Randomness in number theory

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November 29th, 2018

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Slides available at `edgarcosta.org` under Research

Randomness principle in number theory

Number theoretic dichotomy [Sarnak]

Given a problem, either

1. there is a rigid structure $\rightsquigarrow$ rigid solution, or
2. the answer is difficult to determine $\rightsquigarrow$ random behaviour

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- Understanding and/or proving the probability law
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- Real world applications
  - pseudo random numbers
  - cryptography
  - quasi-Monte Carlo methods

Counting roots of polynomials

$f(x) \in \mathbb{Z}[x]$ a monic irreducible polynomial of degree $d > 0$

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For quadratic polynomials, $x^2 + ax + b$, the answer just depends on the sign of $\Delta := a^2 - 4b$.

Counting roots of polynomials over finite fields

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How many roots does f have?

$$\begin{aligned} N_f(p) :=& \#\{x \in \{0, \ldots, p-1\} \ : \ f(x) \equiv 0 \bmod p\} \\ =& \#\{x \in \{0, \ldots, p-1\} \ : \ p \mid f(x)\} \\ =& \#\{x \in \mathbb{F}_p \ : \ f(x) = 0\} \in \{0, 1, \ldots, d\} \end{aligned}$$

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How often does each value occur?

Quadratic polynomials

$$f(x) = x^2 + ax + b, \quad \Delta := a^2 - 4b$$

$$\text{Quadratic formula} \implies N_f(p) = \begin{cases} 0 & \text{if } \Delta \text{ is not a square modulo } p \\ 1 & \text{if } \Delta \equiv 0 \bmod p \\ 2 & \text{if } \Delta \text{ is a square modulo } p \end{cases}$$

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Hence, if $\Delta \in \mathbb{Z}$ isn't a square, then

$$\mathsf{Prob}(\Delta \text{ is a square modulo } p) = 1/2$$

$$\implies \mathsf{Prob}(N_f(p) = 0) = \mathsf{Prob}(N_f(p) = 2) = \frac{1}{2}$$

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It is easy to describe for which primes Δ is a square $\bmod p$.
For example, 5 is a square for $p \equiv 1, 4 \bmod 5$ and $p = 2$.

Cubic polynomials

$$f(x) = x^3 - 2 = \left(x - \sqrt[3]{2}\right)\left(x - \sqrt[3]{2}e^{2\pi i/3}\right)\left(x - \sqrt[3]{2}e^{4\pi i/3}\right)$$

$$\text{Prob}\left(N_f(p) = k\right) = \begin{cases} 1/3 & \text{if } k = 0 \\ 1/2 & \text{if } k = 1 \\ 1/6 & \text{if } k = 3. \end{cases}$$

$$g(x) = x^3 - x^2 - 2x + 1 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

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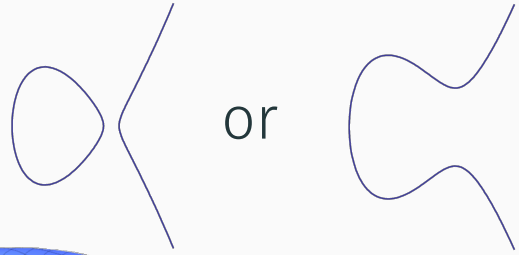
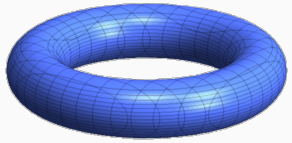
Elliptic curves

An *elliptic curve* is a smooth curve defined by

$$y^2 = x^3 + ax + b$$

Over $\mathbb{R}$ it might look like  or

Over $\mathbb{C}$ this is a torus 

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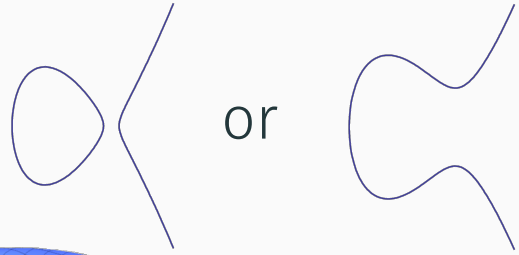
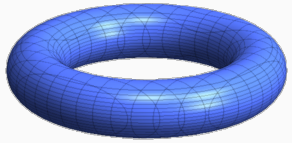
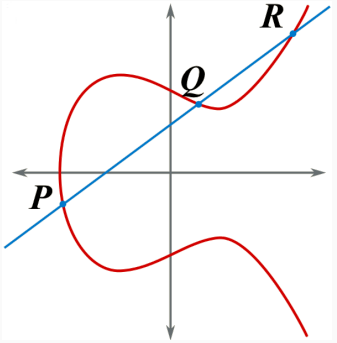
Over $\mathbb{R}$ it might look like  or

Over $\mathbb{C}$ this is a torus 

There is a natural *group structure*!

If P, Q, and R are colinear, then

$$P + Q + R = 0$$



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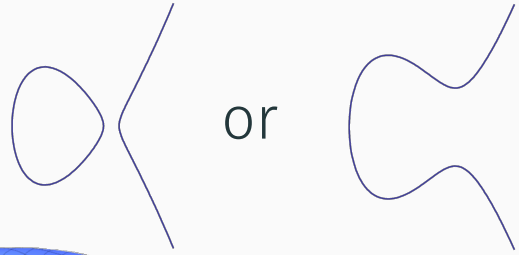
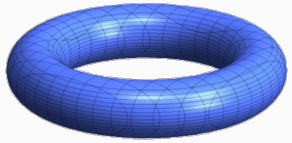
Over $\mathbb{R}$ it might look like  or

Over $\mathbb{C}$ this is a torus 

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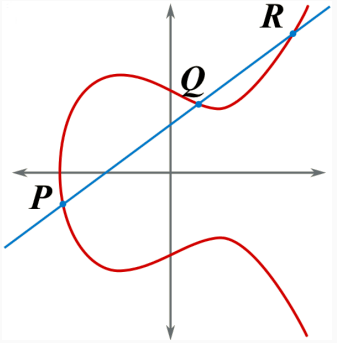

Applications:

- cryptography
- integer factorization
- pseudorandom numbers, ...

Elliptic curves

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Write $E_p := E \bmod p$

- What can we say about $\#E_p$ for an arbitrary p?

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$\rightsquigarrow$ studying the **statistical** properties $\#E_p$.

Hasse's bound

Theorem (Hasse, 1930s)

$$|p + 1 - \#E_p| \leq 2\sqrt{p}.$$

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In other words,

$$\lambda_p := \frac{p + 1 - \#E_p}{\sqrt{p}} \in [-2, 2]$$

What can we say about the error term, λ_p, as $p \to \infty$?

Two types of elliptic curves

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There are two limiting distributions for λ_p

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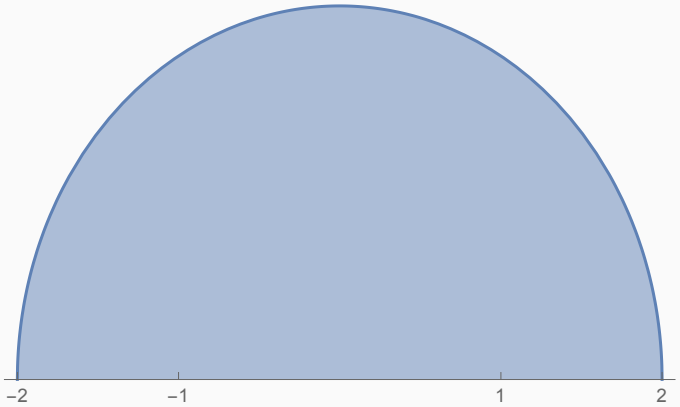

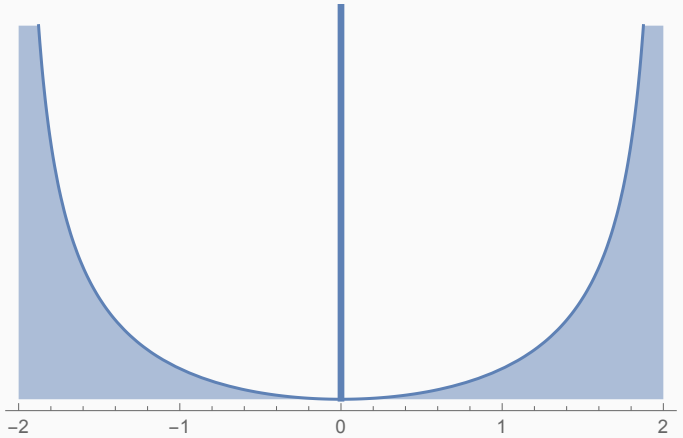

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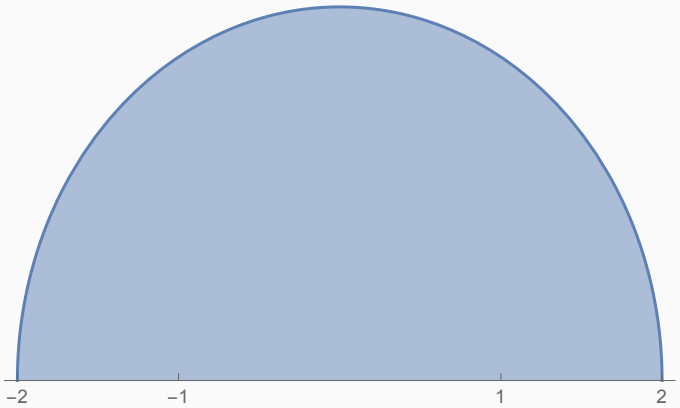

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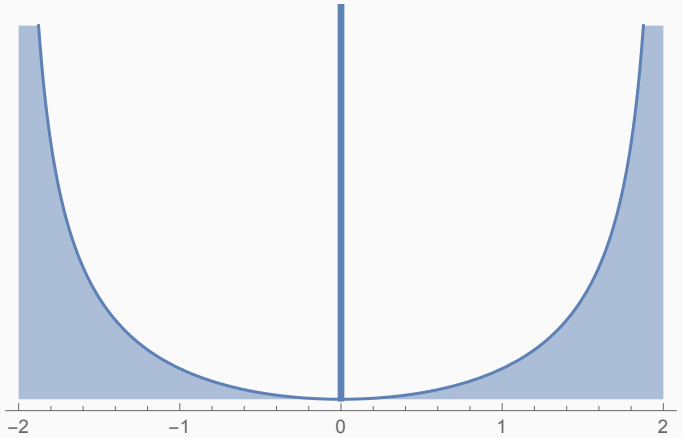

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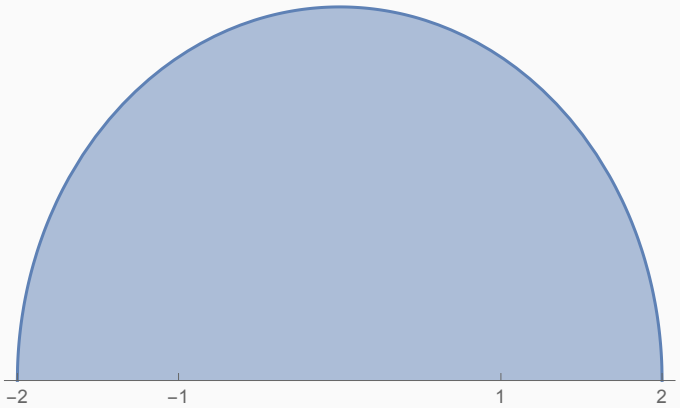

$\mathsf{Prob}(\lambda_p = 0) \overset{?}{\sim} 1/\sqrt{p}$

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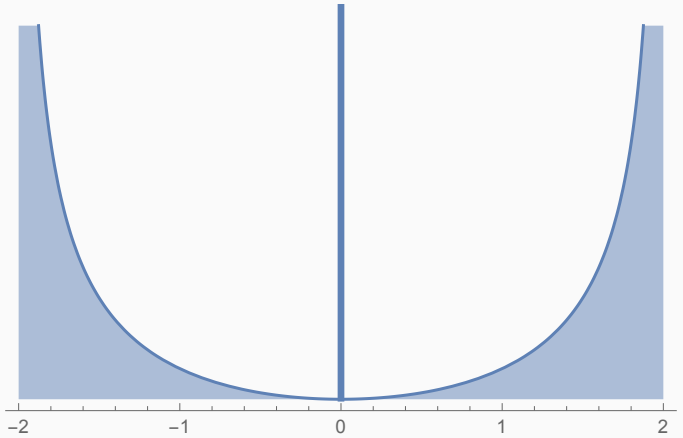

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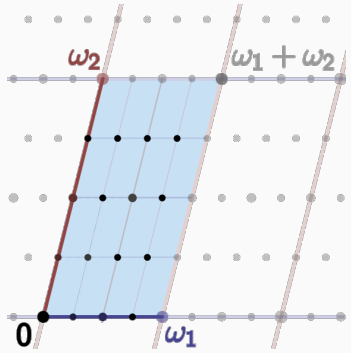
$E_{\mathbb{C}} \simeq \mathbb{C}/\Lambda, \quad$ where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 =$



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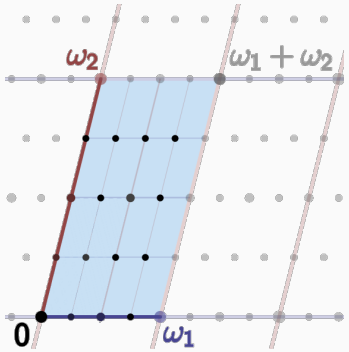

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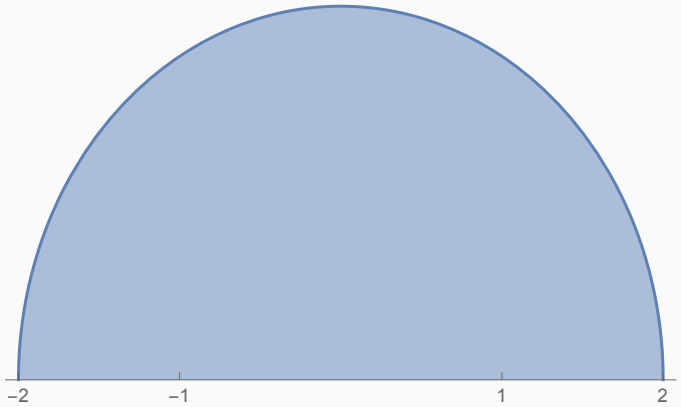
$\mathbb{Z} \subsetneq \mathbf{End}(\Lambda) \subset \mathbb{Q}(\sqrt{-d})$

$\omega_2/\omega_1 \in \mathbb{Q}(\sqrt{-d})$ for some $d > 0$

non-CM

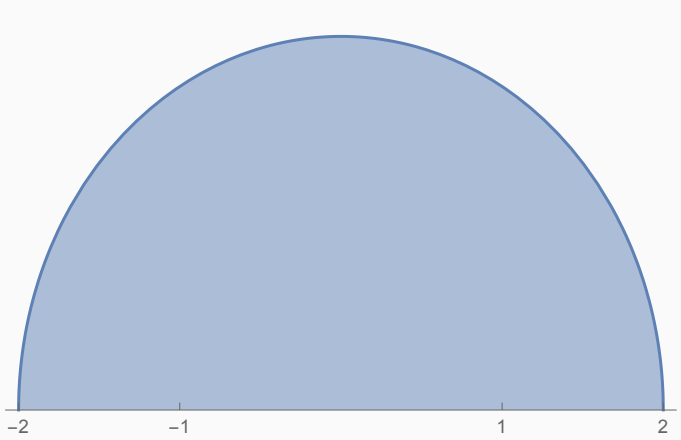


CM

How to distinguish between the two types?

non-CM

$\mathrm{End}_{\mathbb{Q}} E^{\mathrm{al}} = \mathbb{Q}$



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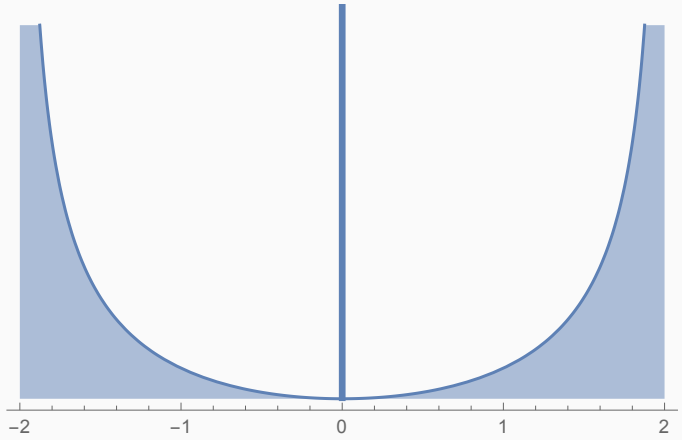

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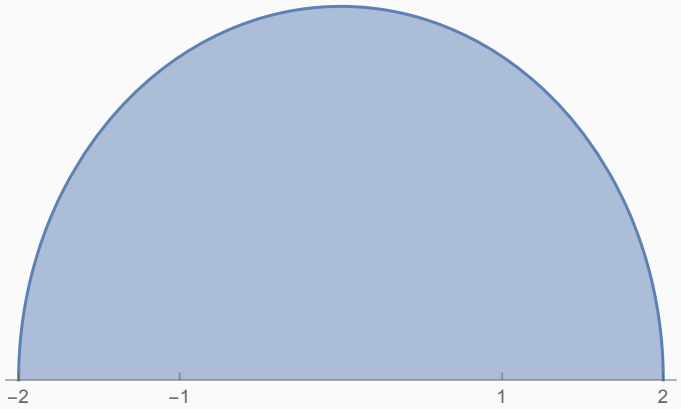

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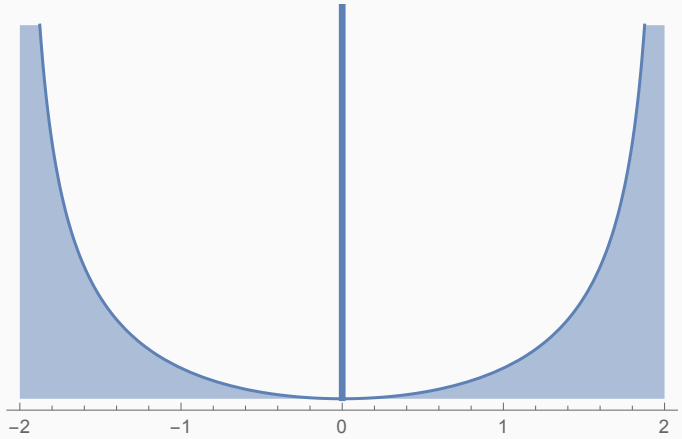

It is enough to count points!

$$p + 1 - \#E_p =: a_p \neq 0 \Longrightarrow \mathrm{End}_{\mathbb{Q}} E^{\mathrm{al}} \subset \mathbb{Q}\left(\sqrt{a_p^2 - 4p}\right)$$

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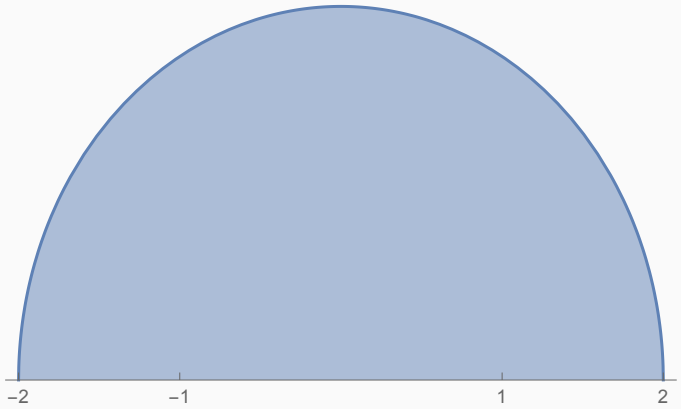

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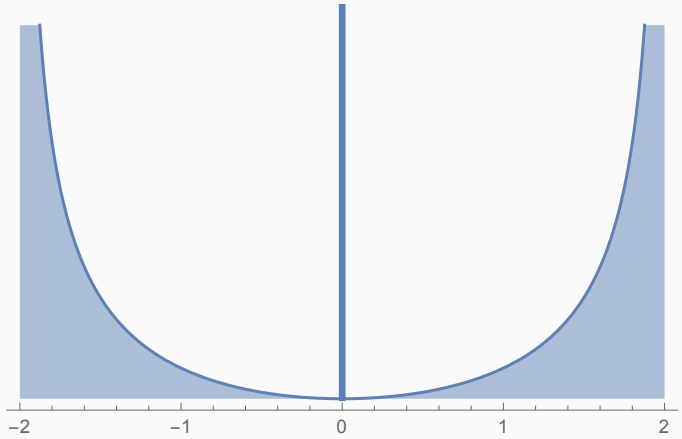

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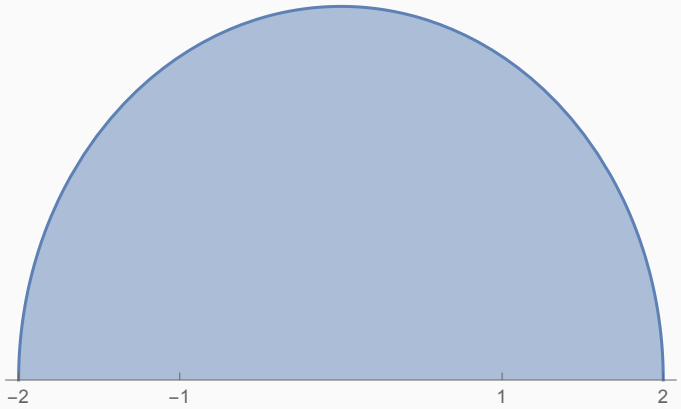

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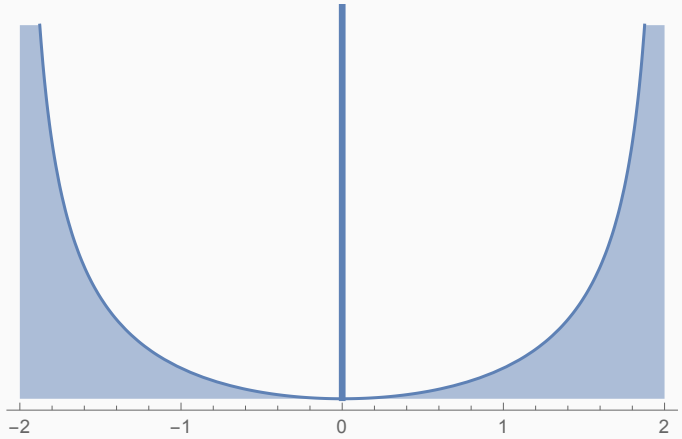

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- If E is non-CM, then $\mathbb{Q}(\sqrt{a_p^2 - 4p}) \not\simeq \mathbb{Q}(\sqrt{a_q^2 - 4q})$ for $p \neq q$ with prob. 1.
- If E has CM, then $\mathbb{Q}(\sqrt{-d}) \simeq \mathbb{Q}(\sqrt{a_p^2 - 4p})$.

Group-theoretic interpretation

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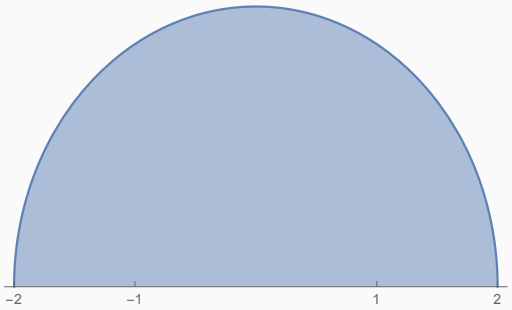
Then, the a_p are distributed as the trace of a matrix chosen at random from ST_E with respect to its Haar measure.

non-CM

$\mathrm{SU}(2)$

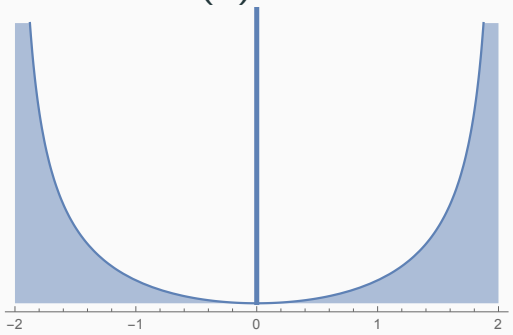


CM

$\mathrm{U}(1)$

CM (with the δ)

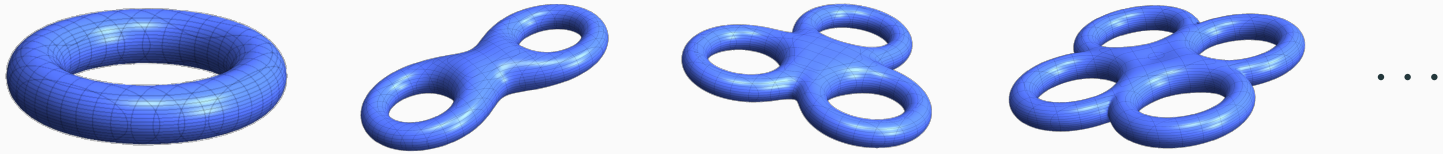
$\mathrm{N}_{\mathrm{SU}(2)}(\mathrm{U}(1))$



Higher Genus curves

Let's now consider curves with higher genus = #handles.

$g = 1$ $g = 2$ $g = 3$ $g = 4$ $\cdots$



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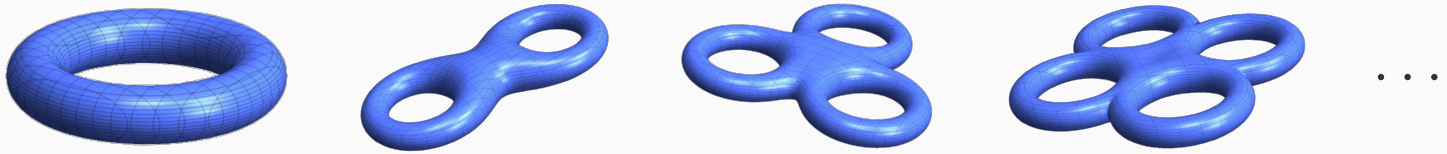

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For example, an hyperelliptic curve:

$$C : y^2 = a_{2g+2}x^{2g+2} + \cdots + a_0$$

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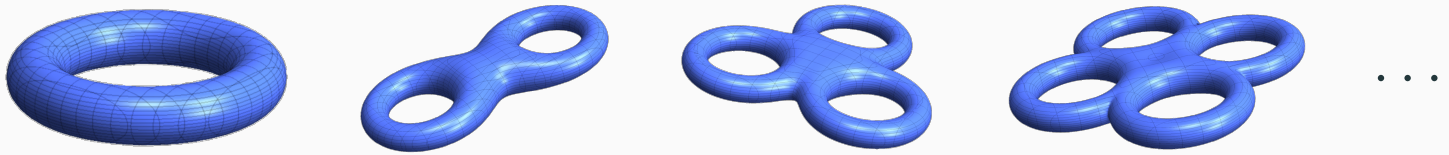

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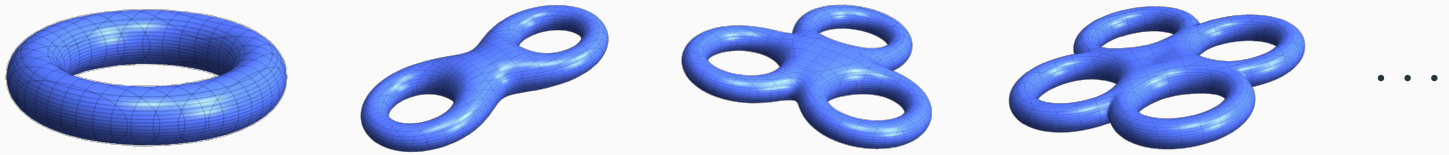

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Now we will need to count solutions over $\mathbb{F}_{p^i}$ for $i = 1, \cdots, g$.

Zeta functions and Frobenius polynomials

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Can we repeat the same experiment?

$$Z_p(T) := \exp\left(\sum_{r=1}^{\infty} \#C(\mathbb{F}_{p^r})T^r/r\right) \in \mathbb{Q}(t)$$

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Sato–Tate conjecture

$L_p(T/\sqrt{p})$ are equidistributed according to $\mathrm{ST}_A \subset \mathrm{USp}(2g)$

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	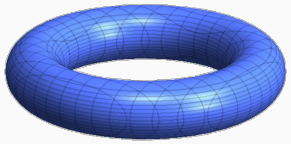	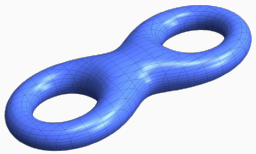	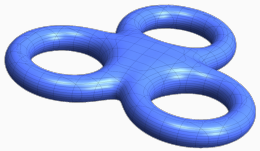	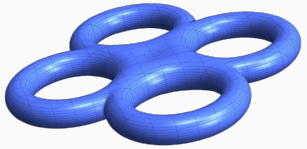	$\cdots$
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	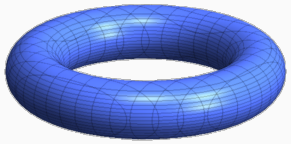	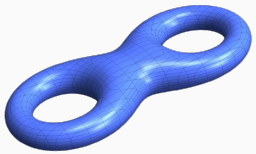	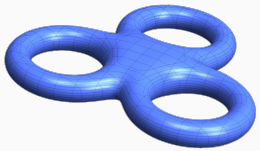	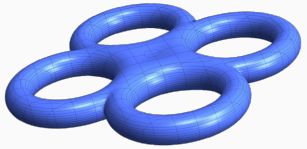	$\cdots$
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	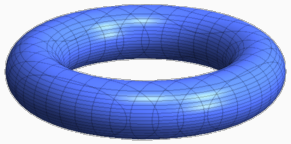	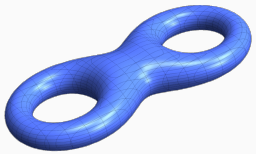	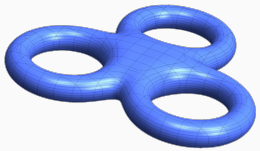	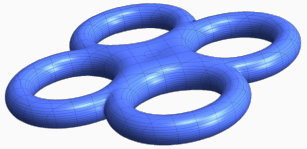	$\cdots$
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	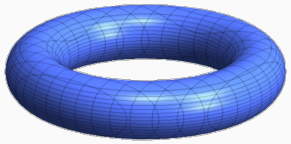	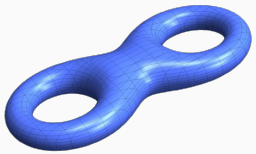	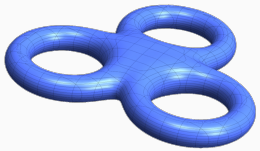	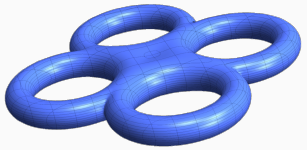	$\cdots$
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Yes, we can compute $\mathbf{End}\, A^{\mathrm{al}}$!

Lower bounds for the endomorphism ring

- C be a nice curve over a number field
- $A := \mathrm{Jac}(C) \simeq_{\mathbb{C}} \mathbb{C}^g/\Lambda$
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Theorem (C–Mascot–Sijsling–Voight)

There exists a deterministic algorithm that, given input $\alpha \in \mathrm{M}_g(\mathbb{Q}^{\mathrm{al}})$, *returns*

$$\begin{cases} \texttt{true} & \alpha \in \mathrm{End}\, A^{\mathrm{al}} \text{ and } \alpha \text{ is nondegenerate}^1, \\ \texttt{false} & \alpha \notin \mathrm{End}\, A^{\mathrm{al}} \text{ or } \alpha \text{ is degenerate.} \end{cases}$$

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In practice, we first compute $\mathsf{End}\, \mathbb{C}^g/\Lambda$ numerically.

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Upper bounds for the endomorphism ring

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This is done by just counting points.

Upshot

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For $g \leq 3$ this is sufficient to determine ST_A.

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Publicly available for you to try out

`github.com/edgarcosta/endomorphisms/`

Already used on more than 250000 curves, coming soon to

`LMFDB.org`

K3 surfaces

These provide another natural generalization of elliptic curves

They may arise in many ways:

- smooth quartic surfaces in $\mathbb{P}^3$

$$X : f(x, y, z, w) = 0, \quad \deg f = 4$$

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Can we play similar game as before?

In this case, instead of studying $\#X_p$ or $a_p := \operatorname{Tr} \mathrm{Frob}_p$ we study

$$p \longmapsto \operatorname{rk} \mathrm{NS}\, X_p{}^{\mathrm{al}} \in \{2, 4, \ldots, 22\}$$

K3 Surfaces

$X/\mathbb{Q}$ a K3 surface

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In the later case,

$$\{p : a_p = 0\} = \{p : p \text{ is ramified or inert in } \mathbb{Q}(\sqrt{-d})\}$$

Néron–Severi group

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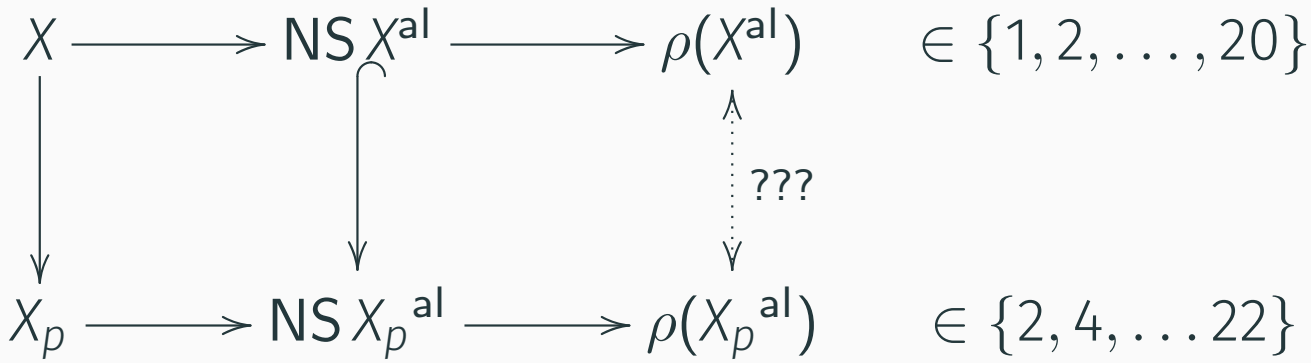
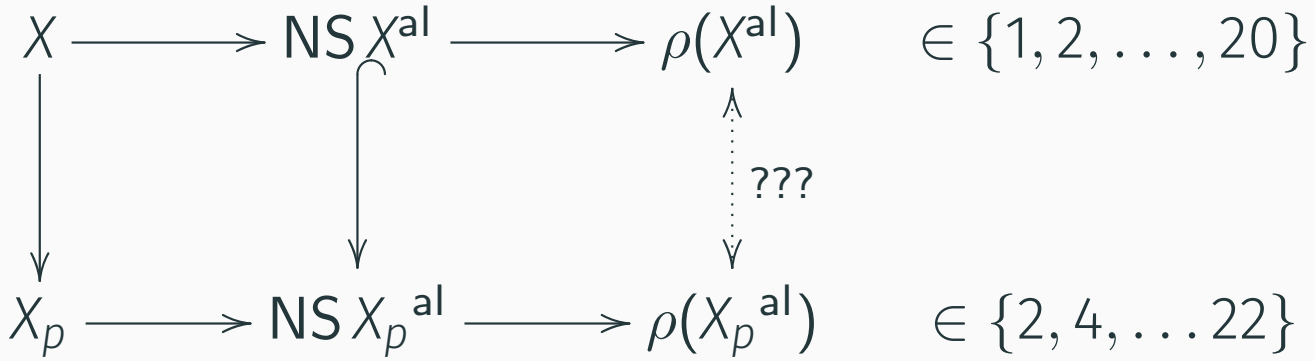


Néron–Severi group

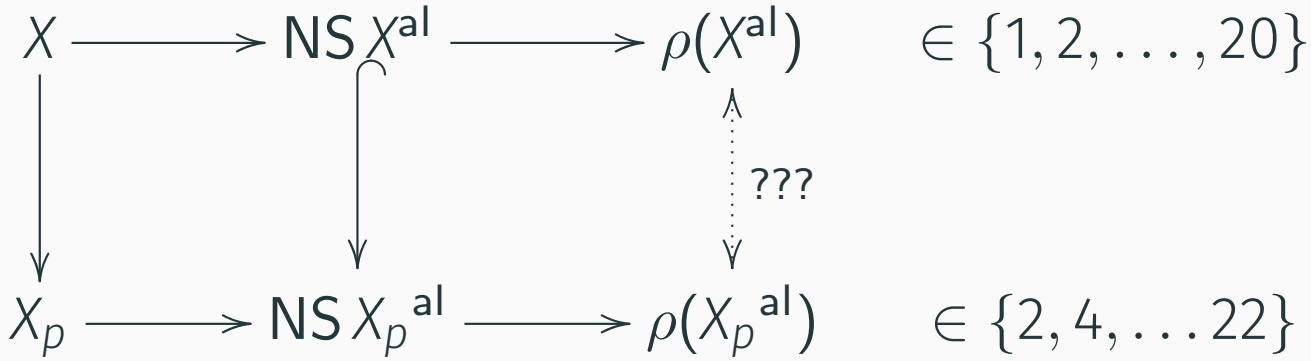
- $\mathsf{NS}\,\bullet$ = Néron–Severi group of $\bullet \simeq \{\text{curves on } \bullet\}/\sim$
- $\rho(\bullet) = \mathsf{rk}\,\mathsf{NS}\,\bullet$
- $X_p := X \bmod p$



Theorem (Charles)

For infinitely many p we have $\rho({X_p}^{\mathsf{al}}) = \mathsf{min}_q\, \rho({X_q}^{\mathsf{al}})$.

The Problem



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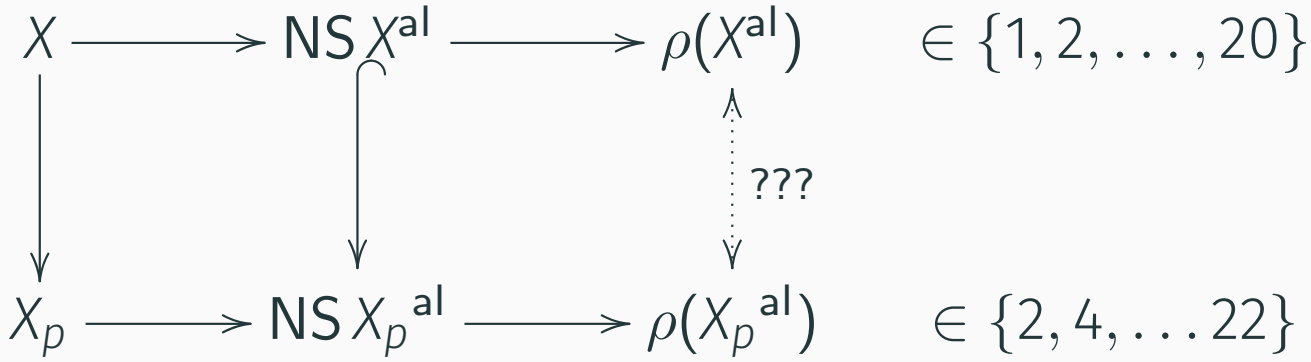

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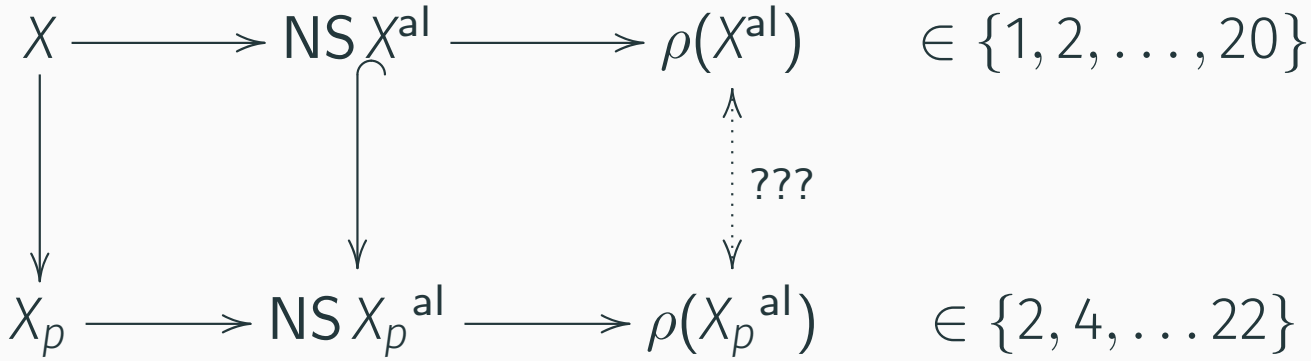

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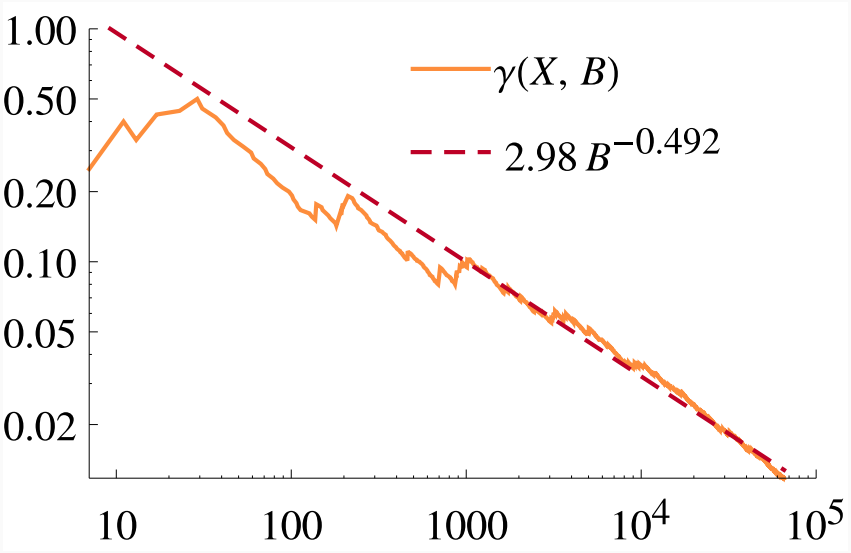
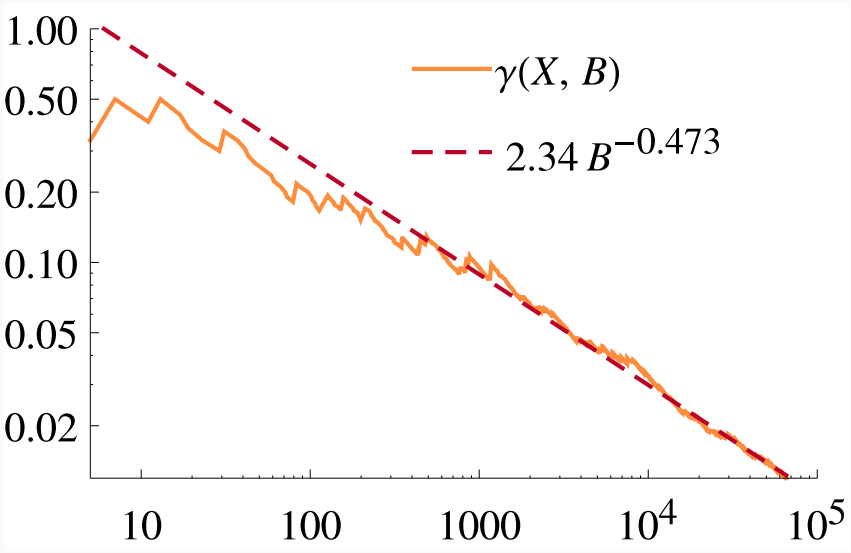
Let's do some numerical experiments!

Two generic K3 surfaces, $\rho(X^{\mathrm{al}}) = 1$





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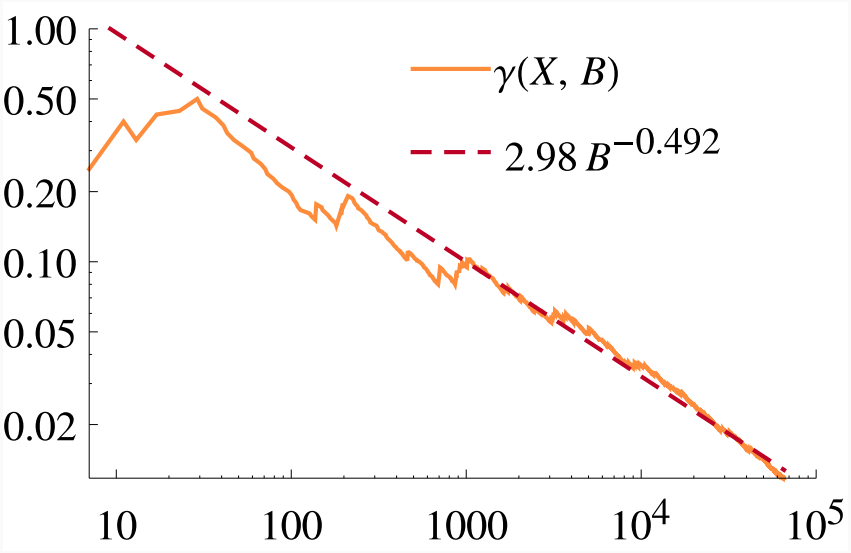

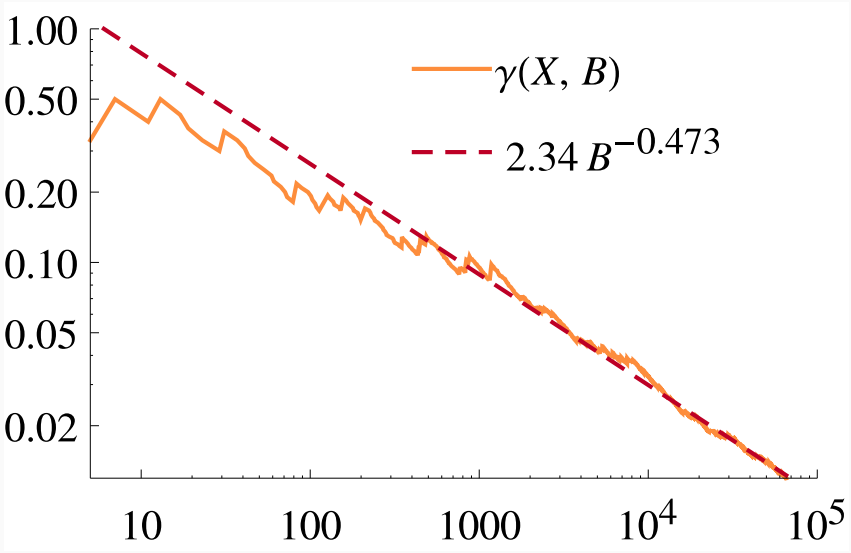

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$$\Longrightarrow \ \mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) \overset{?}{\sim} 1/\sqrt{p}$$

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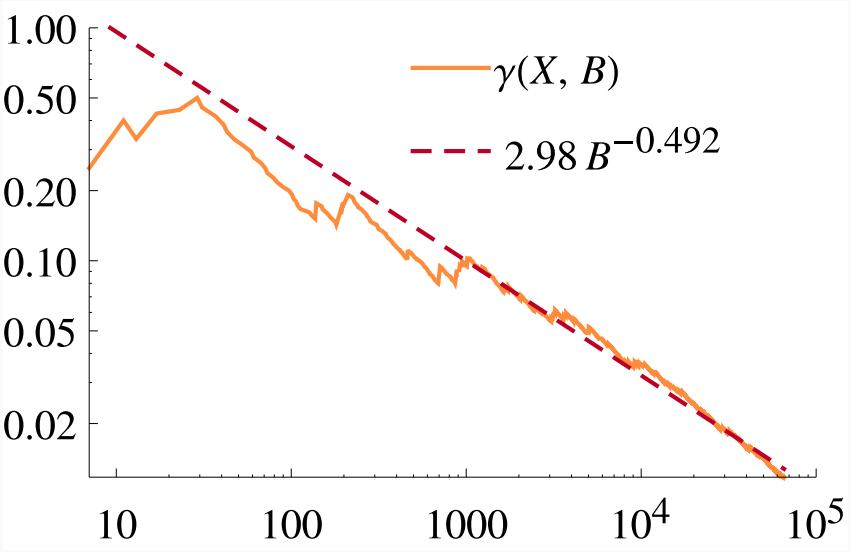

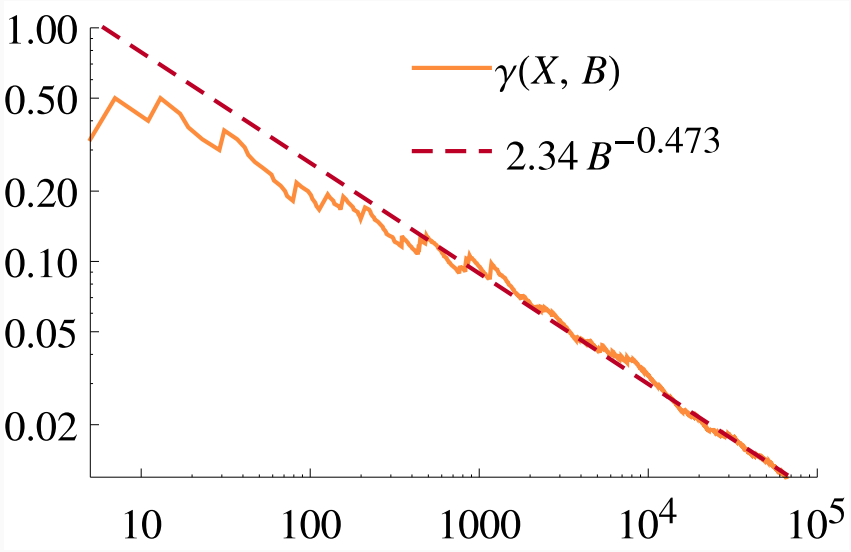

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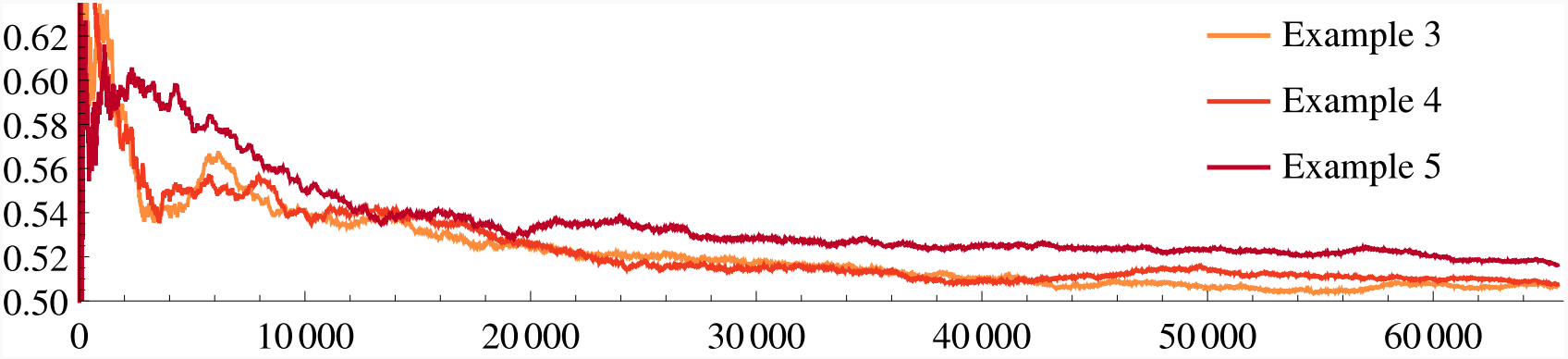
$\sim$ 1 CPU year per example

`github.com/edgarcosta/controlled-reduction/`

Three K3 surfaces with $\rho(X^{\mathrm{al}}) = 2$



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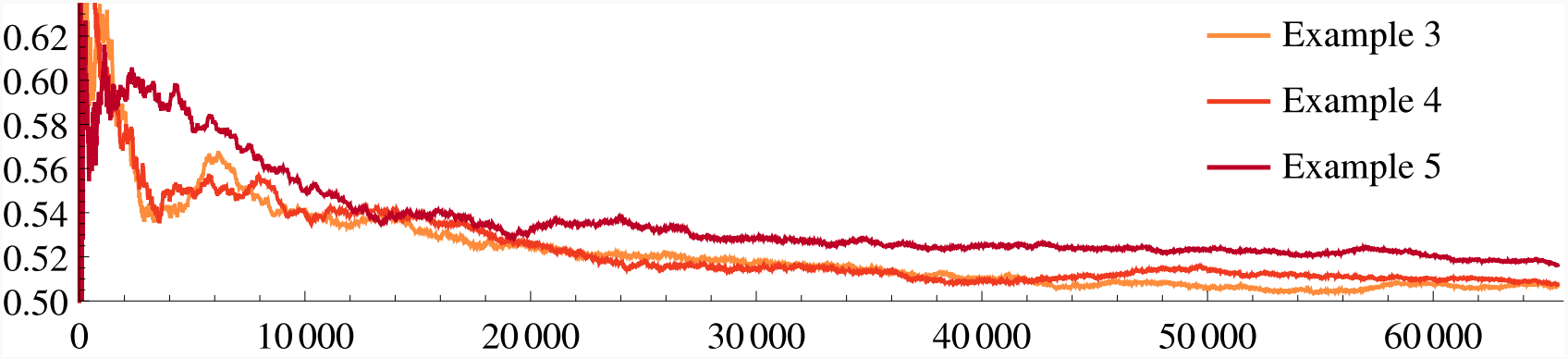

Do you see a trend?

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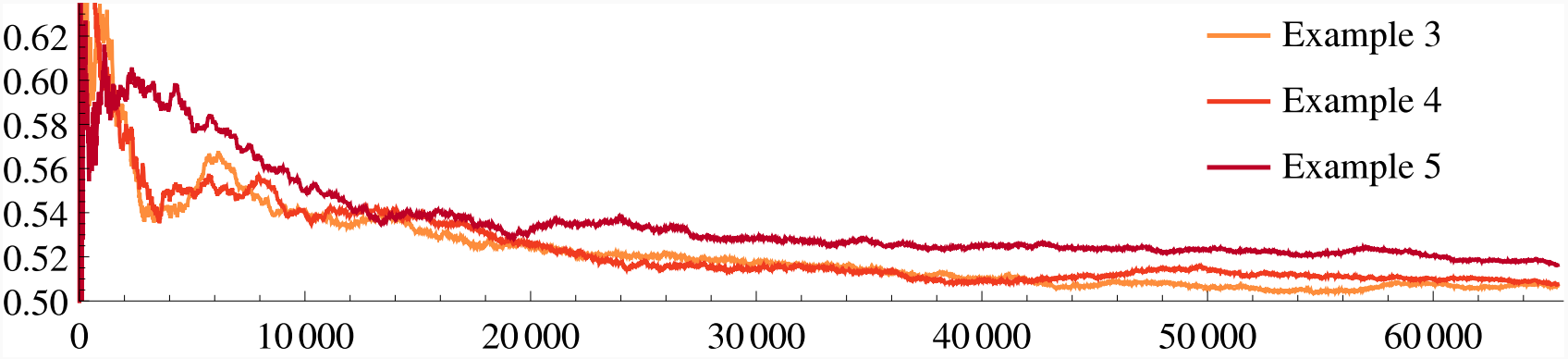

Do you see a trend?

Could it be related to some integer being a square modulo p?

We can explain the 1/2

Theorem (C, C–Elsenhans–Jahnel)

If $\rho(X^{\mathrm{al}}) = \min_q \rho({X_p}^{\mathrm{al}})$, *then there is a* $d_X \in \mathbb{Z}$ *such that:*

$$\left\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\right\} \subset \Pi_{\mathrm{jump}}(X).$$

In general, d_X *is not a square.*

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$d_3 = -1 \cdot 5 \cdot 151 \cdot 2249081735741437104 1 \cdot 3873084974301493372336663$

$d_4 = 53 \cdot 2624174618795407 \cdot 5128545618469648171394942020727783 41 \cdot$

$d_5 = -1 \cdot 47 \cdot 3109 \cdot 4969 \cdot 14857095849982608071 \cdot 44541027766092834$

Experimental data for $\rho(X^{\mathrm{al}}) = 2$ (again)

What if we ignore $\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\} \subset \Pi_{\mathrm{jump}}(X)$?

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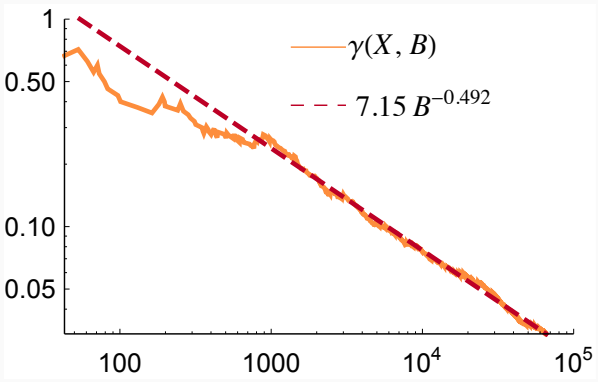

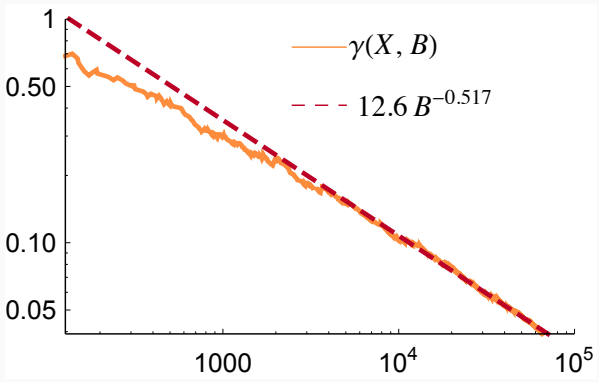

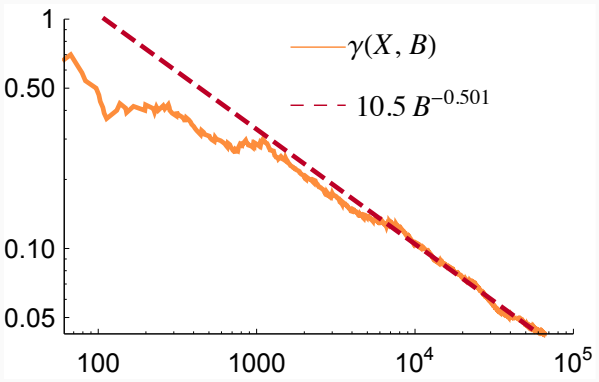

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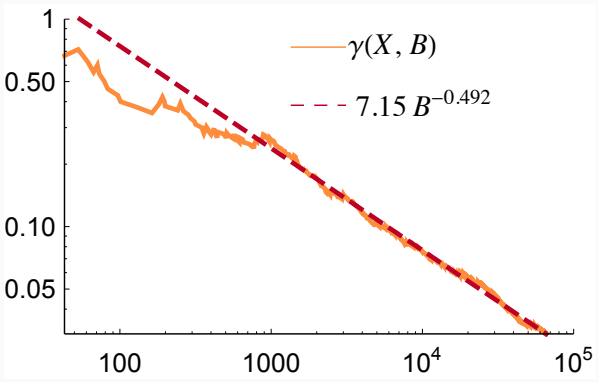

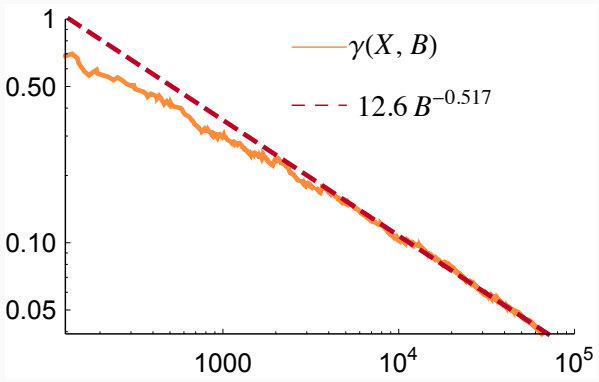

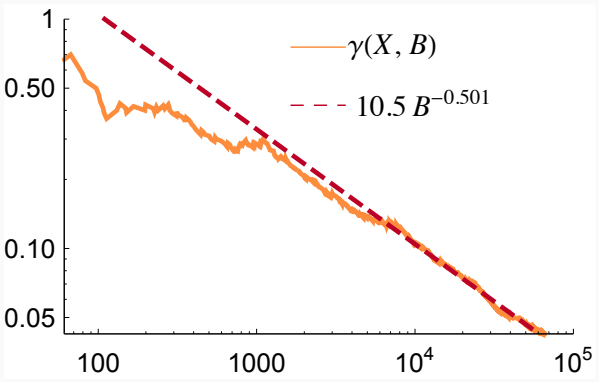

$$\mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) = \begin{cases} 1 & \text{if } d_X \text{ is not a square modulo } p \\ \overset{?}{\sim} \frac{1}{\sqrt{p}} & \textit{otherwise} \end{cases}$$