The Derived Hecke Algebra - Complex Realization

March 7, 2019

1 Brief Reminder and Orientation

1.1 The setup

Let $\mathbb{G}$ be a reductive group defined over $\mathbb{Q}$ which has no central split torus.
Let $K \subseteq \mathbb{G}(\mathbb{A}_f)$ be a compact open subgroup, let K_∞^0 be a maximal connected compact subgroup of $G := \mathbb{G}(\mathbb{R})$ and let

$$Y = Y(K) := \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{R})\times\mathbb{G}(\mathbb{A}_f)/KK_\infty^0$$

be the associated arithmetic manifold.
Let S be the set of places v such that K_v is hyperspecial, and let $\chi = (\chi_v)_{v\in S}$, with $\chi_v : \mathscr{H}(\mathbb{G}(\mathbb{Q}_v), K_v) \to \mathbb{Q}$ be a Hecke character with values in $\mathbb{Q}$.
We consider the set Π of automorphic representations $\pi = \otimes_v \pi_v$ of $\mathbb{G}(\mathbb{A})$ such that:

- $\pi^K \neq 0$
- π_∞ has nonvanishing $(\mathfrak{g}, K_\infty^0)$-cohomology
- For $v \in S$, π_v is spherical and corresponds to the character χ_v.

1.1.1 Remark

Π is a finite set, any two representations in π are nearly equivalent (by the third condition), and therefore belong to a single Arthur packet. (By definition)

1.1.2 Assumptions

We assume that $\Pi - \{\pi_1, \ldots, \pi_h\}$ is nonempty, and that each π_i is cuspidal and tempered at ∞.

1.1.3 Remark

If Arthur's conjecture holds, and χ_v is tempered for one place v, then all the π_i are tempered at ∞.
The cuspidality assumption is in order to avoid non-compactness issues.
Denote

$$H^*(Y,\mathbb{Q})_\Pi = \{h \in H^*(Y,\mathbb{Q}) : Th = \chi_v(T)h \quad \forall T \in \mathscr{H}(\mathbb{G}(\mathbb{Q}_v),K_v) \quad \forall v \in S\}$$

Let $\delta = l_0(\mathbb{G}) = rank(G) - rank(K_\infty)$, and let $q = q_0(\mathbb{G}) = \frac{1}{2}(\dim Y(K) - l_0(\mathbb{G}))$.

1.2 The motivic hidden action conjecture

In this situation, it is conjectured that Π admits a compatible system of Galois representations
(* There is a slight modification when G is not simply connected - it is not needed for the adjoint representations)

$$\rho_l : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to^L G(\overline{\mathbb{Q}}_l)$$

then post-composing with the adjoint and dualizing, we get compatible systems of adjoint and co-adjoint representations

$$\mathrm{Ad}\rho_l : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(\hat{\mathfrak{g}} \otimes \mathbb{Q}_l), \quad \mathrm{Ad}^*\rho_l : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(\tilde{\mathfrak{g}} \otimes \mathbb{Q}_l)$$

where $\hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G})$ and $\tilde{\mathfrak{g}} = \mathrm{Hom}(\hat{\mathfrak{g}},\mathbb{Q})$.
Last talk, Alex formulated the following conjecture (in much more details)

1.2.1 Conjecture

There exist Grothendieck motives of weight 0 associated to Π, $\mathrm{Ad}\Pi$ (M_{ad}), $\mathrm{Ad}^*\Pi$ (M_{coad}) , underlying these representations.
Venkatesh then defines

$$\mathscr{H}_\Pi = H^1_{mot}((\mathrm{Ad}\Pi)_\mathbb{Z}, \mathbb{Q}(1))$$

and we recall the following conjecture

1.2.2 Conjecture (Motivic Hidden action conjecture)

There exists a natural action of $\mathscr{H}_\Pi$ on $H^*(Y(K),\mathbb{Q})_\Pi$ inducing an isomorphism

$$\bigwedge^* \mathscr{H}_\Pi \otimes_\mathbb{Q} H^{q_0}(Y_K,\mathbb{Q})_\Pi \to H^*(Y_K,\mathbb{Q})_\Pi$$

In order to gain evidence for the conjecture, he then proceeds to construct the images of this algebra under the different realizations.

We have already seen the realization in the l-adic case, when $l \neq p$ (p-adic étale), where we have actually constructed a derived Hecke algebra $\mathscr{H}$, and showed that when the localization $\mathscr{H}_\Pi = \mathbb{Z}_p$, it induces the correct action on the étale realization, and next week Anna will talk about the realization in the case $l = p$ (crystalline).

In this talk we will construct the complex (Hodge) realization.

2 Plan of the Talk

2.1 Construction of $\mathscr{H}_\Pi \otimes \mathbb{C}$

First, we will construct $\mathscr{H}_\Pi$ in three different ways:

- As the split part of a fundamental Cartan subalgebra.
- As invariant elements of the Lie group of the torus of the dual group, under the longest Weyl element.
- As the centralizer of a tempered cohomological Langlands parameter.

We will identify the three constructions and show that they are canonical. In each of these constructions we will consider in particular the real structures obtained.

2.2 Description of the action

We will describe the action of $\bigwedge^* \mathscr{H}_\Pi \otimes \mathbb{C}$ on the cohomology of a tempered cohomological representation $H^*(\mathfrak{g}, K_\infty^0; \Pi)$

- Directly, by using the theory of Vogan and Zuckerman parameterizing these representations.
- Using Langlands' classification theorem of such representations as parabolic induction of a discrete series. (Maybe only a sketch)

We will then show that the Beilinson regulator takes values in a real structure in $\mathscr{H}_\Pi \otimes \mathbb{C}$, and use that to define an action on $H^*(Y(K), \mathbb{C})$.

Then we can formulate the main conjecture for this realization.

2.3 Addendum

Venkatesh proceeds to several predictions stemming from this conjecture, which he is able to prove. That is, the conjectured preservation of rational structures should give rise to some invariants visible in the real structures. Then one can compute and see that these agree as expected.

If time permits, I will state some of these properties (probably without proof)

3 Construction(s)

3.1 Construction via fundamental Cartan subalgebra

Let Let $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G)$ be the real Lie algebra, and let $\mathfrak{k}_{\mathbb{R}}$ be the Lie algebra of K_∞. Write $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$.

3.1.1 Running Example

In some sense, the smallest relevant example will be taking $G = SL_4$. In that case, $K_\infty = SO_4$, and we have $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_4(\mathbb{R})$ and $\mathfrak{k}_{\mathbb{R}} = \mathfrak{so}_4(\mathbb{R})$.

3.1.2 Short version (upshot)

Let $\mathfrak{a}_G$ be the split part of a fundamental Cartan subalgebra of $\mathfrak{g}$. Then $\mathscr{H}_\Pi \otimes \mathbb{C} = \mathfrak{a}_G^*$.
(Note that it does not depend on Π) .
But we will want to be more canonical, and also recall some facts and terminology that would be of use.

3.2 Digression on Lie algebras

3.2.1 Definition (Cartan subalgebra)

A subalgebra $\mathfrak{b} \subset \mathfrak{g}$ of a Lie algebra is called a *Cartan subalgebra* of $\mathfrak{g}$ if:
(a) $\mathfrak{b}$ is nilpotent
(b) $\mathfrak{b}$ is its own normalizer ($\mathfrak{b} = \mathfrak{n}(\mathfrak{b}) := \{x \in \mathfrak{g} \mid \mathrm{ad}(x)(\mathfrak{b}) \subseteq \mathfrak{b}\}$)

3.2.2 Fact ([6], Ch. 1)

Any Lie algebra has a Cartan subalgebra. (the nilspace of any regular element $\mathfrak{g}_x^0$), and its dimension is equal to the rank of the Lie algebra.
Thus we can take a Cartan subalgebra $\mathfrak{b}_{\mathbb{R}} \subseteq \mathfrak{k}_{\mathbb{R}}$. Let $\mathfrak{t}_{\mathbb{R}}$ be its centralizer in $\mathfrak{g}_{\mathbb{R}}$, i.e.

$$\mathfrak{t}_{\mathbb{R}} = \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{b}_{\mathbb{R}}) = \{x \in \mathfrak{g}_{\mathbb{R}} \mid \mathrm{ad}(x)(\mathfrak{b}_{\mathbb{R}}) = 0\}$$

3.2.3 Facts ([6], Ch. 1)

Let $B_{\mathbb{R}}$ be a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}_{\mathbb{R}}$.
There exists a (Cartan) decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ such that

$$[\mathfrak{k}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}] \subseteq \mathfrak{k}_{\mathbb{R}}, [\mathfrak{k}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}] \subseteq \mathfrak{p}_{\mathbb{R}}, [\mathfrak{p}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}] \subseteq \mathfrak{k}_{\mathbb{R}}$$

Moreover, $B_{\mathbb{R}}$ is negative definite on $\mathfrak{k}_{\mathbb{R}}$, and positive definite on $\mathfrak{p}_{\mathbb{R}}$, and $B_{\mathbb{R}}(\mathfrak{k}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}) = 0$.

In addition, there exists an involution (a Cartan involution) $\theta \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}})$ such that $B_{\mathbb{R}}$ is θ-invariant, $\theta|_{\mathfrak{k}_{\mathbb{R}}} = 1$, $\theta|_{\mathfrak{p}_{\mathbb{R}}} = -1$, and the bilinear form

$$B_\theta(X,Y) = -B_{\mathbb{R}}(X, \theta(Y))$$

is symmetric and positive definite.

3.2.4 Lemma

There exists a subalgebra $\mathfrak{a}_{\mathbb{R}} \subseteq \mathfrak{t}_{\mathbb{R}}$ such that $\mathfrak{t}_{\mathbb{R}} = \mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{b}_{\mathbb{R}}$.

Proof Note that $\mathfrak{t}_{\mathbb{R}}$ is invariant under θ:
If $t \in \mathfrak{t}_{\mathbb{R}}$, and $b \in \mathfrak{b}_{\mathbb{R}}$ then

$$[\theta(t), b] = [\theta(t), \theta(b)] = [t, b] = 0$$

so that eigenspace decomposition gives $\mathfrak{t}_{\mathbb{R}} = (\mathfrak{t}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{R}}) \oplus (\mathfrak{t}_{\mathbb{R}} \cap \mathfrak{p}_{\mathbb{R}})$. Note also that

$$\mathfrak{b}_{\mathbb{R}} \subseteq \mathfrak{t}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{R}} = \mathfrak{z}_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{b}_{\mathbb{R}}) \subseteq \mathfrak{n}_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{b}_{\mathbb{R}}) = \mathfrak{b}_{\mathbb{R}}$$

Thus, setting $\mathfrak{a}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} \cap \mathfrak{p}_{\mathbb{R}}$, we are done. ■
We denote $\mathfrak{a}_G = \mathfrak{a} := \mathfrak{a}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification, and let $\mathfrak{a}_G^* = Hom_{\mathbb{C}}(\mathfrak{a}_G, \mathbb{C})$.

3.2.5 Running Example

In the case of $\mathfrak{sl}_w(\mathbb{R})$, a Cartan subalgebra of $\mathfrak{so}_4(\mathbb{R})$ can be taken to be

$$\mathfrak{b}_{\mathbb{R}} = \left\{ \begin{pmatrix} & & & x \\ & & y & \\ & -y & & \\ -x & & & \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$$

and then

$$\mathfrak{a}_{\mathbb{R}} = \left\{ \begin{pmatrix} z & & & \\ & -z & & \\ & & -z & \\ & & & z \end{pmatrix} \mid z \in \mathbb{R} \right\}$$

3.2.6 Lemma

For $x \in i \cdot \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$, $\mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g})$ admits only real eigenvalues.

Proof Note that B_θ is positive definite, hence its complexification $B_{\theta,\mathbb{C}}$ is an inner product on $\mathfrak{g}$. Since $B_\mathbb{R}$ was invariant, for any $x, y, z \in \mathfrak{g}_\mathbb{R}$ one has

$$B_\theta([x,y],z) = -B_\mathbb{R}([x,y],\theta(z)) = B_\mathbb{R}(y,[x,\theta(z)]) =$$

$$= -B_\theta(y,[\theta(x),z])$$

In particular, if x is the -1 eigenspace of θ ($i \cdot \mathfrak{k}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$), then we would get that

$$B_\theta(\mathrm{ad}(x)(y),z) = B_\theta(y,\mathrm{ad}(x)(z))$$

hence also that

$$B_{\theta,\mathbb{C}}(\mathrm{ad}(x)(y),z) = B(y,\mathrm{ad}(x)(z))$$

so that $\mathrm{ad}(x)$ is a Hermitian operator on $\mathfrak{g}$, hence has only real eigenvalues.■

3.3 First construction - uniqueness

3.3.1 Proposition ([4], 3.1.1)

Let $\mathfrak{a}, \mathfrak{b}$ be as above. Let $x \in i\mathfrak{b}_\mathbb{R}$ be a regular element (can think of it as a choice of a positive system of roots). Denote

$$\mathfrak{g}_x^\lambda = \{y \in \mathfrak{g} \mid \exists m \quad (\mathrm{ad}(x) - \lambda)^m y = 0\}$$

and let $\mathfrak{q} = \bigoplus_{\lambda \geq 0} \mathfrak{g}_x^\lambda$ be the Borel subalgebras corresponding to $\mathfrak{g}_x^0$ and $\{\lambda \mid \lambda \geq 0\}$.
(Note that $\mathfrak{g}_x^0$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$, hence $\mathfrak{g}_x^0 = \mathfrak{t}$)
Then $\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{b}$ can be identified as the torus quotient of $\mathfrak{q}$. The action described on $\mathfrak{a}, \mathfrak{b}$ is as this quotient.
Suppose $(\mathfrak{a}, \mathfrak{b}, \mathfrak{q})$ and $(\mathfrak{a}', \mathfrak{b}', \mathfrak{q}')$ arise from $(\mathfrak{b}, x)$ and $(\mathfrak{b}', x')$. Then there exists $g \in \mathbb{G}_\mathbb{R}(\mathbb{C})$ such that $\mathrm{Ad}(g)(\mathfrak{a}, \mathfrak{b}, \mathfrak{q}) = (\mathfrak{a}', \mathfrak{b}', \mathfrak{q}')$ and $\mathrm{Ad}(g)(\mathfrak{a}_\mathbb{R}) = \mathfrak{a}'_\mathbb{R}$. Moreover, any two g, g' induce the same isomorphism $\mathfrak{a} \to \mathfrak{a}'$.

Proof (Run along with the example)

1. For uniqueness, note that $g^{-1}g' \in N_G(\mathfrak{q}) = Q$, the Borel subgroup, which acts trivially on its torus quotient (since it is abelian).

2. Different choices of $\mathfrak{b}$ and positive systems for $\Delta(\mathfrak{k} : \mathfrak{b})$ are conjugate in K_∞^0 (all Cartan subgroups are conjugate, and so are all the Borel with a specified maximal torus), so we may assume that $\mathfrak{b} = \mathfrak{b}'$ and that $\mathfrak{q}, \mathfrak{q}'$ induce the same positive system on $\mathfrak{k}$. Then also $\mathfrak{a} = \mathfrak{a}'$.

3. Prove that there exists $w \in \mathbb{G}_\mathbb{R}(\mathbb{C})$ such that

$$w\mathfrak{q}' = \mathfrak{q}, w\mathfrak{b} = \mathfrak{b}, w\mid_a = Id$$

   (a) Let $M = Z_{G_\mathbb{C}}(\mathfrak{a})$. As a centralizer of a torus, it is a Levi subgroup. Let $\mathfrak{m} = \mathrm{Lie}(M)$.

$$\mathfrak{m} = \left\{ \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix} \right\}$$

(b) One checks that $\Delta(\mathfrak{g}:\mathfrak{b})$ is a (not necessarily reduced) root system, inside the dual of ${}^{i\mathfrak{b}_{\mathbb{R}}}/_{i(\mathfrak{z}_{\mathbb{R}}\cap\mathfrak{b}_{\mathbb{R}})}$, which one regards as elements in $i\mathfrak{b}_{\mathbb{R}}^*$ that vanish on the centre.

$$\Delta(\mathfrak{g}:\mathfrak{b}) = \{(\pm 2, 0), (\pm 1, \pm 1), (0, \pm 2)\}$$

(c) Lemma 3.1.1: For any $w \in W_M$, the Weyl group of $\Delta(\mathfrak{m}:\mathfrak{b})$, there is $\tilde{w} \in N_M(\mathfrak{b})$ representing it: Enough to consider G semisimple and $w = s_\beta$. Then there is some $\beta^* \in \Delta(\mathfrak{m}:\mathfrak{t})$ such that $\beta = \beta^* |_{\mathfrak{b}}$. Since this is a maximal torus, there is $\tilde{w} \in N_M(\mathfrak{t})$ representing it. Since it preserves $\mathfrak{a}$ ($M = Z_G(\mathfrak{a})$), by using orthogonality via B, it preserves $\mathfrak{b}$.

$$\Delta(\mathfrak{m}:\mathfrak{b}) = \{(\pm 2, 0), (0, \pm 2)\}$$

If $\beta = (2, 0)$ for example, then we get $\beta^* = (2, 0, 0)$ coming from the root space $X_{\beta^*} = E_{1,1} + iE_{1,4} + iE_{4,1} - E_{4,4}$, with

$$Y_{\beta^*} = E_{1,1} - iE_{1,4} - iE_{4,1} - E_{4,4}$$

Then we take

$$\tilde{w} = \begin{pmatrix} i & & & \\ & 1 & & \\ & & 1 & \\ & & & -i \end{pmatrix}$$

(d) Lemma 3.1.2: If $\mathscr{C}, \mathscr{C}'$ are chambers for $\Delta(\mathfrak{g}:\mathfrak{b})$ that lie in a fixed chamber for $\Delta(\mathfrak{k}:\mathfrak{b})$, there is a $w_M \in W_M$ such that $w_M\mathscr{C} = \mathscr{C}'$. In particular, the map

$$W_M \longrightarrow W_G/W_K$$

is surjective:
Positive chamber for $\Delta(\mathfrak{k}:\mathfrak{b})$ is divided by hyperplanes orthogonal to $\Delta(\mathfrak{g}:\mathfrak{b})$. If each is orthogonal to a root in $\Delta(\mathfrak{m}:\mathfrak{b})$, the corresponding reflection will allow us to move between the chambers. Thus, it suffices to check that each root line of $\Delta(\mathfrak{g}:\mathfrak{b})$ is either in $\Delta(\mathfrak{k}:\mathfrak{b})$ or in $\Delta(\mathfrak{m}:\mathfrak{b})$. Reduces to simple groups with $\delta > 0$ - not many, only inner forms of SL_n and E_6^{split}, $SO_{p,q}$ with both odd, or the restriction from a complex group. Needs only consider cases where $W_K \neq W_G$, which reduces only to $SL_{2n}^{split}, E_6^{split}$ or $SO_{p.q}$ with p, q odd. In each of these three cases, there is a single W_K-orbit for root lines in $\Delta(\mathfrak{g}:\mathfrak{b}) - \Delta(\mathfrak{k}:\mathfrak{b})$, hence it suffices to produce a single root in $\Delta(\mathfrak{m}:\mathfrak{b})$ which is not in $\Delta(\mathfrak{k}:\mathfrak{b})$. Using Vogan diagrams, one constructs it (it is a shaded vertex, which exists in any of these cases - simple root which vanishes on $\mathfrak{a}$, and its root space is in $\mathfrak{p}$).

$$\Delta(\mathfrak{k}:\mathfrak{b}) = \{(\pm 1, \pm 1)\}$$

If one wishes to see an example with more than one line, can consider $G = SO_{3,5}$, where the root groups are $B_2 \times B_1$ inside B_3 (attempt to draw) - the chamber is then divided to three by hyperplanes, and not all of the edges are given by root lines. But the hyperplanes are perpendicular to the new root lines.

(e) The Borel $\mathfrak{q}$ depends only on the chamber of x w.r.t. $\Delta(\mathfrak{g}:\mathfrak{b})$ so $\mathfrak{q}, \mathfrak{q}'$ correspond to $\mathscr{C}, \mathscr{C}'$. But they lie in a fixed chamber for $\Delta(\mathfrak{k}:\mathfrak{b})$, hence , and there is $w \in W_M$ with $w\mathscr{C} = \mathscr{C}'$ by Lemma 3.1.2., whence by Lemma 3.1.1. there is a $\tilde{w} \in N_M(\mathfrak{b})$ representing it. But then $\tilde{w}$ is what we wanted.

■

Therefore, we see that $\mathfrak{a}$ and $\mathfrak{a}_{\mathbb{R}}$ are well defined up to a unique isomorphism. Note further that $\mathfrak{a}_G$ does not depend on the isogeny class of $\mathbb{G}_{\mathbb{R}}$- it depends only on Lie($\mathbb{G}_{\mathbb{R}}$).

3.3.2 Remark

Our running example shows that $(\mathfrak{a}, \mathfrak{b}, \mathfrak{q})$ and $(\mathfrak{a}', \mathfrak{b}', \mathfrak{q}')$ need not be conjugate in G. Indeed, take $\mathfrak{a} = \mathfrak{a}'$, $\mathfrak{b} = \mathfrak{b}'$ and $\mathfrak{q}, \mathfrak{q}'$ corresponding to the two distinct chambers. Any element $g \in G$ conjugating the triples, should satisfy $g \in M$ (it centralizes $\mathfrak{a}$), $g \in N_M(\mathfrak{b})$, hence it acts on $\mathfrak{q}$ as its representative in the Weyl group, and to move $\mathfrak{q}$ to $\mathfrak{q}'$ it must be a representative of the above w. Explicit check shows that $B \cdot g \cap G(\mathbb{R}) = \emptyset$.

There is one more thing we will need to consider later.

3.3.3 Lemma (3.1.3)

Let $n_K \in K_\infty^0$ normalize $\mathfrak{b}$ and carry $\mathfrak{q} \cap \mathfrak{k} \subseteq \mathfrak{k}$ to its opposite wrt to $\mathfrak{b}$. Let $n_G \in G(\mathbb{C})$ normalize $\mathfrak{t}$ and carry $\mathfrak{q}$ to its opposite. Then n_G and n_K both preserve $\mathfrak{a}$, and coincide on it.

Proof n_K preserves $\mathfrak{t}$, by the fact that $Ad(n_K)$ is a Lie algebra endomorphism. Since B is invariant, we get also that it preserves $\mathfrak{a}$. Let $\mathscr{C}$ be the positive chamber for $\mathfrak{q}$. Let $w_G \in W_G$ be such that $w_G\mathscr{C} = -\mathscr{C}$. Then $w_K w_G \mathscr{C}$ and $\mathscr{C}$ both lie in the positive chamber for $\Delta(\mathfrak{k} : \mathfrak{b})$. Thus, there is a $w_M \in W_M$ such that $w_M\mathscr{C} = w_K w_G \mathscr{C}$, thus $w_M^{-1} w_K w_G$ stabilizes $\mathscr{C}$, but the action is simple, hence $w_K w_G = w_M \in W_M$. Thus, there is some $n_M \in N_M(\mathfrak{b})$ representing $w_K w_G$. Let $n := n_K^{-1} \cdot n_M$. Then it normalizes $\mathfrak{b}$, hence also $\mathfrak{a}$. Further,

$$n\mathscr{C} = n_K^{-1} \cdot n_M \mathscr{C} = w_K^{-1} w_M \mathscr{C} = w_G \mathscr{C} = -\mathscr{C}$$

and so it takes $\mathfrak{q}$ to $\mathfrak{q}^{op}$. Then may suppose that $n = n_G$, hence the result. ■

3.3.4 Definition (long Weyl element)

The *long Weyl element* is the involution of $\mathfrak{a}_G$ induced by the common action of $n_G$or n_K from the prior Lemma.

3.3.5 Definition (twisted real structure)

The *twisted real structure* $\mathfrak{a}'_{G.\mathbb{R}}$ on $\mathfrak{a}_G$ is the fixed points of the involution

$$(X \mapsto \overline{X}) \otimes w$$

where $X \mapsto \overline{X}$ is the antilinear involution defined by $\mathfrak{a}_{G,\mathbb{R}}$ and w is the long Weyl element for $\mathfrak{a}_G$.

3.4 Construction via the dual group

We begin with a simple Lemma

3.4.1 Lemma

Let $\mathbb{S}$ be a complex torus. Then

$$\mathrm{Lie}(\mathbb{S}) = X_*(\mathbb{S}) \otimes \mathrm{Lie}(\mathbb{G}_m) \cong X_*(\mathbb{S}) \otimes \mathbb{C}$$

Proof Since $\mathbb{S}$ is a torus, we have$\mathbb{C}[\mathbb{S}] = X^*(\mathbb{S}) \otimes \mathbb{C}$. Then there is a natural isomorphism

$$\mathrm{Lie}(\mathbb{S}) = \mathrm{Der}_L(\mathbb{C}[\mathbb{S}]) \cong \mathrm{Hom}(X^*(\mathbb{S}), \mathrm{Lie}(\mathbb{G}_m)) =$$

$$= \mathrm{Hom}(X^*(\mathbb{S}), \mathbb{Z}) \otimes \mathbb{C} = X_*(\mathbb{S}) \otimes \mathbb{C}$$

where we identify $\mathbb{G}_a = \mathrm{Lie}(\mathbb{G}_m) \cong \mathbb{C}$ by choosing the dual to $\frac{dz}{z}$ as basis. ■

Let $\widehat{G}$ be the Langlands dual group of G. Let $\widehat{T}$ and $\widehat{B}$ be a maximal torus and a Borel in $\widehat{G}$.

A choice of $(\mathfrak{a}, \mathfrak{b}, \mathfrak{q})$ as before gives us the torus $\mathfrak{a} \oplus \mathfrak{b}$ and the Borel $\mathfrak{q}$. Thus, we obtain a map

$$\mathrm{Lie}(\widehat{T}) \simeq X_*(\widehat{T}) \otimes \mathbb{C} = X^*(T) \otimes \mathbb{C} = (X_*(T) \otimes \mathbb{C})^* \simeq \mathrm{Lie}(T)^* = (\mathfrak{a} \oplus \mathfrak{b})^* \longrightarrow \mathfrak{a}^* \tag{1}$$

Note that the identification of $X_*(\widehat{T})$ with $X^*(T)$ is done such that $\hat{B}$ corresponds to$\mathfrak{q}$ (a choice of a positive system roots), so the isomorphism depends also on $\mathfrak{q}$.

Also for a different triple, conjugated by $g \in \mathbb{G}_{\mathbb{R}}(\mathbb{C})$, then the above map just differs by $\mathrm{Ad}(g)$.

Let ${}^L G := \widehat{G} \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Let

$${}^L W := N_{{}^L G}(\widehat{T}) / \widehat{T}$$

There exists a unique lift $w_0 \in^L W$ of the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, unique up to $\widehat{T}(\mathbb{R})$, sending $\widehat{B}$ to its opposite wrt to $\widehat{T}$.

The following Lemma justifies the use of $\mathrm{Lie}(\widehat{T})^{w_0}$ as an alternate definition for $\mathfrak{a}^*_G$.

3.4.2 Lemma ([4] 3.2.1)

The map above carries $\mathrm{Lie}(\widehat{T})^{w_0}$ isomorphically onto $\mathfrak{a}^*_G$, and preserves real structures. Moreover, the long Weyl group element $w_{\widehat{G}}$ for $\widehat{T}$, carrying $\widehat{B}$ to its opposite, preserves $\mathrm{Lie}(\widehat{T})^{w_0}$, and is carried under this identification to the long Weyl element acting on $\mathfrak{a}^*_G$.

Proof Recall that the action of $Gal(\mathbb{C}/\mathbb{R})$ on $\widehat{T}$ corresponds to the complex conjugation on T for a certain choice of isomorphism, hence the action of w_0 is carried under (1) to an automorphism in the outer class of complex conjugation, carrying $\mathfrak{q}$ to $\mathfrak{q}^{op}$ relative to $\mathfrak{a} \oplus \mathfrak{b}$. But we have constructed $\mathfrak{q}$ using $x \in i \cdot \mathfrak{b}_{\mathbb{R}}$, hence complex conjugation has precisely that effect. Thus w_0 corresponds to the action of complex conjugation, c, on $X^*(\mathbb{T}) \otimes \mathbb{C}$ (on the left component). By Lemma 3.2.6, the weights in $\mathfrak{b}^*$ are complex, and in $\mathfrak{a}^*$ are real, hence it acts by -1 on the first and 1 on the second, proving the first assertion.

Note that we also have an involution $c \otimes (z \mapsto \overline{z})$ (diagonal action) on $X^*(\mathbb{T}) \otimes \mathbb{C} = (\mathfrak{a} \oplus \mathfrak{b})^*$, which has $\mathfrak{a}^*_{\mathbb{R}} \oplus \mathfrak{b}^*_{\mathbb{R}}$ as its fixed points. The map $z \mapsto \overline{z}$ on $\mathbb{C}$ corresponds under (1) to the involution associated with the split real torus $\hat{T}(\mathbb{R})$, σ. Thus, this real structure corresponds to the fixed points of $c' = w_0 \circ \sigma$. Looking at the w_0-fixed points, this is reduced to σ, thus we have preservation of real structures.

Finally, note that both$w_0 w_{\widehat{G}}$ and $w_{\hat{G}} w_0$ fix $\widehat{B}$, and lift the nontrivial element of $Gal(\mathbb{C}/\mathbb{R})$, hence they commute. Then $w_{\widehat{G}}$ preserves that space.

Under (1), it corresponds to an element of the Weyl group of T sending $\mathfrak{q}$ to $\mathfrak{q}^{op}$. By definition, this is the long Weyl element. ■

3.5 Construction via a tempered cohomological parameter

Let $W_{\mathbb{R}} = \mathbb{C}^{\times} \rtimes \langle j \rangle$ be the real Weil group (here $j^2 = -1$).

3.5.1 Definition (tempered Langlands parameter)

A Langlands parameter represented by$\rho : W_{\mathbb{R}} \to^{L} G = \widehat{G} \rtimes Gal(\mathbb{C}/\mathbb{R})$ is *tempered* if the image of the projection $\rho_1 : W_{\mathbb{R}} \to \widehat{G}$ has compact closure.

3.5.2 Definition (tempered representation)

An admissible representation of G is *tempered* if its K_∞-finite matrix coefficients are in $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$.

We have the following theorem (we quote only the properties we need):

3.5.3 Theorem ([3], associated L-packet)

There is a nonempty set $\Pi_\rho$of infinitesimal equivalence classes of irreducible representations of G, having infinitesimal characters, such that:

(i) If $\pi \in \Pi_\rho$, then $\chi_\pi \in X_\rho = \{\chi_\varphi \mid \varphi \sim \rho\}$. Here, if $\rho(z) = \mu(z)\nu(\overline{z})$ for some $\mu, \nu \in X_*(\widehat{T}) = X^*(\widehat{T})$ and $z \in \mathbb{C}^{\times}$, then $\chi_\rho = (\mu - \eta) \cdot \sigma(\nu + \eta)$, where σ is complex conjugation, and η is half the sum of the positive roots of $\widehat{G}$.

(ii) If ρ is tempered, then all the elements of Π_ρ are tempered.

We call Π_ρ the$L$-packet associated to ρ.

We also recall the definition of $(\mathfrak{g}, K)$-cohomology, from Juaquin's talk.

3.5.4 Definition ($(\mathfrak{g}, K_\infty^0)$-cohomology)

Let $C^*(\mathfrak{g}, K_\infty^0; \Pi) = \mathrm{Hom}_{K_\infty^0}(\bigwedge^*(\mathfrak{g}/\mathfrak{k}), \Pi)$. Then $H^*(\mathfrak{g}, K_\infty^0; \Pi) = H^*(C^\bullet(\mathfrak{g}, K_\infty^0; \Pi)$.

3.5.5 Lemma (Wigner's Lemma, [2]I.5.3)

Let U, V be two $(\mathfrak{g}, K)$-modules. Assume that they have infinitesimal characters χ_U, χ_V. If $\chi_U \neq \chi_V$, then $H^*(\mathfrak{g}, K; \tilde{U} \otimes V) = 0$.

Proof If $\chi_U \neq \chi_V$, we can find $z \in Z(\mathfrak{g})$ such that $\chi_U(z) = 1$ and $\chi_V(z) = 0$ (by shifting and scaling, as this is an algebra). Since $z \in Z(\mathfrak{g})$, the homomorphisms induced on $Ext^*_{\mathfrak{g},K}(U,V)$ by $z : U \to U$ and $z : V \to V$ are identical, showing that $1 = 0$ on this space. But

$$H^*(\mathfrak{g},K;\tilde{U} \otimes V) = H^*(\mathfrak{g},K;\mathrm{Hom}_{\mathbb{C}}(U,V)) = Ext^*_{\mathfrak{g},K}(\mathbb{C},\mathrm{Hom}_{\mathbb{C}}(U,V)) = Ext^*_{\mathfrak{g},K}(U,V) = 0$$

■

Let $\rho : W_{\mathbb{R}} \to^L G$ be a tempered Langlands parameter whose associated L-packet contains a representation Π with non-vanishing $(\mathfrak{g}, K^0_\infty)$-cohomology.

Then by Wigner's Lemma, its infinitesimal character must be trivial, hence by Theorem 3.5.3, we can conjugate ρ to some ρ_0 such that $\rho_0 |_{\mathbb{C}^\times} = \eta(z/\overline{z})$, where $\eta = \sum_{\alpha \in R_+} \alpha$.

The connected centralizer of $\rho_0 |_{\mathbb{C}^\times}$ is then exactly $\widehat{T}$. Indeed, as the image is contained in the commutative $\widehat{T}$, it is certainly contained in the centralizer.

Since $\langle \eta, \alpha \rangle \neq 0$ for any root α, we see that the centralizer must be exactly $\hat{T}$.

3.5.6 Lemma

$\rho_0(j)$ normalizes $\widehat{T}$ and sends $\widehat{B}$ to $\widehat{B}^{op}$.

Proof If $t \in \widehat{T}$, then for any $z \in \mathbb{C}^\times$, $Ad(\rho_0(\overline{z}))(t) = t$, hence

$$\mathrm{Ad}(\rho_0(z))(\mathrm{Ad}(\rho_0(j))(t)) = (\mathrm{Ad}\rho_0(zj))(t) =$$

$$= \mathrm{Ad}(\rho_0(j\overline{z}))(t) = \mathrm{Ad}(\rho_0(j))(\mathrm{Ad}(\rho_0(\overline{z}))(t)) = \mathrm{Ad}(\rho_0(j))(t)$$

showing that $\mathrm{Ad}\rho_0(j)(t) \in Z_{\widehat{G}}(\rho_0(\mathbb{C}^\times)) = \widehat{T}$. Therefore $\rho_0(j)$ normalizes $\widehat{T}$. Also, in particular, if $t = \rho_0(z) = \eta(z/\overline{z})$, then

$$\mathrm{Ad}\rho_0(j)(t) = \rho_0(\overline{z}) = \eta(\overline{z}/z) = t^{-1}$$

Since $\rho_0(j) \in N_{\widehat{G}}(\widehat{T})$, it acts on the root spaces. Therefore, if $X \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta(G : B)$, then one has $\mathrm{Ad}\rho_0(j)(X) \in \mathfrak{g}_{j(\alpha)}$for some $j(\alpha) \in \Delta(G : B)$. But that would mean for any $t \in \widehat{T}$

$$e^{j(\alpha)}(t) \cdot \mathrm{Ad}\rho_0(j)(X) = \mathrm{Ad}t\mathrm{Ad}\rho_0(j)(X) = \mathrm{Ad}\rho_0(j)\mathrm{Ad}(\mathrm{Ad}(\rho_0(j))^{-1}(t))(X) =$$

$$= e^{\alpha}\left(\mathrm{Ad}(\rho_0(j))^{-1}(t)\right) \cdot \mathrm{Ad}\rho_0(j)(X)$$

Thus we should have

$$e^{j(\alpha)}(t) = e^{\alpha}\left(\mathrm{Ad}(\rho_0(j))^{-1}(t)\right)$$

In particular for $t = \rho_0(z) = \eta(z/\overline{z})$, we get

$$e^{j(\alpha)}(t) = e^{\alpha}(t^{-1}) = e^{-\alpha}(t)$$

so that $j(\alpha) = -\alpha$. Thus $\rho_0(j)$ sends every root to its negation, hence it sends $\widehat{B}$ to $\widehat{B}^{op}$. ■

Therefore, $\rho_0(j)$ and $w_0$from the previous construction determine the same class in ${}^LW$. (induce the same action on the Borel, and lift the nontrivial element of the Galois group). Therefore

$$\mathfrak{a}_G^* = \mathrm{Lie}(\widehat{T})^{w_0} = \mathrm{Lie}(Z_{\widehat{G}}(\rho_0(\mathbb{C}^\times)))^{\rho_0(j)} = \mathrm{Lie}(Z_{\widehat{G}}(\rho_0))$$

Now $\rho = \mathrm{Ad}(g)\rho_0$ for some $g \in \widehat{G}$. This g is specified up to right translation by $\widehat{T}$, since $Z_{\widehat{G}}(\rho_0) \subseteq \widehat{T}$. It follows that the induced isomorphism

$$\mathrm{Lie}(Z_{\widehat{G}}(\rho_0)) \simeq \mathrm{Lie}(Z_{\widehat{G}}(\rho))$$

is independent of the choice of g.

4 Description of the action on cohomology

4.1 Description when $\mathbb{G}_{\mathbb{R}}$ is simply connected

Note that in this case $K_\infty = K_\infty^0$.

The main ingredient that we will need in order to consruct the action on cohomology will be the following result of Vogan and Zuckerman (see [5]):

4.1.1 Theorem ([5])

Let $\mathfrak{q}$ be as before, and let $\mathscr{C}$ be the corresponding positive chamber of $\Delta(\mathfrak{g} : \mathfrak{b})$. Let $\mathfrak{u}$ be its unipotent radical. Then

(a) there is a unique irreducible tempered module $\pi_{\mathfrak{q}} = \pi(\mathscr{C})$ for $\mathfrak{g}$ with the following properties: ([5], Theorem 2.5)

1. The restriction of $\pi(\mathscr{C})$ to $\mathfrak{k}$ contains the irreducible representation, $V_{\mathscr{C}}$ of K_∞ of highest weight $\mu = \mu_{\mathscr{C}} = \sum_{\mathfrak{g}_\alpha \subseteq \mathfrak{u}\cap\mathfrak{p}} \alpha$.

2. $Z(\mathfrak{g})$ acts on $\pi(\mathscr{C})$ trivially.

3. some numerical criterion for which irreducible representations can occur in $A_{\mathfrak{q}}$.

(b) $H^*(\mathfrak{g}, K_\infty; \pi(\mathscr{C})) \cong \mathrm{Hom}_{K_\infty}(\bigwedge^* \mathfrak{p}, \pi(\mathscr{C}))$ (the differentials of the complex vanish - it is in fact true for any unitary representation). ([5], Proposition 3.2)

(c) $V_{\mathscr{C}}$ is the only irreducible representation occuring in both $\bigwedge^* \mathfrak{p}$ and $\pi(\mathscr{C})$. ([5], proof of Corollary 3.7)

(d) If π is an irreducible unitary representation of G, and X its Harish-Chandra module. Suppose π is cohomological, then there exists $\mathfrak{q}$ such that $X \cong \pi_{\mathfrak{q}}$. ([5], Theorem 4.1)

In effect, this theorem parametrizes the tempered cohomological representations by the chambers of $\Delta(\mathfrak{g} : \mathfrak{b})$.

4.1.2 Definition (weights)

We let $V_{-\mathscr{C}}$ be the dual representation to $V_{\mathscr{C}}$, with lowest weight $-\mu_{\mathscr{C}}$. Let $v^+ \in V_{\mathscr{C}}$ be a highest weight vector, and $v^- \in V_{-\mathscr{C}}$ a lowest weight vector.

For a representation Π, and a vector v, we say that v is *of weight* μ if it is of weight μ with respect to the entire$\mathfrak{q} \cap \mathfrak{k}$ (not only the Cartan subalgebra $\mathfrak{b}$), and similarly for $-\mu$.

We denote by Π^μ the vectors of weight μ in Π.

4.1.3 Lemma

There is an action $\bigwedge^* \mathfrak{a}_G^* \times H^*(\mathfrak{g}, K_\infty; \pi(\mathscr{C})) \to H^*(\mathfrak{g}, K_\infty; \pi(\mathscr{C}))$.

Proof By Theorem 4.1.1 (c), we have

$$H^*(\mathfrak{g}, K_\infty; \pi(\mathscr{C})) = \mathrm{Hom}_{K_\infty}\left(\bigwedge^* \mathfrak{p}, \pi(\mathscr{C})\right) = \left(\bigwedge^* \mathfrak{p}^* \otimes \pi(\mathscr{C})\right)^{K_\infty} =$$

and by (d), we get the isotypic part

$$= \mathrm{Hom}_{K_\infty}\left(V_{-\mathscr{C}}, \bigwedge^* \mathfrak{p}^*\right) \otimes \mathrm{Hom}_{K_\infty}(V_{\mathscr{C}}, \pi(\mathscr{C})) \cong \left(\bigwedge^* \mathfrak{p}^*\right)^{-\mu} \otimes \mathrm{Hom}_{K_\infty}(V_{\mathscr{C}}, \pi(\mathscr{C})) \tag{2}$$

where the last isomorphism is given by $f \mapsto f(v^-)$.
Therefore, it's enough to construct an action on $(\bigwedge^* \mathfrak{p}^*)^{-\mu}$. But the splitting

$$\mathfrak{p} = \mathfrak{a} \oplus (\mathfrak{u} \cap \mathfrak{p}) \oplus (\overline{\mathfrak{u}} \cap \mathfrak{p})$$

yields a tensor product decomposition

$$\bigwedge^* \mathfrak{p}^* = \bigoplus_{i,j,k} \left(\bigwedge^i \mathfrak{a}^*\right) \otimes \left(\bigwedge^j (\mathfrak{u} \cap \mathfrak{p})^*\right) \otimes \left(\bigwedge^k (\overline{\mathfrak{u}} \cap \mathfrak{p})^*\right) = \bigoplus_{i,j,k} T_{i,j,k}$$

where a pure element in $T_{i,j,k}$ is of the form

$$t = a \otimes (v_1 \wedge \ldots \wedge v_j) \otimes (w_1 \wedge \cdots \wedge w_k)$$

with v_i of weight $\alpha_i > 0$, and w_l of weight $\beta_l < 0$. It follows that t is of weight

$$\sum_{i=1}^{j} \alpha_i + \sum_{l=1}^{k} \beta_l \geq \sum_{\beta < 0} \beta = -\mu$$

and equality iff $\alpha_i = 0$ for all i, and the $\beta_l$cover all possible $\beta < 0$, so that $k = \dim(\overline{\mathfrak{u}} \cap \mathfrak{p})$, establishing

$$\left(\bigwedge^* \mathfrak{p}^*\right)^{-\mu} = \bigwedge^* \mathfrak{a}^* \otimes \det(\overline{\mathfrak{u}} \cap \mathfrak{p})^*$$

In particular, we get natural actions of $\bigwedge^* \mathfrak{a}^*$ on $(\bigwedge^* \mathfrak{p}^*)^{-\mu}$ from both sides (differing by $(-1)^{\deg}$) making it a free module of rank 1. We will consider the right action. Thus, we are done. ■
In order to explicitly characterize this action on the cohomology, we note that similarly we have

$$\left(\bigwedge^* \mathfrak{p}\right)^{\mu} = \bigwedge^* \mathfrak{a} \otimes \det(\mathfrak{u} \cap \mathfrak{p})$$

so there is a contraction action:

For $X \in \mathfrak{a}^*, Y = \bigwedge_i Y_i \in \bigwedge^* \mathfrak{a}$, the rule $Y \mapsto X$ (\contract) Y can be viewed as

$$X(contract) \bigwedge_i Y_i = \sum_i (-1)^{i-1} \cdot X(Y_i) \cdot \bigwedge_{j \neq i} Y_j$$

or if one considers Y as an alternating form on $\mathfrak{a}^*$, then

$$(X(contract)Y)(a_1, \ldots, a_n) = Y(X, a_1, \ldots, a_n)$$

This extends, and again we get a free module of rank 1 (this time generated by the top degree element).

4.1.4 Exercise

These two actions are adjoint - for $X \in \bigwedge^* \mathfrak{a}^*$, $A \in \bigwedge^* \mathfrak{p}^*$ and $B \in \bigwedge^* \mathfrak{p}$, we have

$$\langle X \wedge A, B \rangle = \langle A, X(contract)B \rangle$$

4.1.5 Lemma

The action described in Lemma 4.1.3 can be characterized as follows. For $f \in \mathrm{Hom}_{K_\infty}(\bigwedge^* \mathfrak{p}, \pi(\mathscr{C}))$, any vector $v \in \bigwedge^* \mathfrak{p}$ of weight $\mu_{\mathscr{C}}$ and for $X \in \bigwedge^* \mathfrak{a}^*$, we have

$$(f \cdot X)(v) = f(X(contract)v)$$

The left action is related via $X \cdot f = (-1)^{\deg(X)} \cdot (f \cdot X)$.

Proof By (2), f factors through $V_{\mathscr{C}} \subseteq \pi(\mathscr{C})$, so we may regard f as a map to $V_{\mathscr{C}}$. Let $f^t : V_{-\mathscr{C}} \to \bigwedge^* \mathfrak{p}^*$ be its transpose. Now

$$\left\langle v^-, (fX)(v^+) \right\rangle = \left\langle (fX)^t(v^-), v^+ \right\rangle = \left\langle X f^t(v^-), v^+ \right\rangle =$$
$$= \left\langle X \wedge f^t(v^-), v^+ \right\rangle = \left\langle f^t(v^-), X(contract)v^+ \right\rangle = \left\langle v^-, f\left(X(contract)v^+\right) \right\rangle$$

Since $(f \cdot X)(v^+)$ is determined by its pairing with v^-(it generates the entire dual $V_{-\mathscr{C}}$), and $f \cdot X$ is determined by its evaluation on v^+, we are done. ■

4.1.6 Remark

Should verify the action does not depend on the choice of $(\mathfrak{b}, \mathfrak{q})$. This is not difficult to show.

This extends in the obvious way to an arbitrary finite length tempered cohomological representation.

4.2 Construction through parabolic induction

We also know from Langlands' classification that any tempered cohomological representation π is of the form $\pi = Ind_P^G \sigma$, where P is a parabolic subgroup whose Levi is $M = Z_G(\mathfrak{a})$ and σ is a discrete series.
Note that we have by Shapiro's Lemma

$$H^*(\mathfrak{g}, K_\infty; \pi) = Ext^*_{\mathfrak{g},K_\infty}(1, \pi) = Ext^*_{\mathfrak{p},K_P}(1, \sigma)$$

equipped with a canonical action of

$$\bigwedge^* \mathfrak{a}_G^* = Ext^*_{\mathfrak{a}_G}(1, 1) \to Ext^*_{\mathfrak{p},K_p}(1, 1)$$

. It turns out that this is the same action as above. This follows from Vogan Zuckerman description of $\pi(\mathscr{C})$ in terms of the Langlands classification. Since the differentials vanish, this is just an action on cochains and it is not difficult to see that it is identified with the one we have defined.

4.3 Interaction with automorphisms

4.3.1 Definition

Assume still that G is semisimple and simply connected. Let $\alpha \in Ad(\mathbb{G}^{ad}) \subset Aut(G)$ be an automorphism preserving K_∞. Let Π be a representation of G, and let

$${}^\alpha\Pi(g) = \Pi(\alpha^{-1}(g))$$

be its α-twist.

4.3.2 Lemma ([4] 3.4.1)

Let Π be a tempered cohomological representation of finite length. Then ${}^\alpha\Pi$ is also such.
The natural map

$$\mathrm{Hom}_{K_\infty}\left(\bigwedge^* \mathfrak{p}, \Pi\right) \to \mathrm{Hom}_{K_\infty}\left(\bigwedge^* \mathfrak{p}, {}^\alpha \Pi\right)$$

sending f to

$$\bigwedge^* \mathfrak{p} \cong \bigwedge^{*\alpha} \mathfrak{p} \to^\alpha \Pi$$

commutes the $\bigwedge^* \mathfrak{a}_G^*$ actions on both spaces. (Here the left arrow is $Y \mapsto \alpha^{-1}(Y)$ intertwining $\mathfrak{p}$ and ${}^\alpha\mathfrak{p}$, and the right arrow is f).

Proof Reduce to irreducible, so $\pi = \pi(\mathscr{C})$. by conjugating with an element in K_∞, may assume α preserves $\mathfrak{b}$ and $\mathfrak{q} \cap \mathfrak{k}$. Now, ${}^{\alpha}\pi(\mathscr{C})$ contains the irreducible representation of highest weight $\mu_{\mathscr{C}} \circ \alpha^{-1}$, associated to $\alpha(\mathscr{C})$, $\alpha(\mathfrak{q})$.

We can lift α to $\mathbb{G}_{\mathbb{R}}(\mathbb{C})$, which shows that the identification with $\mathfrak{a}$ is changed by α sending $(\mathfrak{a}, \mathfrak{b}, \mathfrak{q})$ to $(\mathfrak{a}, \mathfrak{b}, \alpha(\mathfrak{q}))$.

The map $Y \mapsto \alpha^{-1}(Y)$ carries $(\bigwedge^* \mathfrak{p})^{\mu\alpha^{-1}} \to (\bigwedge^* \mathfrak{p})^{\mu}$. Thus, if $f \in \mathrm{Hom}_{K_\infty}(\bigwedge^* \mathfrak{p}, \Pi)$ factors through highest weight μ, $f \circ \alpha^{-1} \in \mathrm{Hom}_{K_\infty}(\bigwedge^* \mathfrak{p}, {}^{\alpha}\Pi)$ factors through highest weight $\mu \circ \alpha^{-1}$.

For $v \in (\bigwedge^* \mathfrak{p})^{\mu}$, and $X \in \bigwedge^* \mathfrak{a}_G^*$, we have $\alpha(v) \in (\bigwedge^* \mathfrak{p})^{\mu\alpha^{-1}}$, and (with $f' = f \circ \alpha^{-1}$)

$$(Xf)'(\alpha v) = (Xf)(v) = f(X(contract)v)$$

$$(\alpha(X)f')(\alpha v) = f'(\alpha(X)(contract)\alpha(v)) = f(X(contract)v)$$

■

4.4 Interaction with duality and complex conjugation

4.4.1 Lemma (duality, [4] 3.4.2)

Let $\langle \cdot, \cdot \rangle : H^j(\mathfrak{g}, K_\infty; \Pi) \times H^{d-j}(\mathfrak{g}, K_\infty; \tilde{\Pi}) \to \det \mathfrak{p}^*$ be the natural pairing induced from cup product and duality on coefficients (here $d = \dim Y(K)$). Then for $X \in \bigwedge^* \mathfrak{a}_G^*$, and w the long Weyl group element, we have

$$\langle f_1 \cdot X, f_2 \rangle = \langle f_1, (wX) \cdot f_2 \rangle$$

Proof Let $\Pi = \pi(\mathscr{C})$ be irreducible. then one has to show adjointness for the map

$$Hom(V_{-\mathscr{C}}, \bigwedge^{j} \mathfrak{p}^*) \otimes Hom(V_{\mathscr{C}}, \bigwedge^{d-j} \mathfrak{p}^*) \to Hom(V_{\mathscr{C}} \otimes V_{-\mathscr{C}}, \det \mathfrak{p}^*) \to \det \mathfrak{p}^*$$

Enough to consider $(f_1, f_2) \mapsto f_1(v^-) \wedge f_2(v^+)$. ($v^- \otimes v^+$ has a nonzero projection on the space of invariants). We then have

$$(f_1 \cdot X)(v^-) \wedge f_2(v^+) = f_1(v^-) \wedge X \wedge f_2(v^+) = f_1(v^-) \wedge (Xf_2)(v^+)$$

The only thing to note is that the identification of $\mathfrak{a}$ with $\mathfrak{a}_G$ arising from $(\mathfrak{a}, \mathfrak{b}, \mathfrak{q})$ and $(\mathfrak{a}, \mathfrak{b}, \mathfrak{q}^{op})$ differ by w.

■

4.4.2 Lemma (complex conjugation, [4] 3.4.3)

Let $\overline{\Pi} = \Pi \otimes_c \mathbb{C}$ be the conjugate representation, and let $H^*(\mathfrak{g}, K_\infty; \Pi) \to H^*(\mathfrak{g}, K_\infty; \overline{\Pi})$ the map induced from the real structure on $\mathfrak{p}$. Then the following diagram commutes:

$$\begin{array}{ccc} H^*(\mathfrak{g}, K_\infty; \Pi) \otimes \bigwedge^* \mathfrak{a}_G^* & \longrightarrow & H^*(\mathfrak{g}, K_\infty; \Pi) \\ \downarrow & & \downarrow \\ H^*(\mathfrak{g}, K_\infty; \overline{\Pi}) \otimes \bigwedge^* \mathfrak{a}_G^* & \longrightarrow & H^*(\mathfrak{g}, K_\infty; \overline{\Pi}) \end{array}$$

where the complex conjugation on $\mathfrak{a}_G^*$is that corresponding to the twisted real structure.

Proof Reduce to irredcucible. Since $\Pi = \pi(\mathscr{C})$ is unitary, can fix a Hermitian form, and identify $V_{-\mathscr{C}} = \overline{V}_{\mathscr{C}}$ such that $\overline{v^+} = v^-$.
For $S \in Hom_K(V_{-\mathscr{C}}, \bigwedge^* \mathfrak{p})$, Define $S \in Hom_K(V_{\mathscr{C}}, \bigwedge^* \mathfrak{p})$ by $\overline{S}(\overline{v}) = \overline{S(v)}$. Then by definition, we have a commutative diagram

$$\begin{array}{ccc} Hom_K(V_{-\mathscr{C}}, \bigwedge^* \mathfrak{p}) & \longrightarrow & (\bigwedge^* \mathfrak{p})^{-\mu} \\ \downarrow & & \downarrow \\ Hom_K(V_{\mathscr{C}}, \bigwedge^* \mathfrak{p}) & \longrightarrow & (\bigwedge^* \mathfrak{p})^{\mu} \end{array}$$

We have an induced complex conjugation obtained by tensoring $S \mapsto \overline{S}$ with conjugation on $V_{-\mathscr{C}}$:

$$\bigwedge^* \mathfrak{p}[-\mathscr{C}] = Hom_K(V_{-\mathscr{C}, \bigwedge^* \mathfrak{p}}) \otimes V_{-\mathscr{C}} \to Hom_K(V_{-\mathscr{C}, \bigwedge^* \mathfrak{p}}) \otimes V_{\mathscr{C}} = \bigwedge^* \mathfrak{p}[\mathscr{C}]$$

giving rise to

$$\begin{array}{ccc} \bigwedge^* \mathfrak{p}[-\mathscr{C}] \otimes \bigwedge^* \mathfrak{a}^* & \longrightarrow & \bigwedge^* \mathfrak{p}[-\mathscr{C}] \\ \downarrow & & \downarrow \\ \bigwedge^* \mathfrak{p}[\mathscr{C}] \otimes \bigwedge^* \mathfrak{a}^* & \longrightarrow & \bigwedge^* \mathfrak{p}[\mathscr{C}] \end{array}$$

where the conjugation on $\mathfrak{a}^*$ is the one fixing $\mathfrak{a}^*_{\mathbb{R}}$. Again, the identifications differ by the long Weyl element, wo we should take the twisted real structure. ■

4.5 Construction for general $\mathbb{G}_{\mathbb{R}}$

Let G_{sc} be the simply connected cover of the derived group, and Z_G the center. Then we have the central isogeny $G_{sc} \times Z_G \to G$. then we have

$$\mathfrak{a}_G = \mathfrak{a}_{sc} \oplus \mathfrak{a}_Z$$

For any representation $\Pi$of G, let Π_{sc} be its pullback to G_{sc}. It is a tempered representation of finite length, and we have

$$H^*(\mathfrak{g}, K^0_\infty; \Pi) = \bigwedge^* \mathfrak{a}_Z \otimes H^*(\mathfrak{g}_{sc}, K_{\infty,sc}; \Pi_{sc})$$

We have an action of $\bigwedge^* \mathfrak{a}^*_{sc}$ on the second factor, so we get an action of

$$\bigwedge^* \mathfrak{a}^*_{sc} \otimes \bigwedge^* \mathfrak{a}^*_Z = \bigwedge^* (\mathfrak{a}_{sc} \oplus \mathfrak{a}_Z)^* = \bigwedge^* \mathfrak{a}^*_G$$

on $H^*(\mathfrak{g}, K^0_\infty; \Pi)$, with the previous Lemmas still holding. Finally, we note that the action of $K_\infty / K^0_\infty$by automorphisms commute with the action, in view of Lemma 4.3.2.

4.6 Metrization

Briefly - complexification of $B_{\mathbb{R}}$ induces a positive definite hermitian form on $\mathfrak{a}^*_G$. We then have

4.6.1 Lemma (isometry, [5] 3.5.1)

Let $X \in \bigwedge^* \mathfrak{a}_G^*$. Let Π be a finite length cohomological tempered representation. Let $T \in H^q(\mathfrak{g}, K_\infty; \Pi)$, where $q = q_0(G)$ is the minimal cohomological degree. Equip $H^*(\mathfrak{g}, K_\infty^0; \Pi)$ with the natural Hermitian metric arising from a fixed inner product on Π and the bilinear form $B_\mathbb{R}$. Then $\|T \cdot X\| = \|T\| \cdot \|X\|$.

Proof This is quite clear - reducing to an irreducible $\pi(\mathscr{C})$ again, reduces to the weight space $(\bigwedge^* \mathfrak{p}^*)^{-\mu}$, since the map

$$Hom\left(V_{-\mathscr{C}}, \left(\bigwedge^* \mathfrak{p}^*\right)\right) \to \left(\bigwedge^* \mathfrak{p}^*\right)^{-\mu}$$

is an isometry. But this is clear, since the factors $\mathfrak{a}, (\mathfrak{u} \oplus \overline{\mathfrak{u}}) \cap \mathfrak{p}$ are orthogonal to one another under B. ■

5 Conjectures

Recall that we wanted to construct an action of $\bigwedge^* \mathfrak{a}_G^*$ on $H^*(Y(K), \mathbb{C})_\Pi$, in order to formulate a conjecture.

5.1 The Beilinson Regulator

Recall some things from Alex's talk.

We asserted the existence of a motive $\mathrm{Ad}\Pi$, which has weight zero, with an isomorphism to an inner form

$$\iota : H_B(\mathrm{Ad}\Pi_\mathbb{C}, \mathbb{Q}) \to \hat{\mathfrak{g}}_{\mathbb{Q},*} \subseteq \hat{\mathfrak{g}}_\mathbb{Q}$$

such that the isomorphism

$$H_{dR}(\mathrm{Ad}\Pi) \otimes_\mathbb{Q} \mathbb{C} \simeq H_B(\mathrm{Ad}\Pi_\mathbb{C}, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C} \to \hat{\mathfrak{g}}_{\mathbb{Q},*} \otimes \mathbb{C} \simeq \hat{\mathfrak{g}}$$

identifies the action of the Weil group $W_\mathbb{R}$on the de Rham cohomology of $\mathrm{Ad}\Pi$ with a representation in the class of$\mathrm{Ad}\rho : W_\mathbb{R} \to \mathrm{Aut}(\hat{\mathfrak{g}})$.

Recall that we also have

$$H^1_\mathscr{D}(\mathrm{Ad}\Pi_\mathbb{R}, \mathbb{R}(1)) \cong H^0_B(\mathrm{Ad}\Pi_\mathbb{R}, \mathbb{C}) / H^0_B(\mathrm{Ad}\Pi_\mathbb{R}, \mathbb{R}(1)) + F^1 H^0_{DR}(\mathrm{Ad}\Pi_\mathbb{R}) \cong$$

$$\cong H^0_B(\mathrm{Ad}\Pi_\mathbb{R}, \mathbb{R}) / F^1 H^0_{DR}(\mathrm{Ad}\Pi_\mathbb{R})$$

using the splitting $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}(1)$. This gives the Beilinson's exact sequence:

$$0 \to F^1 H_{dR}(\mathrm{Ad}\Pi) \otimes_\mathbb{Q} \mathbb{R} \to H^0_B(\mathrm{Ad}\Pi_\mathbb{R}, \mathbb{R}) \to H^1_\mathscr{D}(\mathrm{Ad}\Pi_\mathbb{R}, \mathbb{R}(1)) \to 0$$

Here, the left map is just taking the real part.

Note that the Weil group $W_\mathbb{R}$ acts naturally on $H_B(\mathrm{Ad}\Pi_\mathbb{C}, \mathbb{R})$, letting j act as ϕ_∞, the complex conjugation on the motive, and $\mathbb{C}^\times$ acts on the $(p, -p)$ part via multiplication by $(z/\overline{z})^p$.

The fixed points are then the subspace of the $(0,0)$-Hodge part of $H_B(\mathrm{Ad}\Pi_{\mathbb{C}}, \mathbb{C})$ fixed by $\phi_\infty$and complex conjugation of coefficients, c.

Assuming a weak polarization on $\mathrm{Ad}\Pi$ (will later show it exists), one can show that $H_B(\mathrm{Ad}\Pi_{\mathbb{C}}, \mathbb{R})^{W_{\mathbb{R}}}$ is the orthogonal complement to $F^1 H_{dR}(\mathrm{Ad}\Pi) \otimes_{\mathbb{Q}} \mathbb{R}$, hence we have an isomorphism

$$H_B(\mathrm{Ad}\Pi_{\mathbb{C}}, \mathbb{R})^{W_{\mathbb{R}}} \simeq H^1_{\mathscr{D}}(\mathrm{Ad}\Pi_{\mathbb{R}}, \mathbb{R}(1))$$

Therefore, the Beilinson regulator gives us a map

$$H^1_{\mathscr{M}}(\mathrm{Ad}\Pi, \mathbb{Q}(1)) \to H_B(\mathrm{Ad}\Pi_{\mathbb{C}}, \mathbb{R})^{W_{\mathbb{R}}} \hookrightarrow H_B(\mathrm{Ad}\Pi_{\mathbb{C}}, \mathbb{C})^{W_{\mathbb{R}}} \to \hat{\mathfrak{g}}^{W_{\mathbb{R}}} \to \mathfrak{a}_G^* \tag{3}$$

where the last isomorphism comes from our third construction.

Dually, we have

$$H^1_{\mathscr{M}}(\mathrm{Ad}^*\Pi, \mathbb{Q}(1)) \to \mathfrak{a}_G$$

5.1.1 Remark

If one accepts Beilinson's conjecture, then the image is a $\mathbb{Q}$–structure on $\mathfrak{a}_G$.

5.1.2 Lemma ([5], 5.1.1)

The map $H_B(\mathrm{Ad}\Pi_{\mathbb{C}}, \mathbb{R})^{W_{\mathbb{R}}} \to \mathfrak{a}_G^*$ has image equal to the twisted real structure on $\mathfrak{a}_G^*$.

Proof We may assume that (3) carries the $W_{\mathbb{R}}$-action to the action of $\rho_0 : W_{\mathbb{R}} \to \mathrm{Aut}(\hat{\mathfrak{g}})$, where ρ_0 is normalized as before.

The complex conjugation of coefficients c in the Betti cohomology can be seen as acting on $\hat{\mathfrak{g}}$.

By our hypothesis, its fixed points are given by the inner form $\hat{\mathfrak{g}}_{\mathbb{Q},*} \otimes \mathbb{R}$ and so c is an inner twist of the involution of $\hat{\mathfrak{g}}$ with respect to the Chevalley real form.

Since for $z \in S^1$, we have

$$\rho_0(z) = \eta(z/\overline{z}) = \eta(z^2) = \prod_{\alpha>0} \alpha(z)$$

we see that it preserves real cohomology, hence commutes with c. Define $\iota(X) = \mathrm{Ad}(w_{\hat{G}})\overline{X}$ on $\hat{\mathfrak{g}}$. Then it also commutes with $\rho_0(S^1)$. (changing $\overline{z}/z$ to its inverse and the conjugating, yields the same result).

The composition ιc is an inner automorphism of $\hat{\mathfrak{g}}$, commuting with $\rho_0(S^1)$, hence is given by conjugation by an element of $\widehat{T}$. Therefore they act in the same way on $\mathrm{Lie}(\widehat{T})$.

The image is the fixed points of c, hence the fixed points of ι, which give the twisted real structure. ■

5.2 Trace forms

If $\hat{B}$ is a nondegenerate ${}^L G_{\mathbb{Q}}$-invariant, $\mathbb{Q}$-valued bilinear form on $\hat{\mathfrak{g}}_{\mathbb{Q}}$, scalar extension induce a complex values one one on $\hat{\mathfrak{g}}$. Its pullback under the hypothesized map

$$H_B((\mathrm{Ad}\Pi)_{\mathbb{C}}, \mathbb{Q}) \cong \hat{\mathfrak{g}}_*$$

defines a weak polarization Q on $\mathrm{Ad}\Pi$. Since $\hat{\mathfrak{g}}_*$ is an inner form, the restriction of $\hat{B}$ is $\mathbb{Q}$-valued. We may form the corresponding Hermitian form $Q(x, c(y))$ on $H_B((\mathrm{Ad}\Pi)_\mathbb{C}, \mathbb{C})$. When restricting to the $W_\mathbb{R}$-invariants, we get

$$(X, Y) \in \mathfrak{a}_G^* \times \mathfrak{a}_G^* \mapsto \hat{B}(X, \overline{\mathrm{Ad}(w_{\hat{G}})Y})$$

due to the previous Lemma. This is real valued when restricted to the twisted real structure, since

$$\overline{\hat{B}(X, \overline{w_{\hat{G}}Y})} = \hat{B}(\overline{X}, w_{\hat{G}}Y) = \hat{B}(\overline{w_{\hat{G}}^{-1}X}, Y) = \hat{B}(\overline{w_{\hat{G}}X}, Y)$$

and on the twisted real structure, this gives

$$\overline{\hat{B}(X, Y)} = \hat{B}(X, Y)$$

5.2.1 Remark

This form need not be positive definite.

5.3 Main Conjecture

5.3.1 Definition

Let π be a spherical cohomological automorphic representation of $\mathbb{G}$. Then for $g \in \mathbb{G}(\mathbb{A})$ and $X \in \mathfrak{g}/\mathfrak{k}$, we denote by $[g, X]$ the tangent vector to $Y(K)$ at the point $\mathbb{G}(\mathbb{Q})gK_\infty^0 K$ given by the derivative of the curve

$$\mathbb{G}(\mathbb{Q})ge^{tX}K_\infty^0 K$$

at $t = 0$.
Consider the natural map

$$\Omega : \mathrm{Hom}_{K_\infty^0}\left(\bigwedge^p \mathfrak{g}/\mathfrak{k}, \pi^K\right) \to \Omega^p(Y(K))$$

given by

$$\Omega(f)\left([g, X_1 \wedge \ldots \wedge X_p]\right) = f\left(X_1 \wedge \ldots \wedge X_p\right)(g)$$

where

$$[g, X_1 \wedge \ldots \wedge X_p] = [g, X_1] \wedge \ldots \wedge [g, X_p]$$

Then it induces a map on cohomology

$$\Omega : H^p(\mathfrak{g}, K_\infty^0; \pi^K) \to H^*(Y(K), \mathbb{C})$$

5.3.2 Theorem (Borel, [1])

If $\Pi = \{\pi_1, \ldots, \pi_r\}$ is as before (spherical tempered cohomological representations). Then Ω induces an isomorphism

$$\Omega : \bigoplus_{i=1}^r H^*(\mathfrak{g}, K_\infty^0; \pi_i^K) \to H^*(Y(K), \mathbb{C})_\Pi$$

We are now ready to state the main conjecture for the complex realization

5.3.3 Conjecture (Motivic classes preserve rational automorphic cohomology, [4] 5.4)

Assume Π is as before, and the existence of the motive $\mathrm{Ad}\Pi$, in the sense formalized last week.
Assume the first part of Beilinson's conjecture, namely that the Beilinson's regulator is an isomorphism:

$$r_{\mathscr{D}} : H^1_{\mathscr{M}}(\mathrm{Ad}\Pi, \mathbb{Q}(1)) \otimes \mathbb{R} \to H^1_{\mathscr{D}}(\mathrm{Ad}\Pi_{\mathbb{R}}, \mathbb{R}(1))$$

Then the induced $\mathbb{Q}$–structure on $\bigwedge^* \mathfrak{a}_G^*$ preserves $H^*(Y(K), \mathbb{Q})_\Pi \subseteq H^*(Y(K), \mathbb{C})_\Pi$.

5.4 Properties of the action

5.4.1 Proposition ([4] 5.5.1)

The action has the following properties:

1. It is isometric when acting on the minimal degree (in the natural way). The hermitian metric on $H^*(Y(K), \mathbb{C})_\Pi$is that obtained by identification with harmonic forms.
2. It satisfies adjointness as in Lemma 3.4.2 (duality) wrt the Poincare duality pairing.
3. If χ is real valued, the twisted real structure preserves real cohomology.

Proof sketch This just follows from the Lemmas for the action on the Harish-Chandra modules

5.4.2 Lemma ([4] 5.5.1)

If there is an identification of motives $\mathfrak{d} : \mathrm{Ad}\Pi \to \mathrm{Ad}\tilde{\Pi}$, for which the action on $\hat{\mathfrak{g}}$ differs by an inner twist of the Chevalley involution (composition of inversion and long Weyl), then the actions are adjoint to one another, up to sign, with respect to Poincare duality pairing:

$$\langle f_1 \cdot X, f_2 \rangle = -\langle f_1, \mathfrak{d}(X) \cdot f_2 \rangle$$

for $X \in H^1_{\mathscr{M}}(\mathrm{Ad}^*\Pi, \mathbb{Q}(1))^\vee$ and $f_1 \in H^*(Y(K), \mathbb{C})_\Pi$, $f_2 \in H^*(Y(K), \mathbb{C})_{\tilde{\Pi}}$.

Proof again follows from previous lemmas, only note that the C action (inner twist of C_0) corresponds to $-w$. ■

5.4.3 Corollary ([4], 5.5.2)

If $\Pi \cong \tilde{\Pi}$,then the image of $H^1_{\mathscr{M}}(\mathrm{Ad}^*\Pi, \mathbb{Q}(1))$ inside $\mathfrak{a}_G$ is stable by w_G.

5.5 Example ([4], 1.3)

Let $\mathbb{T}$ be an anisotropic $\mathbb{Q}$-torus. You might want to think of $Res_{F/\mathbb{Q}}\mathbb{G}_m$ for F a quadratic imaginary field. Let $\mathfrak{a}_T^*$ be the canonical vector space we have attached to it. Then

$$\mathfrak{a}_T = \mathrm{Lie}(\mathbb{S}) \otimes \mathbb{C}$$

where $\mathbb{S}$ is the maximal $\mathbb{R}$-split subtorus of T. We may then define $\log : T \to \mathfrak{a}_T$, letting it be trivial on K_∞, and the usual logarithm on the connected component of S.

The symmetric space

$$Y = \mathbb{T}(\mathbb{Q})\backslash\mathbb{T}(\mathbb{R}) \times \mathbb{T}(\mathbb{A}_f)/KK_\infty^0$$

has the structure of a compact abelian Lie group. Each component is of the form

$$\Gamma\backslash T^\circ/K_\infty^0 \simeq \Gamma\backslash\mathfrak{a}_T$$

where $\Gamma = \mathbb{T}(\mathbb{Q}) \cap K$ is a discrete cocompact subgroup of T.

The natural action we have defined of $\bigwedge^* \mathfrak{a}_T^*$ on the cohomology of Y is simply taking cup products. That is

$$\Omega : \bigwedge^* \mathfrak{a}_T^* = Hom(\bigwedge^* \mathfrak{a}_T, \mathbb{C}) \to \Omega^*(Y, \mathbb{C})$$

comes from identifying the tangent space of T/K_∞ at the identity with $\mathfrak{a}_T$.

Then for $v \in \bigwedge^* \mathfrak{a}_T^*$, the cohomology class of $\Omega(v)$ is rational iff $\langle \log(\gamma), v\rangle \in \mathbb{Q}$ for all $\gamma \in \Gamma$.

On the other hand, if Π is cohomological, it has an assoicated motive $\mathrm{Ad}^*\Pi$ of dimension $\dim(T)$. In this case, we can actually "point" to such a motive - we want the Galois realization to be the Galois representation on $X_*(\mathbb{T}) \otimes \mathbb{Q}$. Then

$$H^1_{\mathcal{M}}(\mathrm{Ad}^*\Pi, \mathbb{Q}(1)) = \mathbb{T}(\mathbb{Q}) \otimes \mathbb{Q}$$

and the subspace of integral classes should be

$$H^1_{\mathcal{M}}((\mathrm{Ad}^*\Pi)_{\mathbb{Z}}, \mathbb{Q}(1)) = \Gamma \otimes \mathbb{Q}$$

The regulator map is simply the logarithm map.

In this case, the conjecture says that if v attains rational values on $\log \Gamma$, then cup product with $\Omega(v)$ preserves $H^*(Y, \mathbb{Q})$. But this is obvious, since cup products with $H^1(Y, \mathbb{Q})$ remain in $H^*(Y, \mathbb{Q})$.

5.6 Final remarks

Based on this conjecture, Venkatesh and Prasanna continue to make several predictions and to gain evidence for the conjecture by observing some invariants that preservation of the rational structure should preserve. For that reason they define a metric on the relevant spacesm and demand that the conjectures would be equivalent with the natural bilinear forms, so that they will be able to calculate some period integrals and see that they satisfy the expected predictions. The predictions are as follows:

5.6.1 Prediction ([4] 1.4.1)

If $\dim H^q(Y,\mathbb{C})_\Pi = 1$, and ω is a harmonic q-form on Y whose cohomology generates $H^q(Y,\mathbb{Q})_\Pi$. Then

$$\langle \omega, \omega \rangle \in vol(L)\mathbb{Q}^\times$$

where $L \subseteq \mathfrak{a}_G$ is the rational structure from Beilinson's conjecture, and the volume is measures wrt to the metric induced from $\mathfrak{a}_G$.

5.6.2 Prediction 2 ([4] 1.4.2)

Suppose that $\mathbb{G}, \mathbb{G}'$ are inner forms of one another, and Π, Π' are almost eqivalent (matching characters) automorphic tempered cuspidal representations, contributing to both cohomologies of Y and Y'. Equip Y, Y' with metrics arising from invariant bilinear forms on $\mathfrak{g}$, $\mathfrak{g}'$ which induce the same form on the base change to the algebraic closure. Then if ω, ω' are bases for harmonic forms which give $\mathbb{Q}$-rational bases for cohomology in degree q, then

$$\det\left(\left\langle \omega_i, \omega_j \right\rangle\right)^{d'} = \det\left(\left\langle \omega_i^{'}, \omega_j^{'} \right\rangle\right)^{d}$$

where $d = \dim H^q(Y,\mathbb{Q})_\Pi$, and $d' = \dim H^q(Y',\mathbb{Q})_{\Pi'}$.

These two predictions are shown to be compatible in various cases with the Ichino-Ikeda conjectures on periods.

5.6.3 Prediction 3 ([4] 1.4.3)

If L/F is Galois, split at all infinite primes, choose a level structure for $\mathbb{G}_F$ and a $Gal(L/F)$-invariant level structure for $\mathbb{G}_L$, yielding Y_F and Y_L. Fix compatible metrics on them. Suppose that

$$\dim H^{q_F}(Y_F,\mathbb{Q})_{\Pi_F} = \dim H^{q_L}(Y_L,\mathbb{Q})_{\Pi_L} = 1$$

Then there exist harmonic representatives $\omega_F, \omega_L, \omega_{L'}$ for nonzero classes in

$$H^{q_F}(Y_F,\mathbb{Q})_{\Pi_F}, H^{q_L}(Y_L,\mathbb{Q})_{\Pi_L}, H^{q_L+\delta_F}(Y_L,\mathbb{Q})_{\Pi_L}^{Gal_{L/F}}$$

such that

$$\frac{\left\|\omega_L^{'}\right\| \|\omega_F\|^2}{\|\omega_L\|} \in \sqrt{[L:F]}\cdot\mathbb{Q}^\times$$

This is proved for F an imagianry quadratic field, L a cyclic (unramified) extension of degree 3, $\mathbb{G}_F$ the underlying group of the multiplicative group $D^\times$ of the non split quaternion algebra D, when Π_F is trivial at the ramified places of D, level structure $\Gamma_0(\mathfrak{n})$ where $\mathfrak{n} = \prod \mathfrak{p}^{f(\mathfrak{p})}$is the conductor of π, under the additional assumption that $\dim H^3_{cusp}(Y,\mathbb{C}) = 1$.

References

[1] Armand Borel. Stable real cohomology of arithmetic groups ii. In *Manifolds and Lie groups*, pages 21–55. Springer, 1981.

[2] Armand Borel, Nolan R Wallach, and Nolan Wallach. *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*. Number 67. American Mathematical Soc., 2000.

[3] Robert P Langlands. Irreducible representations of real algebraic groups. *Representation theory and harmonic analysis on semisimple Lie groups*, (31):101, 1989.

[4] Kartik Prasanna and Akshay Venkatesh. Automorphic cohomology, motivic cohomology, and the adjoint l-function. *arXiv preprint arXiv:1609.06370*, 2016.

[5] David A Vogan and Gregg J Zuckerman. Unitary representations with non-zero cohomology. *Compositio Mathematica*, 53(1):51–90, 1984.

[6] Nolan R Wallach. *Real reductive groups I*, volume 132. Academic press, 1988.