Bounding Picard numbers

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Joint work with Emre Sertoz

Slides available at `edgarcosta.org` under Research

Picard numbers of surfaces over $\mathbb{Q}^{\mathsf{al}}$

Let $X := Z(f) \subset \mathbb{P}^3_{\mathbb{Q}^{\mathsf{al}}}$ be a smooth surface of degree d.

Picard group $\mathrm{Pic}(X) \simeq \mathbb{Z}^\rho$ and Picard number $\rho(X) := \operatorname{rank} \mathrm{Pic}(X)$

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- $d = 2$: $\rho(X) = 2$ and $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$
- $d = 3$: $\rho(X) = 7$ and $X \simeq \operatorname{Bl}_{p_1,\dots,p_6} \mathbb{P}^2$

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- $d = 4$: $\rho(X) \in \{1, 2, \ldots, 20\}$ and X is a K3 surface
- $d = 5$: $\rho(X) \in \{1, 2, \ldots, 45\}$

Problem

Compute $\rho(X)$ from $f \in \mathbb{Z}[x, y, z, w]$

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This is very analogous to computing $\mathrm{rk}\,\mathrm{End}(A)$.

In positive characteristic

Let $X := Z(f) \subset \mathbb{P}^3_{\mathbb{F}_p^{\mathrm{al}}}$ be a smooth surface of degree d.

- $d = 1$: $\rho(X) = 1$
- $d = 2$: $\rho(X) = 2$
- $d = 3$: $\rho(X) = 7$
- $d = 4$: $\rho(X) \in \{2, 4, \ldots, 22\}$ (Over $\mathbb{Q}^{\mathrm{al}}$ it was $\{1, 2, \ldots, 20\}$)
- $d = 5$: $\rho(X) \in \{1, 3, \ldots, 53\}$ (Over $\mathbb{Q}^{\mathrm{al}}$ it was $\{1, 2, \ldots, 45\}$)

Problem

Compute $\rho(X)$ from $f \in \mathbb{F}_p[x, y, z, w]$

By Tate conjecture (known for $d \leq 4$) one only needs to compute the Hasse–Weil zeta function of X, i.e, count points.

Also solved in practice.

Reduction to finite characteristic

Take $f \in \mathbb{Z}[x, y, z, w]$ and $X := Z(f) \subset \mathbb{P}^3_{\mathbb{Z}}$.

We may consider the surface $X_{\mathbb{F}_p} := Z(f \bmod p) \subset \mathbb{P}^3(\mathbb{F}_p)$.

Theorem

If X and $X_{\mathbb{F}_p}$ are smooth then $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$.

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For a given f and p, improve the inequality above.

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For a given f and p, improve the inequality above.

Idea, try to lift algebraic cycles (curves) from $\mathbb{F}_p^{\text{al}}$ to $\mathbb{Q}_p^{\text{al}}$.

We will do this by considering the thickenings

$$Z(f \bmod p^i) \subset P^3(\mathbb{Z}/(p)^i) \quad i = 1, 2, \ldots$$

1st ingredient: Cohomology

Over characteristic zero we have:

- $H^2_{dR}(X/\mathbb{Q}) = F^0(X) \supset F^1(X) \supset F^2(X)$, the Hodge filtration
- $\mathrm{Pic}(X) \hookrightarrow F^1(X)$
- For $d = 4$, $\dim F^i(X) = 22, 21, 1$.

Over characteristic p we have:

- $\mathrm{Pic}(X_{\mathbb{F}_p}) \hookrightarrow H^2_{crys}(X_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes \mathbb{Q}_p \simeq H^2_{dR}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$

Theorem (Berthelot, Ogus 1978; Raynaud 1979)

$$\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}} \cap F^1_{\mathbb{Q}_p}$$

2nd ingredient: Approximate $\mathrm{Pic}(X_{\mathbb{F}_p}) \otimes_{\mathbb{Z}} \mathbb{Q}_p$

Via the isomorphism $\mathrm{H}^2_{\mathrm{crys}}(X_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes \mathbb{Q}_p \simeq \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q})$, we have

$$\mathrm{Frob}_p : \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p) \to \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p).$$

Tate conjecture

$$\mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_p} = \ker(\mathrm{Frob}_p - p \cdot \mathrm{id} \,|\mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p))$$

By computing a p-adic approximation of Frob_p, we may compute a p-adic approximation of

$$\pi : \mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_p} \to \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p)/\,\mathrm{F}^1(X)_{\mathbb{Q}_p}$$

thus

$$\dim_{\mathbb{Q}} \mathrm{Pic}(X) \leq \dim_{\mathbb{Q}_p} \ker \pi.$$

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$$\dim_{\mathbb{Q}} \mathrm{Pic}(X) \leq \dim_{\mathbb{Q}_p} \ker \pi.$$

By picking a basis that respects the Hodge filtration, the map $\mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p) \to \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p)/\,\mathrm{F}^1(X)_{\mathbb{Q}_p}$ is a coordinate projection.

Abelian surface

$$X = \mathrm{Jac}(y^2 = 4x^5 - 36x^4 + 56x^3 - 76x^2 + 44x - 23)$$

$$\mathrm{Frob}|_{\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q}_p)} \equiv \begin{pmatrix} 31\cdot 482 & 31\cdot 284 & 16241 & 3075 \\ 31\cdot 386 & 31\cdot 886 & 2644 & 12126 \\ 31\cdot 284 & 31\cdot 659 & 6336 & 9750 \\ 31\cdot 194 & 31\cdot 876 & 27408 & 10841 \end{pmatrix} \pmod{31^3},$$

$$L(t) = \det(1 - t\,\mathrm{Frob}\,|\mathrm{H}^1) = 1 - 3t + 14t^2 - 93t^3 + 961t^4.$$

From this deduce $\mathrm{Frob}|_{\mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p)}$, and $\det(1 - t31^{-1}\,\mathrm{Frob}\,|\mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p)) = (t-1)^2(31t^4 + 48t^3 + 43t^2 + 48t + 31)/31$ Compute 2 eigenvectors

$$v_1 \equiv (356,\ 37,\ 831,\ 0,\ 295,\ 31) \pmod{31^2}$$

$$v_2 \equiv (4,\ 957,\ 3,\ 1,\ 0,\ 0) \pmod{31^2}.$$

and observe that the last coordinate of v_1 is nonzero, hence, it cannot lift to $\mathbb{Q}_p$. Thus, $\mathrm{rank}\,\mathrm{NS}(A_{\mathbb{Q}^{\mathrm{al}}}) = 1 \implies \mathrm{rank}\,\mathrm{End}(A_{\mathbb{Q}^{\mathrm{al}}}) = 1$.

Quartic surface

$$X := Z(y^4 - x^3z + yz^3 + zw^3 + w^4) \subset \mathbb{P}^3_{\mathbb{C}}$$

```
sage: crystalline_obstruction(f, p=89, precision=3)
(4,
 {'rank T(X_Fpbar)': 10,
  'factors': [(t - 1, 1), (t + 1, 1), (t - 1, 4), (t^4 + 1, 1)],
  'dim Ti': [1, 1, 4, 4],
  'dim Li': [1, 0, 3, 0]},
  'precision': 3, 'p': 89})
```

- $\rho(X_{\mathbb{F}_{89}^{\mathrm{al}}}) = 10$
- $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq 4$
- In fact, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \geq 4$ as there are four lines in $z = 0$.

Quartic surface

$$X = Z(y^4 - x^3 z + yz^3 + zw^3 + w^4) \subset \mathbb{P}^3_{\mathbb{C}}$$

```
sage: crystalline_obstruction(f, p=31, precision=5)
(4,
 {'rank T(X_Fpbar)': 4,
  'factors': [(t - 1, 1), (t - 1, 1), (t + 1, 2)],
  'dim Ti': [1, 1, 2],
  'dim Li': [1, 1, 2]},
  'precision': 5, 'p': 31})
```

- $\rho(X_{\mathbb{F}_{31}^{\mathsf{al}}}) = 4$
- $\rho(X_{\mathbb{Q}^{\mathsf{al}}}) \leq 4$, with some confidence that the equality might hold.

Abelian variety

$A := \mathrm{Jac}(y^4 + x^3z + 2y^3z - yz^3)$

We may also replace $\mathrm{H}^2(A)$ with $\mathrm{H}^1(A) \otimes \mathrm{H}^1(A)$.

```
sage: crystalline_obstruction(f=f, p=31, precision=5)
(4, {'rank T(X_Fpbar)': 5,
     'factors': [(t - 1, 3), (t^2 + t + 1, 1)],
     'dim Ti': [3, 2],
     'dim Li': [2, 2]},
     'precision': 5, 'p': 31})
sage: crystalline_obstruction(f=f, p=31, precision=5, tensor=True)
(6, {'rank T(X_Fpbar)': 10,
     'factors': [(t - 1, 6), (t^2 + t + 1, 2)],
     'dim Ti': [6, 4],
     'dim Li': [4, 2],
     'precision': 5, 'p': 31})
```

- $\operatorname{rank} \mathrm{NS}(A_{\mathbb{Q}^{\mathrm{al}}}) \leq 4$
- $\operatorname{rank} \mathrm{End}(A_{\mathbb{Q}^{\mathrm{al}}}) \leq 5$
- indeed, $\mathrm{End}(A_{\mathbb{Q}^{\mathrm{al}}})_{\mathbb{Q}} = \mathbb{Q}(\sqrt{-3}) \times B$, B is a quaternion algebra with $\operatorname{disc} B = 6$.

Quintic surface

$X := Z(9xy^4 + 3x^4z + 9y^2z^3 + z^5 + 5w^5) \subset \mathbb{P}^3$

```
sage: crystalline_obstruction(f, p=23, precision=6)
(1, {'rank T(X_Fpbar)': 5,
     'factors': [(t - 1, 1), (t - 1, 1), (t + 1, 1), (t^2 + 1, 1)],
     'dim Ti': [1, 1, 1, 2],
     'dim Li': [1, 0, 0, 0],
    'precision': 6, 'p': 23})
sage: crystalline_obstruction(f, p=29, precision=20)
(3, {'rank T(X_Fpbar)': 5,
     'factors': [(t - 1, 1), (t - 1, 2), (t + 1, 2)],
     'dim Ti': [1, 2, 2],
     'dim Li': [1, 1, 1]})
     'precision': 20, 'p': 29})
```

- $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 1$
- However, we can not deduce this from $p = 29$, not even with infinite precision.
- The surface has *CM* by $\mathbb{Q}(\zeta_5)$

Theoretical example

$X\colon w^2 = \left(-y^2/8 + yz - z^2\right)\left(7x^2/8 + 5xz + 7z^2\right)\left(2x^2 + 3xy + y^2\right)$

Take $p = 83$.

By counting points on the singular model, we deduce $\det(1 - t83^{-1}\,\mathrm{Frob}\,|\mathrm{H}^2) = \chi_1 \cdot \chi_2$ where

$$\chi_1 = (t-1)^{10}(t+1)^6, \quad \chi_2 = (t^2+1)(83 - 158t^2 + 83t^4)/83.$$

Thus, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 16$ or 18.

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Thus, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 16$ or 18.

Indeed, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 16$ and X has RM by $\mathbb{Q}(\sqrt{2})$.

So given a good enough approximation to Frob we will obstruct 2 extra cycles.