General Shimura Varieties

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Abelian Motives

Definition (Abelian motive)

An abelian motive over $\mathbb{C}$ *is a triple* (V, e, m) *such that* V *is a variety over* $\mathbb{C}$ *whose connected components are abelian varieties,* $e \in \mathrm{Corr}^0(V, V)$ *is an idempotent, and* $m \in \mathbb{Z}$*.*

Conjecture ([Mil13, Conjecture C], Murre, 1993)

In the ring $\mathrm{End}(hX) = \mathrm{Corr}^0(X, X) = A_{\dim X}(X \times X)$*, the diagonal* Δ_X *has a canonical decomposition into a sum of orthogonal idempotents*

$$\Delta_X = \pi_0 + \ldots + \pi_{2n}$$

This induces a decomposition

$$hX = h^0X \oplus h^1X \oplus \ldots \oplus h^{2n}X$$

which maps to

$$H^\bullet(X) = H^0(X) \oplus H^1(X) \oplus \ldots \oplus H^{2n}(X)$$

Category of abelian motives

Example (Projectors)

Let A be an abelian variety. Let $\pi_i \in \mathrm{Corr}^0(X,X)$ be the idempotent from Murre's conjecture. Then it induces a projection

$$H^\bullet(A,\mathbb{Q}) \to H^i(A,\mathbb{Q}) \subseteq H^\bullet(A,\mathbb{Q})$$

Denote $h^i(A) = (A, \pi_i, 0)$.

Proposition (Properties)

The category of abelian motives, AM, admits biproducts, tensor products and duals, which satisfy

$$(V,e,m) \oplus (V',e',m) = \left(V \sqcup V', e+e', m\right)$$

$$(V,e,m) \otimes (V',e',m') = \left(V \times V', e\cdot e', m+m'\right)$$

$$(V,e,m)^\vee = (V, e^t, d-m)$$

Polarizable Hodge Structures

Proposition

Hod($\mathbb{Q}$) *the category of polarizable rational Hodge structures is abelian, closed under tensor products and duals. Moreover, it is semisimple.* $\implies$ Hod($\mathbb{Q}$) $\simeq \mathrm{Rep}_{\mathbb{Q}}(G_{\mathrm{Hod}})$, $h_{\mathrm{Hod}} : \mathbb{S} \to G_{\mathrm{Hod}}$.

Proof.

Let $\mathcal{C} = \mathrm{Rep}_{\mathbb{Q}}(\mathbb{S})$ be the category of all rational Hodge structures.
As a category of rep., it is abelian with tensor products and duals.
$(0, \phi)$ is the zero object. (polarizable condition is empty).
Biproducts are polarizable by $\psi_V + \psi_W$, kernels are polarizable by restriction. Have to check cokernels. But polarization induces $f(V)^{\perp} \cong W/f(V)$. This also shows semisimplicity.
Tensor products by taking $\psi_V \otimes \psi_W$ on pure weights.
Duals since polarization induces $V^{\vee} \simeq V$. □

Abelian Hodge Structures

Definition

(V, e, m) *abelian motive.* $H(V, e, m) = eH^\bullet(V, \mathbb{Q})(m)$.

Proposition

The functor

$$(V, e, m) \rightsquigarrow H(V, e, m) : AM \to \mathrm{Hod}(\mathbb{Q})$$

commutes with $\oplus, \otimes, {}^\vee$.

Proof.

If V is connected,

$$H^\bullet(V, \mathbb{Q}) \simeq \bigwedge H^1(V, \mathbb{Q}) \simeq \mathrm{Hom}_{\mathbb{Q}}\left(\bigwedge H_1(V, \mathbb{Q}), \mathbb{Q}\right)$$

inducing a polarizable Hodge structure.

The rest is additivity, Künneth and Poincare for cohomology. □

Abelian Hodge Structures

Definition (Abelian Hodge structure)

(W, h) is abelian if it is in the essential image, iso. to $H(V, e, m)$.

Example (Tate)

E elliptic curve, then $\bigwedge^2 H_1(E, \mathbb{Q}) \simeq \mathbb{Q}(1)$, hence $\mathbb{Q}(1)$ is abelian.

Proposition ([Mil05, Proposition 9.1])

The category $\mathrm{Hod}^{ab}(\mathbb{Q})$ is the smallest strictly full subcategory of $\mathrm{Hod}(\mathbb{Q})$ containing $H_1(A, \mathbb{Q})$ for each abelian variety A and closed under direct sums, subquotients, duals and tensor products.
Moreover, $H : AM \to \mathrm{Hod}^{ab}(\mathbb{Q})$ is an equivalence of categories.
$\implies \mathrm{Hod}^{ab}(\mathbb{Q}) \simeq \mathrm{Rep}_{\mathbb{Q}}(G_{Mab})$, $\rho : G_{\mathrm{Hod}} \to G_{Mab}$.

Hodge Structures of CM-type

Definition (CM-type)

$(V, h) \in \mathrm{Hod}(\mathbb{Q})$ *is of CM-type if* $MT(V, h)$ *is a torus.*

Proposition

The category $\mathrm{Hod}^{cm}(\mathbb{Q})$ *is a Tannakian subcategory of* $\mathrm{Hod}(\mathbb{Q})$*.*

Proposition ([Mil94a, Proposition 4.6])

Every Hodge structure of CM-type is abelian.

Corollary

$\mathrm{Ker}\, \rho : G_{Hod} \to G_{Mab} \subseteq G_{Hod}^{\mathrm{der}}$

$$\mathrm{Hod}_{\mathbb{Q}}^{cm} \hookrightarrow \mathrm{Hod}_{\mathbb{Q}}^{ab} \hookrightarrow \mathrm{Hod}_{\mathbb{Q}}$$

$$S \twoheadleftarrow G_{Mab} \twoheadleftarrow G_{\mathrm{Hod}}$$

Shimura varieties of abelian type

Definition (abelian type)

1. (H, X^+) *is of primitive abelian type if* H *is simple,* $\exists(V, \psi)$, $H \hookrightarrow S(\psi)$ *mapping* X^+ *to* $X(\psi)$.
2. (H, X^+) *is of abelian type if* $\exists(H_i, X_i^+)$ *primitive abelian, isogeny* $\prod_i H_i \to H$, *mapping* $\prod_i X_i^+$ *to* X^+.
3. (G, X) *is of abelian type if* (G^{der}, X^+) *is of abelian type.*

Theorem ([Mil94b, Theorem 1.27])

$h : \mathbb{S} \to \mathbb{G}_{\mathbb{R}}$ *s.t.*

- *(SV1)* $\mathrm{Ad} \circ h$ *is of type* $\{(1, -1), (0, 0), (-1, 1)\}$.
- *(SV2*)* $\mathrm{ad}\, h(i)$ *is a Cartan involution of* $G / w_h(\mathbb{G}_m)$.
- *(SV4)* $w_h : \mathbb{G}_m \to G_{\mathbb{R}}$ *is defined over* $\mathbb{Q}$, *and maps to* $Z(G)$.

$G = MT(V, h)$ *for* $(V, h) \in \mathrm{Hod}^{ab}\,\mathbb{Q}$ *iff* (G, h) *is of abelian type.*

Proof.

Since h satisfies (SV2*), (SV4), $\exists! \rho(h) : G_{\text{Hod}} \to G$ s.t. $h = \rho(h)_{\mathbb{R}} \circ h_{\text{Hod}}$.
$\rho(h)$ factors through G_{Mab} iff $\rho(h)|_{G^{\text{der}}}$ factors through G^{der}_{Mab}.
If (G,h) abelian, $\exists \alpha : \prod G_i^{\text{der}} \to G^{\text{der}} : G_i^{\text{der}} \hookrightarrow S_i(\psi)$, $\alpha \circ \prod h_i = h$. $\rho(h_i)$ factors through G_{Mab}, so $\rho(h_i)|_{G_i^{\text{der}}}$ factors through G^{der}_{Mab}, hence so does $\rho(h)|_{G^{\text{der}}} = \alpha \circ \rho(\prod h_i)|_{\prod G_i^{\text{der}}}$.
Category where G_{Mab} action factors through G is in $\langle h_1(A) \rangle$, so can replace G by $MT(A)$ showing ($\Leftarrow$), ($\Rightarrow$) holds for $MT(A)$. □

Proposition ([Mil05, Proposition 9.3])

1. *(SV4)* $w_X : \mathbb{G}_m \to G$ *is rational.*
2. *(SV6)* $Z(G)^\circ$ *splits over a CM-field.*
3. $\exists \nu : G \to \mathbb{G}_m$ *s.t.* $\nu \circ w_X = -2$. *(so* $\mathbb{Q}(1) \in \langle (V,h) \rangle$*)*

If (G,X) *abelian,* $(V, \rho \circ h)$ *abelian* $\forall (V,\rho) \in \text{Rep}(G), h \in X$.
If $(V, \rho \circ h)$ *abelian,* ρ *faithful,* (G,X) *abelian.*

Proof.

$G' = \langle X \rangle$. (SV3) $\implies$ $\{\mathrm{ad} \circ h \mid h \in X\}$ generates G^{ad}, hence $G'/G' \cap Z(G) = G^{\mathrm{ad}}$, so $[G^{\mathrm{der}} : G'^{\mathrm{der}}] < \infty$.
Both connected $\implies$ $G^{\mathrm{der}} = G'^{\mathrm{der}}$.
By Prop. 5.9, $(V, \rho \circ h)$ is a VHS, so $G' = MT(V, \rho \circ h_0)$.
By the theorem, $(V, \rho \circ h_0) \in \mathrm{Hod}^{ab}(\mathbb{Q})$ iff (G', h_0) abelian iff (G, X) abelian. Can eliminate ρ.
$h = \mathrm{ad}\, g \circ h_0$, $g \in G'(\mathbb{R})^+$, $(V, h_0) \in \langle h_1(A) \rangle$. $(G_1, h_1) = MT(A)$.

$$(G', h_0) \twoheadleftarrow (G_1, h_1) \hookrightarrow (G(\psi), X(\psi))$$

Lift g to $g_1 \in G_1(\mathbb{R})^+$. Then

$$(G', h) \twoheadleftarrow (G_1, \mathrm{ad}(g_1) \circ h_1) \hookrightarrow (G(\psi), X(\psi))$$

Set $G_h = MT(V, h)$, $G_{h,1}$ its preimage in G_1. So G_h is a quotient of $G_{h,1}$, the MT of an abelian variety. □

Moduli Space

Theorem ([Mil05, Theorem 9.4])

*(G, X) abelian s.t. (SV4), (SV6), $\exists \nu : G \to \mathbb{G}_m$ with $\nu \circ w_X = -2$. $\rho : G \hookrightarrow GL(V)$, $\exists \psi : V \times V \to \mathbb{Q}$ s.t. $g\psi = \nu(g)^m \psi$, ψ is a polarization of $(V, \rho \circ h)$.
Fix $t_i : V \times \ldots \times V \to \mathbb{Q}(r_i), 1 \leqslant i \leqslant n$ s.t.*

$$G = \{g\ inGL(V) \mid g\psi = \nu(g)^m \psi, gt_i = t_i\}$$

$\mathrm{Sh}_K(G, X)$ classifies $(A, (s_i)_{i=0}^n, \eta K)/\sim$ s.t.

- *A is an abelian motive.*
- *$\pm s_0$ is a polarization for $H(A)$.*
- *$s_1, \ldots, s_n$ are Hodge tensors for A.*
- *ηK is a K-orbit of $\mathbb{A}_f$-linear isom. $V(\mathbb{A}_f) \to V_f(A)$, sending ψ to an $\mathbb{A}_f^\times$ multiple of s_0, and t_i to s_i.*
- *$\exists a : H(A) \to V$ sending s_0 to a $\mathbb{Q}^\times$-multiple of ψ, each s_i to t_i, and h onto an element of X.*

Classification

Theorem (Deligne, 1979)

(G, X^+) connected, G simple.
If G^{ad} of type A, B, C, then (G, X^+) abelian.
If G^{ad} of type E_6, E_7, then (G, X^+) not abelian.
G^{ad} of type D, can have both. (no $G \to S(\psi)$ or none injective.)

Conjecture (Deligne, 1979)

If (G, X) satisfies (SV4), $\mathrm{Sh}_K(G, X)$ classifies isom. classes of motives with additional structure.

Shimura Varieties of Type A_1

Example (Hilbert Modular Variety)

B quat. over F tot.real. $G = B^\times$.

$$G(\mathbb{R}) \approx \prod_{v \in I_c} \mathbb{H}^\times \times \prod_{v \in I_{nc}} GL_2(\mathbb{R})$$

- $B = M_2(F)$, then (G, X) is of PEL-type (Type (C)):

$$W = F^2, \phi = 1, \alpha^\star = \alpha^T, V_0 = F^2, \psi_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

  $\mathrm{End}_B(W \otimes V_0) = End_F(V_0) = B$, so $G = B^\times$.
  Classifies $(A, i, t, \eta K)$, A abelian variety of dim. $d = [F : \mathbb{Q}]$, $i : F \to \mathrm{End}_{\mathbb{Q}}(A)$ is RM.

Example

- B division algebra, $I_c = \emptyset$ (split at all infinite places). again PEL-type - if L is the splitting field then $V = V(M_2(L))$. Classifies $(A, i, t, \eta K)$, A abelian variety of dim. $d = 2[F : \mathbb{Q}]$, $i : B \to \mathrm{End}_{\mathbb{Q}}(A)$ is QM.
- B division algebra, $I_c \neq \emptyset$. Then (G, X) abelian, not (SV4).

$$w_{X,\mathbb{R}} : \mathbb{R} \to (F \otimes \mathbb{R})^{\times} \cong \prod_{v : F \to \mathbb{R}} \mathbb{R}$$

$$a \mapsto (\ldots, a_i, \ldots)_{i \in I}, \quad a_i = \begin{cases} 1 & i \in I_c \\ a & i \in I_{nc} \end{cases}$$

  $T = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$, so $w_X : \mathbb{G}_m \to T_{\mathbb{R}}$ is defined over $\bar{\mathbb{Q}}^{G_{I_c}}$. Then $\mathrm{Sh}_K(G, X)$ classifies Hodge structures, but not motivic.
- When $|I_{nc}| = 1$, Shimura curves.

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