Variation of Néron–Severi ranks of reductions of K3 surfaces

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Slides available at `edgarcosta.org` under Research

Elliptic curves

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

Write $E_p := E \bmod p$

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$\rightsquigarrow$ studying the **statistical** properties $\#E_p$.

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Question

What can we say about the error term $a_p/\sqrt{p}$ as $p \to \infty$?

Two types of elliptic curves

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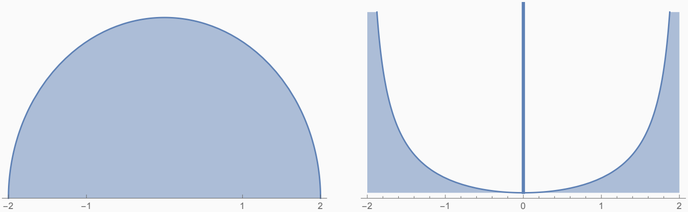

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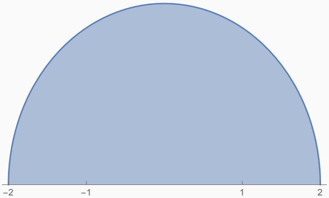

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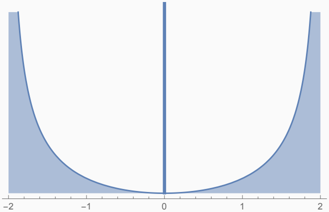

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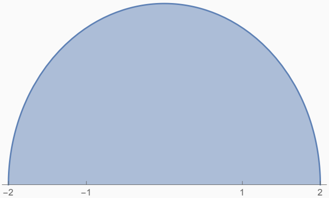
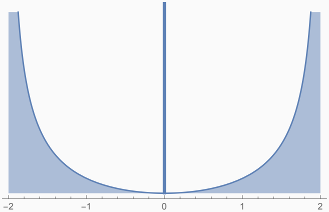
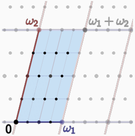




Over $\mathbb{C}$ an elliptic curve E is a torus

$E_{\mathbb{C}} \simeq \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 =$



and we have $\operatorname{End} E^{\mathrm{al}} = \operatorname{End}\Lambda$

How to distinguish between the two types?

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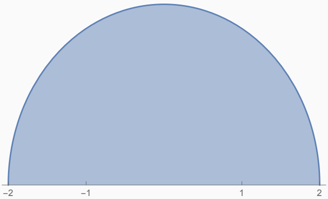

- $\mathrm{End}_{\mathbb{Q}} E^{\mathrm{al}} \hookrightarrow \mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}} \hookleftarrow \mathbb{Q}(\mathrm{Frob}_p)$
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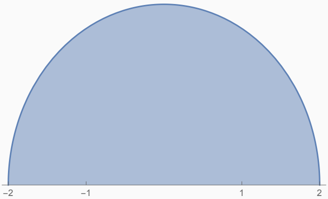

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- If E has CM and $a_p \neq 0$, then $\mathrm{End}_{\mathbb{Q}} E^{\mathrm{al}} \simeq \mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}}$.

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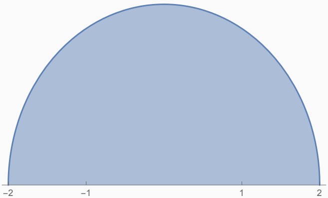

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- If E has CM and $a_p \neq 0$, then $\mathrm{End}_{\mathbb{Q}} E^{\mathrm{al}} \simeq \mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}}$.
- If E is non-CM, then $\mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}} \cap \mathrm{End}_{\mathbb{Q}} E_q^{\mathrm{al}} \simeq \mathbb{Q}$ with prob. 1.

Examples

$E : y^2 + y = x^3 - x^2 - 10x - 20$ (11.a2)

- $\operatorname{End}_{\mathbb{Q}} E_2^{\mathrm{al}} \simeq \mathbb{Q}(\sqrt{-1})$
- $\operatorname{End}_{\mathbb{Q}} E_3^{\mathrm{al}} \simeq \mathbb{Q}(\sqrt{-11})$
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$E : y^2 + y = x^3 - 7$ **(27.a2)**

- $p = 2 \bmod 3 \Rightarrow a_p = 0 \Rightarrow \mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}}$ is a Quaternion algebra
- $p = 1 \bmod 3 \Rightarrow \mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}} \simeq \mathbb{Q}(\sqrt{-3})$
- $\rightsquigarrow \mathrm{End}_{\mathbb{Q}} E^{\mathrm{al}} = \mathbb{Q}(\sqrt{-3})$

K3 surfaces

K3 surfaces are a possible generalization of elliptic curves

They may arise in many ways:

- smooth quartic surfaces in $\mathbb{P}^3$

$$X : f(x, y, z, w) = 0, \quad \deg f = 4$$

- double cover of $\mathbb{P}^2$ branched over a sextic curve

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Can we play similar game as before?

In this case, instead of studying $\#X_p$ or $\operatorname{Tr}\operatorname{Frob}_p$ we study

$$p \longmapsto \operatorname{rk}\operatorname{NS} {X_p}^{\mathrm{al}} \in \{2, 4, \ldots, 22\}$$

K3 Surfaces

$X/\mathbb{Q}$ a K3 surface

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Recall that:

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- $\operatorname{Prob}(a_p = 0) = \begin{cases} \overset{?}{\sim} \frac{1}{\sqrt{p}} & \text{if } E \text{ is non-CM (Lang–Trotter)} \\ 1/2 & \text{if } E \text{ has CM by } \mathbb{Q}(\sqrt{-d}) \end{cases}$

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In the later case,

$$\{p : a_p = 0\} = \{p : p \text{ is ramified or inert in } \mathbb{Q}(\sqrt{-d})\}$$

Néron–Severi group

- $\mathrm{NS}\,\bullet =$ Néron–Severi group of $\bullet \simeq \{$curves on $\bullet\}/\sim$
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For infinitely many p we have $\rho({X_p}^{\mathrm{al}}) = \min_q \rho({X_q}^{\mathrm{al}})$.

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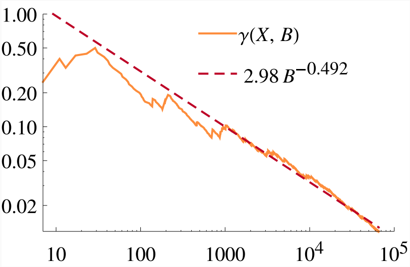
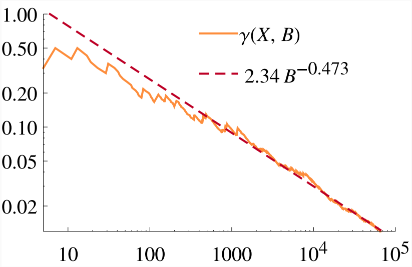
Let's do some numerical experiments!

Two generic K3 surfaces, $\rho(X^{\text{al}}) = 1$





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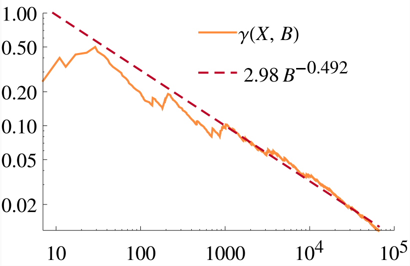

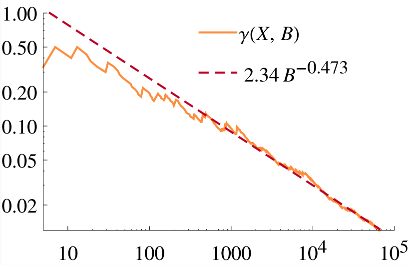

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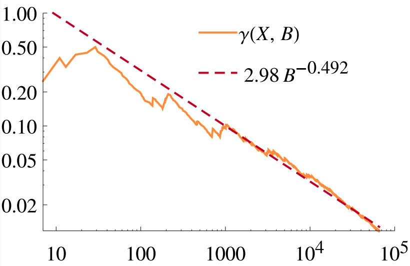

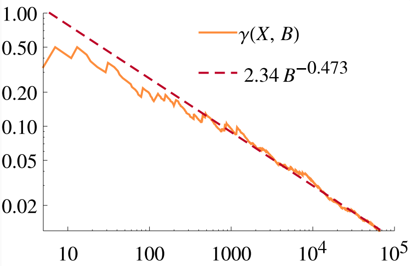

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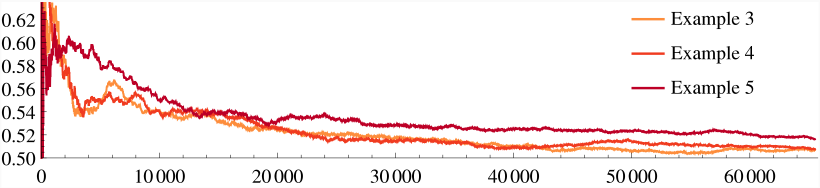
Why?

Three K3 surfaces with $\rho(X^{al}) = 2$



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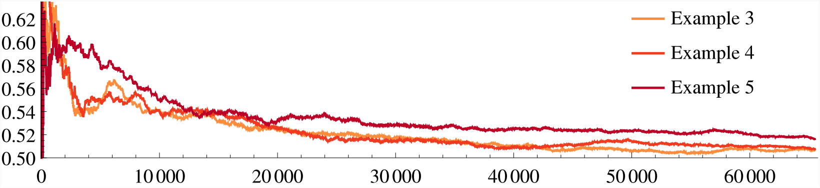

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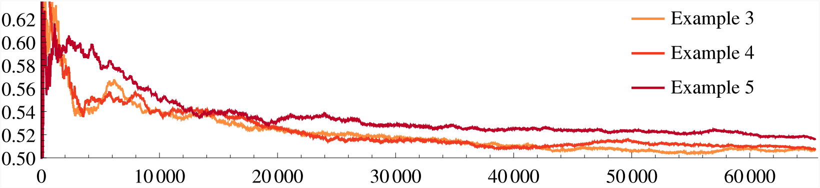

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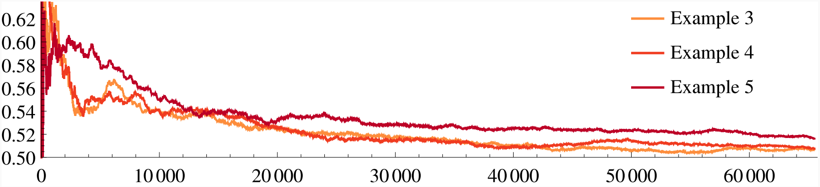
Could it be related to some integer being a square modulo p?

We can explain the $1/2$



Theorem (C, C–Elsenhans–Jahnel)

If $\rho(X^{\mathrm{al}}) = \min_q \rho(X_p{}^{\mathrm{al}})$*, then there is a* $d_X \in \mathbb{Z}$ *such that:*

$$\left\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\right\} \subset \Pi_{\mathrm{jump}}(X).$$

In general, d_X *is not a square.*

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If d_X is not a square:

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$D_3 = -1 \cdot 5 \cdot 151 \cdot 22490817357414371041 \cdot 3873084974301493372336663588079962607808750567408509842132769703432789353$

$D_4 = 53 \cdot 2624174618795407 \cdot 512854561846964817139494202072778341 \cdot 1215218370089028769076718102126921744353362873 \cdot 68$

$D_5 = -1 \cdot 47 \cdot 3109 \cdot 4969 \cdot 14857095849982608071 \cdot 4454102776609283477625867643318744322025846880161 49 \cdot 65865270852$

Experimental data for $\rho(X^{\mathrm{al}}) = 2$ (again)

What if we ignore $\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\} \subset \Pi_{\mathrm{jump}}(X)$?

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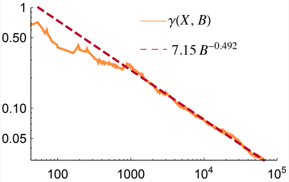

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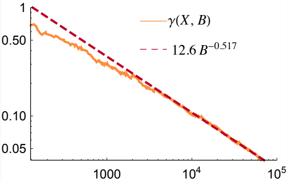

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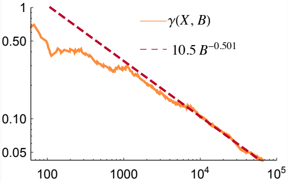

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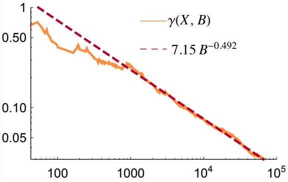
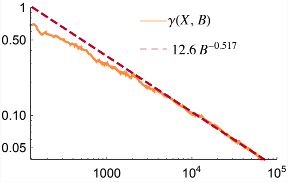
$$\gamma\left(X_{\mathbb{Q}(\sqrt{d_X})}, B\right) \overset{?}{\sim} \frac{c_X}{\sqrt{B}}, \quad B \to \infty$$

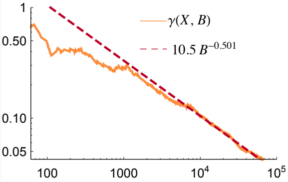
Example 3



Example 4



Example 5



$$\mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) = \begin{cases} 1 & \text{if } d_X \text{ is not a square modulo } p \\ \overset{?}{\sim} \frac{1}{\sqrt{p}} & \textit{otherwise} \end{cases}$$

Why?