Effective obstruction to lifting Tate classes from positive characteristic

Edgar Costa (MIT)
Simons Collab. on Arithmetic Geometry, Number Theory, and Computation

December 9th, 2020

Joint work with Emre Sertoz

Slides available at `edgarcosta.org` under Research

Picard numbers of surfaces over $\mathbb{Q}^{\mathrm{al}}$

Let $X := Z(f) \subset \mathbb{P}^3$ be a smooth surface of degree d.

Picard group $\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \simeq \mathbb{Z}^\rho$ and Picard number $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) := \operatorname{rank} \mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}})$

Picard numbers of surfaces over $\mathbb{Q}^{\mathrm{al}}$

Let $X := Z(f) \subset \mathbb{P}^3$ be a smooth surface of degree d.

Picard group $\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \simeq \mathbb{Z}^{\rho}$ and Picard number $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) := \mathrm{rank}\,\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}})$

- $d = 1$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 1$ and $X \simeq \mathbb{P}^2$
- $d = 2$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 2$ and $X_{\mathbb{Q}^{\mathrm{al}}} \simeq \mathbb{P}^1 \times \mathbb{P}^1$
- $d = 3$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 7$ and $X_{\mathbb{Q}^{\mathrm{al}}} \simeq \mathrm{Bl}_{p_1,\ldots,p_6}\,\mathbb{P}^2$

Picard numbers of surfaces over $\mathbb{Q}^{\mathrm{al}}$

Let $X := Z(f) \subset \mathbb{P}^3$ be a smooth surface of degree d.

Picard group $\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \simeq \mathbb{Z}^\rho$ and Picard number $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) := \mathrm{rank}\,\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}})$

- $d = 1$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 1$ and $X \simeq \mathbb{P}^2$
- $d = 2$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 2$ and $X_{\mathbb{Q}^{\mathrm{al}}} \simeq \mathbb{P}^1 \times \mathbb{P}^1$
- $d = 3$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 7$ and $X_{\mathbb{Q}^{\mathrm{al}}} \simeq \mathrm{Bl}_{p_1,\ldots,p_6}\,\mathbb{P}^2$
- $d = 4$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \in \{1, 2, \ldots, 20\}$ and X is a K3 surface
- $d = 5$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \in \{1, 2, \ldots, 45\}$

Problem

Compute $\rho(X)$ from $f \in \mathbb{Z}[x, y, z, w]$

Picard numbers of surfaces over $\mathbb{Q}^{\mathrm{al}}$

Let $X := Z(f) \subset \mathbb{P}^3$ be a smooth surface of degree d.

Picard group $\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \simeq \mathbb{Z}^\rho$ and Picard number $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) := \mathrm{rank}\,\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}})$

- $d = 1$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 1$ and $X \simeq \mathbb{P}^2$
- $d = 2$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 2$ and $X_{\mathbb{Q}^{\mathrm{al}}} \simeq \mathbb{P}^1 \times \mathbb{P}^1$
- $d = 3$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 7$ and $X_{\mathbb{Q}^{\mathrm{al}}} \simeq \mathrm{Bl}_{p_1,\ldots,p_6} \mathbb{P}^2$
- $d = 4$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \in \{1, 2, \ldots, 20\}$ and X is a K3 surface
- $d = 5$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \in \{1, 2, \ldots, 45\}$

Problem

Compute $\rho(X)$ from $f \in \mathbb{Z}[x, y, z, w]$

In principle, solved, if given the Tate conjecture.
[Charles, Poonen–Testa–van Luijk, Hassett–Kresch–Tschinkel, Shioda, Lairez–Sertöz]

Picard numbers of surfaces over $\mathbb{Q}^{\mathrm{al}}$

Let $X := Z(f) \subset \mathbb{P}^3$ be a smooth surface of degree d.

Picard group $\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \simeq \mathbb{Z}^\rho$ and Picard number $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) := \operatorname{rank} \mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}})$

- $d = 1$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 1$ and $X \simeq \mathbb{P}^2$
- $d = 2$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 2$ and $X_{\mathbb{Q}^{\mathrm{al}}} \simeq \mathbb{P}^1 \times \mathbb{P}^1$
- $d = 3$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 7$ and $X_{\mathbb{Q}^{\mathrm{al}}} \simeq \mathrm{Bl}_{p_1,\ldots,p_6} \mathbb{P}^2$
- $d = 4$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \in \{1, 2, \ldots, 20\}$ and X is a K3 surface
- $d = 5$: $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \in \{1, 2, \ldots, 45\}$

Problem

Compute $\rho(X)$ from $f \in \mathbb{Z}[x, y, z, w]$

In principle, solved, if given the Tate conjecture.
[Charles, Poonen–Testa–van Luijk, Hassett–Kresch–Tschinkel, Shioda, Lairez–Sertöz]
In practice, the generic known algorithms will not terminate.

In positive characteristic

Let $X := Z(f) \subset \mathbb{P}^3$ be a smooth surface of degree d, with $f \in \mathbb{F}_p[x, y, z, w]$.

Tate conjecture

$$\mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_\ell} = \ker(\mathrm{Frob}_p - p \cdot \mathrm{id} \,|\mathrm{H}^2_{\mathrm{et}}(X_{\mathbb{F}_p^{\mathrm{al}}}, \mathbb{Q}_\ell))$$

In positive characteristic

Let $X := Z(f) \subset \mathbb{P}^3$ be a smooth surface of degree d, with $f \in \mathbb{F}_p[x, y, z, w]$.

Tate conjecture

$$\mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_\ell} = \ker(\mathrm{Frob}_p - p \cdot \mathrm{id} \,|\mathrm{H}^2_{\mathrm{et}}(X_{\mathbb{F}_p^{\mathrm{al}}}, \mathbb{Q}_\ell))$$

The Hasse–Weil zeta function $Z_X(t)$ for a surface X can be written as

$$Z_X(t) := \exp\left(\sum_{m=1}^{\infty} \frac{\#X(\mathbb{F}_{p^m})}{m} t^m\right) = \frac{1}{(1-t)\chi(t)(1-p^2t)},$$

where $\chi(t) = \det(1 - t\,\mathrm{Frob}\,|\mathrm{H}^2_{\mathrm{et}}(X_{\mathbb{F}_p^{\mathrm{al}}}, \mathbb{Q}_\ell)) \in \mathbb{Z}[t]$.

Thus, by counting points, one can compute $\rho(X_{\mathbb{F}_{p^n}})$.

Since acts Frob_p acts semisimply, we have

$$\rho(X_{\mathbb{F}_{p^n}}) = \#\{z : \chi(1/z) = 0 \text{ and } z^n = p^n\}.$$

In positive characteristic

Let $X := Z(f) \subset \mathbb{P}^3_{\mathbb{F}_p^{\mathrm{al}}}$ be a smooth surface of degree d.

- $d = 1$: $\rho(X) = 1$
- $d = 2$: $\rho(X) = 2$
- $d = 3$: $\rho(X) = 7$
- $d = 4$: $\rho(X) \in \{2, 4, \ldots, 22\}$ (Over $\mathbb{Q}^{\mathrm{al}}$ it was $\{1, 2, \ldots, 20\}$)
- $d = 5$: $\rho(X) \in \{1, 3, \ldots, 53\}$ (Over $\mathbb{Q}^{\mathrm{al}}$ it was $\{1, 2, \ldots, 45\}$)

Problem

Compute $\rho(X)$ from $f \in \mathbb{F}_p[x, y, z, w]$

By Tate conjecture (known for $d \leq 4$) one only needs to compute the Hasse–Weil zeta function of X

$$Z_X(t) := \exp\left(\sum_{m=1}^{\infty} \frac{\#X(\mathbb{F}_{p^m})}{m} t^m\right) = \frac{1}{(1-t)\chi(t)(1-p^2 t)},$$

[Abbott–Kedlaya–Roe, C, C–Harvey–Kedlaya, Tuitman–Pancratz]

Reduction to finite characteristic

Take $f \in \mathbb{Z}[x, y, z, w]$ and $X := Z(f) \subset \mathbb{P}^3_{\mathbb{Z}}$.

We may consider the surface $X_{\mathbb{F}_p} := Z(f \bmod p) \subset \mathbb{P}^3(\mathbb{F}_p)$.

Theorem

If X and $X_{\mathbb{F}_p}$ are smooth then the specialization map is injective

$$\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \hookrightarrow \mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}}) \quad \text{and} \quad \rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}}).$$

Reduction to finite characteristic

Take $f \in \mathbb{Z}[x, y, z, w]$ and $X := Z(f) \subset \mathbb{P}^3_{\mathbb{Z}}$.

We may consider the surface $X_{\mathbb{F}_p} := Z(f \bmod p) \subset \mathbb{P}^3(\mathbb{F}_p)$.

Theorem

If X and $X_{\mathbb{F}_p}$ are smooth then the specialization map is injective

$$\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \hookrightarrow \mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}}) \quad \text{and} \quad \rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}}).$$

Goal

For a given f and p, improve the inequality $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$.

Reduction to finite characteristic

Take $f \in \mathbb{Z}[x,y,z,w]$ and $X := Z(f) \subset \mathbb{P}^3_{\mathbb{Z}}$.

We may consider the surface $X_{\mathbb{F}_p} := Z(f \bmod p) \subset \mathbb{P}^3(\mathbb{F}_p)$.

Theorem

If X and $X_{\mathbb{F}_p}$ are smooth then the specialization map is injective

$$\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \hookrightarrow \mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}}) \quad \text{and} \quad \rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}}).$$

Goal

For a given f and p, improve the inequality $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$.

Parity reasons might already force the inequality to not be sharp.

Endomorphisms of the transcendental lattice can complicate things even further.

Previous approaches for K3 surfaces (van Luijk)

Given $\chi(t) := \det(1 - t\,\mathrm{Frob}_p\,|\mathrm{H}^2)$ we may compute $\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_{p^n}})$ modulo squares via the Artin–Tate formula:

$$\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_q}) = \lim_{s\to 1} \frac{(-1)^{\rho(X_{\mathbb{F}_q})-1}\det(1 - q^{-s}\,\mathrm{Frob}_q\,|\mathrm{H}^2)}{q(1-q^{1-s})^{\rho(X_{\mathbb{F}_q})}} \bmod \mathbb{Q}^{\times 2}.$$

Previous approaches for K3 surfaces (van Luijk)

Given $\chi(t) := \det(1 - t\,\mathrm{Frob}_p\,|\mathrm{H}^2)$ we may compute $\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_{p^n}})$ modulo squares via the Artin–Tate formula:

$$\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_q}) = \lim_{s\to 1} \frac{(-1)^{\rho(X_{\mathbb{F}_q})-1}\det(1 - q^{-s}\,\mathrm{Frob}_q\,|\mathrm{H}^2)}{q(1-q^{1-s})^{\rho(X_{\mathbb{F}_q})}} \bmod \mathbb{Q}^{\times 2}.$$

Combining this with the Tate conjecture we obtain $\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$ modulo squares.

Previous approaches for K3 surfaces (van Luijk)

Given $\chi(t) := \det(1 - t\,\mathrm{Frob}_p\,|\mathrm{H}^2)$ we may compute $\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_{p^n}})$ modulo squares via the Artin–Tate formula:

$$\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_q}) = \lim_{s\to 1} \frac{(-1)^{\rho(X_{\mathbb{F}_q})-1}\det(1 - q^{-s}\,\mathrm{Frob}_q\,|\mathrm{H}^2)}{q(1-q^{1-s})^{\rho(X_{\mathbb{F}_q})}} \bmod \mathbb{Q}^{\times 2}.$$

Combining this with the Tate conjecture we obtain $\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$ modulo squares.

Thus, if we have two primes p_1 and p_2 such that

- $\rho(X_{\mathbb{F}_{p_1}^{\mathrm{al}}}) = \rho(X_{\mathbb{F}_{p_2}^{\mathrm{al}}})$ and
- $\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_{p_1}^{\mathrm{al}}}) \not\equiv \operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_{p_2}^{\mathrm{al}}}) \bmod \mathbb{Q}^{\times 2}$

then $\operatorname{Pic}(X_{\mathbb{F}_{p_1}^{\mathrm{al}}}) \neq \operatorname{Pic}(X_{\mathbb{F}_{p_2}^{\mathrm{al}}})$, and $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) < \rho(X_{\mathbb{F}_{p_1}^{\mathrm{al}}})$

Previous approaches for K3 surfaces (Elsenhans–Jahnel)

They show that the specialization map

$$\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \hookrightarrow \mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$$

has torsion-free cokernel for $p \neq 2$.

Thus, if $\rho(X_{\mathbb{F}_p^{\mathrm{al}}}) = \rho(X_{\mathbb{Q}^{\mathrm{al}}})$ every invertible sheaf lifts.

Previous approaches for K3 surfaces (Elsenhans–Jahnel)

They show that the specialization map

$$\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \hookrightarrow \mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$$

has torsion-free cokernel for $p \neq 2$.

Thus, if $\rho(X_{\mathbb{F}_p^{\mathrm{al}}}) = \rho(X_{\mathbb{Q}^{\mathrm{al}}})$ every invertible sheaf lifts.

For example, if $\rho(X_{\mathbb{F}_p^{\mathrm{al}}}) = 2$, their approach is

1. compute $\mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$
2. estimate the degree of a hypothetical effective divisor of the lift
3. use Gröbner bases to verify that such a divisor does or does not exist

Previous approaches for K3 surfaces (Elsenhans–Jahnel)

They show that the specialization map

$$\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}}) \hookrightarrow \mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$$

has torsion-free cokernel for $p \neq 2$.

Thus, if $\rho(X_{\mathbb{F}_p^{\mathrm{al}}}) = \rho(X_{\mathbb{Q}^{\mathrm{al}}})$ every invertible sheaf lifts.

For example, if $\rho(X_{\mathbb{F}_p^{\mathrm{al}}}) = 2$, their approach is

1. compute $\mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$
2. estimate the degree of a hypothetical effective divisor of the lift
3. use Gröbner bases to verify that such a divisor does or does not exist

This approach is only practical if one can compute $\mathrm{Pic}(X_{\mathbb{F}_p^{\mathrm{al}}})$ and if the obtained estimates are low.

Reduction to finite characteristic

Take $f \in \mathbb{Z}[x, y, z, w]$ and $X := Z(f) \subset \mathbb{P}^3_{\mathbb{Z}}$.

We may consider the surface $X_{\mathbb{F}_p} := Z(f \bmod p) \subset \mathbb{P}^3(\mathbb{F}_p)$.

Theorem

If X and $X_{\mathbb{F}_p}$ are smooth then $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$.

Goal

For a given f and p, improve the inequality above.

Reduction to finite characteristic

Take $f \in \mathbb{Z}[x, y, z, w]$ and $X := Z(f) \subset \mathbb{P}^3_{\mathbb{Z}}$.

We may consider the surface $X_{\mathbb{F}_p} := Z(f \bmod p) \subset \mathbb{P}^3(\mathbb{F}_p)$.

Theorem

If X and $X_{\mathbb{F}_p}$ are smooth then $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = \rho(X_{\mathbb{Q}_p^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$.

Goal

For a given f and p, improve the inequality above.

Reduction to finite characteristic

Take $f \in \mathbb{Z}[x, y, z, w]$ and $X := Z(f) \subset \mathbb{P}^3_{\mathbb{Z}}$.

We may consider the surface $X_{\mathbb{F}_p} := Z(f \bmod p) \subset \mathbb{P}^3(\mathbb{F}_p)$.

Theorem

If X and $X_{\mathbb{F}_p}$ are smooth then $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = \rho(X_{\mathbb{Q}_p^{\mathrm{al}}}) \leq \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$.

Goal

For a given f and p, improve the inequality above.

Idea, try to lift algebraic cycles (curves) from $\mathbb{F}_p^{\mathrm{al}}$ to $\mathbb{Q}_p^{\mathrm{al}}$.

We will do this by considering the thickenings

$$Z(f \bmod p^i) \subset P^3_{\mathbb{Z}/(p)^i} \quad i = 1, 2, \ldots$$

1st ingredient: Cohomology

For simplicity, assume that all curve classes are defined over the base field, i.e.,

$$\rho(X) = \rho(X_{\mathbb{Q}^{\mathrm{al}}}) \quad \text{and} \quad \rho(X_{\mathbb{F}_p}) = \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$$

1st ingredient: Cohomology

For simplicity, assume that all curve classes are defined over the base field, i.e.,

$$\rho(X) = \rho(X_{\mathbb{Q}^{\mathrm{al}}}) \quad \text{and} \quad \rho(X_{\mathbb{F}_p}) = \rho(X_{\mathbb{F}_p^{\mathrm{al}}})$$

Over characteristic zero we have:

- $\mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}) = \mathrm{F}^0(X) \supset \mathrm{F}^1(X) \supset \mathrm{F}^2(X)$, the Hodge filtration
- $\mathrm{Pic}(X) \hookrightarrow \mathrm{F}^1(X)$
- For $d = 4$, $\dim \mathrm{F}^i(X) = 22, 21, 1$.

Over characteristic p we have:

- $\mathrm{Pic}(X_{\mathbb{F}_p}) \hookrightarrow \mathrm{H}^2_{\mathrm{crys}}(X_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes \mathbb{Q}_p \simeq \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$

Theorem (Berthelot, Ogus 1978; Raynaud 1979)

$$\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}} \cap \mathrm{F}^1_{\mathbb{Q}_p}$$

2nd ingredient: Approximate $\mathrm{Pic}(X_{\mathbb{F}_p}) \otimes_{\mathbb{Z}} \mathbb{Q}_p$

Via the isomorphism $\mathrm{H}^2_{\mathrm{crys}}(X_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes \mathbb{Q}_p \simeq \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q})$, we have

$$\mathrm{Frob}_p : \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p) \to \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p).$$

Tate conjecture

$$\mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_p} = \ker(\mathrm{Frob}_p - p \cdot \mathrm{id} \,|\mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p))$$

By computing a p-adic approximation of Frob_p, we may compute a p-adic approximation of

$$\pi : \mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_p} \to \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p)/\, \mathrm{F}^1(X)_{\mathbb{Q}_p}$$

thus

$$\dim_{\mathbb{Q}} \mathrm{Pic}(X) \leq \dim_{\mathbb{Q}_p} \ker \pi.$$

2nd ingredient: Approximate $\mathrm{Pic}(X_{\mathbb{F}_p}) \otimes_{\mathbb{Z}} \mathbb{Q}_p$

Via the isomorphism $\mathrm{H}^2_{\mathrm{crys}}(X_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes \mathbb{Q}_p \simeq \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q})$, we have

$$\mathrm{Frob}_p : \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p) \to \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p).$$

Tate conjecture

$$\mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_p} = \ker(\mathrm{Frob}_p - p \cdot \mathrm{id} \,|\, \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p))$$

By computing a p-adic approximation of Frob_p, we may compute a p-adic approximation of

$$\pi : \mathrm{Pic}(X_{\mathbb{F}_p})_{\mathbb{Q}_p} \to \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p)/\,\mathrm{F}^1(X)_{\mathbb{Q}_p}$$

thus

$$\dim_{\mathbb{Q}} \mathrm{Pic}(X) \leq \dim_{\mathbb{Q}_p} \ker \pi.$$

By picking a basis that respects the Hodge filtration, the map

$$\mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p) \to \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p)/\,\mathrm{F}^1(X)_{\mathbb{Q}_p}$$

is a coordinate projection.

Abelian surface

$A = \mathrm{Jac}(y^2 = 4x^5 - 36x^4 + 56x^3 - 76x^2 + 44x - 23)$

$$\mathrm{Frob}\,|_{\mathrm{H}^1_{\mathrm{dR}}(A/\mathbb{Q}_p)} \equiv \begin{pmatrix} 31\cdot 482 & 31\cdot 284 & 16241 & 3075 \\ 31\cdot 386 & 31\cdot 886 & 2644 & 12126 \\ 31\cdot 284 & 31\cdot 659 & 6336 & 9750 \\ 31\cdot 194 & 31\cdot 876 & 27408 & 10841 \end{pmatrix} \pmod{31^3},$$

$$L(t) = \det(1 - t\,\mathrm{Frob}\,|\mathrm{H}^1) = 1 - 3t + 14t^2 - 93t^3 + 961t^4.$$

From this we deduce $\mathrm{Frob}\,|_{\mathrm{H}^2_{\mathrm{dR}}(A/\mathbb{Q}_p)}$ and

$$\det(1 - t31^{-1}\,\mathrm{Frob}\,|\mathrm{H}^2_{\mathrm{dR}}(A/\mathbb{Q}_p)) = (t-1)^2(31t^4 + 48t^3 + 43t^2 + 48t + 31)/31$$

Thus, $\rho(A_{\mathbb{F}_p^{\mathrm{al}}}) = 2$.

Since the basis of H^1 respects the Hodge filtration, the induced basis in H^2 will also respect it.

Abelian Surface

$A = \mathrm{Jac}(y^2 = 4x^5 - 36x^4 + 56x^3 - 76x^2 + 44x - 23)$

$$\det(1 - t31^{-1}\,\mathrm{Frob}\,|\mathrm{H}^2_{\mathrm{dR}}(A/\mathbb{Q}_p)) = (t-1)^2(31t^4 + 48t^3 + 43t^2 + 48t + 31)/31$$

Thus, $\rho(A_{\mathbb{F}_p^{\mathrm{al}}}) = 2$.

Compute 2 eigenvectors

$$v_1 \equiv (356,\ 37,\ 831,\ 0,\ 295,\ 31) \pmod{31^2}$$
$$v_2 \equiv (4,\ 957,\ 3,\ 1,\ 0,\ 0) \pmod{31^2}.$$

The last coordinate of the vectors above gives the projection to H^2/F_1.

Therefore, $v_1 \notin F^1$ and the corresponding algebraic cycle cannot lift to $\mathbb{Q}_p$.

Thus, we improved $\rho(A_{\mathbb{Q}^{\mathrm{al}}}) \leq 2$ to $\rho(A_{\mathbb{Q}^{\mathrm{al}}}) \leq 2$, and therefore $\mathrm{End}(A_{\mathbb{Q}^{\mathrm{al}}}) = \mathbb{Z}$.

Abelian Surface

$A = \text{Jac}(y^2 = 4x^5 - 36x^4 + 56x^3 - 76x^2 + 44x - 23)$

$$\det(1 - t31^{-1}\,\text{Frob}\,|\text{H}^2_{\text{dR}}(A/\mathbb{Q}_p)) = (t-1)^2(31t^4 + 48t^3 + 43t^2 + 48t + 31)/31$$

Thus, $\rho(A_{\mathbb{F}_p^{\text{al}}}) = 2$.

Compute 2 eigenvectors

$$v_1 \equiv (356,\ 37,\ 831,\ 0,\ 295,\ 31) \pmod{31^2}$$
$$v_2 \equiv (4,\ 957,\ 3,\ 1,\ 0,\ 0) \pmod{31^2}.$$

The last coordinate of the vectors above gives the projection to H^2/F_1.

Therefore, $v_1 \notin F^1$ and the corresponding algebraic cycle cannot lift to $\mathbb{Q}_p$.

Thus, we improved $\rho(A_{\mathbb{Q}^{\text{al}}}) \leq 2$ to $\rho(A_{\mathbb{Q}^{\text{al}}}) \leq 2$, and therefore $\text{End}(A_{\mathbb{Q}^{\text{al}}}) = \mathbb{Z}$.

van Luijk's method would have succeed in this example by using a second prime.

Abelian surface

$A = \mathrm{Jac}(y^2 = 4x^5 - 36x^4 + 56x^3 - 76x^2 + 44x - 23)$

We have automated this process and the SageMath package is available at

`github.com/edgarcosta/crystalline_obstruction`

Abelian surface

$A = \mathrm{Jac}(y^2 = 4x^5 - 36x^4 + 56x^3 - 76x^2 + 44x - 23)$

We have automated this process and the SageMath package is available at

`github.com/edgarcosta/crystalline_obstruction`

```
sage: f = ZZ['x,y']('4*x^5 - 36*x^4 + 56*x^3 - 76*x^2 + 44*x - 23 -y^2')
sage: crystalline_obstruction(f=f, p=31, precision=3)
(1,
 {'precision': 3,
  'p': 31,
  'rank T(X_Fpbar)': 2,
  'factors': [(t - 1, 2)],
  'dim Ti': [2],
  'dim Li': [1]})
```

As we had observed before:

- $\rho\left(A_{\mathbb{F}_p^{\mathrm{al}}}\right) = 2$
- $\rho(A_{\mathbb{Q}^{\mathrm{al}}}) \leq 1 \implies \mathrm{rank}\,\mathrm{End}(A_{\mathbb{Q}^{\mathrm{al}}}) = 1.$

Abelian threefold $A := \mathrm{Jac}(y^4 + x^3z + 2y^3z - yz^3)$

We may also bound $\operatorname{rank}\mathrm{End}(A)$ directly by replacing $\mathrm{H}^1(A) \otimes \mathrm{H}^1(A)$.

```
sage: crystalline_obstruction(f=f, p=31, precision=5)
(4, {'rank T(X_Fpbar)': 5,
     'factors': [(t - 1, 3), (t^2 + t + 1, 1)],
     'dim Ti': [3, 2],
     'dim Li': [2, 2]},
     'precision': 5, 'p': 31})

sage: crystalline_obstruction(f=f, p=31, precision=5, tensor=True)
(6, {'rank T(X_Fpbar)': 10,
     'factors': [(t - 1, 6), (t^2 + t + 1, 2)],
     'dim Ti': [6, 4],
     'dim Li': [4, 2],
     'precision': 5, 'p': 31})
```

- Improved, $\rho(A_{\mathbb{Q}^{\mathrm{al}}}) \leq 5$ to $\rho(A_{\mathbb{Q}^{\mathrm{al}}}) \leq 4$
- Improved, $\operatorname{rank}\mathrm{End}(A_{\mathbb{Q}^{\mathrm{al}}}) \leq 10$ to $\operatorname{rank}\mathrm{End}(A_{\mathbb{Q}^{\mathrm{al}}}) \leq 6$
- indeed, $\mathrm{End}(A_{\mathbb{Q}^{\mathrm{al}}})_{\mathbb{Q}} = \mathbb{Q}(\sqrt{-3}) \times B$, B is a quaternion algebra with $\operatorname{disc} B = 6$.

Quartic surface

$X := Z(y^4 - x^3z + yz^3 + zw^3 + w^4) \subset \mathbb{P}^3_{\mathbb{C}}$

```
sage: crystalline_obstruction(f, p=89, precision=3)
(4,
 {'rank T(X_Fpbar)': 10,
  'factors': [(t - 1, 1), (t + 1, 1), (t - 1, 4), (t^4 + 1, 1)],
  'dim Ti': [1, 1, 4, 4],
  'dim Li': [1, 0, 3, 0]},
  'precision': 3, 'p': 89})
```

- $\rho(X_{\mathbb{F}_{89}^{\mathrm{al}}}) = 10$
- $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq 4$
- In fact, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 4$ as there are four lines in $z = 0$.
- previous approaches could have not used this $p = 89$

Quartic surface

$X = Z(y^4 - x^3z + yz^3 + zw^3 + w^4) \subset \mathbb{P}^3_{\mathbb{C}}$

```
sage: crystalline_obstruction(f, p=31, precision=5)
(4,
 {'rank T(X_Fpbar)': 4,
  'factors': [(t - 1, 1), (t - 1, 1), (t + 1, 2)],
  'dim Ti': [1, 1, 2],
  'dim Li': [1, 1, 2]},
  'precision': 5, 'p': 31})
```

- $\rho(X_{\mathbb{F}_{31}^{\mathrm{al}}}) = 4$
- no cycle obstruction found while working $\mathbb{Z}/(p)^5$
- $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) \leq 4$, with some extra confidence that the equality might hold.
- by searching for lines Elsenhans–Jahnel's method would have succeeded in this example

Quintic surface

$X := Z(9xy^4 + 3x^4z + 9y^2z^3 + z^5 + 5w^5) \subset \mathbb{P}^3$

```
sage: crystalline_obstruction(f, p=23, precision=6)
(1, {'rank T(X_Fpbar)': 5,
     'factors': [(t - 1, 1), (t - 1, 1), (t + 1, 1), (t^2 + 1, 1)],
     'dim Ti': [1, 1, 1, 2],
     'dim Li': [1, 0, 0, 0],
    'precision': 6, 'p': 23})

sage: crystalline_obstruction(f, p=29, precision=20)
(3, {'rank T(X_Fpbar)': 5,
     'factors': [(t - 1, 1), (t - 1, 2), (t + 1, 2)],
     'dim Ti': [1, 2, 2],
     'dim Li': [1, 1, 1]})
     'precision': 20, 'p': 29})
```

- $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 1$
- However, we can not deduce this from $p = 29$, not even with infinite precision.
- The surface has *CM* by $\mathbb{Q}(\zeta_5)$

Theoretical example

$X\colon w^2 = \left(-y^2/8 + yz - z^2\right)\left(7x^2/8 + 5xz + 7z^2\right)\left(2x^2 + 3xy + y^2\right)$

Take $p = 83$.

By counting points on the singular model, we deduce
$\det(1 - t83^{-1}\,\mathrm{Frob}\,|\mathrm{H}^2) = \chi_1 \cdot \chi_2$ where

$$\chi_1 = (t-1)^{10}(t+1)^6, \quad \chi_2 = (t^2+1)(83 - 158t^2 + 83t^4)/83.$$

Where $\chi_1(t)$ is the characteristic polynomial of Frobenius acting on the known sublattice of $\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}})$ generated by the polarization and the 15 singular points.

Thus, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 16$ or 18.

Theoretical example

$X\colon w^2 = \left(-y^2/8 + yz - z^2\right)\left(7x^2/8 + 5xz + 7z^2\right)\left(2x^2 + 3xy + y^2\right)$

Take $p = 83$.

By counting points on the singular model, we deduce
$\det(1 - t83^{-1}\,\mathrm{Frob}\,|\,\mathrm{H}^2) = \chi_1 \cdot \chi_2$ where

$$\chi_1 = (t-1)^{10}(t+1)^6, \quad \chi_2 = (t^2+1)(83 - 158t^2 + 83t^4)/83.$$

Where $\chi_1(t)$ is the characteristic polynomial of Frobenius acting on the known sublattice of $\mathrm{Pic}(X_{\mathbb{Q}^{\mathrm{al}}})$ generated by the polarization and the 15 singular points.

Thus, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 16$ or 18.

Indeed, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 16$ and X has RM by $\mathbb{Q}(\sqrt{2})$.

Theoretical example

$X\colon w^2 = \left(-y^2/8 + yz - z^2\right)\left(7x^2/8 + 5xz + 7z^2\right)\left(2x^2 + 3xy + y^2\right)$

Take $p = 83$.

By counting points on the singular model, we deduce
$\det(1 - t83^{-1}\operatorname{Frob}|\mathrm{H}^2) = \chi_1 \cdot \chi_2$ where

$$\chi_1 = (t-1)^{10}(t+1)^6, \quad \chi_2 = (t^2+1)(83 - 158t^2 + 83t^4)/83.$$

Where $\chi_1(t)$ is the characteristic polynomial of Frobenius acting on the known sublattice of $\operatorname{Pic}(X_{\mathbb{Q}^{\mathrm{al}}})$ generated by the polarization and the 15 singular points.

Thus, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 16$ or 18.

Indeed, $\rho(X_{\mathbb{Q}^{\mathrm{al}}}) = 16$ and X has RM by $\mathbb{Q}(\sqrt{2})$.

Given a good enough approximation to Frob_p we will obstruct 2 extra cycles.

However, we don't yet know how to compute such approximation!