The Canonical Model

Eran Assaf

Dartmouth College

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Model - analyst approach

Theorem (Shimura, Petri, ...)

If $\Gamma \subseteq PSL_2(\mathbb{Z})$ *is a neat congruence subgroup, then the algebraic curve* $X(\Gamma) = (\Gamma \backslash \mathcal{H})^*$ *has a model defined over a number field.*

Sketch of proof # 1.

$X(\Gamma)$ compact Riemann surface $\implies$ algebraic curve over $\mathbb{C}$.
If $g > 1$, not hyperelliptic, ω_X very ample $\implies X \hookrightarrow \mathbb{P}^{g-1}$.
But $\Gamma(X, \omega_X) = S_2(\Gamma)$!
If $f(\tau) = \sum_{n=1}^{\infty} a_n(f) q^n \in S_2(\Gamma)$ is an eigenform, $a_p(f)$ are eigenvalues of T_p, so all lie in a number field K.
All relations $R_i(f_1, \ldots, f_g)$ defined over K
$\implies X(\mathbb{C}) = V(R_1, \ldots, R_m) = X_0(\mathbb{C})$, with X_0/K. □

Model - geometer approach

Theorem (Shimura, Petri, ...)

If $\Gamma \subseteq PSL_2(\mathbb{Z})$ is a neat congruence subgroup, then the algebraic curve $X(\Gamma) = (\Gamma\backslash\mathcal{H})^*$ has a model defined over a number field.

Sketch of proof # 2.

E elliptic curve, $H \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$, $\phi : E[N] \to \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. Let

$$(E, \phi) \sim_H (E', \phi') \iff \exists h \in H, \iota : E \xrightarrow{\sim} E' \text{ s.t. } h \circ \phi = \phi' \circ \iota$$

Then

- $S(H) = \{(E, \phi)\}/\sim_H$.
- $(E, \phi)^\sigma = (E^\sigma, \phi \circ \sigma^{-1})$ for $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.
- (E, ϕ) is K-rational iff E/K and $\phi \circ \mathrm{Gal}(\bar{\mathbb{Q}}/K) \circ \phi^{-1} \subseteq H$.
- $S(\Gamma)(\mathbb{C}) = \Gamma_H\backslash\mathcal{H}$.

Take maximal K such that the above holds. □

Model - the Shimura way

Theorem (Shimura, [Shi67])

F totally real of degree g, B/F quaternion algebra s.t. $B \otimes \mathbb{R} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}^{g-1}$, $\mathfrak{o}$ maximal order, $\Gamma(\mathfrak{o}) = \mathfrak{o}^\times \cap B^+$. Then $X(\Gamma) = \Gamma(\mathfrak{o}) \backslash \mathcal{H}$ has a model over H_F^+.

Remark

- *This is only a special case of the theorem.*
- *When $g > 1$, this is not(!) the field of moduli of abelian varieties. Roughly - for any tot. imaginary quadratic K/F, there is a family Σ_K of a.v. A_x s.t. $B \otimes_F K \subseteq \mathrm{End}_{\mathbb{Q}}(A_x)$. Each has a field of moduli k_Σ which is not abelian over F, but over some K'. Then $C_F = \bigcap_{K'} C_{K'}$.*
- *Main point - models over $C_{K'} \implies$ model over C_F.*
- *Involves choices, but reciprocity laws uniquely determine it.*

Canonical model

Reminder - Connected components

Recall $1 \longrightarrow G' \longrightarrow G \xrightarrow{\nu} T \longrightarrow 1$ induces

$$f_K : \mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \to T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/\nu(K)$$

and if $Y = T(\mathbb{R})/T(\mathbb{R})^\dagger$, then

$$T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/\nu(K) = T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/\nu(K) = \mathrm{Sh}_{\nu(K)}(T, Y).$$

Moreover, it induces

$$\pi_K = \pi_0(\mathrm{Sh}_K(G, X)) \simeq \mathrm{Sh}_{\nu(K)}(T, Y),\ \mathrm{Sh}_K(G, X)^\circ = \mathrm{Sh}'_K(G', X^+).$$

Today

- Reflex field $E = E(G, X)$, algebraic number field.
- (Canonical) model $(f_K)_0 : \mathrm{Sh}_K(G, X)_0 \to (\pi_K)_0$ of f_K over E.
- Reciprocity laws determining the model uniquely.
- $\mathrm{Aut}(\mathbb{C}/E) \circlearrowright \pi_K \implies$ model of $\mathrm{Sh}'_K(G', X^+)$ over E'/E.

Models of varieties

Example (Elliptic Curve)

$E/\mathbb{C}$ elliptic curve has a model over K iff $j(E) \in K$. Twists by different elements of $K^\times/(K^\times)^2$ give different models.

Existence - Why? [Pet17]

- Spreading.
  Algebraic $\implies$ defined over $k = k_0(\alpha_1, \ldots, \alpha_r)$, $k_0 = \mathbb{Q}[x_0]/P$. Replace α_j by x_j to get a family over S' with $k(S') = k$, and the fiber at $x_j = \alpha_j$ is X. Replace S' by an open subset S to get a smooth fibration over $\bar{\mathbb{Q}}$.
- Rigidity.
  Countably many Shimura varieties $\implies$ fibers of small deformations are isomorphic. But $S(\bar{\mathbb{Q}})$ is dense in S (!).

Cocharacters

Definition (Conjugacy classes of cocharacters)

$G/\mathbb{Q}$ reductive, $k \subseteq \mathbb{C}$. Write

$$\mathcal{C}(k) = G(k)\backslash \operatorname{Hom}(\mathbb{G}_m, G_k).$$

Example (Unitary group)

Let $G = U_{K/\mathbb{Q}}(2)$ for some quadratic extensions $K/\mathbb{Q}$.

Let $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & \sigma(a)^{-1} \end{pmatrix} \mid a \in K^\times \right\}$. Then

$$X_*(T)_K = \left\{ t \mapsto \begin{pmatrix} t^m \sigma(t)^n & 0 \\ 0 & t^{-n}\sigma(t)^{-m} \end{pmatrix} \right\}_{m,n\in\mathbb{Z}}$$

and

$$X_*(T)_\mathbb{Q} = \left\{ t \mapsto \begin{pmatrix} \operatorname{Nm}_{K/\mathbb{Q}}(t)^n & 0 \\ 0 & \operatorname{Nm}_{K/\mathbb{Q}}(t)^{-n} \end{pmatrix} \right\}_{n\in\mathbb{Z}}$$

Conjugacy classes of cocharacters

Galois action

$\sigma(\mathrm{Ad}(g) \circ \mu) = \mathrm{Ad}(\sigma(g)) \circ \sigma(\mu) \implies \mathrm{Aut}(k'/k) \circlearrowright \mathcal{C}(k')$

Lemma

Assume G splits over k. Let T be a split maximal torus. Then

$$W \backslash \mathrm{Hom}(\mathbb{G}_m, T_k) \to G(k) \backslash \mathrm{Hom}(\mathbb{G}_m, G_k)$$

is bijective. Here $W = W(G_k, T)$ is the Weyl group.

Proof.

All maximal split tori are conjugate $\implies$ surjective. If $\mu, \mu' \in X_*(T)$ are such that $\mu = \mathrm{Ad}(g) \circ \mu'$, let $C = C_G(\mu(\mathbb{G}_m))$. Then $T \subseteq C$ and

$$\mathrm{Ad}(g)(T) \subseteq \mathrm{Ad}(g)(C_G(\mu'(\mathbb{G}_m))) = C_G(\mathrm{Ad}(g)\mu'(\mathbb{G}_m)) = C$$

are maximal split tori in a connected reductive group C.
$\exists c \in C(k) : \mathrm{Ad}(cg)T = T \implies cg \in N_G(T), \mathrm{Ad}(cg) \circ \mu' = \mu.$ □

Hodge cocharacter

Definition (Hodge cocharacter)

(G, X) *Shimura datum. The Hodge character is*

$$x \in X \rightsquigarrow h_x : \mathbb{S} \to G \rightsquigarrow h_{x\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \to G_{\mathbb{C}}$$

$$\rightsquigarrow \mu_x : \mathbb{G}_m \xrightarrow{z \mapsto (z,1)} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\sim} \mathbb{S}_{\mathbb{C}} \to G_{\mathbb{C}}$$

where $\mathbb{S}_{\mathbb{C}} \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{G}_m$ *is* $r \otimes z \mapsto (rz, r\bar{z})$.

Example (Modular curve)

Let $G = GL_2(\mathbb{Q})$, $X = \mathrm{Ad}(G(\mathbb{R})) \cdot h$, $h(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Since $\frac{z+1}{2} \otimes 1 - \frac{i(z-1)}{2} \otimes i \mapsto (z,1)$, we have

$$\mu_h(z) = \begin{pmatrix} \frac{z+1}{2} & -\frac{i(z-1)}{2} \\ \frac{i(z-1)}{2} & \frac{z+1}{2} \end{pmatrix}, \ \mathrm{Tr}(\mu_h(z)) = z+1, \ \det(\mu_h(z)) = z$$

Thus $\mu_h(z) \sim \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$.

Reflex field - definition

Hodge cocharacter is algebraic

The Hodge cocharacters $\{\mu_x\}_{x\in X}$ define $c(X) \in \mathcal{C}(\mathbb{C})$. From the lemma

$$\mathcal{C}(\bar{\mathbb{Q}}) \simeq W(\bar{\mathbb{Q}})\backslash \mathrm{Hom}(\mathbb{G}_m, T_{\bar{\mathbb{Q}}}) = W(\mathbb{C})\backslash \mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}}) \simeq \mathcal{C}(\mathbb{C}).$$

Definition (Reflex field)

$E(G,X)$ is the field of definition of $c(X)$ in $\bar{\mathbb{Q}}$. Explicitly, if $H_X = \mathrm{Stab}_{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}\, c(X)$, then $E(G,X) = \bar{\mathbb{Q}}^{H_X}$.

Remark (Finiteness of the reflex field)

$E(G,X) \subseteq k$, since $W(k) = W(\bar{\mathbb{Q}})$ and $X_*(T)_k = X_*(T)_{\bar{\mathbb{Q}}}$.

Reflex field and Hodge cocharacters

Lemma ([Kot84])

If $\mu \in c(X)$ is defined over k, then $E(G,X) \subseteq k$. If G is quasi-split over k, and $E(G,X) \subseteq k$, then $c(X)$ contains a μ defined over k.

Proof.

$(\Rightarrow)$ If $\sigma \in \mathrm{Gal}(\bar{Q}/k)$ then $\sigma(\mu) = \mu$, hence $\sigma(c(X)) = c(X)$, so $\mathrm{Gal}(\bar{Q}/k) \subseteq H_X$. Taking fixed fields $E(G,X) \subseteq k$.
$(\Leftarrow)$ S maximal k-split torus, $T = C_G(S)$. G q.s. $\implies$ T maximal torus. B a k-Borel containing T, C the B-positive Weyl chamber of $X_*(T) \otimes \mathbb{R}$. Since $\bar{C}$ is a fundamental domain for W, $\exists \mu \in \bar{C} \cap c(X)$. If $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/k)$, then $\sigma\mu \in \bar{C}$ since B is a k-group, and since $c(X)$ is fixed by σ, $\mu, \sigma\mu$ are in the same W-orbit $\implies$ $\sigma\mu = \mu$. □

Reflex fields of tori

Example (Torus)

T torus over $\mathbb{Q}$, $h : \mathbb{S} \to T_{\mathbb{R}}$. $\mu_h : \mathbb{G}_{m\mathbb{C}} \to T_{\mathbb{C}}$ is defined over $\bar{\mathbb{Q}}$, $E(T, \{h\})$ fixed field of the sub. of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ fixing $\mu_h \in X_*(T)$.

Example (CM Torus)

(E, Φ) CM type, $T = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$. $T(\mathbb{R}) = (E \otimes \mathbb{R})^\times \simeq (\mathbb{C}^\Phi)^\times$.
Let $h_\Phi : \mathbb{S} \to T_{\mathbb{R}}$ be s.t.

$$h_\Phi(\mathbb{R}) = z \mapsto (z, z, \ldots, z) : \mathbb{C}^\times \to (\mathbb{C}^\Phi)^\times.$$

On $\mathbb{C}$-points, $h_{\Phi,\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \to T_{\mathbb{C}}$ is the map

$$(z_1, z_2) \mapsto (z_1, \ldots, z_1, z_2, \ldots, z_2) : \mathbb{C} \times \mathbb{C} \to (\mathbb{C}^\Phi)^\times \times (\mathbb{C}^{\bar{\Phi}})^\times.$$

The Hodge cocharacter is

$$\mu_\Phi = z \mapsto (z, \ldots, z, 1, \ldots, 1) : \mathbb{C}^\times \to (\mathbb{C}^\Phi)^\times \times (\mathbb{C}^{\bar{\Phi}})^\times.$$

Thus $E(T, \{h_\Phi\})$ is the reflex field of (E, Φ).

PEL Shimura varieties

Example (PEL Shimura varieties)

(G, X) simple PEL datum of type (A) or (C). Then σ fixes the conjugacy class of h iff it fixes $\mathrm{Tr} \circ h$. Since B acts on $V = W \otimes V_0$ via W, case by case analysis shows that $F_0(\{\mathrm{Tr} \circ h(z)\}_{z \in \mathbb{C}}) = F_0(\{\mathrm{Tr}_X(b)\}_{b \in B})$. Then $E(G, X)$ is the field generated by $\{\mathrm{Tr}_X(b)\}_{b \in B}$.

Remark

Note that this is the field of definition of the moduli problem - $(A, i, s, \eta K)$ s.t. A abelian variety, $\pm s$ polarization, $i : B \to \mathrm{End}^0(A)$, ηK level, $\mathrm{Tr}(i(b)|T_0 A) = \mathrm{Tr}_X(b)$.

Shimura curves

Example (Quaternion algebra)

F totally real, B/F quaternions, $G = B^\times$, I_{nc} the split places, I_c the non-split places. Let $h(\mathbb{R}) : \mathbb{C}^\times \to \mathbb{H}^{I_c} \times GL_2(\mathbb{R})^{I_{nc}}$ be

$$a + bi \mapsto \left(1, 1, \ldots, 1, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \ldots, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right).$$

Then $c(X)$ contains $\mu : \mathbb{G}_{m\mathbb{C}} \to G_\mathbb{C}$,

$$z \mapsto (1, 1, \ldots, 1) \times \left(\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \ldots \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\right) \in GL_{2\mathbb{C}}^{I_c} \times GL_{2\mathbb{C}}^{I_{nc}},$$

so $E(G, X)$ is the fixed field of the subgroup stabilizing I_{nc}.
If $I_{nc} = \{v\}$, the case of Shimura curve, $E(G, X) = v(F)$.

Reflex field from Dynkin diagram

Example (Adjoint groups)

Assume G adjoint. T maximal torus in $G_{\bar{Q}}$, Δ a base for the roots of $\Phi(G,T)_{\bar{\mathbb{Q}}}$. If $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, it acts on $X^*(T)$, hence on Φ, and $\sigma(\Delta)$ is also a base for Φ. W acts simply transitively on the bases of Φ, so $\exists! w_\sigma \in W$ s.t. $w_\sigma(\sigma(\Delta)) = \Delta$. Then $\sigma * \alpha = w_\sigma(\sigma\alpha)$ is an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on Δ.

If G is split, choose T split and B a $\mathbb{Q}$-Borel, so $\sigma(\Delta) = \Delta$, and the action is trivial. Since G splits over a finite extension, the action is continuous.

Every $c \in \mathcal{C}(\bar{\mathbb{Q}})$ contains a unique $\mu : \mathbb{G}_m \to G_{\bar{Q}}$ such that $\langle \alpha, \mu \rangle \geqslant 0$ for all $\alpha \in \Delta$, because W acts simply transitively on the chambers, and the map

$$c \mapsto (\langle \alpha, \mu \rangle)_{\alpha \in \Delta} : \mathcal{C}(\bar{\mathbb{Q}}) \to \mathbb{N}^{\Delta}$$

is bijective. Therefore $E(G,X)$ is the fixed field of the subgroup fixing $(\langle \alpha, \mu \rangle)_{\alpha \in \Delta} \in \mathbb{N}^{\Delta}$. Complex conjugation acts as $\alpha \mapsto -w_0\alpha$.

Special points

Definition (Special point)

$x \in X$ is special if $\exists T \subseteq G$ such that $h_x(\mathbb{C}^\times) \subseteq T(\mathbb{R})$. (T, x) or (T, h_x) is a special pair in (G, X). When the weight is rational (SV4) and $Z(G)^\circ$ splits over a CM field (SV6), they are called CM-points and CM-pairs.

Remark

- *If (T, x) is special then $T(\mathbb{R})$ fixes x. If T maximal and fixes x, $h(\mathbb{C}^\times) \subseteq C_G(T)(\mathbb{R}) = T(\mathbb{R})$, so x is special.*
- *$Z(G)^\circ$ is isogenous to G^{ab} $\implies$ (SV6) means G^{ab} splitting over a CM field. With (SV2), G splits over a CM field. (SV4) shows $h_x : \mathbb{S} \to T$ factors through G_{Hod}, and as it factors through a CM torus, it also factors through the Serre group, so for any rep. (V, ρ) of T, $(V, \rho \circ h_x)$ is the Hodge structure of a CM motive.*

Special points and CM

Example (Modular curve)

$G = GL_2$, $X = \mathcal{H}^{\pm} = \mathbb{C} \backslash \mathbb{R}$. Let $z \in \mathbb{C} \backslash \mathbb{R}$ is s.t. $E = \mathbb{Q}[z]$ is quadratic imaginary. Embed $E \hookrightarrow M_2(\mathbb{Q})$ using the basis $\{1, -z\}$, to get a maximal subtorus $\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m \subseteq G$. The kernel of $e \otimes z \mapsto z : E \otimes \mathbb{C} \to \mathbb{C}$ is spanned by $z \otimes 1 + 1 \otimes (-z)$, which is $(z, 1)$ in our basis, representing z. The map is $E \otimes \mathbb{R}$-linear, hence $(E \otimes \mathbb{R})^{\times}$ fixes z. Thus z is special.
If z is special, it is fixed by some $t \in G(\mathbb{Q})$, so $\mathbb{Q}[z]$ is quadratic.

Remark

If $\mathrm{Sh}_K(G, X) = \{(A, \ldots)\}/\sim$*, special points are a.v.s of CM-type. The theory of CM describes how an open subgroup of* $\mathrm{Aut}(\mathbb{C})$ *acts on such points. Shimura constructs an action on the special points, that agrees with it.*

The homomorphism r_x

Definition (action on special points)

T torus over $\mathbb{Q}$, $\mu \in X_*(T)$ defined over E. For $Q \in T(E)$, $\sum_{\rho:E\to\bar{\mathbb{Q}}} \rho(Q) \in T(\bar{\mathbb{Q}})$ is stable under the Galois action, so is in $T(\mathbb{Q})$. $r(T,\mu) : \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m \to T$ is s.t.

$$r(T,\mu)(P) = \sum_{\rho:E\to\bar{Q}} \rho(\mu(P)) \ \forall P \in E^\times.$$

(T,x) special pair, $E(x)$ the field of definition of μ_x.

$$r_x : \mathbb{A}_{E(x)}^\times \xrightarrow{r(T,\mu)} T(\mathbb{A}_\mathbb{Q}) \to T(\mathbb{A}_{\mathbb{Q},f}).$$

Explicitly, if $a = (a_\infty, a_f) \in \mathbb{A}_{E(x)}^\times$, then

$$r_x(a) = \sum_{\rho:E\to\bar{\mathbb{Q}}} \rho(\mu_x(a_f)).$$

Shimura reciprocity

Definition (Canonical model)

A model $M_K(G,X)$ *of* $\mathrm{Sh}_K(G,X)$ *over* $E(G,X)$ *is canonical if for every special pair* (T,x) *and* $a \in G(\mathbb{A}_f)$, $[x,a]$ *has coordinates in* $E(x)^{\mathrm{ab}}$ *and*

$$\sigma[x,a]_K = [x, r_x(s)a]_K$$

for all $\sigma \in \mathrm{Gal}(E(x)^{\mathrm{ab}}/E(x))$, $s \in \mathbb{A}_{E(x)}^{\times}$ *with* $\mathrm{art}_{E(x)}(s) = \sigma$.

Example (Tori)

T torus over $\mathbb{Q}$, $h : \mathbb{S} \to T_{\mathbb{R}}$. (T,h) is a Shimura datum, $E = E(T,h)$ is the field of definition of μ_h. $\mathrm{Sh}_K(T,H)$ is a finite set, and Shimura reciprocity defines a continuous action of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ on $\mathrm{Sh}_k(T,h)$. This defines a model of $\mathrm{Sh}_K(T,h)$ over E, which is, by definition, canonical.

Complex multiplication

Example (CM tori)

(E, Φ) CM-type. (T, h_Φ) as before. Then $E(T, h_\Phi) = E^*$, and $r(T, \mu_\Phi) : \mathrm{Res}_{E*/\mathbb{Q}} \mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ is given by

$$r(T, \mu_\Phi)(z) = \prod_{\rho : E* \to \bar{\mathbb{Q}}} \rho(z, z, \ldots, z, 1, 1, \ldots, 1)$$
$$= \left(\mathrm{Nm}_{E*/\mathbb{Q}}(z), \ldots, \mathrm{Nm}_{E*/\mathbb{Q}}(z), 1, 1, \ldots, 1\right)$$

and we want $z' \in E$ mapping to it. However, recall that $V = E \otimes \mathbb{R} \simeq \mathbb{C}^\Phi$ is stable by $\mathrm{Aut}(\mathbb{C}/E^*)$, hence there is an E^*-vector space V_0 such that $V = V_0 \otimes \mathbb{C}$. $z \in E^*$ acts on V via multiplication by $\mathrm{Nm}_{E*/\mathbb{Q}}(z)$, which on $\mathbb{C}^\Phi$ is $\mathrm{Nm}_{E*/\mathbb{Q}}(z) \times 1_\Phi$. But as an E-vector space, this is a line, so there is $z' \in E$ such that this is multiplication by z'. This is how we defined the reflex norm $N_{\Phi*}(z)$. Therefore $r(T, \mu_\Phi) = N_{\Phi*}$.

CM tori as moduli spaces

Lemma (CM tori)

$\mathrm{Sh}_K(T, h_\Phi)$ *classifies triples* $(A, i, \eta K)$ *s.t. A is a.v. of CM type* (E, Φ) *and* $\eta : V(\mathbb{A}_f) \to V_f(A)$ *is an* $E \otimes \mathbb{A}_f$*-linear isomorphism.*

Proof.

$\dim_E V = 1$. E-action on V gives $T \hookrightarrow GL_{\mathbb{Q}}(V)$. If (A, i) is of CM-type (E, Φ), there is $a : H_1(A, \mathbb{Q}) \to V$ carrying i_A to i_Φ. The isomorphism $a \circ \eta : V(\mathbb{A}_f) \to V(\mathbb{A}_f)$ is $E \otimes \mathbb{A}_f$-linear, so is multiplication by $g \in (E \otimes \mathbb{A}_f)^\times = T(\mathbb{A}_f)$.
The map $(A, i, \eta) \mapsto [g]$ is the bijection. □

Galois action

Same as classes over $\bar{\mathbb{Q}}$. Let $\mathcal{M}_K$ be the set of such triples. Then $\mathrm{Gal}(\bar{\mathbb{Q}}/E^*)$ acts on $\mathcal{M}_K$, via $\sigma(A, i, \eta K) = (\sigma A, \sigma \circ i, \sigma \circ \eta)$.

Reciprocity law and CM

Proposition (Reciprocity law = CM)

The map $(A, i, \eta) \mapsto [a \circ \eta]_K : \mathcal{M}_K \to \mathrm{Sh}_K(T, h_\Phi)$ *commutes with the Galois actions on both sides.*

Proof.

Main theorem of CM $\implies$ E-linear $\alpha : A \to \sigma A$ s.t. $\alpha(N_{\Phi *}(s) \cdot x) = \sigma x$ for $x \in V_f(A)$, where $s \in \mathbb{A}_{E*}$ is such that $\mathrm{art}_{E*}(s) = \sigma|E^*$. Then $a \circ V_f(\alpha)^{-1} : V_f(\sigma A) \to V(\mathbb{A}_f)$ is an E-isomorphism, and so

$$\sigma(A, i, \eta) = (\sigma A, \sigma \circ i, \sigma \circ \eta) \mapsto [a \circ V_f(\alpha)^{-1} \circ \sigma \circ \eta]_K.$$

But $V_f(\alpha)^{-1} \circ \sigma = (N_{\Phi *})_f(s) = r_{h_\Phi}(s)$, so

$$[a \circ V_f(\alpha)^{-1} \circ \sigma \circ \eta]_K = [r_{h_\Phi}(s) \cdot (a \circ \eta)]_K. \qquad \square$$

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