A criterion for integral structures in p-adic representations of $U_3(F)$

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Abstract

Let F be a p-adic field, R a commutative complete discrete valuation ring, and $\mathcal{L}$ a $U_3(F)$-equivariant coefficient system of free R-modules of finite type over the tree of $U_3(F)$. In this talk, extending the work of Vigneras on the group $GL_2(F)$, we give a necessary and sufficient criterion for the degree 0 homology of $\mathcal{L}$ to be a free R-module. This allows us to construct integral structures in locally algebraic representations of $U_3(F)$, and by reduction to show that representations of $U_3(F)$ over a finite field of characteristic p are isomorphic to the degree 0 homology of a system of coefficients. For example, we take a moderately ramified p-adic character $\chi \otimes \chi_1$ of the Levi subgroup $M(F)$ of $U_3(F)$, such that $\chi^{-1}(\pi)$ and $q_E^2\chi(\pi)$ are p-adic integers, π is a uniformizer of E, a quadratic extension of F, and q_E is the order of the residue field of E; Then the principal series of $U_3(F)$ induced, by a smooth non-normalized induction, from $\chi \otimes \chi_1$ is integral with a remarkably explicit integral structure.

1. Self introduction

Hello!

First, I would like to thank the organizers for inviting me to give a talk about my research.

My name is Eran Assaf, a PhD student at the Hebrew University of Jerusalem, under the guidance of Prof. de-Shalit.

This will be a talk about a work in progress - I will present the results obtained thus far, and say something about other expected results.

First, let us state briefly the main result.

2. Statement of the main result

We begin by introducing some notations -

2.1. Notations

Let F be a finite extension of $\mathbb{Q}_p$, E a quadratic extension of F, and V a 3 dimensional vector space over E.

Let $\sigma \in Gal(E/F)$ be the nontrivial involution. We shall often denote $\overline{x} = \sigma(x)$ for $x \in E$.

$\mathcal{O}_E$ is the ring of integers in E, π a uniformizer, and q the cardinality of the residue field.

We shall denote by E^1 the norm one elements in E, i.e. $E^1 = U_1(F) = \{x \in E \mid x\overline{x} = 1\}$.

We shall further denote by C a finite extension of $\mathbb{Q}_p$, and by $\mathcal{O}_C$ its ring of integers.

Denote by θ the Hermitian form on E^3 repersented by the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ with respect to the standard basis, which we will denote by e_1, e_2, e_3.

Let

$$G = U_3(F) = U(\theta) = \{g \in GL_3(E) \mid^t \overline{g}\theta g = \theta\}$$

be the unitary group, that is all the invertible linear transformations on V which preserve θ.

We denote by P the standard Borel subgroup of upper triangular matrices, M is its Levi subgroup, which is the non-split torus

$$M = \left\{ m(t,s) := \begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \overline{t}^{-1} \end{pmatrix} \mid t \in E^\times, s \in E^1 \right\}$$

We denote by N the unipotent radical so that $P = MN$ and

$$N = \left\{ n(b,z) := \begin{pmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{pmatrix} \mid b, z \in E, \quad z + \bar{z} + b\bar{b} = 0 \right\}$$

2.2. *Definitions*

In order to state the results, we will also need the notion of parabolic induction and principal series representation.

Definition 2.1. **(parabolic induction)** Let (ρ, V) be a representation of P. The *smooth parabolic induction* $Ind_P^G \rho$ (sometimes denoted also $Ind_P^G V$) is defined as

$$Ind_P^G \rho = \{ f : G \to V \mid \forall p \in P \quad f(pg) = \rho(p) \cdot f(g), \quad f \text{ smooth} \}$$

This space is equipped with a natural G-action by right translation, explicitly $(gf)(x) = f(xg)$. The functions are smooth with respect to this action.

Definition 2.2. (**principal series representaions**) Let $\chi : E^\times \to C^\times$, $\chi_1 : E^1 \to C^\times$ be multiplicative characters. They define a character, which we denote $\chi \otimes \chi_1$ of P by letting N act trivially, and

$$(\chi \otimes \chi_1)(m(t,s)) = \chi(t) \cdot \chi_1(s)$$

The associated *principal series* representation is $PS(\chi, \chi_1) = Ind_P^G(\chi \otimes \chi_1)$.

We shall also need the definition of an integral structure (equivalently, a separated lattice) in a representation of G.

Definition 2.3. Let V be a representation of G over C. Let L be an $\mathcal{O}_C[G]$-submodule which spans V over C and contains no C-line. We say that V is *integral* and L is an *integral structure.*

Note 2.4. Note that all the representations considered are vector spaces over C, which is a p-adic field!

2.3. *Results*

Theorem 2.5. *(A.) Let $\chi : E^\times \to C^\times$, $\chi_1 : E^1 \to C^\times$ be tamely ramified characters. Then the following are equivalent:*

1. $\chi(\pi)^{-1}, \quad q^2\chi(\pi) \in \mathcal{O}_C$ (equivalently $1 \le |\chi(\pi)| \le |q^{-2}|$)
2. $V = Ind_P^G(\chi \otimes \chi_1)$ is integral.

Note that the field C is p-adic, not complex!

Remark 2.6. If one considers the pro-p Iwahori subgroup $I(1)$ (kernel of the reduction map on I), then V is a module for the pro-p Iwahori-Hecke algebra $\mathcal{H}_C(G, I(1))$, and condition 1 is equivalent to the integrality of this module (namely, V is spanned by a sub $\mathcal{O}_C$-module, stable by $\mathcal{H}_{\mathcal{O}_C}(G, I(1))$) .

2.3.1. *Locally algebraic representations*

Let ρ be a rational algebraic representation of $U_3(F)$ over C. One may consider the locally algebraic representations $V = Ind_P^G(\chi \otimes \chi_1) \otimes \rho$.

Note that this is possible since we are working over a p-adic field (!).

Further, in such a representation, we note that every vector has a compact open neighbourhood which acts on it polynomially. Such a vector (in arbitrary representaion) will be called locally algebraic.

This is a work in progress, and we are currently working on the locally algebraic case, and expect to have in a short time a criterion for the existence of integral structures when $\chi \otimes \chi_1$ is unramified, and under some smallness condition on ρ. Furthermore, in the case of unramified characters we shall be able to show that the resulting completion is admissible, namely

Definition 2.7. (admissible unitary Banach representations) Let B be a unitary Banach representation of G. Denote by B^0 its unit ball with respect to an invariant norm. Then B is said to be admissible if $B^0 \otimes_{\mathcal{O}_C} k_C$ is admissible in the sense of smooth represntations over k_C, i.e. its subspace of invariant elements under any open compact subgroup of G is finite dimensional. This definition is independent of the choice of B^0.

3. Background

3.1. The p-adic local Langlands programme

The original aim of the local p-adic Langlands programme is to look for a possible p-adic analogue of the classical and l-adic correspondence.

The local p-adic correspondence for $GL_2(\mathbb{Q}_p)$ was fully developed, essentially by Berger, Breuil and Colmez in (Berger et al. [2, 3]) using the theory of (φ, Γ)-modules.

If we restrict ourselves to certain representations, called "potentially semistable", on the Galois side, it is possible to attach to it a smooth representation $\pi_{sm}(\rho)$ on the automorphic side, as in the classical case. However, $\rho \rightsquigarrow \pi_{sm}(\rho)$ is no longer reversible.

The notion of potentially semistable, coming from Fontaine's theory is very technical, hence I will not define it here as my work is mainly on the automorphic side, but in a sense, these are the "good" representaions.

Essentially, represenations coming from geometry are potentially semistable, but this is very involved.

We may also construct irreducible algebraic representation $\pi_{alg}(\rho)$ of $GL_2(\mathbb{Q}_p)$ from some data associated to ρ. Still, one cannot reconstruct ρ from $\pi_{sm}(\rho)$ and $\pi_{alg}(\rho)$.

The problem is that in p-adic Hodge theory, these certain ρ are classified by some linear algebra data which includes a certain filtration, called the "Hodge filtration" - and this data is lost when constructing the $\pi_{alg}(\rho)$ and the $\pi_{sm}(\rho)$.

Remark 3.1. Note that as the coefficient field is p-adic, these two representations "live" in the same universe, and it makes sense to consider the representation $\pi_{sm}(\rho) \otimes \pi_{alg}(\rho)$.

The p-adic local Langlands correspondence takes any continuous representation $\rho : Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(C)$ and attaches a Banach C-space $\Pi(\rho)$ with a unitary $GL_2(\mathbb{Q}_p)$-action. This map $\rho \rightsquigarrow \Pi(\rho)$ is reversible, and compatible with classical local Langlands in the following sense: When ρ is potentially semistable,

$$\Pi(\rho)^{alg} = \pi_{alg}(\rho) \otimes_C \pi_{sm}(\rho)$$

Furthermore, $\Pi(\rho)^{alg} = 0$ otherwise.

This means that the existence of locally algebraic vectors is equivalent (!) to the fact that our representation comes from geometry.

Here, the superscript *alg* indicates taking the locally algebraic vectors, i.e. vectors on which some compact open subgroup acts polynomially.

When ρ is irreducible, $\Pi(\rho)$ is known to be the completion of $\pi_{alg}(\rho) \otimes_C \pi_{sm}(\rho)$ relative to a suitable $GL_2(\mathbb{Q}_p)$-invariant norm $\|\cdot\|$ which somehow corresponds to the lost filtration.

3.2. The Breuil Schneider conjecture and integral structures

For groups other than $GL_2(\mathbb{Q}_p)$ very little is known. One of the main conjectures was stated in (Breuil and Schneider [4]) by Breuil and Schneider, and in some sense it is a "first approximation" - for certain $\rho : Gal(\overline{\mathbb{Q}}_p/F) \to GL_n(C)$, one can define the representation $BS(\rho) := \pi_{alg}(\rho) \otimes_C \pi_{sm}(\rho)$, and if it is indeed a subrepresentation of some larger, conjectural, Banach C-space $\Pi(\rho)$ with a unitary $GL_n(F)$-invariant norm, it should admit an invariant $GL_n(F)$-invariant norm. The resulting completions should be closely related to the yet undefined $\Pi(\rho)$ - at least in the irreducible cases.

We say that a representation V is locally algebraic if $V = V^{alg}$.

Conjecture 3.2. *(Breuil, Schneider Breuil and Schneider [4]) The representation V arises from a (potentially semistable) Galois representation if and only if V admits a $GL_n(F)$-invariant norm.*

The "if" part is completely known for $GL_n(F)$ (Hu [8]), and is due to Y. Hu. The "only if" part remains open.

Note that asking for a norm amounts to asking for a lattice: Given a norm $||\cdot||$, the unit ball is a lattice. Conversely, given a lattice Λ, its gauge $||x|| = q_C^{-v_\Lambda(x)}$, where $v_\Lambda(x) = \sup\{v \mid x \in \pi_C^v \Lambda\}$ is a norm. Thus we are looking for integral structures in locally algebraic representations of G.

3.3. Progress on the BS conjecture

- Note that the central character of $BS(\rho)$ always attains values in $\mathcal{O}_C^\times$. Sorensen (Sorensen [12]) has proved for any connected reductive group G defined over $\mathbb{Q}_p$, that if π_{alg} is an irreducible algebraic representation of $G(\mathbb{Q}_p)$, and π_{sm} is an essentially discrete series representation of $G(\mathbb{Q}_p)$, both defined over C, then $\pi_{alg} \otimes_C \pi_{sm}$ admits a $G(\mathbb{Q}_p)$-invariant norm if and only if its central character is unitary.

- Recently there has been spectacular progress on the BS conjecture in the principal series case, which is the deepest, by joint work of Caraiani, Emerton, Gee, Geraghty, Paskunas ans Shin (Caraiani et al. [5]). Using global methods, they construct a candidate $\Pi(\rho)$ for a p-adic local Langlands correspondence for $GL_n(F)$ and are able to say enough about it to prove new cases of the conjecture. Their conclusion is even somewhat stronger than the existence of a norm on $BS(\rho)$, in that it asserts admissibility.

Both works employ the usage of global methods, and as this is a question of local nature, we believe that there must be some local method to recover these results. There has also been some progress employing local methods, which yields results also for finite extensions of $\mathbb{Q}_p$, namely:

- For $GL_2(F)$, de Ieso (De Ieso [6]), following the methods of Breuil for $\mathbb{Q}_p$, used compact induction together with the action of the spherical Hecke algebra to produce a separated lattice in $BS(\rho)$ where ρ is an unramified locally algebraic principal series representation, under some technical p-smallness condition on the weight. In his thesis, de Ieso also shows that these lattices form an admissible completion.

- For $GL_2(F)$, in a joint work with Kazhdan and de Shalit (Assaf et al. [1]), we have used p-adic Fourier theory for the Kirillov model to get integral structures if ρ is tamely ramified smooth principal series or unramified locally algebraic principal series.

- For general split reductive groups, Grosse-Klonne (Große-Klönne [7]) looked at the universal module for the spherical Hecke algebra, and was able to show some cases of the conjecture for unramified principal series, again under some p-smallness condition on the Coxeter number (when $F = \mathbb{Q}_p$) plus other technical assumptions.

3.4. The case of $G = U_3(F)$

As many attempts were made in order to find criteria for the existence of integral structures in representations of $GL_2(F)$, where F is a finite extension of $\mathbb{Q}_p$, and towards the proof of the Breuil-Schneider conjecture, which concerns the case of $GL_n(F)$, and somewhat more generally, the case of split reductive groups, very little is known about the correspondence for non-split reductive groups, and the unitary group, in particular.

We restrict ourselves only to the case of a smooth tamely ramified principal series representation, and give a necessary and sufficient criterion for the existence of such a norm.

We shall use the method of coefficient systems on the Bruhat-Tits building, which is a tree for U_3, introduced by Vigneras in (Vignéras [13]).

This is still a work in progress, as we currently extend the methods of Breuil and de Ieso to obtain similar results for the unramified locally algebraic principal series representations.

4. Coefficient systems on the tree

4.1. The Bruhat-Tits tree of $U_3(F)$

Definition 4.1. Let $L \subset E^3$. The hermitian form $(\cdot, \cdot)$ induces a dual lattice

$$L^\sharp = \{v \in V \mid (v, l) \in \mathcal{O}_E \quad \forall l \in L\}$$

If L is a lattice satisfying $L \subseteq L^\# \subseteq \pi^{-1} L$, we say that L is a *standard lattice.*

Remark 4.2. The vertices of the tree $\mathcal{T}$ consist of equivalence classes (under $\#$) of standard lattices.

We then have two types of vertices - the vertices represented by standard lattices with $L^\sharp = L$, and the vertices represented by pairs of standard lattices $L^\sharp \supsetneq L \supsetneq \pi L^\sharp$.

Let

$$\mathcal{T}_0^0 = \{\sigma \in \mathcal{T}_0 \mid \sigma = [L], \quad L = L^\sharp\}, \quad \mathcal{T}_0^1 = \{\sigma \in \mathcal{T}_0 \mid \sigma = [L], \quad L \neq L^\sharp\}$$

We call $\mathcal{T}_0^0$ vertices of type 0, and $\mathcal{T}_0^1$ vertices of type 1.

Two such vertices, $L_0 = L_0^\sharp$, and $L_1^\sharp \supseteq L_1 \supseteq \pi L_1^\sharp$ are connected with an edge if $L_1^\sharp \supseteq L_0 \supseteq L_1$.

Definition 4.3. We set $\sigma_0 = [L_0]$ where $L_0 = \mathcal{O}_E e_1 + \mathcal{O}_E e_2 + \mathcal{O}_E e_3$, and $\sigma_1 = [L_1]$, where

$$L_1 = \mathcal{O}_E \cdot e_1 + \mathcal{O}_E \cdot e_2 + \pi \mathcal{O}_E \cdot e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} \cdot L_0$$

and $\sigma_{01} = (\sigma_0, \sigma_1) \in \mathcal{T}_1$ is an edge - in fact, the standard chamber.

4.2. Coefficient systems

Coefficient systems were introduced over $\mathbb{C}$ by Schneider and Stuhler [10]. In this section, we follow Paskunas [9] and translate the language of coefficient systems to the group G.

Let $\mathcal{T}$ be the Bruhat-Tits tree of G. Let R be a commutative ring.

Definition 4.4. An *R-coefficient system* $\mathcal{V} = \{V_\sigma\}_\sigma$ is a contravariant functor from the category of simplices in $\mathcal{T}$ (with inclusions as morphisms) to the category of R-modules.

Let $\mathcal{V} = (\{V_\sigma\}_{\sigma \subset \mathcal{T}}, \{r_\sigma^\tau\}_{\sigma \subset \tau})$ be a R-coefficient system on $\mathcal{T}$. We say that $\mathcal{V}$ is *G-equivariant* if for every $g \in G$ and every simplex $\sigma \subset \mathcal{T}$, we have linear maps $g_\sigma : V_\sigma \to V_{g\sigma}$ satisfying the following properties:

- For every $g, h \in G$ and every simplex $\sigma \subset \mathcal{T}$, we have $(gh)_\sigma = g_{h\sigma} \cdot h_\sigma$
- For every simplex $\sigma \subset \mathcal{T}$, we have $1_\sigma = id_{V_\sigma}$.
- For every $g \in G$ and every inclusion $\sigma \subset \tau$, the following diagram commutes:

$$\begin{tikzcd} V_\tau \arrow[r, "g_\tau"] \arrow[d, "r_\sigma^\tau"] & V_{g\tau} \arrow[d, "r_{g\sigma}^{g\tau}"] \\ V_\sigma \arrow[r, "g_\sigma"] & V_{g\sigma} \end{tikzcd}$$

Let the denote the stabilizers of σ_0, σ_1 by K_0, K_1 respectively. These are the maximal compacts of G (up to conjugacy). The stabilizer of the edge (σ_0, σ_1) is the Iwahori subgroup of G, $I = K_0 \cap K_1$.

We further note that each G-equivariant coefficient system is, in fact, equivalent to a diagram, i.e.

Definition 4.5. Let R be a commutative ring. An *$R[G]$-diagram* consists of the following data:

- A representation of I on an R-module L_{01}.
- A representation of K_0 on an R-module L_0.
- A representation of K_1 on an R-module L_1.
- $R[I]$-equivariant maps $r_0 : L_{01} \to L_0$, $r_1 : L_{01} \to L_1$.

We will refer to a diagram as a quintuple $(L_{01}, L_0, L_1, r_0, r_1)$, and depict such a diagram as

$$\begin{tikzcd} & L_0 \\ L_{01} \arrow[ur, "r_0"] \arrow[dr, "r_1"'] & \\ & L_1 \end{tikzcd}$$

Remark 4.6. The word "diagram" was introduced by Paskunas Paskunas [9] in his construction of supersingular irreducible representations of $GL_2(F)$ on finite fields of characteristic p, and there is an equivalence of categories between $R[G]$-diagrams and G-equivariant coefficient systems on $\mathcal{T}$.

We may now consider the complex of chains with finite support - $C_i(\mathcal{V})$, with the obvious boundary maps, and a natural G-action, and its 0-th homology, which will be denoted by $H_0(\mathcal{V})$.

5. Integrality of irreducible locally algebraic representations

Let V be an irreducible locally algebraic C-representation of G. Then by (Schneider et al. [11], Appendix, Thm 1), $V = V_{sm} \otimes_C V_{alg}$, where V_{sm} is a uniquely determined irreducible smooth representation and V_{alg} is a uniquely determined algebraic one. We will present the first local integrality criterion for $V_{sm} \otimes V_{alg}$, by a purely representation theoretic method. The idea, due to Vigneras (Vignéras [13]) is to realize $V_{sm} \otimes V_{alg}$ as the 0-homology of a G-equivariant coefficent system on the tree.

First, we consider any representation V, and formulate a criterion in terms of a given coefficient system.

Corollary 5.1. *Let R be a complete discrete valuation ring of fraction field S, and let $r_0 : V_{01} \to V_0$, $r_1 : V_{01} \to V_1$ be the maps in the $R[G]$-diagram corresponding to a G-equivariant R-coefficient system $\mathcal{V}$. Assume r_0, r_1 are injective. The S-representation $H_0(\mathcal{V})$ of G is R-integral if and only if there exist R-integral structures L_0, L_1 of the representations V_0 of K_0, V_1 of K_1, such that $L_{01} = r_0^{-1}(L_0) = r_1^{-1}(L_1)$.*
When this is true, the diagram

$$
\begin{tikzcd}
 & L_0 \\
L_{01} \arrow[ur, "r_0"] \arrow[dr, "r_1"'] & \\
 & L_1
\end{tikzcd}
$$

defines a G-equivariant coefficient system $\mathcal{L}$ of R-modules on $\mathcal{T}$, and $H_0(\mathcal{L})$ is an R-integral structure of $H_0(\mathcal{V})$.

The next idea is to begin with some coefficient system and some integral srtucture in one of the modules. We then generate an integral coefficient system as above by a method of zig-zagging.

Definition 5.2. When V_i, for $i = 0, 1$ identified with an element of $\mathbb{Z}/2\mathbb{Z}$, contains an R-integral structure M_i which is a finitely generated R-submodule, one constructs inductively an increasing sequence of finitely generated R-integral structures $(z^n(M_i))_{n \geq 1}$ of V_i, called the *zigzags* of M_i, as follows:
The $R[K_{i+1}]$-module M_{i+1} defined by $M_{i+1} = K_{i+1} \cdot r_{i+1}(r_i^{-1}(M_i))$ is an R-integral structure of the $S[K_{i+1}]$-module V_{i+1} (a finitely generated R-module is free if and only if it is torsion free and does not contain a line). We repeat this construction to get the first zigzag $z(M_i)$:

$$z(M_i) = K_i \cdot r_i \left(r_{i+1}^{-1} \left(K_{i+1} \cdot r_{i+1} \left(r_i^{-1}(M_i) \right) \right) \right)$$

Corollary 5.3. *Let $i \in \mathbb{Z}/2\mathbb{Z}$ and let M_i be an R-integral structure of the $S[K_i]$-module V_i. The representation of G on $H_0(\mathcal{V})$ is R-integral if and only if the sequence of zigzags $(z^n(M_i))_{n \geq 0}$ is finite.*

The main idea allowing us to make use of the above criterion for arbitrary irreducible locally algebraic representations, is the fact that any such representation can be obtained as the 0-homology of some coefficient system on the tree. This was shown for smooth representations over $\mathbb{C}$ by Schneider and Stuhler in Schneider and Stuhler [10], and we will extend the result further here. The proof is the same as in Vignéras [13] for the case $G = GL_2(F)$.

For the sake of the following proposition, we recall that K_0, K_1, I have pro-p subgroups, which will be denotes by $K_0(1), K_1(1), I(1)$.

Proposition 5.4. *Let V_{alg} be an irreducible algebraic C-representation of G, let V_{sm} be a finite length smooth C-representation of G and assume that V_{sm} is generated by its $K_0(1)$-invariants.*

1) The locally algebraic C-representation $V := V_{sm} \otimes_C V_{alg}$ of G is isomorphic to the 0-th homology $H_0(\mathcal{V})$ of the coefficient system $\mathcal{V}$ associated with the inclusions

$$\begin{tikzcd} & V_0 = V_{sm}^{K_0(1)} \otimes_C V_{alg} \\ V_{sm}^{I(1)} \otimes_C V_{alg} \arrow[ur, "r_0"] \arrow[dr, "r_1"'] & \\ & V_1 = V_{sm}^{K_1(1)} \otimes_C V_{alg} \end{tikzcd}$$

2) The representation of G on V is $\mathcal{O}_C$-integral if and only if there exist $\mathcal{O}_C$-integral structures L_0, L_1 of the representations of K_0, K_1 on V_0, V_1 such that $L_{01} := r_0^{-1}(L_0) = r_1^{-1}(L_1)$. Then the 0-th homology L of the G-equivariant coefficient system on $\mathcal{T}$ defined by the diagram

$$\begin{tikzcd} & L_0 \\ L_{01} \arrow[ur] \arrow[dr] & \\ & L_1 \end{tikzcd}$$

is an $\mathcal{O}_C$-integral structure of V.

5.1. *Inflation*

The next idea is to reduce our problem to a problem concerning groups over finite field, which is done by reduction and inflation.

Since our characters are tamely ramified, and the quotients are finite groups, our diagrams are equivalent to diagrams of representations of groups over finite fields which will be easier to handle.

We note that here there is a difference between the ramified and unramified case.

5.2. *Sketch of the proof of the main theorem*

One begins with the coefficient system introduced by Schneider and Stuhler, which is equivalent to a tamely ramified diagram.

We may now consider a natural candidate - the module $L_0 \subset (Ind_P^G \chi)^{K_0(1)}$ of integral-valued functions, and begin the process of zig-zag.

As all the representations are equivalent to representations of groups over finite fields, we have reduced our problem to that of stablizing a certain sequence of representations over finite fields.

Quite surprisingly, it is solved by using the Fourier transform for finite groups.

5.2.1. *Remark - Wild ramification and Locally algebraic representations*

Essentially, this zig-zag method, due to Vigneras, should work also for the wildly ramified cases, by incorporating the $K_i(e)$ and $I(e)$-invariants, for arbitrary e. However, the resulting computations for the groups over the ring $A = \mathcal{O}_E/\pi^e\mathcal{O}_E$ indicate that the zig-zag sequence does not stabilize after a few steps, and calculations become increasingly difficult.

The same could be said about the locally algebraic representations. The method should work, but the calculations become very complicated.

5.2.2. *Remark - higher rank groups*

This example shows that when the Bruhat-Tits building is a tree, one can incorporate the zig-zag method effectively. The question for groups of higher rank remains. It seems that one can develop a similar method of zig-zag for diagrams on the Bruhat-Tits building, but there are still some set-backs to overcome.

5.3. Locally algebraic case

Our work on the locally algebraic case currently follows closely the ideas of Breuil and de Ieso, by considering the action of the spherical Hecke algebras (note that there are two of them) on the coefficient systems on the tree.

As in the work by de Ieso, we believe that in these cases we will also be able to show admissibility.

This is backed also by the recent work of Grosse-Klonne, who uses the universal spherical Hecke algebra to obtain the criterion for split reductive groups over $\mathbb{Q}_p$, and it seems that it will be able to generalize his results to groups over F.

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