# Motives

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# Cohomology theories

- $X$ nonsingular projective algebraic variety over $k = \overline{k}$.
- $H_B^\bullet(X(\mathbb{C}), \mathbb{Z})$, $H_{dR}^\bullet(X(\mathbb{C}))$, $H_{\text{ét}}^\bullet(X, \mathbb{Q}_\ell)$, $H_{\text{cris}}^\bullet(X, W(k))$
- No algebraic cohomology theory with coefficients in $\mathbb{Q}$.
  - In char $p$, dim $H^1(E, \mathbb{Q}) \leqslant 2$, but $[End(E)_{\mathbb{Q}} : \mathbb{Q}] = 4$.
  - In general, choose $k \hookrightarrow \mathbb{C}$, then

$$H^i_{\text{ét}}(X, \mathbb{Q}_\ell) \cong H^i_{\text{ét}}(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_\ell$$
$$H^i_{\text{dR}}(X) \otimes_k \mathbb{C} \cong H^i_B(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

$X_0$ a model of $X$ over $k_0$, $\Gamma = \text{Gal}(k : k_0)$ infinite, uncountable, stabilizing the countable $H^i(X, \mathbb{Q}) \Rightarrow$
Acts through a finite quotient on $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$.

# Grothendieck's idea

## Motives

There should be a $\mathbb{Q}$-category, $\mathcal{M}(k)$, which will be called the category of motives. We would like it to have some nice properties:

- $\mathrm{Hom}_{\mathcal{M}(k)}(A, B) \in \mathrm{Vec}_{\mathbb{Q}}$.
- $\mathcal{M}(k)$ should be abelian.
- Even better, $\mathcal{M}(k)$ should be Tannakian over $\mathbb{Q}$.
- There should be a universal cohomology theory
$$X \rightsquigarrow hX : \mathrm{Var}_k \to \mathcal{M}(k)$$
where $\mathrm{Var}_k$ is the category of non-singular projective varieties.
- Each correspondence* from $X$ to $Y$ (in particular, a regular map $Y \to X$) should define a morphism $hX \to hY$.
- Good cohomology theories factor uniquely through $X \rightsquigarrow hX$.

# Algebraic cycles

## Definition ($k$-cycle)

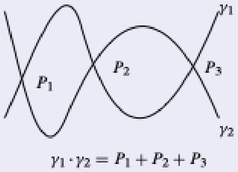
*A $k$-cycle on $X$ is a finite formal sum*

$$\sum n_i[V_i]$$

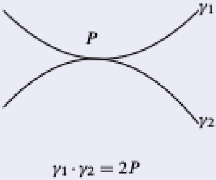
*$V_i$ closed integral $k$-dim. subschemes.*
*The group of $k$-cycles on $X$ is denoted by $Z_k(X)$.*

If two cycles intersect properly, we can define products.

## Example 1




## Example 2



# Rational equivalence

In general, we want to be able to move cycles.

## Definition

*$W$ closed integral $(k+1)$-dim. subscheme of $X$. $f \in K(W)^\times$.*

$$div(f) = \sum_{codim(W,V)=1} \mathrm{ord}_V(f)[V]$$

## Definition (rationally equivalent to zero)

*$\alpha \sim 0$ if $f_i \in K(W_i)^\times$ such that $\alpha = \sum div(f_i)$.*

$$Z_k^0(X) = \{\alpha \in Z_k(X) : \alpha \sim 0\}, A_k(X) = Z_k(X)/Z_k^0(X)$$

## Properties

1. $A_k(X) \cong A_k(X_{red})$.
2. If $X = \bigsqcup X_i$, then $A_\bullet(X) = \bigoplus A_\bullet(X_i)$.
3. $A_k(X_1 \cap X_2) \to A_k(X_1) \oplus A_k(X_2) \to A_k(X_1 \cup X_2) \to 0$.

# Rational equivalence

## Example (Projective space)

$(n-1)$-cycles = hypersurfaces = irreducible homogeneous polynomials.
Equivalent if have same degree. Thus $A_{n-1}(\mathbb{P}^n_k) \cong \mathbb{Z}$.

## Example ($\mathbb{P}^1 \times \mathbb{P}^1$)

1-cycles = irreducible $p(x_0, x_1; y_0, y_1)$ bihomogeneous. Equivalent if both degrees are the same.Thus $A_1(\mathbb{P}^1_k \times \mathbb{P}^1_k) \cong \mathbb{Z} \times \mathbb{Z}$.
Basis - $\{0\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{0\}$. $\Delta_{\mathbb{P}^1} \sim \{0\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{0\}$

# Push-forward for cycles

## Definition (push-forward)

*$f : X \to Y$ proper. $W = f(V)$.*

$$f_\star[V] = \begin{cases} [K(V) : K(W)] \cdot [W] & \dim W = \dim V \\ 0 & \dim W < \dim V \end{cases}$$

## Theorem ( [Ful13], Theorem 1.4)

*$f : X \to Y$ proper, $\alpha \in Z^0_k(X)$, then $f_\star\alpha \in Z^0_k(Y)$.*

## Proposition ( [Ful13], Proposition 1.4)

*$f : X \to Y$ proper surjective of integral schemes. $r \in K(X)^\times$.*

1. $f_\star(\operatorname{div}(r)) = 0$ *if* $\dim(Y) < \dim(X)$.
2. $f_\star(\operatorname{div}(r)) = \operatorname{div}(N(r))$ *if* $\dim(Y) = \dim(X)$.

# Push-forward for cycles

## Example (Different dimensions)

$Y = \operatorname{Spec} k$, $X = \mathbb{P}^1_k$, $r \in k[t]$, $\deg(r) = d = [k[t]/(r) : k]$.

$$div(r) = [P] - d[\infty] \Rightarrow f_\star div(r) = d[Y] - d[Y] = 0$$

## Example (Separated is necessary)

$X$ is the projective line with doubled origin, $Y = \operatorname{Spec} k$, $r = x_1/x_0$.

$$div(r) = [0_1] + [0_2] - [\infty] \Rightarrow f_\star div(r) = [Y] + [Y] - [Y] = [Y]$$

## Definition (Degree)

$\pi : X \to \operatorname{Spec} k$ *proper*, $\alpha \in A_0(X)$.

$\deg \alpha = \pi_\star \alpha \in A_0(\operatorname{Spec} k) \cong \mathbb{Z}$.

# Push forward - illustration

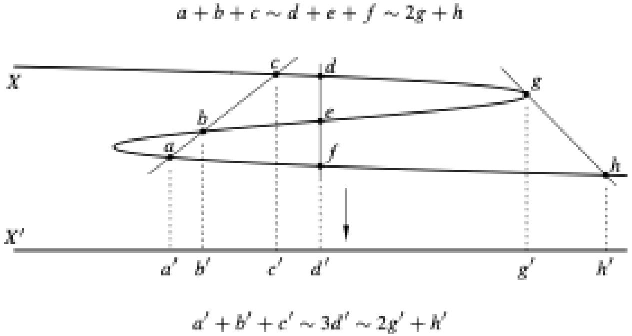



Figure 1.8 Pushforwards of equivalent cycles are equivalent.

# Cycles of subschemes

## Definition (Fundamental cycle)

*$X$ scheme. $X_i$ irred. comp. with local Artinian rings $A_i = \mathcal{O}_{X,X_i}$*

$$[X] = \sum_i \ell_{A_i}(A_i)[X_i]$$

## Example

$f : V \to \mathbb{P}^1$ dominant. Then $div(f) = [f^{-1}(0)] - [f^{-1}(\infty)]$.

## Definition

*Let $V \hookrightarrow X \times \mathbb{P}^1$ be such that $f : V \to \mathbb{P}^1$ is dominant.*
*$P \in \mathbb{P}^1$ rational (degree 1) point.*
*Then $f^{-1}(P)$ is a subscheme of $X \times \{P\}$, mapped isomorphically to a subscheme of $X$. Denote it by $V(P)$.*

# Alternate definition of rational equivalence

## Proposition ( [Ful13], Proposition 1.6)

*$\alpha \in Z_k^0(X)$ iff $\exists V_i \hookrightarrow X \times \mathbb{P}^1$ s.t. $p_i : V_i \to \mathbb{P}^1$ dominant,*

$$\alpha = \sum_i [V_i(0)] - [V_i(\infty)]$$

## Proof.

If $\alpha = div(r)$, $r \in K(W)^\times$, then $r : W \dashrightarrow \mathbb{P}^1$.
Let $V$ be the closure of its graph in $X \times \mathbb{P}^1$.
$p : X \times \mathbb{P}^1 \to X$ is proper and maps $V$ to $W$ birationally.
Let $f : V \to \mathbb{P}^1$ be the second projection. Then by prop. 1.4 (b)

$$div(r) = p_\star div(f) = [V(0)] - [V(\infty)]$$

Conversely, by Theorem 1.4

$$[V(0)] - [V(\infty)] = p_\star div(f) \in Z_k^0(X) \quad \square$$

# Rational Equivalence

## Alternate Definition

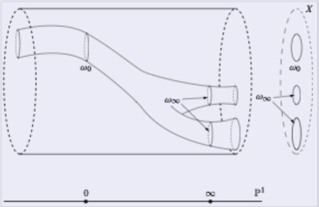



Figure 1.1 Rational equivalence between two cycles $\omega_0$ and $\omega_\infty$ on $X$.

## Hyperbola $\sim$ two lines

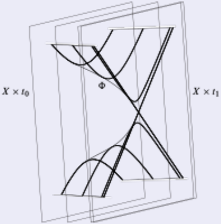



Figure 1.2 Rational equivalence between a hyperbola and the union of two lines in $\mathbb{P}^2$.

# Flat pull-back of cycles

### Definition (Flat pull-back)

*$f : X \to Y$ flat. Then $f^\star[V] = [f^{-1}(V)]$.*

### Proposition ( [Ful13], Prop. 1.7)

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

*Cartesian, $g$ flat, $f$ proper. Then $g'$ flat, $f'$ proper, $f'_\star g'^\star = g^\star f_\star$.*

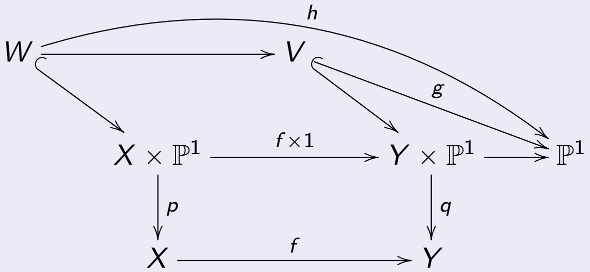
### Theorem ( [Ful13], Theorem 1.7)

*$f : X \to Y$ flat of rel. dim. $n$, $\alpha \in Z^0_k(Y)$, then $f^\star(\alpha) \in Z^0_{k+n}(X)$.*

# Proof of theroem

**Proof.**

Assume $\alpha = [V(0)] - [V(\infty)]$. Consider the diagram




Then

$$f^\star \alpha = f^\star q_\star div(g) = p_\star (f \times 1)^\star ([g^{-1}(0)] - [g^{-1}(\infty)]) =$$
$$= p_\star ([h^{-1}(0)] - [h^{-1}(\infty)]) = p_\star div(h) \qquad \square$$

# Pullback - Illustration

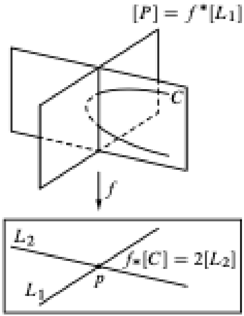



Figure 1.9 $2[p] = f_*([P][C]) = f_*([f^*L_1][C]) = [L_1]f_*[C] = [L_1][2L_2]$.

# Affine bundles

## Proposition ( [Ful13], Proposition 1.8)

*$i : Y \hookrightarrow X$ closed, $j : U = X - Y \hookrightarrow X$. Then*

$$A_k Y \xrightarrow{i_\star} A_k X \xrightarrow{j^\star} A_k U \longrightarrow 0$$

## Proposition ( [Ful13], Proposition 1.9)

*$p : E \to X$ an affine bundle of rank $n$. $p^\star : A_k X \twoheadrightarrow A_{k+n} E$.*

## Proof.

$$\begin{array}{ccccccc} A_\bullet Y & \longrightarrow & A_\bullet X & \longrightarrow & A_\bullet U & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A_\bullet(p^{-1}Y) & \longrightarrow & A_\bullet E & \longrightarrow & A_\bullet(p^{-1}U) & \longrightarrow & 0 \end{array}$$

reduces to $E = X \times \mathbb{A}^n$, then to $n = 1$. □

## Proof.

Replace $X$ by $\overline{p(V)}$, so $X$ integral and $p$ dominant.

$$\begin{array}{ccc} V & \longrightarrow & \overline{p(V)} \\ \downarrow & & \downarrow \\ X \times \mathbb{A}^1 & \xrightarrow{p} & X \end{array}$$

$A = K[X], K = K(X)$, $q = I(V) \subseteq A[t]$.
$V \neq E$, dominant hence $qK[t] = (r)$ is nontrivial. Then

$$[V] - div(r) = \sum n_i[V_i]$$

where $V_i$ don't dominate $X$. Then

$$[V] = div(r) + \sum n_i p^\star[p(V_i)] \quad \square$$

## Example (Affine space)

$$A_k(\mathbb{A}^n) = \begin{cases} 0 & k < n \\ \mathbb{Z} & k = n \end{cases}$$

# Affine bundles

## Example (Projective space)

$L^k = k$-dim. linear subspace of $\mathbb{P}^n$. From

$$A_k([L^{n-1}]) \to A_k(\mathbb{P}^n) \to A_k(\mathbb{A}^n) \to 0$$

we get $\mathbb{Z} \cdot [L^k] \twoheadrightarrow A_k(\mathbb{P}^n)$. Isomorphism for $k = n-1, n$.
If $k < n-1$, and $d[L^k] = \sum n_i div(r_i)$, $r_i \in K(V_i)$, set $Z = \bigcup V_i$ and project from a linear $(n-k-2)$-dimensional subspace disjoint from $Z$ to get $f : Z \to \mathbb{P}^{k+1}$. Using proper push-forward and induction, $A_k(\mathbb{P}^n) = \mathbb{Z} \cdot [L^k]$.

## Example (Hypersurface)

Let $H$ ve a reduced hypersurface of degree $d$ in $\mathbb{P}^n$. Then $[H] = d[L]$, for a hyperplane $L$ and

$$A_{n-1}(\mathbb{P}^n - H) = \mathbb{Z}/d\mathbb{Z}$$

# Exterior Products

## Definition (Exterior Product)

*The exterior product*

$$Z_k X \otimes Z_l Y \xrightarrow{\times} Z_{k+l}(X \times Y)$$

*is defined by* $[V] \times [W] = [V \times W]$.

## Proposition ( [Ful13], Proposition 1.10)

1. *If* $\alpha \sim 0$ *or* $\beta \sim 0$*, then* $\alpha \times \beta \sim 0$.
2. *If* $f : X' \to X, g : Y' \to Y$ *proper, then*
$$(f \times g)_\star(\alpha \times \beta) = f_\star\alpha \times g_\star\beta$$
3. *If* $f : X' \to X, g : Y' \to Y$ *flat, then*
$$(f \times g)^\star(\alpha \times \beta) = f^\star\alpha \times g^\star\beta$$

# Chern class of a line bundle

## Definition (Chern class)

*$L$ line bundle on $X$. $V \subseteq X$, $\dim V = k$. Then $L|_V \cong \mathscr{O}_V(C)$, $C \in Div(V)$.*

$$c_1(L) \cap - : Z_k(X) \to A_{k-1}(X), \quad [V] \mapsto [C]$$

## Proposition ( [Ful13], Proposition 2.5)

1. *$\alpha \sim 0 \Rightarrow c_1(L) \cap \alpha = 0$. Hence $c_1(L) \in Hom(A_k X, A_{k-1} X)$.*
2. *(Commutativity) $c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha)$.*
3. *(Projection formula) $f : X' \to X$ proper*
$$f_\star(c_1(f^\star L) \cap \alpha) = c_1(L) \cap f_\star(\alpha)$$
4. *(Flat pull-back) $f : X' \to X$ flat of rel. dim. $n$.*
$$c_1(f^\star L) \cap f^\star \alpha = f^\star(c_1(L) \cap \alpha)$$
5. *(Additivity) $c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$.*

# Segre classes

## Example (Projective space)

$c_1(\mathscr{O}(1)) \cap [L^k] = [L^{k-1}]$

## Definition (Segre classes)

*$E$ line bundle on $X$ of rank $e+1$, $P = P(E)$, $p : P \to X$, and $\mathscr{O}(1)$ the canonical line bundle on $P$.*

$$s_i(E) \cap \alpha = p_\star(c_1(\mathscr{O}(1))^{e+i} \cap p^\star \alpha)$$

## Proposition ( [Ful13], Proposition 3.1)

1. $s_i(E) \cap \alpha = \begin{cases} 0 & i < 0 \\ \alpha & i = 0 \end{cases}$
2. *If $E$ line bundle, $s_1(E) \cap \alpha = -c_1(E) \cap \alpha$.*
3. *Commutativity, projection formula and flat pull-back.*

# Segre classes

**Proof.**

(3) follows formally from the same for $c_1$.
For (1), consider $[V]$. Use projection formula to reduce to $X = V$.
Then $A_{k-i}X = 0$ for $i < 0$. Also

$$s_0(E) \cap \alpha = p_\star(c_1(\mathscr{O}(1))^e \cap [P]) = m[X]$$

To show $m = 1$, by flat pull-back, reduce to trivial $E$.
Then $P(E) = X \times \mathbb{P}^e$, and $\mathscr{O}(1)$ has sections whose zero scheme is $X \times \mathbb{P}^{e-1}$, so

$$c_1(\mathscr{O}(1)) \cap [X \times \mathbb{P}^e] = [X \times \mathbb{P}^{e-1}]$$

For (2) , note that $P(E) = X$ and $\mathscr{O}(-1) = E$, so $\mathscr{O}(1) = E^\vee$. □

# Rational equivalence on bundles

## Theorem ( [Ful13], Theorem 3.3)

*$\pi : E \to X$ vector bundle of rank $r = e + 1$.*

1. *$\pi^\star : A_{k-r}X \to A_k E$ is an isomorphism.*
2. *The map $\theta_E : \bigoplus_{i=0}^{e} A_{k-e+i}X \to A_k P(E)$ defined by*

$$\theta_E(\bigoplus \alpha_i) = \sum_{i=0}^{e} c_1(\mathscr{O}(1))^i \cap p^\star \alpha_i$$

   *is an isomorphism.*

## Proof.

Start by showing $\theta_E$ is surjective. Noetherian induction reduces to trivial $E$. Induction on the rank reduces to $F = E \oplus 1$, $\theta_E$ surjective. Write $P = P(E)$, $Q = P(F) = P(E \oplus 1)$. □

# Rational equivalence on bundles

## Proof.

$$\begin{array}{ccccc} P & \xrightarrow{i} & Q & \xleftarrow{j} & E \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle q} & \swarrow{\scriptstyle \pi} & \\ & & X & & \end{array} \quad \Longrightarrow \quad \begin{array}{ccccccc} A_kP & \xrightarrow{i_\star} & A_kQ & \xrightarrow{j^\star} & A_kE & \longrightarrow & 0 \\ & & {\scriptstyle q^\star}\uparrow & \nearrow{\scriptstyle \pi^\star} & & & \\ & & A_{k-r}X & & & & \end{array}$$

For $\beta \in A_\bullet Q$ , $j^\star\beta = \pi^\star\alpha$, so by induction and projection formula

$$\beta - q^\star\alpha = i_\star\left(\sum_{i=0}^{e} c_1(\mathscr{O}_E(1))^i \cap p^\star\alpha_i\right) = \sum_{i=0}^{e} c_1(\mathscr{O}_F(1))^i \cap i_\star p^\star\alpha$$

$\mathscr{O}_F(1)$ has a section vanishing on $P$, so

$$c_1(\mathscr{O}_F(1)) \cap [q^{-1}V] = [p^{-1}V] \implies c_1(\mathscr{O}_F(1)) \cap q^\star\alpha = i_\star p^\star\alpha$$

showing $\theta_F$ surjects.

For injectivity, if $\beta = \sum_{i=0}^{l} c_1(\mathscr{O}(1))^i \cap p^\star\alpha = 0$ with $\alpha_l \neq 0$, then $p_\star(c_1(\mathscr{O}(1))^{e-l} \cap \beta) = \sum_{i=0}^{l} s_{i-l}(E) \cap \alpha_i = \alpha_l$. □

# Gysin homomorphism

## Proof.

Finally, to show $\pi^\star$ is injective, let $F = E \oplus 1$, assume $\pi^\star\alpha = 0$. Then $j^\star q^\star\alpha = 0$, so using surjectivity of $\theta_E$

$$q^\star\alpha = i_\star\left(\sum_{i=0}^{e} c_1(\mathscr{O}_E(1))^i \cap p^\star\alpha_i\right) = \sum_{i=0}^{e} c_1(\mathscr{O}_F(1))^{i+1} \cap q^\star\alpha_i$$

contradicting the injectivity of $\theta_F$. □

## Definition (Gysin homomorphism)

*$s$ zero section of $\pi : E \to X$ of rank $r$.*

$$s^\star = (\pi^\star)^{-1} : A_k E \to A_{k-r} X$$

# Deformation to the normal cone

## Definition (Normal cone)

*$X \subseteq Y$ closed, ideal $\mathscr{I}$.* $C_X Y = \operatorname{Spec}\left(\bigoplus_{n=0}^{\infty} \mathscr{I}^n/\mathscr{I}^{n+1}\right)$.

## Theorem

*Let $M = M_X Y$ be the blow-up of $Y \times \mathbb{P}^1$ along $X \times \{\infty\}$.*

$$\begin{array}{ccc} X \times \mathbb{P}^1 & \hookrightarrow & M \\ & {\scriptstyle pr} \searrow \quad \swarrow {\scriptstyle \varrho\text{ - flat}} & \\ & \mathbb{P}^1 & \end{array}$$

*commutes and:*

1. *Over $\mathbb{A}^1$, $\varrho^{-1}(\mathbb{A}^1) = Y \times \mathbb{A}^1$, and $X \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{A}^1$.*
2. *Over $\infty$, the divisor $M_\infty = \varrho^{-1}(\infty) = P(C \oplus 1) + \tilde{Y}$.*
3. *$X \times \{\infty\} \hookrightarrow M_\infty$ is the zero section to $C \hookrightarrow P(C \oplus 1)$.*
4. *$P(C \oplus 1) \cap \tilde{Y} = P(C)$, the hyperplane at $\infty$, exc. divisor.*

# Specialization to the Normal Cone

## Definition (specialization)

$$\sigma : Z_k Y \to Z_k C, \quad \sigma[V] = [C_{V \cap X} V].$$

## Proposition ( [Ful13], Proposition 5.2)

*If* $\alpha \sim 0$, *then* $\sigma(\alpha) \sim 0$.

## Proof.

Let $M^\circ = M - \tilde{Y}$ (deformation of $X \hookrightarrow Y$ to $X \hookrightarrow C_X Y$).

$$\begin{array}{ccccccc} A_{k+1} C & \xrightarrow{i_\star} & A_{k+1} M^\circ & \xrightarrow{j^\star} & A_{k+1}(Y \times \mathbb{A}^1) & \longrightarrow & 0 \\ & & \downarrow{i^\star} & & \uparrow{pr^\star} & & \\ & & A_k C & \dashleftarrow & A_k Y & & \end{array}$$

Enough to show this is the map. Use

$$pr^\star[V] = [V \times \mathbb{A}^1] = j^\star[M^\circ_{V \cap X} V] \quad \square$$

# Intersection products

## The basic construction

$i : X \hookrightarrow Y$ regular embedding of codim. $d$ defined by $\mathscr{I}$.
$V$ pure $k$-dim. $f : V \to Y$, $W = f^{-1}(X)$.
$N = g^\star N_X Y$, $\pi : N \to W$, $C = C_W V$.

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ \downarrow g & & \downarrow f \\ X & \xrightarrow{i} & Y \end{array}$$

Set $\mathscr{J}$ the ideal sheaf of $W$ in $V$.

$$\bigoplus_n f^\star(\mathscr{I}^n/\mathscr{I}^{n+1}) \twoheadrightarrow \bigoplus_n \mathscr{J}^n/\mathscr{J}^{n+1} \implies C \hookrightarrow N$$

Define $X \cdot V = s^\star[C] \in A_{k-d} W$.

## Example

Let $\pi : X \to \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at the origin. Let $D, D'$ be the inverses of the $x$-axis and the $y$-axis. Then $D \cdot D'$ and $D' \cdot D$ are well-defined cycles on $D \cap D' = E$, the exceptional divisor. These cycles are not equal, although they are rationally equivalent.

# Intersection products

## Definition (Canonical decomposition)

*$C_1, \dots, C_r$ irreducible components of $C$. $Z_i = \pi(C_i) \subseteq W$ their support. $Z_1, \dots, Z_r$ are the distinguished varieties of $X \cdot V$. $N_i = N|_{Z_i}$, $s_i$ the zero sections, and $\alpha_i = s_i^\star[C_i] \in A_{k-d}(Z_i)$. Then*

$$[C] = \sum_{i=1}^{r} m_i[C_i] \implies X \cdot V = \sum_{i=1}^{r} m_i \alpha_i$$

## Example

$Y = \mathbb{P}^2$, $X_1 = \{xy = 0\}$, $X_2 = \{x = 0\}$, $P$ the point $(0,0)$. For $X_1 \cdot X_2$, only $X_2$ is distinguished. For $X_2 \cdot X_1$, locally

$$k[C_{X_2}X_1] \cong k[x,y,T]/(x,yT) \cong {}^{k[x,y,T]}/_{(x,y)} \times {}^{k[x,y,T]}/_{(x,T)}$$

so both $X_2$ and $P$ are distinguished.

Similarly for $(X_1 \times X_2) \cdot \Delta_{\mathbb{P}^2}$.

# Refined Gysin Homomorphism

## Example

$A, B$ lines in $\mathbb{P}^2$ meeting at $P$. Set $D_1 = 2A + B$, $D_2 = A + 2B$.
$X = D_1 \times D_2$, $Y = \mathbb{P}^2 \times \mathbb{P}^2$, $V = \mathbb{P}^2$, $f : V \to Y$ the diagonal.
$A, B, P$ distinguished varieties, canonical decomposition is

$$X \cdot V = \alpha + \beta + 3[P], \alpha \in Z_0(A), \beta \in Z_0(B), \deg(\alpha) = \deg(\beta) = 3$$

## Definition (Refined Gysin homomorphism)

*Let* $i^! : A_k V \to A_{k-d} W$ *be the composite*

$$A_k V \xrightarrow{\sigma} A_k C \longrightarrow A_k N \xrightarrow{s^\star} A_{k-d} W$$

*Explicitly,* $i^! \left(\sum n_i [V_i]\right) = \sum n_i X \cdot V_i$.
*If* $V = Y, f = id_Y$, *then we denote* $i^\star : A_k Y \to A_{k-d} X$.

# Refined Gysin Homomorphism

## Theorem ( [Ful13], Theorem 6.2)

$$\begin{array}{ccccc} X'' & \xrightarrow{q} & X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle i''} & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ Y'' & \xrightarrow{p} & Y' & \xrightarrow{f} & Y \end{array}$$

*Cartesian with $i$ regular of codim. $d$. Then*

1. *(Push-forward) If $p$ is proper, $i^! p_\star = q_\star i^!$.*
2. *(Pull-back) If $p$ is flat, $i^! p^\star = q^\star i^!$.*
3. *(Compatibility) If $i'$ regular of codim. $d$, then $i^! = i'^!$.*

## Example

$\delta : \mathbb{P}^n \to \mathbb{P}^n \times \ldots \mathbb{P}^n$. Then

$$\delta^\star([k_1] \times \ldots \times [k_r]) = [k_1 + \ldots + k_r - (r-1)n]$$

# Intersection on smooth varieties

## Definition (Intersection product)

$f : X \to Y$, $Y$ *smooth*, $\dim Y = n$. $p_X : X' \to X, p_Y : Y' \to Y$.

$$\begin{array}{ccc} X' \times_Y Y' & \longrightarrow & X' \times Y' \\ \big\downarrow & & \big\downarrow {\scriptstyle p_X \times p_Y} \\ X & \xrightarrow{\gamma_f} & X \times Y \end{array}$$

*Define* $x \cdot_f y = \gamma_f^!(x \times y) : A_k X' \times A_l Y' \to A_{k+l-n}(X' \times_Y Y')$.

## Example

$X' = X, Y' = Y$. Then $x \cdot_f y = \gamma_f^\star(x \times y)$.

If $X = Y, f = id_Y$, then $\gamma_f = \delta$, and $x \cdot y = \delta^\star(x \times y)$.

Given $x, y$ can take $X', Y'$ to be their supports.

# Refined intersections

## Proposition ( [Ful13], Proposition 8.1.1)

1. *(Associaitivity)* $x \cdot_f (y \cdot_g z) = (x \cdot_f y) \cdot_{gf} z$.
2. *(Commutativity)* $(x \cdot_{f_1} y_1) \cdot_{f_2} y_2 = (x \cdot_{f_2} y_2) \cdot_{f_1} y_1$.
3. *(Projection formula)* If $p_Y f' = fp_X$,
$$(f' \times_Z id_Z)_\star (x \cdot_{gf} z) = f'_\star(x) \cdot_g z$$
4. *(Compatibility)* If $g : V' \to Y'$ a regular, $g^!(x \cdot_f y) = x \cdot_f g^! y$.

## Corollary

1. $j : V \to Y$ regular, $x \cdot [V] = j^!(x)$.
2. $x \cdot_f y = (x \times y) \cdot [\Gamma_f]$. If $X = Y$, $x \cdot y = (x \times y) \cdot [\Delta]$.
3. $x \cdot_f [Y] = x$.

# Algebra of Correspondences

## Definition (Correspondence)

*$X, Y$ varieties. The correspondences of degree $r$ are* $\alpha : X \vdash Y$

$$\mathrm{Corr}^r(X, Y) = A_{\dim Y - r}(X \times Y)$$

## Definition (composition)

*If $\alpha : X \vdash Y$, $\beta : Y \vdash Z$, $X, Y, Z$ smooth, the product is*

$$\beta \circ \alpha = (p_{XZ})_\star(p_{XY}^\star \alpha \cdot p_{YZ}^\star \beta)$$

## Example

$f : Y \to X$ regular. Its graph $\Gamma_f \in A_{\dim Y}(Y \times X)$ hence its transpose $\Gamma_f^t \in A_{\dim Y}(X \times Y) = \mathrm{Corr}^0(X, Y)$.

# Algebra of Correspondences

## Proposition

$\alpha : X \vdash Y, \beta : Y \vdash Z$.

1. *If* $\gamma : Z \vdash W$, *then* $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.
2. $(\beta \circ \alpha)^t = \alpha^t \circ \beta^t$ *and* $(\alpha^t)^t = \alpha$.
3. 3.1 *If* $\beta = \Gamma_g$, *then* $\beta \circ \alpha = (1_X \times g)_\star(\alpha)$.
   3.2 *If* $\alpha = \Gamma_f$, *then* $\beta \circ \alpha = (f \times 1_Z)^\star(\beta)$.
   3.3 *If* $\alpha = \Gamma_f$, $\beta = \Gamma_g$, *then* $\beta \circ \alpha = \Gamma_{gf}$.

## Corollary

*The product* $(\alpha, \beta) \mapsto \alpha \circ \beta$ *makes* $A_\bullet(X \times X)$ *a ring, with unit* $[\Delta_X]$, *and an involution* $\alpha \mapsto \alpha^t$.

# Correspondences and cohomology

## Theorem (cycle class map)

*There exists a map*

$$cl : A_\bullet(X)_\mathbb{Q} \to H^\bullet(X)$$

*which doubles degrees and sends intersection products to cup products.*

## Corollary

*If* $\alpha \in \mathrm{Corr}^0(X, Y)$ *then it defines* $\alpha^\star : H^\bullet(X) \to H^\bullet(Y)$

$$x \mapsto (p_Y)_\star(p_X^\star x \cup cl(\alpha))$$

# First Attempt

Define $\mathcal{M}(k)$ to be the category with objects $hX$ and

$$\mathrm{Hom}(hX, hY) = \mathrm{Corr}^0(X, Y)_{\mathbb{Q}}, h(f) = [\Gamma_f]^t$$

## Example

If $e : V \to V$ is s.t. $e^2 = e$, then $V = \ker(e) \oplus eV$.
If $(W, f)$ is another pair, then

$$\mathrm{Hom}_{\mathbb{Q}}(eV, fW) = f \circ \mathrm{Hom}_{\mathbb{Q}}(V, W) \circ e$$

add images of idempotents in $\mathrm{End}(hX) = \mathrm{Corr}^0(X \times X)_{\mathbb{Q}}$.

## Remark

*When $X$ is not pure dim. $X = \bigcup_i X_i$, we let*

$$\mathrm{Corr}^r(X, Y) = \bigoplus \mathrm{Corr}^r(X_i, Y) \subseteq A_\bullet(X \times Y)$$

# Second Attempt

Define $\mathcal{M}(k)$ to be the category with objects $h(X, e)$ where $e \in \mathrm{Corr}^0(X, X)_{\mathbb{Q}}$ is an idempotent ($e^2 = e$) and

$$\mathrm{Hom}(h(X, e), h(Y, f)) = f \circ \mathrm{Corr}^0(X, Y)_{\mathbb{Q}} \circ e$$

## Example

$\mathrm{End}(h(\mathbb{P}^1, \Delta_{\mathbb{P}^1})) = \mathbb{Z} \oplus \mathbb{Z}$ with $e_0 = (1, 0)$ represented by $\{0\} \times \mathbb{P}^1$ and $e_2 = (0, 1)$ represented by $\mathbb{P}^1 \times \{0\}$.
Since $\Delta_{\mathbb{P}^1} \sim e_0 + e_2$, we get

$$h(\mathbb{P}^1, \Delta_{\mathbb{P}^1}) = h^0\mathbb{P}^1 \oplus h^2\mathbb{P}^1$$

with $h^i\mathbb{P}^1 = h(\mathbb{P}^1, e_i)$. Write $\mathbb{1} = h^0\mathbb{P}^1$ and $\mathbb{L} = h^2\mathbb{P}^1$.

This is the category of effective motives. However, we would like objects to have duals.

# Third attempt

We invert $\mathbb{L}$. Objects - $h(X, e, m)$ with $X, e$ as before, $m \in \mathbb{Z}$.

$$\mathrm{Hom}(h(X, e, m), h(Y, f, n)) = f \circ \mathrm{Corr}^{n-m}(X, Y)_{\mathbb{Q}} \circ e$$

This is the category of (Chow) motives over $k$.

## Definition (pseudoabelian category)

*An additive category is pseudoabelian if every idempotent $f \in \mathrm{End}\, M$ has an image, and the canonical map $\mathrm{Im}(f) \oplus \mathrm{Im}(1 - f) \to M$ is an isomorphism.*

## Theorem ( [Sch94, Theorem 1.6] )

*The category of Chow motives over $k$, $\mathcal{M}_k$, is an additive, $\mathbb{Q}$-linear category, which is pseudoabelian.*

## Remark

*In general, $\mathcal{M}_k$ is not abelian.*

# Direct Sums

We begin by constructing direct sums for equal degrees.

## Definition (Direct sum)

$e \in \mathrm{Corr}^0(X, X), f \in \mathrm{Corr}^0(Y, Y)$. *Recall that*

$$\mathrm{Corr}^0(X \sqcup Y, X \sqcup Y) =$$
$$= \mathrm{Corr}^0(X, X) \oplus \mathrm{Corr}^0(X, Y) \oplus \mathrm{Corr}^0(Y, X) \oplus \mathrm{Corr}^0(Y, Y)$$

*Define*

$$(X, e, m) \oplus (Y, f, m) = (X \sqcup Y, e \oplus f, m)$$

## Exercise

$g \circ \phi_X \circ e : (X, e, m) \to (Z, g, n)$, $g \circ \phi_Y \circ f : (Y, f, m) \to (Z, g, n)$ *then* $g \circ (\phi_X \circ e \oplus \phi_Y \circ f) \circ (e \oplus f) : (X \sqcup Y, e \oplus f, m) \to (Z, g, n)$ *satisfies the universal property.*

# Pseudoabelian

**Example**

$e \circ f \circ e \in \mathrm{End}(X, e, m) = e \circ \mathrm{Corr}^0(X, X) \circ e$ is an idempotent.

$(X, e\circ f\circ e, m) \oplus (X, e - e\circ f\circ e, m) = (X \sqcup X, e\circ f\circ e \oplus (e - e\circ f\circ e), m)$

The morphisms

$$(efe, e - efe) : (X, e, m) \to (X \sqcup X, efe \oplus (e - efe), m)$$

$$(efe, e - efe) : (X \sqcup X, efe \oplus (e - efe) \to (X, e, m)$$

are inverses since

$efe \circ efe + (e - efe) \circ (e - efe) = efe + e - efe = e = \mathrm{id}_{\mathrm{End}(X,e,m)}$

This shows if $\mathcal{M}_k$ is additive, then it is also pseudoabelian.

**Proof.**

Quick check shows that $\mathrm{Im}(e \circ f \circ e) = (X, e \circ f \circ e, m)$, and the above shows that $\mathrm{Im}(efe) \oplus \mathrm{Im}(e - efe) \to (X, e, m)$ is an isomorphism. □

# Tensor Products

## Definition (Tensor product)

*Define*

$$(X, e, m) \otimes (Y, f, n) = (X \times Y, e \times f, m + n)$$

*and on morphisms*

$$f_1 \circ \phi_1 \circ e_1 \otimes f_2 \circ \phi_2 \circ e_2 = (f_1 \times f_2) \circ (\phi_1 \times \phi_2) \circ (e_1 \times e_2)$$

## Example (Unit and Lefschetz motives)

Let $\mathbb{1} = (\operatorname{Spec} k, \mathrm{id}, 0)$. Then $\mathbb{1}$ is a unit for $\otimes$.
Let $\mathbb{L} = (\operatorname{Spec} k, \mathrm{id}, -1)$. Then

$$(X, e, m) = e(hX) \otimes \mathbb{L}^{\otimes -m} \subseteq h(X) \otimes \mathbb{L}^{\otimes -m}$$

# Pullback and Pushforward

## Definition (Pullback)

*For* $\phi : Y \to X$, $\phi^\star = h(\phi) = [\Gamma_\phi] \in \mathrm{Corr}^0(X, Y)$.

## Example (Diagonal)

$\Delta : X \to X \times X$ defines a product structure on $h(X) = (X, \mathrm{id}, 0)$

$$m_X : h(X) \otimes h(X) = h(X \times X) \stackrel{h(\Delta)}{\to} h(X)$$

## Definition (Pushforward)

$\phi : Y \to X$, $X, Y$ *pure of dim.* $d, e$, *then* $\phi_\star$ *is the image*
$[\Gamma_\phi]^t \in A_e(Y \times X) = \mathrm{Corr}^{d-e}(Y, X) = \mathrm{Hom}(h(Y), h(X) \otimes \mathbb{L}^{e-d})$

# Subobjects and Quotients

## Example (finite maps)

$d = e$, $\phi$ finite of degree $r$, then $\phi^\star \circ \phi_\star = [r] \in \operatorname{End} h(X)$. Indeed

$$\phi_\star \circ \phi^\star = p_{XX\star}(p_{XY}^\star([\Gamma_\phi]) \cdot p_{YX}^\star([\Gamma_\phi]^t)) = $$
$$= p_{XX\star}([\Gamma_\phi \times X] \cdot [X \times \Gamma_\phi^t]) = p_{XX\star}(\phi, \mathrm{id}, \phi)_\star[Y] = r[\Delta_X]$$

## Example (subobjects, quotients)

$X$ irred. of dim. $d$. $x \in X(k)$, $\alpha : X \to \operatorname{Spec} k$. Then $x^\star \circ \alpha^\star = 1$, so $\alpha^\star : \mathbb{1} \to h(X)$ is a subobject.

Similarly, $\alpha_\star : h(X) \to \mathbb{L}^d$ is a quotient.

In general, let $k' = \Gamma(X, \mathscr{O}_X), \alpha : X \to \operatorname{Spec} k'$, let $k''/k$ be separable s.t. $\exists x \in X(k'')$. Write $\gamma = \alpha \circ x$. Then it is finite, so $\gamma_\star \circ x^\star \circ \alpha^\star = \gamma_\star \circ \gamma^\star = [k'' : k']$, and $\alpha^\star : h(\operatorname{Spec} k') \to h(X)$ defines a subobject, denoted by $h^0(X)$. Similarly, $\alpha_\star : h(X) \to h(\operatorname{Spec} k') \otimes \mathbb{L}^d$ is a quotient, denoted $h^{2d}(X)$.

# Subobjects and Quotients

## Proposition ( [Sch94, Proposition 1.12] )

*Any motive $M$ can be expressed as a direct factor of some $h(X) \otimes \mathbb{L}^n$, with $X$ equidimensional.*

## Proof.

Enough to consider $M = h(X)$.

$$h(X) = \bigoplus h(X_i) = \bigoplus \left( h(X_i) \otimes h^0(\mathbb{P}^{d_i}) \right)$$

This is a direct factor of $\bigoplus h(X_i) \otimes h(\mathbb{P}^{d_i}) = h\left(\bigsqcup X_i \times \mathbb{P}^{d_i}\right)$. $\square$

## Example (canonical idempotents)

$X$ irred., $Z$ a 0-cycle on $X$ of degree $d > 0$. Then $p_0 = (1/d)[Z \times X] \in A_d(X \times X)$ is an idempotent, inducing

$$h^0(X) \simeq (X, p_0, 0), \quad h^{2d}(X) \simeq (X, p_{2d}, 0) \quad p_{2d} = p_0^t$$

# Direct Sums and Duals

## Definition (Direct Sum)

$(X, e, m), (Y, f, n), m < n$. *Then*

$$(X, e, m) = (X, e, n) \otimes \mathbb{L}^{n-m} = (X, e, n) \otimes h^2(\mathbb{P}^1)^{n-m}$$

*which is a direct factor of* $h(X \times (\mathbb{P}^1)^{n-m}) \otimes \mathbb{L}^{-n}$. *Denote by* $e'$ *the projection, so* $(X, e, m) = (X \times (\mathbb{P}^1)^{n-m}, e', n)$. *Then*

$$(X, e, m) \oplus (Y, f, n) = \left(X \times (\mathbb{P}^1)^{n-m} \sqcup Y, e' \oplus f, n\right)$$

## Definition (Dual)

*X pure of dim.* $d$. *Set* $(X, e, m)^\vee = (X, e^t, d - m)$.
*In particular* $h(X)^\vee = h(X) \otimes \mathbb{L}^{-d}$ *("Poincaré Duality").*

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