Computing modular forms

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Structure of the talk

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1. Elliptic Curves

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2. Modular Forms

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3. Results

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4. Modular Symbols

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2. Modular Forms
3. Results
4. Modular Symbols
5. Hecke Operators
6. Computational Aspects

Elliptic Curves

Diophantine Equations

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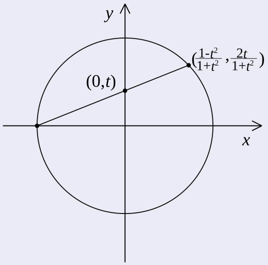
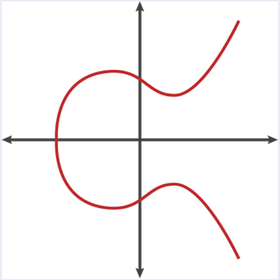
- $ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0$
- $y^2 = ax^3 + bx^2 + cx + d$

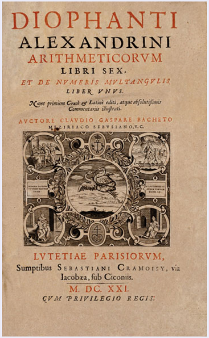
Rational Parameterization



Elliptic Curves



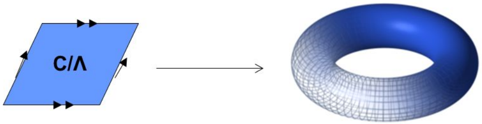
Arithmetica


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primum Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Elliptic Curves

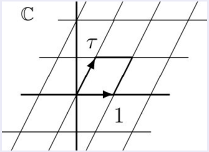
Over $\mathbb{C}$

Elliptic Curves over Complex Numbers

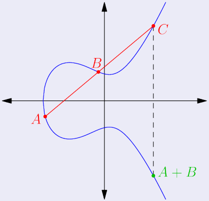


$$\mathbb{C}/\Lambda \longrightarrow E : y^2 = x^3 + Ax + B$$
$$z \longmapsto (\wp(z), -\wp'(z)/2)$$

$\Lambda = \mathbb{Z} + \mathbb{Z}\tau$



Addition Law



Elliptic Curves

Over $\mathbb{Q}$

Theorem (Mordell, 1922)

$E : y^2 = f(x),\ f(x) \in \mathbb{Q}[x] \Rightarrow E(\mathbb{Q})$ *is finitely generated.*

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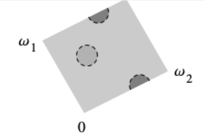
- $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $E(\bar{\mathbb{Q}})$.
- $\rho_{E,p} : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(E[p]) \cong GL_2(\mathbb{F}_p)$.

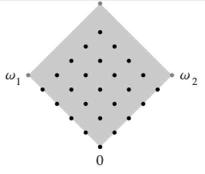
Question

What can we say about $\rho_{E,p}$?

$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$



$E[5]$



Elliptic Curves

Theorem (Serre's Open Image Theorem, 1972)

E defined over $\mathbb{Q}$ without complex multiplication. Then $[GL_2(\mathbb{F}_p) : \operatorname{Im} \rho_{E,p}] \leqslant c_E$.

Conjecture (Serre's uniformity conjecture, 1972)

$\exists c$, independent of E, such that $[GL_2(\mathbb{F}_p) : \operatorname{Im} \rho_{E,p}] \leqslant c$.

Maximal subgroups of $PGL_2(\mathbb{F}_p)$

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- Exceptional - A_4, S_4, A_5

Modular Curves

Moduli Spaces

$$SL_2(\mathbb{Z})\backslash\mathcal{H} \xrightarrow{\sim} \{\Lambda \subseteq \mathbb{C}\}/\sim \longrightarrow \{\text{Elliptic curves over } \mathbb{C}\}/\sim$$
$$\tau \mapsto \Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z} \mapsto E_\tau = \mathbb{C}/\Lambda_\tau$$

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- $X_\Gamma(\mathbb{C}) = \Gamma\backslash\mathcal{H}^*$

Cusps

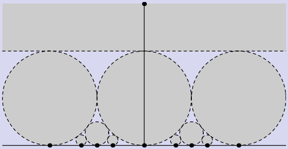
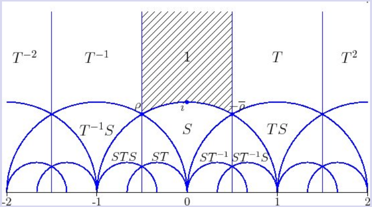


Figure 2.5. Neighborhoods of ∞ and of some rational points

$SL_2(\mathbb{Z})\backslash\mathcal{H}$



Modular Curves

Moduli Spaces Over $\bar{\mathbb{Q}}$

$H \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}), \phi : E[N] \to \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$

$(E, \phi) \sim_H (E', \phi') \iff \exists h \in H, \iota : E \to E' \text{ s.t. } h \circ \phi = \phi' \circ \iota$

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- $\Gamma_H \subseteq SL_2(\mathbb{Z}),\ Y_{\Gamma_H} = S(H)$

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- Borel - $\Gamma_0(p)$
- Normalizer of split (non-split) Cartan - $\Gamma_s^+(p)$, $\Gamma_{ns}^+(p)$

Theorem (Serre, 1972)

For $p > 13$, $H \subseteq GL_2(\mathbb{F}_p)$ exceptional, the modular curve X_{Γ_H} has no rational points.

Theorem (Mazur, 1978)

For $p > 37$, the modular curve $X_0(p)$ has no non-CM, non-cuspidal rational points.

Theorem (Bilu, Parent, Rebolledo, 2011)

For $p > 13$, the modular curve $X_s^+(p)$ has no non-CM, non-cuspidal rational points.

Conjecture (Serre's uniformity conjecture)

For $p > 11$, the only $\mathbb{Q}$-points of the modular curve $X_{ns}^+(p)$ are CM.

Numerical Evidence

Theorem (Balakrishnan, Dogra, Müller, Tuitman, Vonk, 2017)

The modular curve $X_{ns}^{+}(13)$ has exactly 7 rational points, all of which are CM.

Theorem (Mercuri, Schoof, 2018)

For $p = 17, 19, 23$, there are no "small" rational points on $X_{ns}^{+}(p)$, other than the seven CM points.

Explicit equations

Theorem (Baran, 2014)

The modular curve $X_{ns}^{+}(13)$ is defined by the equation

$$(-y-z)x^3 + (2y^2 + zy)x^2 +$$
$$(-y^3 + zy^2 - 2z^2y + z^3)x + (2z^2y^2 - 3z^3y) = 0.$$

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- $\Delta(\tau) = (60 G_4(\tau))^3 - 27(140 G_6(\tau))^2 \in \mathcal{S}_{12}(SL_2(\mathbb{Z}))$

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- $j(\tau) = 1728 \frac{(60 G_4(\tau))^3}{\Delta(\tau)} \in \mathcal{A}_0(SL_2(\mathbb{Z}))$

Results

Problem

Given a group $H \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$, an integer $k \geqslant 2$ and a positive integer L, find the q-expansion of a basis for $S_k(\Gamma_H)$ up to precision q^L. (Known for $\Gamma_0(N), \Gamma_1(N)$ - Stein, Cremona)

Theorem (A., 2020)

There exists an algorithm that given a group $H \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$ satisfying (*), an integer $k \geqslant 2$, and a positive integer L, returns the q-expansions of a basis of $S_k(\Gamma_H)$ up to precision q^L.

Corollary (Banwait, Cremona, 2014)

The modular curve $X_{S_4}(13)$ is a genus 3 curve whose canonical embedding in $\mathbb{P}^2_{\mathbb{Q}}$ has the model

$$4x^3y - 3x^2y^2 + 3xy^3 - x^3z + 16x^2yz - 11xy^2z + \\ +5y^3z + 3x^2z^2 + 9xyz^2 + y^2z^2 + xz^3 + 2yz^3 = 0.$$

Results

Corollary (Baran, 2014)

The modular curves $X_{ns}^{+}(13)$ and $X_{s}^{+}(13)$ are defined by the equation

$$(-y-z)x^3 + (2y^2+zy)x^2 + (-y^3+zy^2-2z^2y+z^3)x + (2z^2y^2-3z^3y) = 0.$$

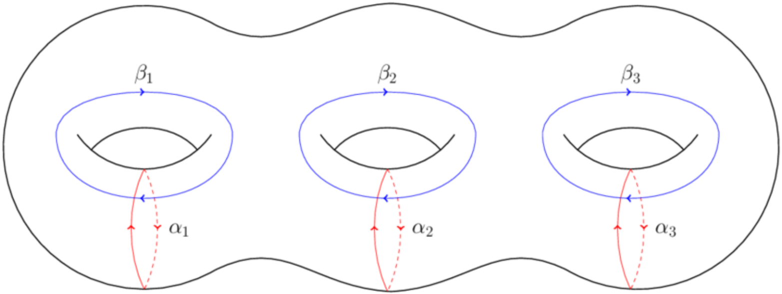
Similar results for $X_{ns}^{+}(17), X_{ns}^{+}(19)$ and $X_{ns}^{+}(23)$.

Corollary (A., 2020)

The Jacobian of the modular curve $X_{ns}^{+}(97)$ decomposes as the direct sum of 13 Hecke irreducible subspaces, of dimensions $3, 4, 4, 6, 7, 7, 12, 14, 24, 24, 24, 56, 168$. *In particular, it has no elliptic curve factor.*

Modular Symbols

$H_1(X_0(39), \mathbb{Z})$



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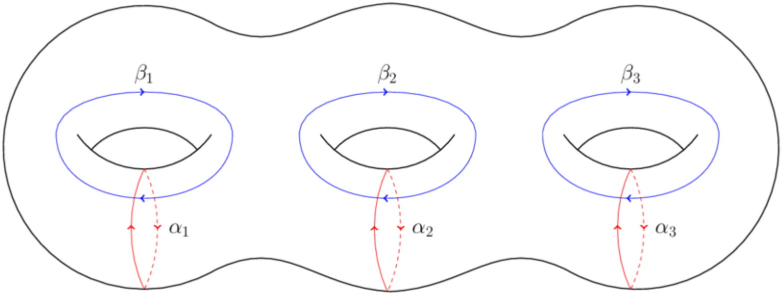

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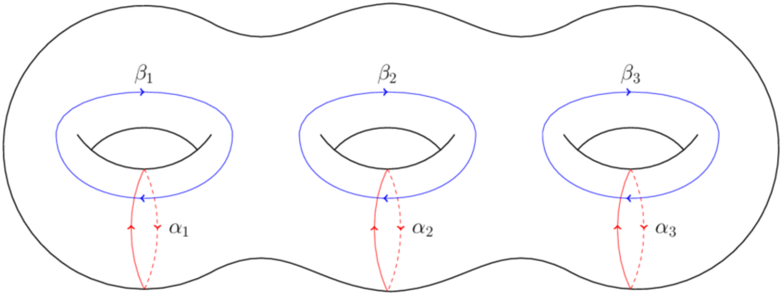

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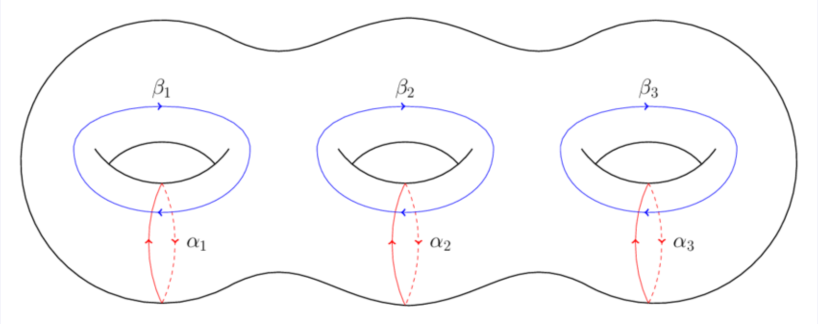

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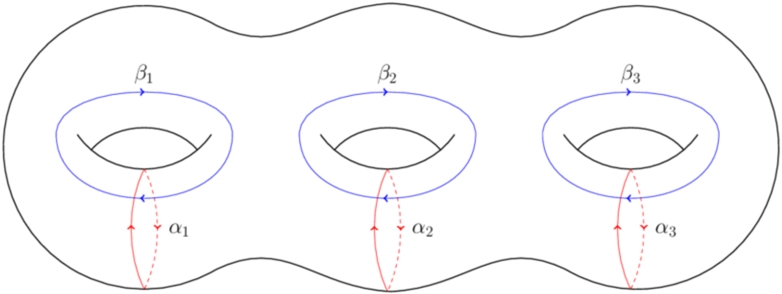

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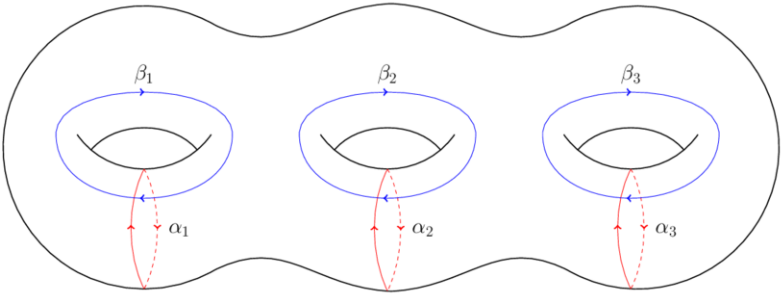

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- $\langle\{\alpha z_1, \alpha z_2\}, \omega\rangle = \langle\{z_1, z_2\}, \omega \circ \alpha\rangle$

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Example

$X^3 \otimes \{0, 1/2\} - 17XY^2 \otimes \{\infty, 1/7\} \in \mathbb{M}_5$

Theorem (Manin, 1972)

$\varphi : \mathbb{M}_2(\Gamma) \to H_1(X_\Gamma, cusps, \mathbb{Z})$ is an isomorphism.

Modular Symbols

Pairing with modular forms

$$\left(\mathcal{S}_k(\Gamma) \oplus \bar{\mathcal{S}}_k(\Gamma)\right) \times \mathbb{M}_k(\Gamma) \to \mathbb{C}$$

$$\langle (f_1, f_2), P\{\alpha, \beta\} \rangle = \int_\alpha^\beta f_1(z) P(z,1) dz + \int_\alpha^\beta f_2(z) P(\bar{z},1) d\bar{z}$$

Cuspidal modular symbols

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- $\mathbb{S}_k(\Gamma) = \ker(\partial : \mathbb{M}_k(\Gamma) \to \mathbb{B}_k(\Gamma))$

Theorem (Shokurov, 1980 + Merel, 1994)

The pairing

$$\langle \cdot, \cdot \rangle : \left(\mathcal{S}_k(\Gamma) \oplus \bar{\mathcal{S}}_k(\Gamma)\right) \times \mathbb{S}_k(\Gamma; \mathbb{C}) \to \mathbb{C}$$

is a nondegenerate pairing of complex vector spaces

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- Great for computation!
- Can compute the vector space $\mathbb{S}_k(\Gamma) = \left(\mathcal{S}_k(\Gamma) \oplus \bar{\mathcal{S}}_k(\Gamma)\right)^\vee$.
- If Γ is of real type, $\mathcal{S}_k(\Gamma) = (\mathbb{S}_k(\Gamma)^+)^\vee$, so also $\mathcal{S}_k(\Gamma)$.

That's great, but what about q-expansions?

Hecke Operators

Hecke operators

$$T_\Delta : \mathcal{M}_k(\Gamma) \to \mathcal{M}_k(\Gamma)$$
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- Commuting normal operators $\Rightarrow$ basis of common eigenvectors
- For $\Gamma = \Gamma_0(N), \Gamma_1(N),\ T_p = T_{\alpha_p}$ satisfies $T_p f = a_p(f) f$

That's great, but what about arbitrary Γ?

Hecke Operators

Proposition

Assume $p \nmid N$. Let $\alpha \in M_2(\mathbb{Z})$ be such that $\det(\alpha) = p$ and $\lambda_N(\alpha) \in H$. Then T_α is independent of α, and if $f = \sum_{n=1}^{\infty} a_n q^n$ is an eigenform of the Hecke algebra then $T_\alpha f = a_p f$.

Question

What about $p \mid N$?

Computational Aspects

Merel pair

Δ, $\phi : \tilde{\Delta} \cdot SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z})$ s.t.

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- $\Gamma \backslash \Delta \hookrightarrow \tilde{\Delta} \cdot SL_2(\mathbb{Z}) / SL_2(\mathbb{Z})$

Theorem (Merel, 1994)

(Δ_n, ϕ_n) Merel pair for Γ with $\Delta_n \subseteq M_2(\mathbb{Z})_n$. Let $\sum_M u_M M$ satisfy

$$\sum_{M \in K} u_M([M\infty] - [M0]) = [\infty] - [0].$$

in $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$. Then in $\mathbb{M}_k(\Gamma)$

$$T^{\vee}_{\Delta_n}([P, g]) = \sum_M u_M [P|_{\tilde{M}}, \phi_n(gM)]$$

where the sum is over M such that $gM \in \tilde{\Delta}_n SL_2(\mathbb{Z})$.

Computational Aspects

Corollary (A., 2020)

Computation of the Hecke operator T_p on $\mathcal{S}_k(\Gamma_H)$, for $p \in \det(H)$, can be done in $O(p \log p)$ basic CosetIndex *operations.*

Complexity of T_α with $(\det(\alpha), N) > 1$ is dominated by the cost of conjugation, done using Farey symbols -

Theorem (A., 2020)

There exists an algorithm that given a congruence subgroup of real type $\Gamma \subseteq SL_2(\mathbb{Z})$ of level N, an element $\alpha \in GL_2^+(\mathbb{Q})$ such that $\eta^{-1}\alpha\eta \in \Gamma\alpha\Gamma$ and an integer $k \geqslant 2$, computes the Hecke operator T_α corresponding to the double coset $\Gamma\alpha\Gamma$, on the space of cusp forms $S_k(\Gamma)$, in complexity

$$O(C \cdot I_{\alpha,\Gamma} \log(N^2 \cdot D(\alpha)) + [SL_2(\mathbb{Z}) : \Gamma]^2 \cdot \mathrm{In}).$$

Computational Aspects

Corollary (A., 2020)

There exists an algorithm that given a group of real type $H \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$ with surjective determinant such that for all $p \mid N$ the Hecke operator is effectively computable, an integer $k \geqslant 2$, and a positive integer L, returns the q-expansions of a basis of eigenforms for $\mathcal{S}_k(\Gamma_H)$ using

$$O(d(C \log N(L \log L + N) + NI_H^2 \cdot \mathrm{In} + kI_H \log(kI_H)) + d^3)$$

field operations, where $d := \dim S_k(\Gamma_H)$, $I_H := [SL_2(\mathbb{Z}) : \Gamma_H]$, C is the cost of a CosetIndex *operation, and* In *is the cost of membership testing in H.*

Other issues

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- Characters, diamond operators
- Degeneracy maps and newforms
- Real type
- Non-surjective determinant. ($p \notin \det(H)$)
- Fast implementation (boundary map, sparse linear algebra, computing intersection and conjugation)

Thanks for listening!