K3 surfaces: From counting points to rational curves

Edgar Costa (MIT)
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VaNTAGe: K3 surfaces

Slides available at `researchseminars.org`

Joint work with Andreas-Stephan Elsenhans, Francesc Fité, Jörg Jahnel, Emre Sertöz, and Andrew Sutherland.

"Dans la seconde partie de mon rapport, il s'agit des variétés kählériennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire." —André Weil *(Photo credit: Waqas Anees)*

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- smooth quartic surface in $\mathbb{P}^3$

$$X : f(x, y, z, w) = 0, \quad \deg f = 4$$

  e.g. Fermat quartic surface $x^4 + y^4 + z^4 + w^4 = 0$.

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$$X : w^2 = f(x, y, z), \quad \deg f = 6$$

e.g. Fermat like surface $w^2 = x^6 + y^6 + z^6$.

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- Kummer surfaces, $\mathrm{Kummer}(A) := \widetilde{A/\pm}$, with A an abelian surface.

K3 surfaces — the sweet spot for surfaces™

In the classification of surfaces, they land in the middle.

Neither too simple nor too complicated, next level of difficulty past ruled surfaces

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- a weaker analogue of Honda–Tate theory for abelian varieties.
- categorical description of ordinary K3 surfaces over a finite field

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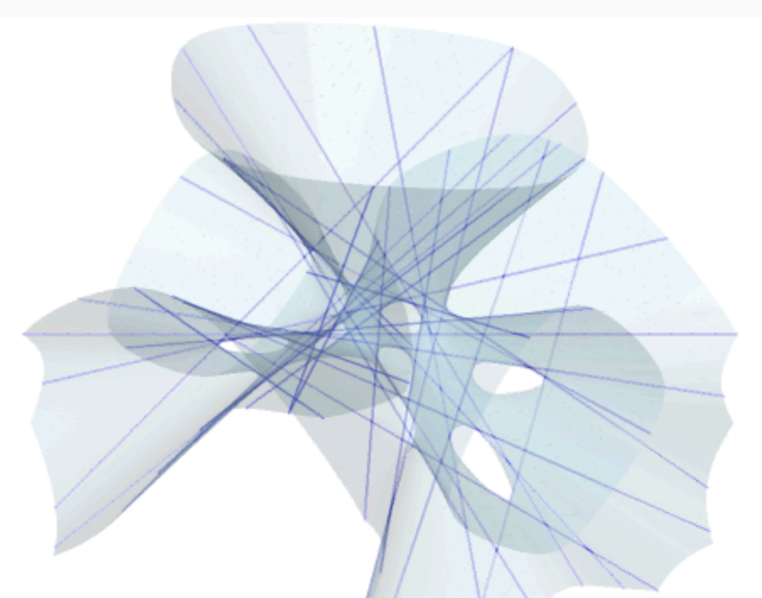
Example: Quartic K3 surface with 42 lines, by Elkies



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- Compute geometric invariants
  - Automorphism group $\mathrm{Aut}(X)$
  - Period map
  - Brauer group $\mathrm{Br}(X)$
  - Picard lattice $\mathrm{Pic}(X) \simeq \mathbb{Z}^\rho$

Picard lattice

A key geometric invariant for an algebraic K3 surface is its Picard lattice

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$$\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \simeq H^{1,1}(X_{\mathbb{C}}) \cap H^2(X_{\mathbb{C}}, \mathbb{Z}) \subsetneq H^2(X_{\mathbb{C}}, \mathbb{Z}) \simeq (-E_8)^2 \oplus U^3 \simeq \mathbb{Z}^{22}$$

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The degree of "difficulty" is negatively correlated with $\rho(X)$

$$H^2(X_{\mathbb{C}}, \mathbb{Q}) \simeq Pic(X_{\mathbb{Q}^{\mathrm{al}}})_{\mathbb{Q}} \oplus T(X)_{\mathbb{Q}}$$

The "new and interesting" Galois representations arise from $T(X)$.

Picard lattice – over finite fields

Over $\mathbb{F}_p^{\mathsf{al}}$ we have $\rho(X_{\mathbb{F}_p^{\mathsf{al}}}) \in \{2, 4, \ldots, 22\}$ (Over $\mathbb{Q}^{\mathsf{al}}$ it was $\{1, 2, \ldots, 20\}$)

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$$Z_X(t) := \exp\left(\sum_{m=1}^{\infty} \frac{\#X(\mathbb{F}_{p^m})}{m} t^m\right) = \frac{1}{(1-t)\chi(t)(1-p^2t)}$$

where $\chi(t) = \det(1 - t\,\mathrm{Frob}\,|H^2_{\mathrm{et}}(X_{\mathbb{F}_p^{\mathrm{al}}}, \mathbb{Q}_\ell)) \in \mathbb{Z}[t]$ and $\deg \chi = 22$.

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$$\mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_\ell} = \mathbf{ker}(\mathbf{Frob}_p - p \cdot \mathbf{id}\,|H^2_{\mathrm{et}}(X_{\mathbb{F}_p^{\mathrm{al}}}, \mathbb{Q}_\ell))$$

Tate conjecture is a theorem for K3 surfaces over finite fields.
[Charles, Madapusi, Kim–Madapusi]

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Tate conjecture is a theorem for K3 surfaces over finite fields.

For $p > 7$ computing $Z_X(t)$ by naive point counting is not practical.

Instead, one relies in a infrastructure of methods in crystalline cohomology [Abbott–Kedlaya–Roe, C, C–Harvey–Kedlaya, Tuitman–Pancratz]

Computing Picard lattice over $\mathbb{Q}^{\mathrm{al}}$

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Various ad hoc methods exist to improve the inequality above.

Improving upper bounds — using two specializations [van Luijk]

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If p and q are two primes of good reduction, and

$$\rho(X_{\mathbb{F}_p^{\mathrm{al}}}) = \rho(X_{\mathbb{F}_q^{\mathrm{al}}}) = 2r,$$
$$\operatorname{disc}\mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}}) \neq \operatorname{disc}\mathrm{Pic}(X_{\mathbb{F}_q^{\mathrm{al}}}).$$

then

$$\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) < 2r.$$

van Luijk, used this technique with $r = 1$, to provide the first known examples of K3 surfaces over $\mathbb{Q}$ such that $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 1$

Improving upper bounds — torsion-free cokernel [Elsenhans–Jahnel]

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For example, if $\rho(X_{\mathbb{F}_p^{\mathrm{al}}}) = 2$, Elsenhans–Jahnel approach is

1. compute $\mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$
2. estimate the degree of a hypothetical effective divisor of the lift
3. use Gröbner bases to verify that such a divisor does or does not exist

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For example, if $\rho(X_{\mathbb{F}_p^{\mathrm{al}}}) = 2$, Elsenhans–Jahnel approach is

1. compute $\mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$
2. estimate the degree of a hypothetical effective divisor of the lift
3. use Gröbner bases to verify that such a divisor does or does not exist

This approach is only practical if one can compute $\mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$ and if the obtained estimates are low.

Improving upper bounds — p-adic obstruction map [C–Sertöz]

Compute an p-adic approximation of the obstruction map

$$\pi : \mathrm{Pic}(X_{\mathbb{F}_p}) \subset H^2_{\mathbf{crys}}(X/\mathbb{Z}_p) \to H^2_{\mathbf{crys}}(X/\mathbb{Z}_p)/F^1 H^2_{\mathbf{crys}}(X/\mathbb{Z}_p)$$

If $\pi(C) \neq 0$, then $C \notin \mathrm{Pic}(X)$. (analogous to $\mathrm{Pic}(X_{\mathbb{C}}) = H^{1,1}(X_{\mathbb{C}}) \cap H^2(X, \mathbb{Z})$)

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1. compute a p-adic approximation of Frob_p
2. compute an approximation of

$$\mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_p} = \ker(\mathrm{Frob}_p - p \cdot \mathrm{id} \,|H^2_{\mathrm{dR}}(X/\mathbb{Q}_p))$$

3. compute an approximation of

$$\pi_{\mathbb{Q}_p} : \mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_p} \to H^2_{\mathrm{dR}}(X/\mathbb{Q}_p)/F^1 H^2_{\mathrm{dR}}(X/\mathbb{Q}_p)$$

4. $\dim \mathrm{Pic}(X) \leq \dim_{\mathbb{Q}_p} \ker \pi_{\mathbb{Q}_p}$

Picard number via Sato–Tate moments

Theorem (C–Fité–Sutherland)

Let X be a K3 surface over a number field k, then we have

$$\begin{aligned}\dim \operatorname{Pic}(X) &= \mathrm{M}_1[a_1] = \mathrm{E}_{\mathrm{ST}_X}[\mathrm{tr}] \\ &\stackrel{?}{=} \mathrm{E}[\mathrm{tr}(\mathrm{Frob}_p \,|H^2(X)(1))] = \lim_{N\to\infty} \pi_k(N)^{-1} \sum_{\mathrm{Nm}(\mathfrak{p})\leq N} \frac{\mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}})}{\mathrm{Nm}(\mathfrak{p})}\end{aligned}$$

The Sato–Tate group of X is a compact Lie group $G \subset O(22)$ containing (as a dense subset) the image of a representation that maps Frobenius elements to conjugacy classes.

K3 surfaces

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Theorem [Li–Liedtke]

If there are infinitely many p primes such that

$$\rho(X_{\mathbb{Q}^{\mathrm{al}}}) < \rho(X_{\mathbb{F}_p^{\mathrm{al}}}) \text{ and } \rho(X_{\mathbb{F}_p^{\mathrm{al}}}) \neq 22,$$

then $X_{\mathbb{Q}^{\mathrm{al}}}$ contains infinitely many rational curves.

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Corollary [Li–Liedtke]

If $\rho(X_{\mathbb{Q}^{\mathrm{al}}})$ is odd, then $X_{\mathbb{Q}^{\mathrm{al}}}$ contains infinitely many rational curves.

Jumping Picard ranks

Theorem [Charles]

We have

$$\rho(X_{\mathbb{Q}^{\mathrm{al}}}) + \eta(X_{\mathbb{Q}^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$$

for some $\eta(X_{\mathbb{Q}^{\mathrm{al}}}) \geq 0$. Equality occurs infinitely often (density 1 after some finite extension).

Consider

$$\Pi_{\mathrm{jump}}(X) := \left\{ p : \rho(X_{\mathbb{F}_p^{\mathrm{al}}}) > \rho(X_{\mathbb{Q}^{\mathrm{al}}}) + \eta(X_{\mathbb{Q}^{\mathrm{al}}}) \right\}$$

Is this set infinite? What is its density?

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What about

$$\gamma(X, B) := \frac{\# \{p \leq B : p \in \Pi_{\mathrm{jump}}(X)\}}{\# \{p \leq B\}} \quad \text{as } B \to \infty \quad ?$$

Jumping Picard ranks for Kummer surfaces

Let $X \simeq \mathrm{Kummer}(A) := \widetilde{A/\pm}$ be a Kummer surface, where A is an abelian surface.

We have

- $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = \rho(A_{\mathbb{Q}^{\mathrm{al}}}) + 16$
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Thus, $\Pi_{\mathrm{jump}}(X) = \Pi_{\mathrm{jump}}(A)$

Moreover

$$\mathrm{Pic}(A_k)/\,\mathrm{Pic}^0(A_k) \simeq \mathrm{NS}(A_k)_{\mathbb{Q}} \simeq \{\phi \in \mathrm{End}(A_k)_{\mathbb{Q}} : \phi^\dagger = \phi\},$$

where $\dagger$ denotes the Rosati involution and

- $\rho(A_{\mathbb{F}_p^{\mathrm{al}}}) \geq 4 \iff A_{\mathbb{F}_p^{\mathrm{al}}} \sim E^2$, E an elliptic curve
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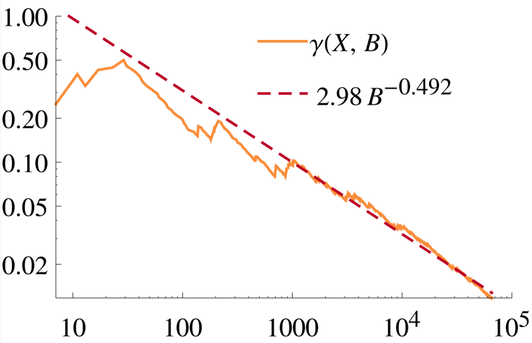
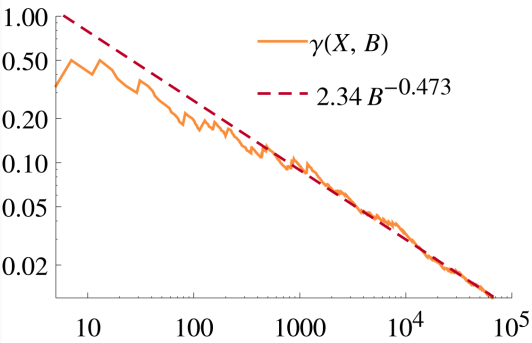
Let's do some numerical experiments for some non Kummer surfaces!

Two generic K3 surfaces with $\rho(X_{\mathbb{Q}^{al}}) = 1$





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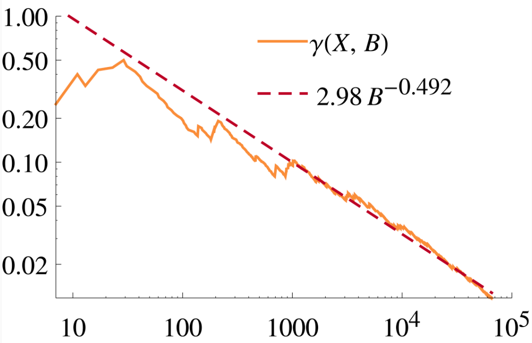

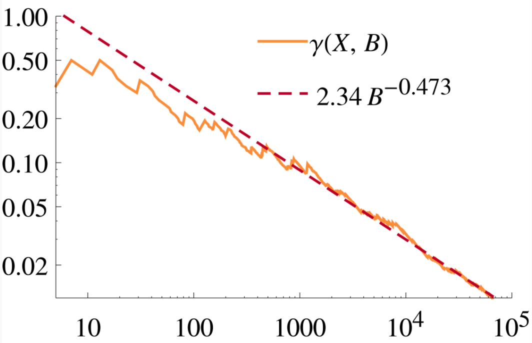

$$\gamma(X, B) \overset{?}{\sim} \frac{c_X}{\sqrt{B}}, \quad B \to \infty$$

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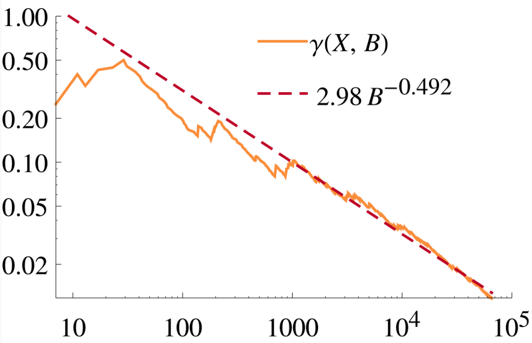

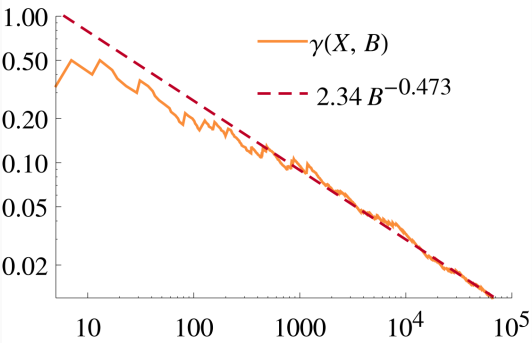

$$\gamma(X, B) \overset{?}{\sim} \frac{c_X}{\sqrt{B}}, \quad B \to \infty$$

$$\Longrightarrow \ \mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) \overset{?}{\sim} 1/\sqrt{p}$$

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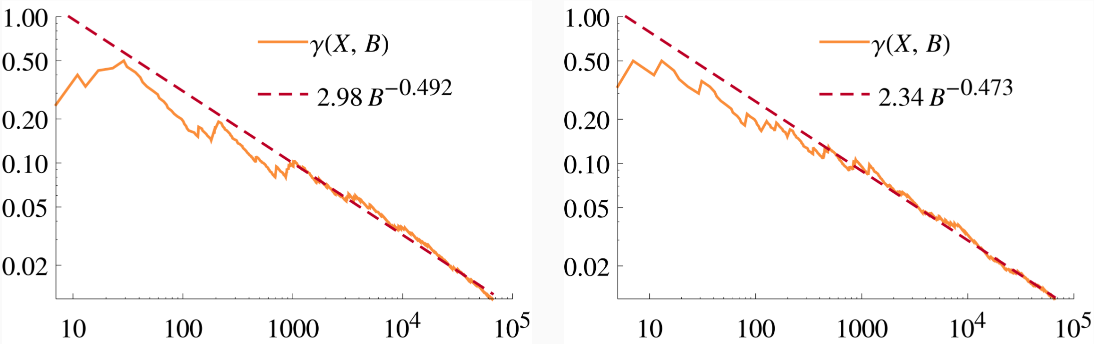

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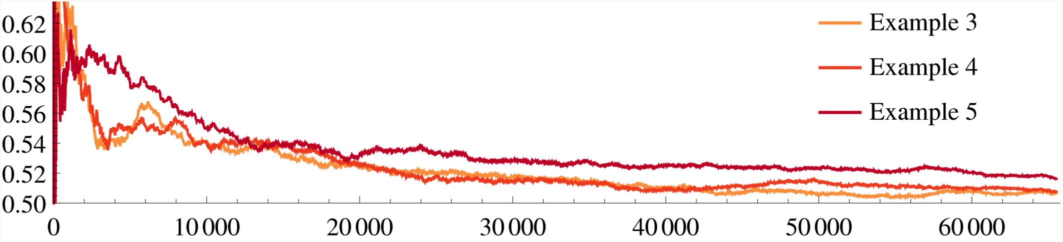
Three K3 surfaces with $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 2$



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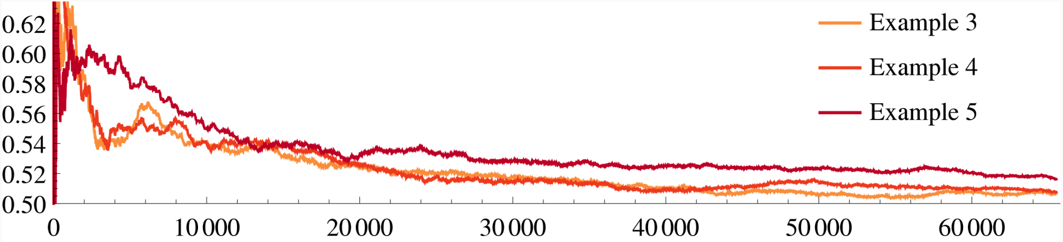

No obvious trend...

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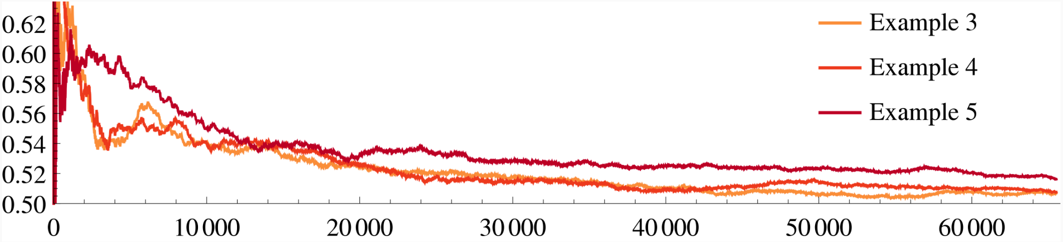

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Could it be related to some integer being a square modulo p?

We can explain the $1/2$

Theorem (C, C–Elsenhans–Jahnel)

If $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = \min_p \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$, *then there is* $d_X \in \mathbb{Z}$ *such that:*

$$\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\} \subset \Pi_{\mathrm{jump}}(X).$$

d_X *represents the quadratic character* $p \mapsto \det(\mathrm{Frob}_p \,|T(X)(1)) \in \pm 1$.

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Corollary

If d_X *is not a square:*

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$d_{X_3} = -1 \cdot 5 \cdot 151 \cdot 22490817357414371041 \cdot 38730849743014933723366635880799626078087505674085098421327697034327893534206888970614673313789$

$d_{X_4} = 53 \cdot 2624174618795407 \cdot 5128545618469648171394942020727783 41 \cdot 1215218370089028769076718102126921744353362873 \cdot 6847124397158950456921300435158$

$d_{X_5} = -1 \cdot 47 \cdot 3109 \cdot 4969 \cdot 14857095849982608071 \cdot 445410277660928347762586764331874432202584688016149 \cdot 65865270852505269999342419873884248599811$

Experimental data for $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 2$ (again)

What if we ignore $\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\} \subset \Pi_{\mathrm{jump}}(X)$?

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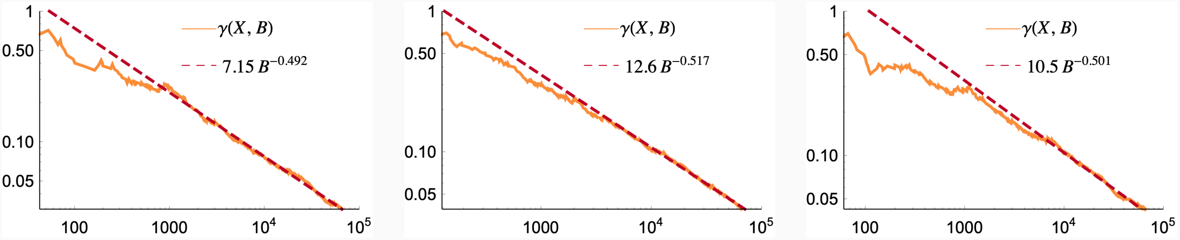

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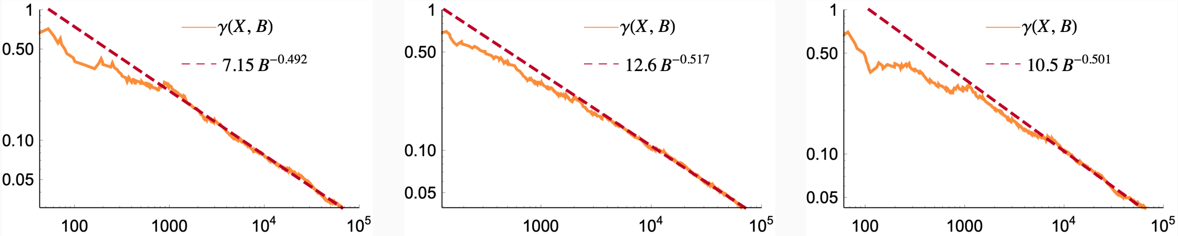

$$\mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) = \begin{cases} 1 & \text{if } d_X \text{ is not a square modulo } p \\ \overset{?}{\sim} \frac{1}{\sqrt{p}} & \textit{otherwise} \end{cases}$$