition 5.6. *([Caraiani and Scholze, 2017], Proposition 3.4.3) There is a natural map (Bialynicki-Birula)*

$$\pi_{G,\mu} : Gr_{G,\mu}^{B_{dR}^+} \to \mathscr{F}l_{G,\mu}$$

where we regard $\mathscr{F}l_{G,\mu}$ as a functor on perfectoid affinoid E-algebras.

Proof. Enough to prove for $G = GL_n$. Write $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. Recall that the functor $\mathrm{Gr}_{GL_n}^{B_{dR}^+}$ parametrizes $\mathbb{B}_{dR,R}^+$-lattices $\Lambda \subset \mathbb{B}_{dR,R}^n$. This gives rise to a filtration on R^n by setting

$$\mathrm{Fil}_m R^n = \left((\mathbb{B}_{dR,R}^+)^n \cap \xi^{-m}\Lambda\right) / \left(\left(\xi\mathbb{B}_{dR,R}^+\right)^n \cap \xi^{-m}\Lambda\right)$$

Recall also that we have

$$\mathbb{B}_{dR,R}^+/\xi\mathbb{B}_{dR,R}^+ \cong \frac{W(R^{\flat+})[1/p]}{\ker(\theta)} \cong R^+[1/p] = R$$

so this is indeed an ascending filtration on R^n. Moreover, the lattice Λ corresponding to μ is given by

$$\mu(\xi)^{-1} = \begin{pmatrix} \xi^{-\mu_1} & & & \\ & \xi^{-\mu_2} & & \\ & & \ddots & \\ & & & \xi^{-\mu_n} \end{pmatrix} \cdot \left(\mathbb{B}_{dR,R}^+\right)^n$$

Therefore, we see that

$$\mathrm{rank}(\mathrm{Fil}_m R^n) = \begin{cases} 0 & m < -\mu_1 \\ i & -\mu_i \leq m < \mu_{i+1} \\ n & -\mu_n \leq m \end{cases}$$

which is a filtration parametrized by $\mathscr{F}l_{G,\mu}$. □

(One should verify also that $R^n/\mathrm{Fil}_m R^n$ is a finite projective R-module (Why?). This is achieved by [Kedlaya and Liu, 2013], proposition 2.8.4 - a finitely generated module M with a continuous rank function $x \mapsto \dim_{k(x)}(M \otimes_R k(x))$ is projective. These are definitley finitely generated, and $\dim_C(R^n/\mathrm{Fil}_m R^n \otimes_R C) = \dim_C(C^n/\mathrm{Fil}_m C^n)$ is constant (depends on μ by the formula above)).

This is nice, but we would like to go the other way. For that we need:

Lemma 5.7. *([Caraiani and Scholze, 2017], Lemma 3.4.4) Assume that μ is minuscule, and that $(R, R^+) = (K, K^+)$, where K/E is a perfectoid field. Then*

$$\pi_{G,\mu} : Gr_{G,\mu}^{B_{dR}^+}(K, K^+) \to \mathscr{F}l_{G,\mu}(K, K^+)$$

is a bijection.

Proof. In this case, $\mathbb{B}_{dR,R}^+$ is a complete DVR with residue field K, hence by the Cohen structure theorem, we may choose an isomorphism $\mathbb{B}_{dR,R}^+ \cong K[[\xi]]$. Then we get the usual affine Grassmannian, and this is known ([NP], Lemma 2.3) basically the fact that μ is minuscule shows that the preimage of the unipotent under the natural map $L^+G \to G$ is contained in G, hence the inverse image of P_μ is $L^+G \cap \mu(\xi)^{-1}(L^+G)\mu(\xi)$, deducing the isomorphism. □

Theorem 5.8. *([Caraiani and Scholze, 2017], Lemma 3.4.5) Assume that μ is minuscule. Then the Bialynicki-Birula morphism*

$$\pi_{G,\mu} : Gr_{G,\mu}^{B_{dR}^+} \to \mathscr{F}l_{G,\mu}$$

is an isomorphism.

Proof. First, we check injectivity. Take $x, y \in \mathrm{Gr}_{G,\mu}^{B_{dR}^+}(R, R^+)$ such that $\pi_{G,\mu}(x) = \pi_{G,\mu}(y)$. We use (again) Tannakian formalism. Thus, for any $(\rho, V) \in \mathrm{Rep}G$, $\rho(x), \rho(y)$ correspond to lattices $\Lambda_{V,x}, \Lambda_{V,y} \subset V \otimes \mathbb{B}_{dR}$. Thus, it is enough that we show $\Lambda_{V,x} = \Lambda_{V,y}$ for all V. However, at any point $z \in Spa(R, R^+)$ with completes residue field $K(z)$, we have by Lemma 5.7 that

$$\Lambda_{V,x} \otimes_{\mathbb{B}_{dR,R}^+} \mathbb{B}_{dR,K(z)}^+ = \Lambda_{V,y} \otimes_{\mathbb{B}_{dR,R}^+} \mathbb{B}_{dR,K(z)}^+$$

which establishes the result. (Indeed, let $a \in \Lambda_x$. Then there is a minimal $m \geq 0$ such that $a \in \xi^{-m}\Lambda_y$. If $m > 0$, a induces a nonzero element $\overline{a} \in \xi^{-m}\Lambda_y/\xi^{-m+1}\Lambda_y$(finite projective R-module), but the specialization of $\overline{a}$ to $K(z)$ vanishes for all z, and R is reduced. contradiction).

Now, for surjectivity, there sre two ways to proceed.

1. (CS way) $\mathrm{Gr}_G^{B_{dR}^+}$ is in fact a sheaf for the pro-etale topology. Then, given a point in the Flag variety, which defines a filtration on G-representations, we will construct, for any representation V of G, a $\mathbb{B}_{dR}^+$-local system $\mathbb{M}_V \subset V \otimes \mathbb{B}_{dR}$ on the pro-etale site of $\mathscr{F}l_{G,\mu}$, compatibly with tensor products and short exact sequences, which maps to the correct filtration. By pullback, this will induce a similar $\mathbb{B}_{dR}^+$-lo

cal system on the pro-etale site of $\mathrm{Spa}(R, R^+)$ for any (R, R^+)-valued point of $\mathscr{F}l_{G,\mu}$. In order to glue them, it is enough to prove that one can glue finite projective $\mathbb{B}_{dR,R}^+$-modules in the pro-etale topology. As $\mathbb{B}_{dR,R}^+$ is ξ-adically complete with ξ a nonzerodivisor and $\mathbb{B}_{dR,R}^+/\xi = R$, we are reduced to gluing finite projective R-modules, which is known ([Kedlaya and Liu, 2013], Theorem 9.2.15.)

We are thus left with construction of $\mathbb{M}_V$. Note that V gives rise to a filtered module with an integrable connection $(V \otimes \mathcal{O}_{\mathscr{F}l_{G,\mu}}, id \otimes \nabla, \mathrm{Fil}_{-\bullet})$, where $\mathrm{Fil}_\bullet$ is the universal ascending filtration parametrized by $\mathscr{F}l_{G,\mu}$ (so that $\mathrm{Fil}_{-\bullet}$ is descending). Because μ is minuscule, this filtered module with integrable connection satisfies Griffiths transversality. Scholze constructs in [Scholze, 2013] the $\mathbb{B}_{dR}^+$-local system $\mathbb{M}_V = \mathrm{Fil}_0\left((V \otimes \mathcal{O}_{\mathscr{F}l_{G,\mu}}) \otimes_{\mathcal{O}} \mathcal{O}\mathbb{B}_{dR}\right)^{id \otimes \nabla = 0}$. The induced filtration is the correct one (check!), and we are done.

2. (SW way) Since μ is minuscule, the source $\mathrm{Gr}_{G,\mu}^{B_{dR}^+} = \mathrm{Gr}_{G,\leq\mu}^{B_{dR}^+}$ and the target (which we now consider as a diamond) $\mathscr{F}l_{G,\mu}^\diamond$ are small v-sheaves which are q.c.q.s. over $\mathrm{Spd}C$, for any algebraically closed field $C/\mathbb{Q}_p$. Thus, (by a result of Scholze) it is enough to check it on (C, C^+)-valued points, which is Lemma 5.7. We recall that a diamond $X^\diamond$ is a functor on perfectoid spaces in characteritic p, sending T to the isomorphism classes of its untilts over X. In the v-topology on perfectoid spaces, we allow open covers and all surjective maps of affinoids. A small v-sheaf is one that admits a surjection from a perfectoid X. Basically, since the datum of G-torsor with a trivialization is a set-theoretically bounded amount of data, it is clear that these are small v-sheaves.

The way to show that Gr_G is separated is by taking $X = \mathrm{Spa}(R, R^+)$ with untilt $R^\sharp$ and two $\mathbb{B}_{dR}^+(R^\sharp)$-lattices $M_1, M_2 \subset \mathbb{B}_{dR}(R^\sharp)^n$, and showing that locus where $\{M_1 = M_2\}$ is representable by a closed subdiamond of $X^\diamond$. It is in fact representable by an affinoid perfectoid space $X_0 \subset X$ that is closed in X. This is done by considering the locus$\{M_1 \subset M_2\}$. Assume by induction that $M_1 \subset \xi^{-1}M_2$. Then it suffices to show that $\{m \mapsto 0 \in \xi^{-1}M_2/M_2\}$ is closed and representable by an affinoid perfectoid. But this quotient is a finite projecive $R^\sharp$-module. Thus, this is the vanishing locus of a tuple in $(R^\sharp)^n$ (writing as a direct summand), hence we are reduced to showing that $\{f = 0\}$ is closed for $f \in R^\sharp$, but $\{f = 0\} = \bigcap\{|f| \leq |\pi|^n\}$, and each of these is a rational. Hence the limit is also affinoid perfectoid, and the complement is open. Now $\mathrm{Gr}_{G\leq\mu}$ is a closed subfunctor of Gr_G, which is proper over $\mathrm{Spd}C$. This follows, as in the classical case, by reducing to GL_n, noting that the relative position condition determines a closed subfunctor, and reducing properness to the case of $\mu = (n, 0, \ldots, 0)$ which defines a successive $\mathbb{P}^{n-1,\diamond}$-bundle on $\mathrm{Spd}C$. □

5.2. **The Fargues-Fontaine curve.** We are now ready to return to the curve. We know how to get from the flag variety to a point on the mixed characteristic affine Grassmannian, which represents a G-bundle on $\mathrm{Spec}\mathbb{B}_{dR,R}^+$ which trivializes on $\mathrm{Spec}\mathbb{B}_{dR,R}$. We still need to get from that to a G-bundle on the curve.

The main result that will help us get there is the following theorem.

Theorem 5.9. *(*[Kedlaya and Liu, 2013]*, Theorem 8.9.6]) Let Z be the image of the canonical closed immersion $i_\infty : \mathrm{Spec}R \to X(R^\flat)$. The category of vector bundles over $X(R^\flat)$ (or over $\mathcal{X}(R^\flat, R^{\flat+})$) is equivalent to the category of triples (M_1, M_2, ι), where M_1 is a vector bundle on $X(R^\flat)\backslash Z$, M_2 is a vector bundle over $\mathrm{Spec}\mathbb{B}_{dR,R}^+$, and $\iota : M_1 |_{\mathrm{Spec}\mathbb{B}_{dR,R}} \cong M_2 |_{\mathrm{Spec}\mathbb{B}_{dR,R}}$. This equivalence is compatible with tensor products and short exact sequences.*

Remark 5.10. Note that by GAGA for the curve, it is enough to define a G-bundle on the scheme version. The idea is that the map i_∞ is induced from θ, so that Z is the locus of $\xi = 0$. Therefore, $\mathrm{Spec}\mathbb{B}_{dR,R}^+$ is the completion of $X(R^\flat)$ along Z, and $\mathrm{Spec}\mathbb{B}_{dR,R} = \mathrm{Spec}\mathbb{B}_{dR,R}^+ \times_{X(R^\flat)} Z^c$. We can now get a functor from $\mathbb{B}_{dR,R}^+$-lattices in $\mathbb{B}_{dR,R}^n$ to vector bundles on the curve by gluing it to a trivial rank n vector bundle on $X(R^\flat)\backslash Z$. This gives us a map

$$\mathcal{E} : \mathrm{Gr}_G^{B_{dR}^+}(R, R^+) \to \{G-\text{bundles over } \mathcal{X}(R^\flat, R^{\flat+})\}$$

in the case $G = GL_n$. It follows from Tannakian formalism for general G.

In particular, if $(R, R^+) = (C, \mathcal{O}_C)$ with $C/\mathbb{Q}_p$ complete and algebraically closed, and $\mathcal{O}_C$ its ring of integers, we may fix an embedding $\overline{\mathbb{Q}}_p \hookrightarrow C$. By Fargues' classification of G-bunles on the curve, we get a map

$$b(\cdot) : \mathrm{Gr}_G^{B_{dR}^+}(C, \mathcal{O}_C) \to B(G) : x \mapsto b(\mathcal{E}(x))$$

When μ is miniscule, defined over E, we obtain a map

$$\mathcal{E} : \mathscr{F}l_{G,\mu}(R,R^+) \xrightarrow{\pi_{G,\mu}^{-1}} Gr_{G,\mu}^{B_{dR}^+}(R,R^+) \to \left\{ G\text{-bundles over } \mathcal{X}(R^\flat, R^{\flat+}) \right\}$$

We can now define a map

$$b(\cdot) : |\mathscr{F}l_{G,\mu}| \to B(G)$$

by sending a (C,C^+)-valued point $x \in \mathscr{F}l_{G,\mu}(C,C^+)$, where C is a complete algebraically closed extension of E, and $C^+ \subset C$ is an open and bounded valuation subring, to the isomorphism class of the associated bundle $\mathcal{E}(x)$.

Definition 5.11. For any $\beta \in B(G)$, the *Newton β-stratum* of the adic flag variety $\mathscr{F}l_{G,\mu}^\beta \subset \mathscr{F}l_{G,\mu}$ is simply $b^{-1}(\beta)$.

Remark 5.12. Recall that if $k' \subset k$ are algebraically closed fields of characteristic p, and $B'(G)$ is the Kottwitz set for k', then $B'(G) \cong B(G)$ via the natural map induced from $G(L') \to G(L)$. Thus this map is well-defined and points of rnak higher than one have the same image as their maximal, rank 1, generalization, and therefore the map factors over the underlying Berkovich space.

In order to get an actual stratification by locallly closed strata, we need to prove the following proposition.

Proposition 5.13. *(1) The image of the map $b(\cdot) : |\mathscr{F}l_{G,\mu}| \to B(G)$ is contained in the set of μ^*-admissible elements $B(G,\mu^*)$.*
(2) The map $b(\cdot) : |\mathscr{F}l_{G,\mu}| \to B(G)$ is lower (!? Shouldn't that be upper - yes, it should !?) semicontinuous.

Proof. We start by proving the first statement. Let $b = b(\mathcal{E}(x))$ for some $x \in \mathrm{Gr}_{G,\mu}^{B_{dR}^+}(C,\mathcal{O}_C)$. We have to prove (a) $\nu_b \preceq \overline{\mu^*}$ as elements of $(X_*(T)\otimes\mathbb{Q})_{dom}^\Gamma$ and that (b) $\kappa_b = \mu^{*\flat} = -\mu^\flat$.
(a) This reduces to the case of $G = GL_n$ by [RR, Lemma 2.2] (Change to location in these notes). Let $\mathcal{E} = \mathcal{E}(x)$ be the vector bundle of rank n over $X_{C^\flat}$, together with a trivialization outside the point ∞ (the closed subset Z). Then $\mu = \mu(\mathcal{E})$ is its relative position. Let $\nu = \nu_\mathcal{E} \in (X_*(GL_n)\otimes\mathbb{Q})_{dom}$ be the Newton polygon of $\mathcal{E}$, with slopes $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, i.e. $\mathcal{E} \cong \bigoplus_{i=1}^n \mathcal{O}_{X_{C^\flat}}(\lambda_i)$. Then, since $\mathcal{O}(\lambda)$ corresponds to the isocrystal $V_{-\lambda}$, we see that $\nu_b = \nu_\mathcal{E}^*$, and we need to prove $\nu_\mathcal{E}^* \preceq \mu^*$. However, this is equivalent to $\nu_\mathcal{E} \preceq \mu$. (Write equations if needed). We note that

$$\bigwedge^r \mathcal{E} \cong \bigoplus_{i_1<i_2<\ldots<i_r} \mathcal{O}_{X_{C^\flat}}\left(\sum_{j=1}^r \lambda_{i_j}\right)$$

has smallest (first) Newton slope $\sum_{i=n+1-r}^n \lambda_i$, and

$$\mu\left(\bigwedge^r \mathcal{E}\right) = \left\{\sum_{j=1}^r \mu_{i_j} : i_1 < i_2 < \ldots < i_r\right\}$$

has smallest (first) Hodge slope $\sum_{i=n+1-r}^n \mu_i$. Therefore, if we prove that $\nu_\mathcal{E} = \mu$ when $\mathcal{E}$ is a line bundle, we get that the top exterior powers agree, and if we show that the first slope λ_n of the Newton polygon of $\mathcal{E}$ lies above the first slope of the Hodge polygon, i.e. $\lambda_n \geq \mu_n$, then applying it to $\bigwedge^r \mathcal{E}$, we get the result.

Verification for the case of line bundles is direct - the modification $\mathcal{E}$ is given by the lattice $\mathcal{E}\otimes_{\mathcal{O}_{X^\flat}} \mathbb{B}_{dR,C}^+ = \xi^{-d}\mathbb{B}_{dR,C}$ for a unique $d \in \mathbb{Z}$, and in that case $\mu(\mathcal{E}) = d \in X_*(\mathbb{G}_m) = \mathbb{Z}$. The resulting line bundle is given by $\mathcal{O}_{X^\flat}(d)$ which is of slope d, as desired.

For the first slope, we may twist to assume that $\mu_n = 0$. This implies $\left(\mathbb{B}_{dR,C}^+\right)^n \subseteq \mathcal{E}\otimes_{\mathcal{O}_{X_{C^\flat}}} \mathbb{B}_{dR,C}^+$, so that the trivialization of $\mathcal{E}$ extends to an injection $\mathcal{O}_{X_{C^\flat}}^n \hookrightarrow \mathcal{E}$. Then we have a quotient map $\mathcal{E} \to \mathcal{O}(\lambda_n)$, inducing a nonzero map $\mathcal{O}_X^n \to \mathcal{O}(\lambda_n)$ by rank considerations, but then we get a nonzero map $\mathcal{O} \to \mathcal{O}(\lambda_n)$, hence $\lambda_n \geq 0$ (more or less by definition - the graded ring has only nonnegative grading).

(b) The map $\mathrm{Gr}_{G,\mu}^{B_{dR}^+}(C,\mathcal{O}_C) \to B(G) \to \pi_1(G)_\Gamma$ is functorial in (G,μ). If $\tilde{G} \to G$ is a central extension with a simply connected derived group (e.g. $\tilde{G} = Z \times G_{sc}$, where G_{sc} is the simply connected cover of G_{der}), and $\tilde{\mu}$ is any lift of μ, the map on Grassmannians is surjective, and by functoriality it's enough to prove the result for $(\tilde{G},\tilde{\mu})$. (draw the commutative diagram!) Next, if G_{der} is simply connected, then $T = G/G_{der}$ is a torus such that $\pi_1(G)_\Gamma \to \pi_1(T)_\Gamma$ is an isomorphism ($\pi_1(G_{der}) = 0$, π_1 is exact). If $G = T$ is a torus, can find a surjection $\tilde{T} \to T$ where $\tilde{T}$ is a product of induced tori $\mathrm{Res}_{K/\mathbb{Q}_p}\mathbb{G}_m$, and enough to prove for $\tilde{T}$. Then it is enough to prove for a single factor in the product, $\tilde{T} = \mathrm{Res}_{K/\mathbb{Q}_p}\mathbb{G}_m$. But then $\pi_1(\tilde{T})_\Gamma = \mathbb{Z}$ (since $X_*(\mathrm{Res}_{K/\mathbb{Q}_p}\mathbb{G}_m) \cong \mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q}_p)]$ is the regular representation), which is torsion-free, so because of the norm map $\mathrm{Norm}_{K/\mathbb{Q}_p} : \tilde{T} \to \mathbb{G}_m$, it is enough to consider the case $\mathbb{G}_m$, but this is the case of line bundles, which we have already covered.

For the second statement, from the fact that the Kottwitz map is constant on $|\mathscr{F}l_{G,\mu}|$, it is enough to show that the Newton map is lower semicontinuous. Consider a topological quotient map $\pi : \mathrm{Spa}(R, R^+) \to \mathscr{F}l_{G,\mu}$ (use a pro-etale cover). Since π is an open and surjective map,

$$\{y : b(y) \preceq \alpha\} = \pi(\{x : (b \circ \pi)(x) \preceq \alpha\})$$

and so it is enough to prove the $b \circ \pi$ is upper (lower) semicontinuous. By Lemma 3.9, this can be checked on representations of G. Such a representation gives rise to a vector bundle on $\mathcal{X}(R^\flat, R^{\flat+})$. This, in turn, translates to a φ-module on the Robba ring $\tilde{\mathcal{R}}_R$, for which the result is the next theorem. □

Theorem 5.14. *([Kedlaya and Liu, 2013], Theorem 7.4.5) For any φ-module M over $\tilde{\mathcal{R}}_R$, the function mapping $\beta \in Spa(R, R^+)$ to the slope polygon of $M \otimes_{\tilde{\mathcal{R}}_R} \tilde{\mathcal{R}}_{k(\beta)}$ is upper (lower - mistake(?) in the original) semicontinuous.*

Proof. (Sketch) Let L be a complete algebraic closure of $k(\beta)$. We know that for some positive integer d, there is a basis for $M \otimes_{\tilde{\mathcal{R}}_R} \tilde{\mathcal{R}}_L$ on which φ^d acts via a diagonal matrix D with values in $p^{\mathbb{Z}}$. We can approximate the basis by a basis for $M \otimes_{\tilde{\mathcal{R}}_R} \tilde{\mathcal{R}}_E$ where E is a finite Galois extension of $k(\beta)$. Since R_β is a Henselian local ring, Then one can find a rational localization $R \to R'$ encircling β and a faithfully finite etale R'-algebra S such that S is Galois over R', S admits a unique extension γ of β, and γ has residue field E. Then we can approximate these by generators of $M \otimes_{\tilde{\mathcal{R}}_R} \tilde{\mathcal{R}}_S$. These, in turn, can be approximated by a basis on which φ^d acts via an invertible matrix of the form FD, where $F - 1$ is p-integral. The slope polygon of M at a given point of $Spa(R')$ is the same as at any point of $Spa(S)$ restricting to the given point. Thus, the negation of p-adic valuations on the diagonal entries of D give the slopes in the slope polygon at β. But the above shows that this is also the generic slope polygon at each $\gamma \in Spa(S)$. But the special slope polygon lies on or above the generic one, and we are done. □

Corollary 5.15. *([Caraiani and Scholze, 2017], Corollary 3.5.9) The strata $\mathscr{F}l^b_{G,\mu}$ are locally closed in $\mathscr{F}l_{G,\mu}$. More precisely, the stratum corresponding to the basic element is open in $\mathscr{F}l_{G,\mu}$, and the strata*

$$\mathscr{F}l^{\succeq b}_{G,\mu} := \coprod_{b \preceq b'} \mathscr{F}l^{b'}_{G,\mu}$$

are closed.

Remark 5.16. Note that this is in a stark contrast to the Newton stratification on classical Shimura varieties, where the basic stratum is closed.

5.3. Example - on $\mathbb{P}^{n-1}$ in the HT case.

Example 5.17. Let $G = GU(1, n-1)$. More precisely, let

$$J = \begin{pmatrix} 1 & \\ & -I_{n-1} \end{pmatrix}$$

Then, setting a quadratic extension $K/\mathbb{Q}_p$, write

$$GU(1, n-1) = \left\{ g \in GL_n(K) \mid^t \overline{g} J g = J \right\}$$

We can consider the Hodge cocharacter

$$\mu(z) = \begin{pmatrix} z & \\ & \overline{z} \cdot I_{n-1} \end{pmatrix} \Rightarrow \mu_{\mathbb{C}}(z, 1) = \begin{pmatrix} z & \\ & I_{n-1} \end{pmatrix}$$

It follows that for $b \in B(G, \mu)$, if $\nu_b = (\nu_1, \ldots, \nu_n)$ with $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_n$, then, as $\mu = (1, 0, 0, \ldots, 0)$, we get

$$\begin{aligned} \nu_1 &\leq 1 \\ \nu_1 + \nu_2 &\leq 1 \\ &\vdots \\ \nu_1 + \nu_2 + \ldots + \nu_{n-1} &\leq 1 \\ \nu_1 + \nu_2 + \ldots + \nu_{n-1} + \nu_n &= 1 \end{aligned}$$

However, the last two equations imply that $\nu_n \geq 0$. If $\nu_n > 0$, we further know (by dimension cosideration) that $\nu_n \geq \frac{1}{n}$, and hence $\sum \nu_i \geq 1$ with equality iff $\nu_i = \frac{1}{n}$ for all i.

It follows that either $\nu_n = 0$ or $\nu_b = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$. If $\nu_n = 0$, then we get the same equations for $(\nu_1, \dots, \nu_{n-1})$, so either $\nu_{n-1} = 0$ or $\nu_b = \left(\frac{1}{n-1}, \dots, \frac{1}{n-1}, 0\right)$. Thus we see that the possible μ-admissible elements in $B(G, \mu)$ are precisely

$$\nu = \left(\underbrace{\frac{1}{n-i}, \dots, \frac{1}{n-i}}_{n-i}, \underbrace{0, \dots, 0}_{i} \right)$$

(Draw Newton polygons!!!!) and the corresponding (to $b^* \in B(G, \mu^*)$) vector bundles are $\mathcal{E}_i = \mathcal{O}^i \oplus \mathcal{O}\left(\frac{1}{n-i}\right)$, with $\mathcal{E}_0$ the basic (minimal) one, and $\mathcal{E}_{n-1} = \mathcal{O}^{n-1} \oplus \mathcal{O}(1)$ the μ-ordinary (maximal) one.
Then, looking for example at the representation $V = \overline{\mathbb{Q}}_p^n$, one sees that $V^{\mu=1} = \langle e_1 \rangle$, $V^{\mu=0} = \langle e_2, \dots, e_n \rangle$, and the filtration is

$$\mathrm{Fil}_m(\mu)(V) = \begin{cases} 0 & m \leq -2 \\ \overline{\mathbb{Q}}_p \cdot e_1 & m = -1 \\ V & m \geq 0 \end{cases}$$

corresponding to the parabolic

$$P_\mu = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

hence $\mathscr{F}l_{G,\mu} = G/P_\mu = \mathbb{P}^{n-1}$ is simply projective space. By Theorem 5.9, we know that for an algebraically closed complete extension C of E, the G-bundles on $\mathcal{X}_{C^\flat}$ are in bijection with $G(E)\backslash G(\mathbb{B}_{dR,C})/G(\mathbb{B}^+_{dR,C})$, and in particular those with relative position μ are in bijection with $G(E)\backslash G(\mathbb{B}^+_{dR,C}) \cdot \mu(\xi)^{-1} \cdot G(\mathbb{B}^+_{dR,C})/G(\mathbb{B}^+_{dR,C})$.
Stated differently, a point $x \in \mathbb{P}^{n-1}$ defines a filtration, which in our case is simply a subspace Fil_x on every G-representation, and in particular on $k(x)^n$. The isomorphism classes of G-bundles correspond to the $G(E)$-orbit of this filtration.
One can see (exercise) that the orbits under $G(\mathbb{Q}_p)$ correspond to the dimension of the maximal rational subspace:$\dim_E(E^n \cap \mathrm{Fil}_x k(x)^n)$. Moreover, from the description of the isomorphism classes of G-bundles, we see that

$$\dim_E(E^n \cap \mathrm{Fil}_x k(x)^n) = i \iff \mathcal{E}(x) \cong \mathcal{O}^i \oplus \mathcal{O}\left(\frac{1}{n-i}\right)$$

Therefore, we get a stratification via (for $i = 0, \dots, n-1$)

$$(\mathbb{P}^{n-1})^{(i)} = \left\{ x \in \mathbb{P}^{n-1} \mid \dim_E E^n \cap \mathrm{Fil}_x k(x)^n = i \right\}$$

In particular, the open (basic) stratum is

$$\left(\mathbb{P}^{n-1}\right)^{(0)} = \Omega^{n-1}$$

Drinfeld space! (removing all rational hyperplanes). For each $i > 0$, the i-th stratum is fibered over the Grassmannian Gr^i of i-dimensional subspaces of $\mathbb{Q}_p^n$ via $x \mapsto \mathbb{Q}_p^n \cap \mathrm{Fil}_x k(x)^n$, with fibers the Drinfeld spaces Ω^{n-1-i}. In particular, we get the description of the stratification for the modular curve by considering $n = 2$.

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