# Beta Jacobi Ensembles, (ghosts) and the GSVD

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Aprl 11, 2016

MIT MATHEMATICS

Machine Learning @amazon

# Two Big Ideas I enjoy mulling over

Hermite, Laguerre, and Jacobi

Hermite 1822-1901 | Laguerre 1834-1886 | Jacobi 1804-1851

| 1. Deep significance of Hermite/Laguerre/Jacobi | | |
|---|---|---|
| Hermite | eig | symmetric eigenvalue problem |
| Laguerre | svd | singular value decomposition |
| Jacobi | gsvd | generalized singular value decomposition |


**2. Ghosts**

$\beta$ not 1,2,4 or even integer has a **ghost** like existence

Today want to emphasize its Appearance in

- Selberg Constants
- Differential Geometry
- Jacobian Computations

# Beta Ensembles

| | | Real dimensions |
|---|---|---|
| $\beta = 1$ | One **real** dimension | 1 |
| $\beta = 2$ | One **complex** dimension | 2 |
| $\beta = 4$ | One **quaternion** dimension | 4 |
| $\beta = \boldsymbol{\beta}$ | One **ghost beta** dimension | $\boldsymbol{\beta}$ |

# Do you recognize this sequence?

$$2, 2\pi$$

# Do you recognize this sequence?

$$2, 2\pi, 4\pi$$

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$$2, 2\pi, 4\pi, 2\pi^2, \ldots$$

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$$\frac{8\pi^2}{3}, \pi^3, \frac{16\pi^3}{15}, \frac{\pi^4}{3}, \frac{32\pi^4}{105}, \frac{\pi^5}{12}, \frac{64\pi^5}{945}$$

# Do you recognize this sequence?

$$2, 2\pi, 4\pi, 2\pi^2, \ldots$$

Volume (surface area) of the unit hypersphere in **n** **real** dimensions

$$V_{\text{sphere}}(n; \beta = 1) = 2\pi^{n/2}/\Gamma(n/2) \qquad \beta = 1$$

# Do you recognize this sequence?

$$2, 2\pi, 4\pi, 2\pi^2, \ldots$$

$\mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$

Volume (surface area) of the unit hypersphere in **n** **real** dimensions

$$V_{\text{sphere}}(n;\beta=1) = 2\pi^{n/2}/\Gamma(n/2)$$ $\beta = 1$

Volume (surface area) of the unit hypersphere in **1** **beta** dimension

$$2\pi^{\boldsymbol{\beta}/2}/\Gamma(\boldsymbol{\beta}/2)$$ $\beta = \boldsymbol{\beta}$

# Do you recognize this sequence?

$$2, 2\pi, 4\pi, 2\pi^2, \ldots$$

$\mathbb{R}$ $\mathbb{C}$ $\mathbb{H}$

Volume (surface area) of the unit hypersphere in **n** **real** dimensions

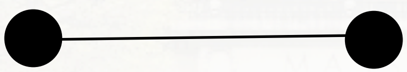
$$V_{\text{sphere}}(n; \beta = 1) = 2\pi^{n/2}/\Gamma(n/2) \qquad \beta = 1$$

Volume (surface area) of the unit hypersphere in **1** **beta** dimension

$$V_{\text{sphere}}(1; \beta) = 2\pi^{\beta/2}/\Gamma(\beta/2) \qquad \beta = \boldsymbol{\beta}$$

**Scalar**

# Pictures of $\mathcal{S}_\beta^0$ (Same as $\mathcal{S}_1^{\beta-1}$)

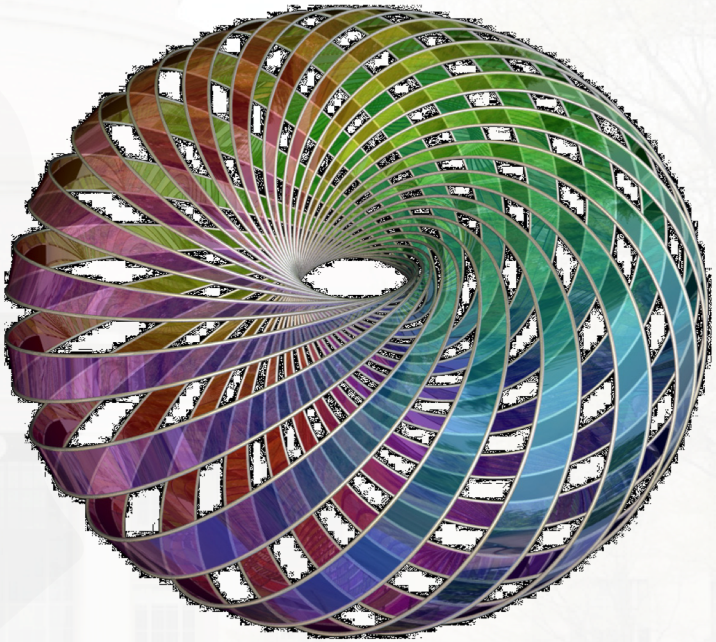


$\beta = 1$

$\mathbb{R}$

$\beta = 2$

$\mathbb{C}$

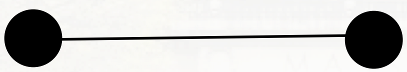


$\beta = 4$

$\mathbb{H}$

Rendition: **Hopf Fibration**

# Pictures of $\mathcal{S}_\beta^0$ (Same as $\mathcal{S}_1^{\beta-1}$)

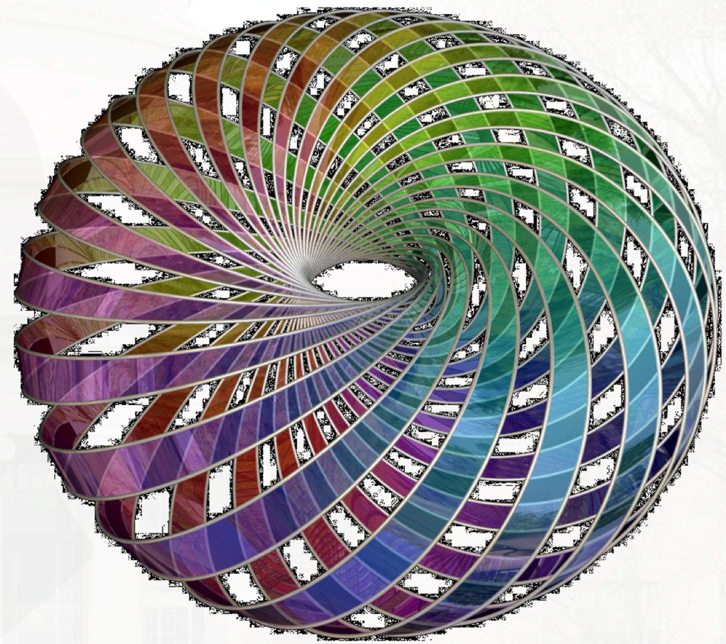


$\beta = 1$

$\mathbb{R}$

$\beta = 2$

$\mathbb{C}$



$\beta = 4$

$\mathbb{H}$

Rendition: **Hopf Fibration**

(See E and Mangoubi Ghost Hopf fibration)
Mangoubi thesis (being finished now)

# Volumes

Volume (surface area) of the unit hypersphere in **n** **real** dimensions

$$2\pi^{n/2}/\Gamma(n/2)$$

$\beta = 1$

Volume (surface area) of the unit hypersphere in **1** **beta** dimension

$$2\pi^{\boldsymbol{\beta}/2}/\Gamma(\boldsymbol{\beta}/2)$$

$\beta = \boldsymbol{\beta}$

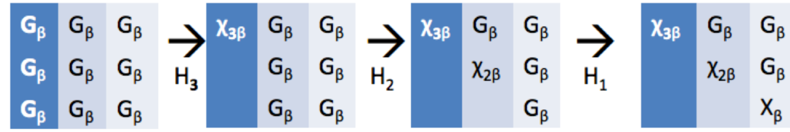
Volume (surface area) of the unit hypersphere in **n** **beta** dimensions

$$2\pi^{\boldsymbol{\beta}\cdot n/2}/\Gamma(\boldsymbol{\beta}\cdot n/2)$$

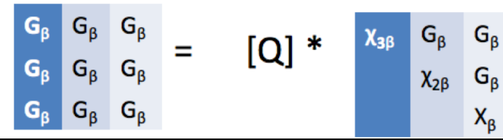
$\beta = \boldsymbol{\beta}$

- $G_1$ is a standard normal N(0,1)
- $G_2$ is a complex normal ($G_1$ +i$G_1$)
- $G_4$ is a quaternion normal ($G_1$ +i$G_1$+j$G_1$+k$G_1$)
- $G_\beta$ (β>0) seems to often work just fine
  → "Ghost Gaussian"

- Gram-Schmidt:




or




Q has β Haar Measure!
We have computed moments!

# Ghosts and Shadows "Collage"

- $|G_1|$ is $\chi_1$, $|G_2|$ is $\chi_2$, $|G_4|$ is $\chi_4$
- So why not $|G_\beta|$ is $\chi_\beta$ ?
- I call $\chi_\beta$ the shadow of $G_\beta$

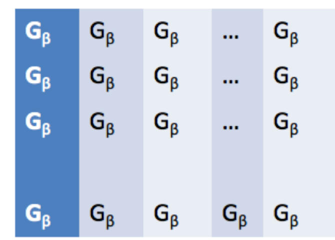
$$c_{m,n,\Sigma,\beta}\prod_{i=1}^{n}\lambda_i^{\frac{m-n+1}{2}\beta-1}\prod_{j<k}|\lambda_k-\lambda_j|^{\beta}\cdot{}_0F_0^{(\beta)}(-\frac{\beta}{2}\Lambda,\Sigma^{-1}),$$

where $\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$.

- Eigenvalue density of G'G Σ ( similar to $A=\Sigma^{1/2}G'G\Sigma^{-1/2}$ )
- Present an algorithm for sampling from this density
- Show how the method of Ghosts and Shadows can be used to derive this algorithm

## Tridiagonalizing Example

Symmetric Part of




$$\rightarrow \quad H_n^\beta=\frac{1}{\sqrt{\beta}}\begin{bmatrix} g_1 & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & g_2 & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & g_{n-1} & \chi_\beta \\ & & & \chi_\beta & g_n \end{bmatrix}$$

Stochastic Operator Limit 2003

$$\frac{d^2}{dx^2}-x+\frac{2}{\sqrt{\beta}}dW\ ,$$

# Jacobi Ensembles Traditional Form

- Suppose A and B are randn(m1,n) and randn(m2,n)
  - iid standard normals, let's assume m1 and m2 >= n
- The eigenvalues of the "MANOVA" matrix

$$(A'A + B'B)^{-1}A'A$$

- Or in Symmetric Form

$$(A'A + B'B)^{-1/2}A'A(A'A + B'B)^{-1/2}$$

# Jacobi Ensembles Traditional Form

- Suppose A and B are randn(m1,n) and randn(m2,n)
  - iid standard normals, let's assume m1 and m2 >= n
- The eigenvalues of

$$(A'A + B'B)^{-1} A'A$$

- Joint Eigenvalue Distribution (1939):

$$c_J \prod_{i<j} |\lambda_i - \lambda_j| \times \prod_i \lambda_i^{(m_1-n-1)/2} (1-\lambda_i)^{(m_2-n-1)/2}$$

# Jacobi Ensembles

- Compute the eigenvalues of

$$(A'A + B'B)^{-1} A'A$$

```
[~,~,~,c,s] = gsvd(randn(m1,n),randn(m2,n))
eigs = c^2
```

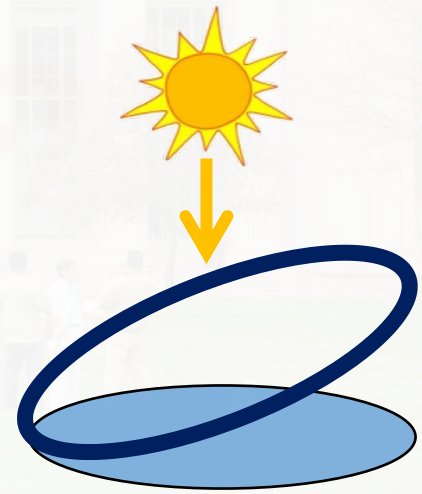
# The Jacobi Ensemble: Geometric Interpretation

- Take reference $n \leqslant m$ dimensional subspace of $\mathbb{R}^m$
- Take **random** $n \leqslant m$ dimensional subspace of $\mathbb{R}^m$

**Jacobi ensemble**

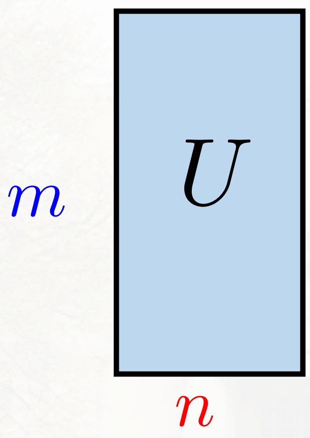
The shadow of the unit ball in the random subspace when projected onto the reference subspace is an ellipsoid

The semi-axes lengths are the Jacobi ensemble cosines.
(MANOVA Convention=Squared cosines)

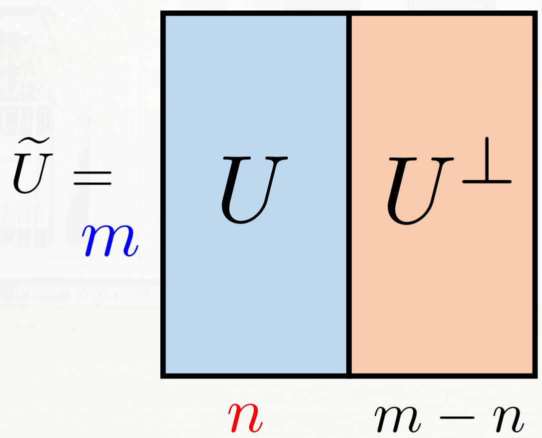


# *Stiefel Manifold :*

**set of n-dim (orthonormal) frames in** $\mathbb{R}^m$ $n \leqslant m$




$$U'U = \mathbb{I}_n$$

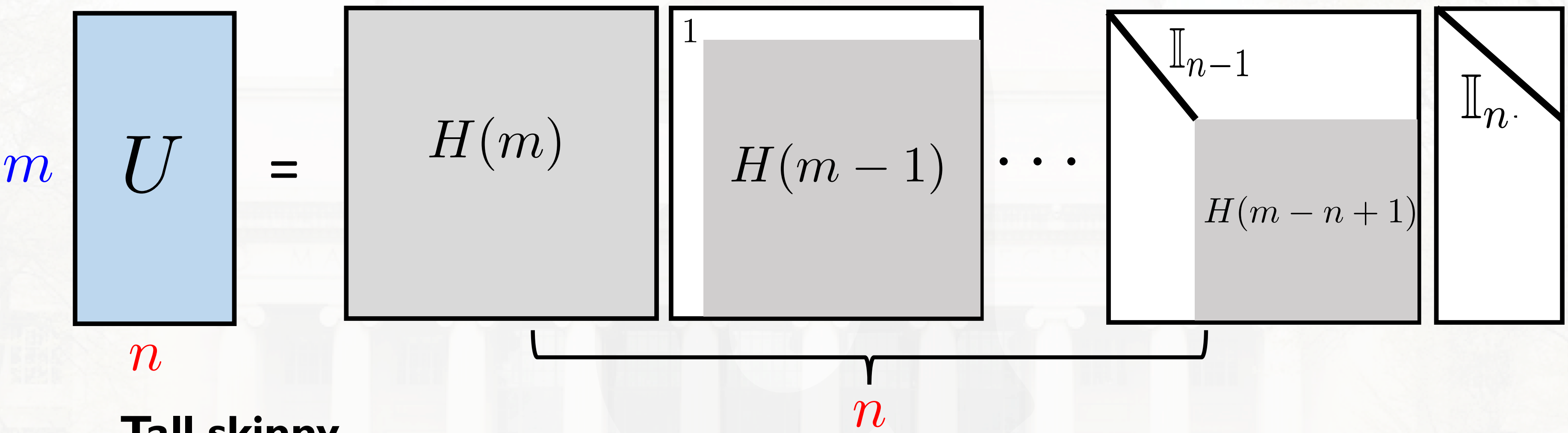



Equivalence Class = $\left\{ \widetilde{U} \begin{bmatrix} \mathbb{I}_n & \\ & W_2 \end{bmatrix} \right\}$

$$W_2 \in \mathcal{O}(m-n)$$

# *Stiefel Manifold (Num. Lin. Alg.)*

**set of n-dim (orthonormal) frames in $\mathbb{R}^m$** $n \leqslant m$




**Tall skinny orthogonal matrix**

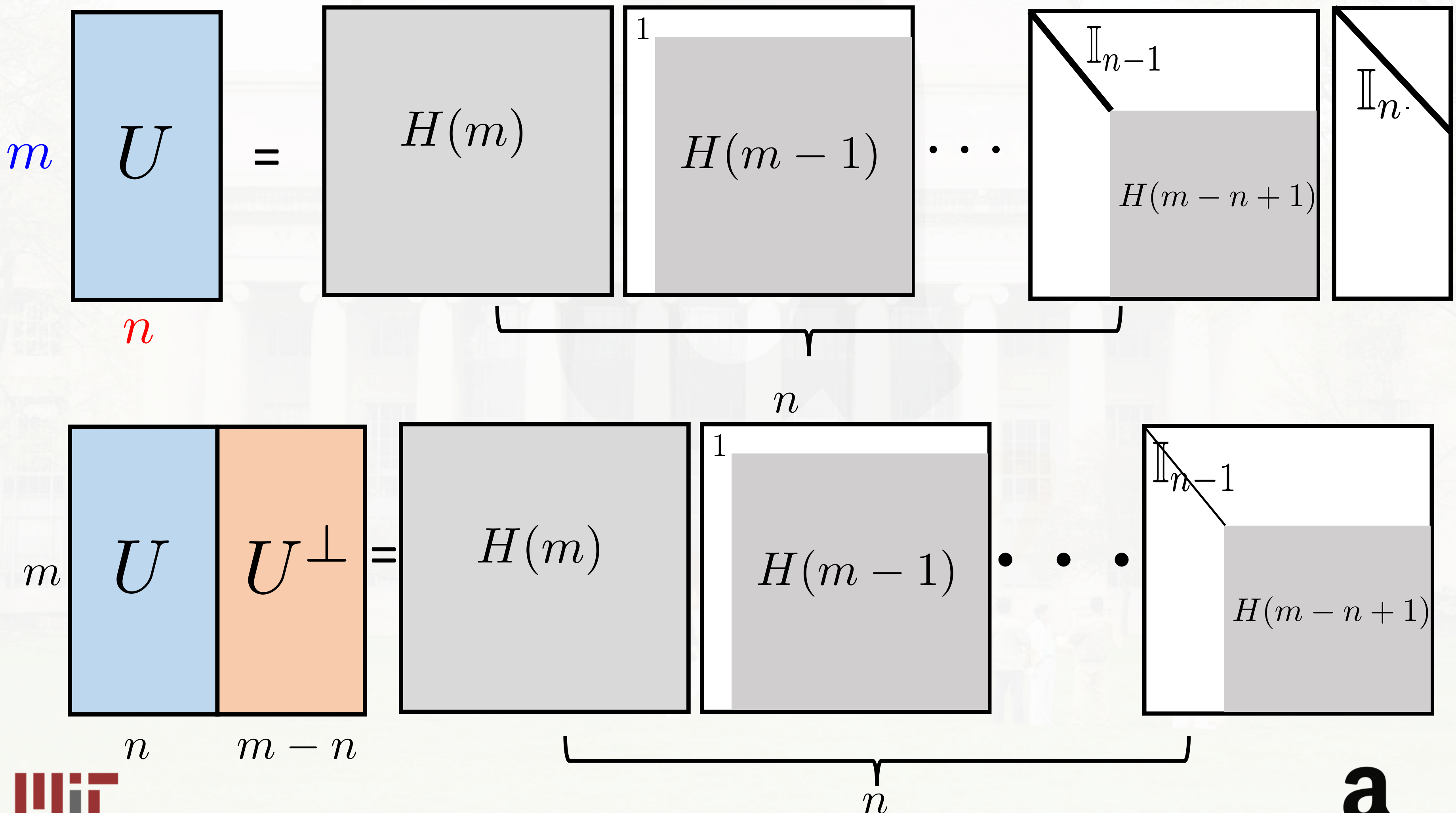
$$U'U = \mathbb{I}_n$$

$H(m)$ Householder matrices

From the Numerical Linear Algebra point of view …

# *Stiefel Manifold (Num. Lin. Alg.)*

**set of n-dim (orthonormal) frames in $\mathbb{R}^m$ $n \leqslant m$**



# Q = Stiefel point

```
In [1]: Q = qrfact(randn(5,3))[:Q]
        full(Q)

Out[1]: 5x3 Array{Float64,2}:
         -0.282794   0.602928  -0.280206
         -0.5741    -0.361804   0.0599935
          0.705835   0.179593  -0.188227
         -0.138295  -0.133773  -0.927542
          0.270385  -0.674853  -0.148735

In [2]: Q

Out[2]: 5x5 Base.LinAlg.QRCompactWYQ{Float64,Array{Float64,2}}:
         -0.282794   0.602928  -0.280206   -0.436969   0.535768
         -0.5741    -0.361804   0.0599935  -0.627802  -0.376525
          0.705835   0.179593  -0.188227   -0.544023  -0.371689
         -0.138295  -0.133773  -0.927542    0.241315  -0.210743
          0.270385  -0.674853  -0.148735   -0.246429   0.62339

In [3]: Q * randn(5)

Out[3]: 5-element Array{Float64,1}:
         -2.067
          0.212148
         -0.642629
          0.0884528
          0.166425
```

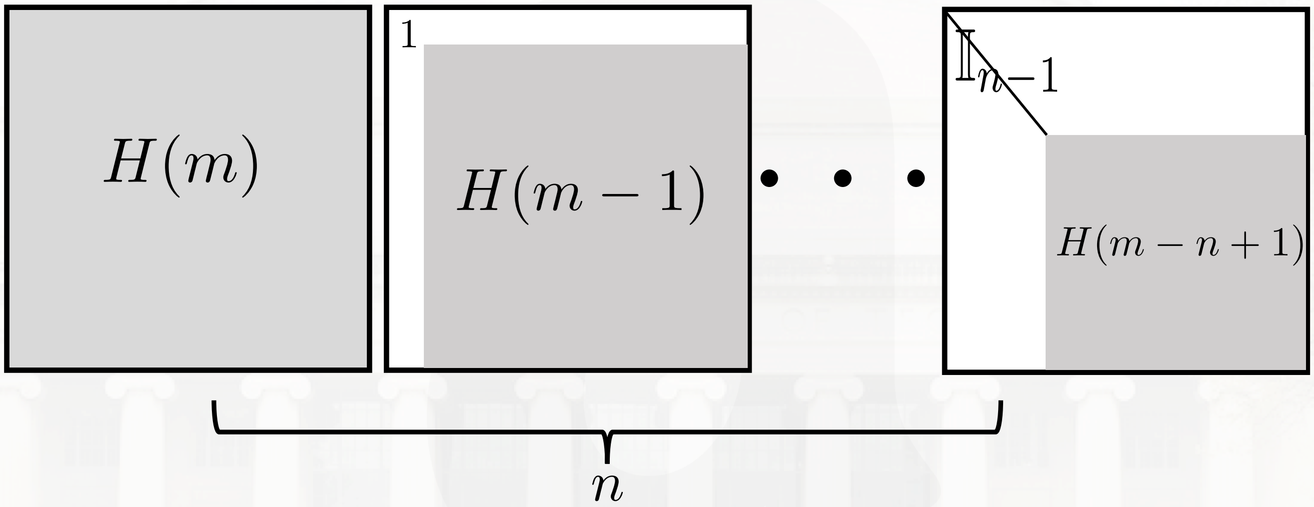
Details:
WY Representation
Schreiber and van Loan
1989

```
In [4]: Q * randn(3)

Out[4]: 5-element Array{Float64,1}:
         -0.771428
          0.490106
         -0.169623
          0.567871
          1.04986
```

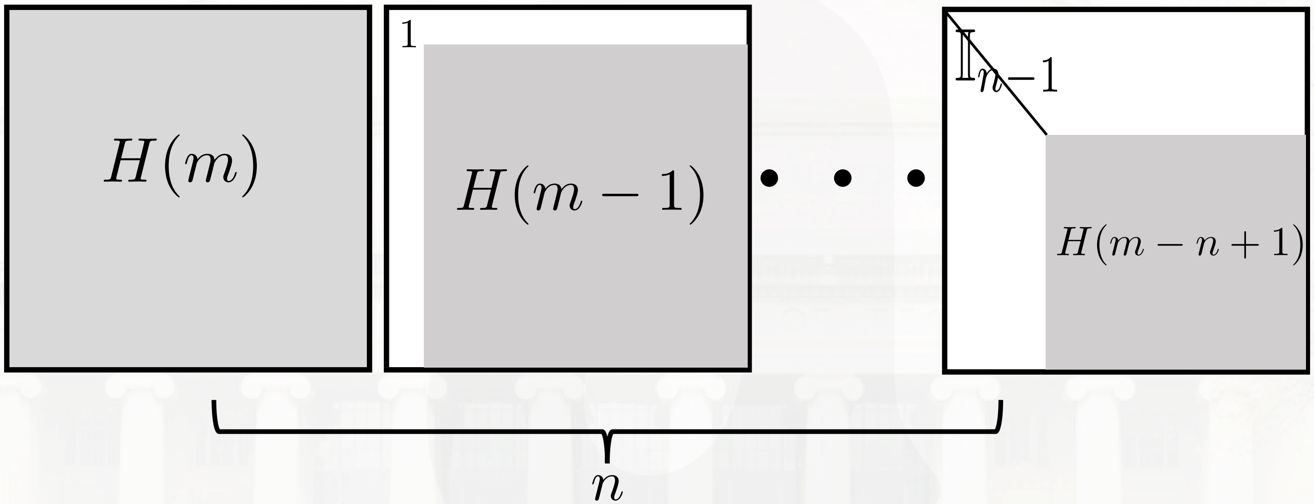
But product of Householders
Is much older

```
In [5]: Q * randn(4)

LoadError: DimensionMismatch("vector must have length either 5 or 3")
while loading In[5], in expression starting on line 1

 in * at linalg/qr.jl:173
```

# Volume of the Stiefel Manifold

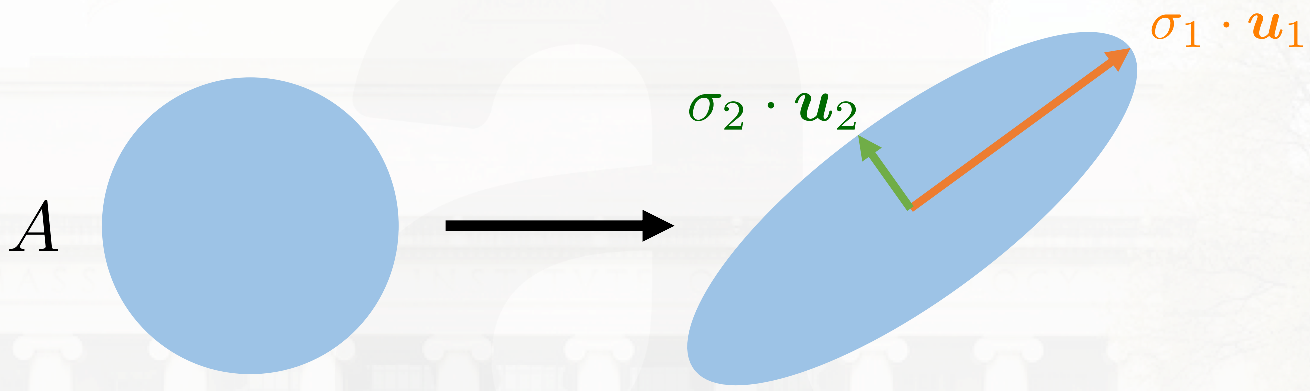


# Volume of the Stiefel Manifold




$$V_{\text{Stiefel}}(m,n) = \prod_{i=1}^{n} V_{\text{Sphere}}(m-i+1)$$

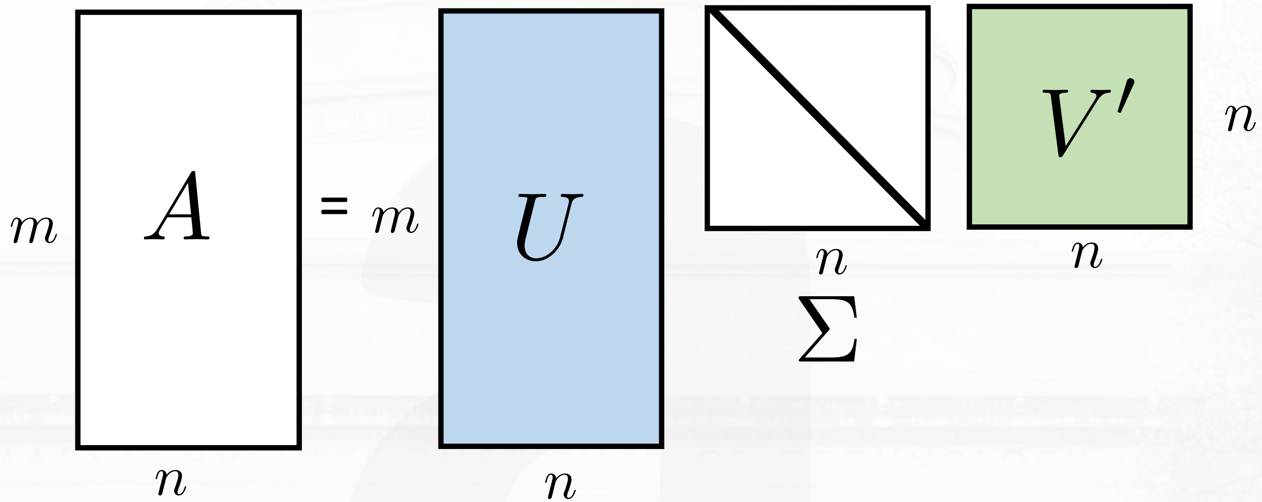
$$A = U\Sigma V'$$




**AVOID** square root eigs of $AA'$ !

- Laguerre $A =$ `svdvals(randn(m, n))`

$$A = U\Sigma V'$$

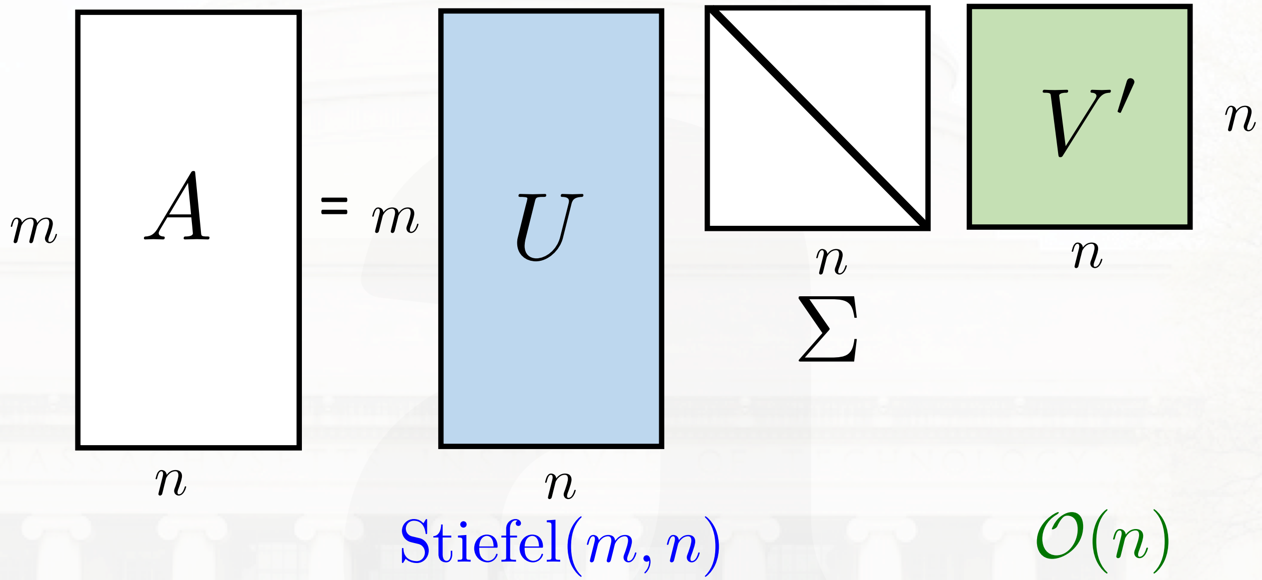



$\text{Stiefel}(m,n)$ $\mathcal{O}(n)$

Given a set of distinct singular values $\Sigma$

$$\text{Volume}\,\{A : A \in \mathbb{R}^{m,n}, \texttt{svdvals}(A) = \Sigma\} = ?$$

$$A = U\Sigma V'$$

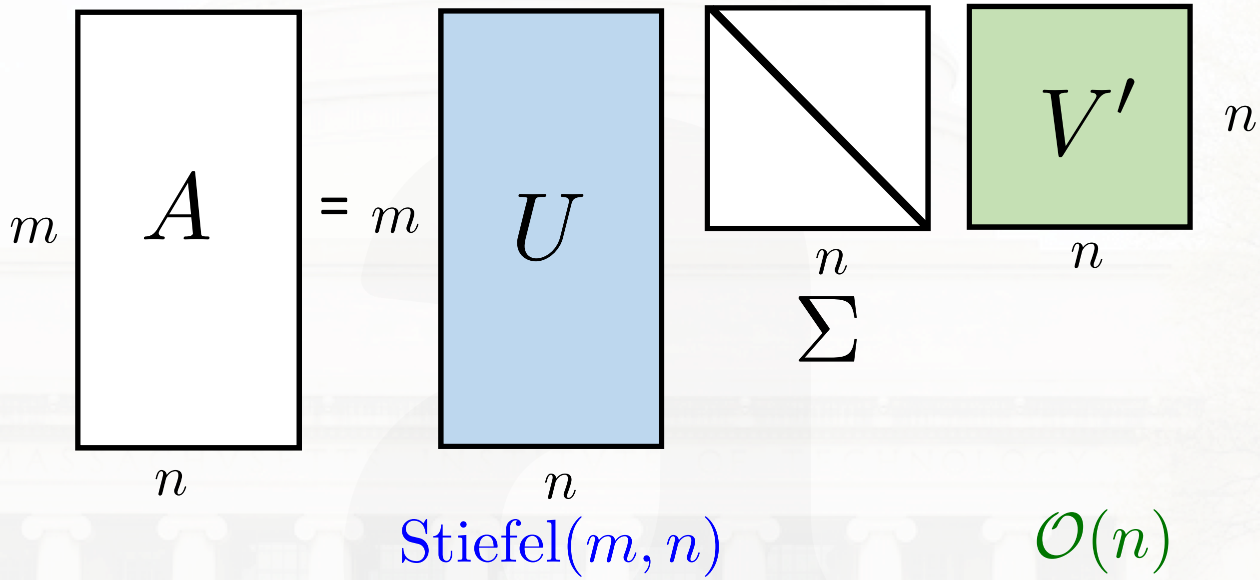



Given a set of distinct singular values $\Sigma$

$$\text{Volume}\,\{A : A \in \mathbb{R}^{m,n}, \texttt{svdvals}(A) = \Sigma\} = ?$$

$$V_{\text{Stiefel}}(m, n) \cdot V_{\text{Orthogonal}}(n)$$

$$A = U\Sigma V'$$

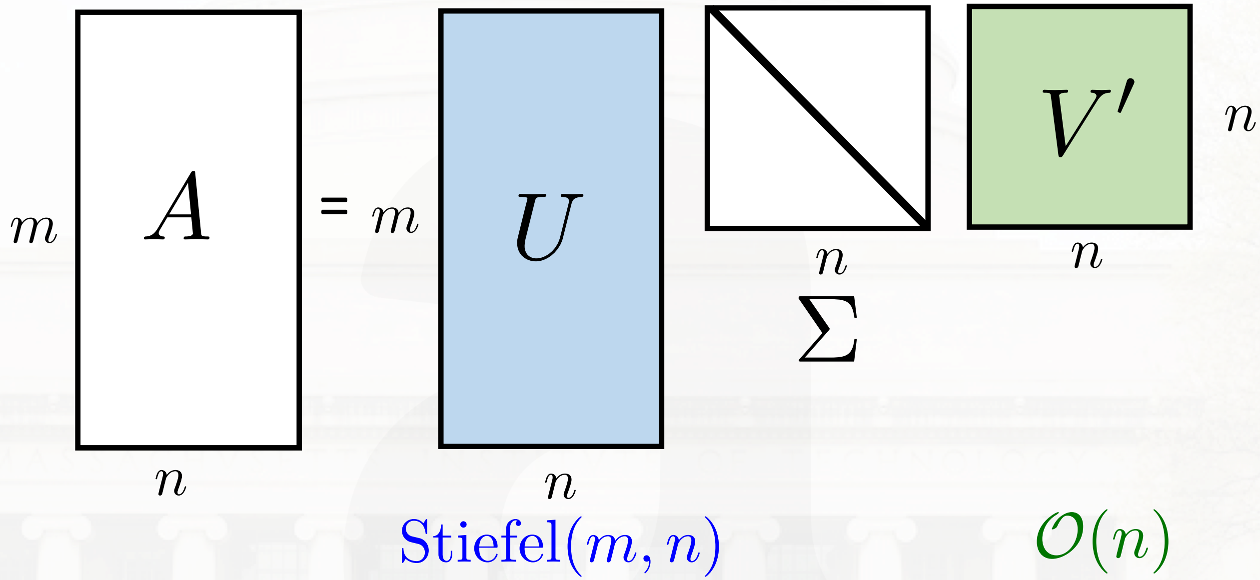



Given a set of distinct singular values $\Sigma$

$$\text{Volume}\,\{A : A \in \mathbb{R}^{m,n}, \texttt{svdvals}(A) = \Sigma\} = \,?$$

$$V_{\text{Stiefel}}(m,n) \cdot V_{\text{Orthogonal}}(n)\;/2^n$$

$$A = U\Sigma V'$$

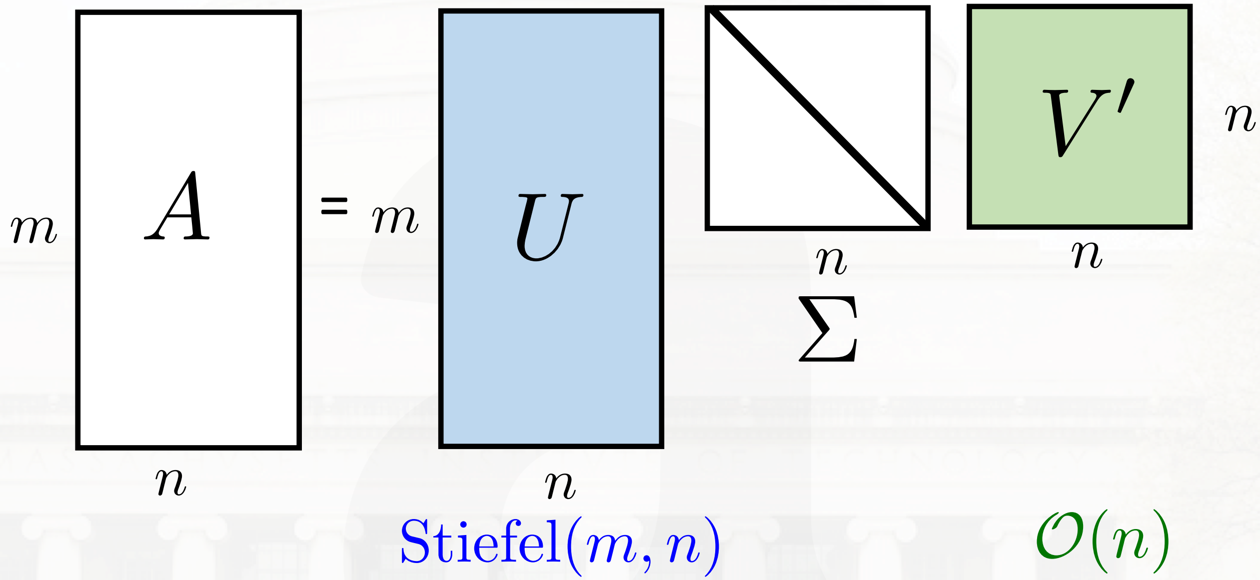



Given a set of distinct singular values $\Sigma$

$$\text{Volume}\,\{A : A \in \mathbb{R}^{m,n}, \texttt{svdvals}(A) = \Sigma\} = \,?$$

$$V_{\text{Stiefel}}(m,n) \cdot V_{\text{Orthogonal}}(n) \;/ V_{\text{sphere}}(1)^n$$

$$A = U\Sigma V'$$




Given a set of distinct singular values $\Sigma$

$$\text{Volume}\,\{A : A \in \mathbb{R}^{m,n}, \texttt{svdvals}(A) = \Sigma\} = \,?$$

$$V_{\text{Stiefel}}(m,n) \cdot V_{\text{Orthogonal}}(n) / V_{\text{phases}}(n)$$

# Volume Cheat Sheet ($\beta = 1$)

$$V_{\text{sphere}}(n) = 2\pi^{n/2}/\Gamma(n/2)$$

$$V_{\text{phases}}(n) = V_{\text{sphere}}(1)^n \ (= 2^n)$$

$$V_{\text{Stiefel}}(m,n) = \prod_{i=1}^{n} V_{\text{sphere}}(m-i+1)$$

$$V_{\text{orthogonal}}(n) = V_{\text{Stiefel}}(n,n)$$

# SVD Jacobian

The Players

$$A = U\Sigma V'$$

$$\widetilde{U} = [U, U^{\perp}] \in \mathcal{O}(m)$$

$$V \in \mathcal{O}(n)$$

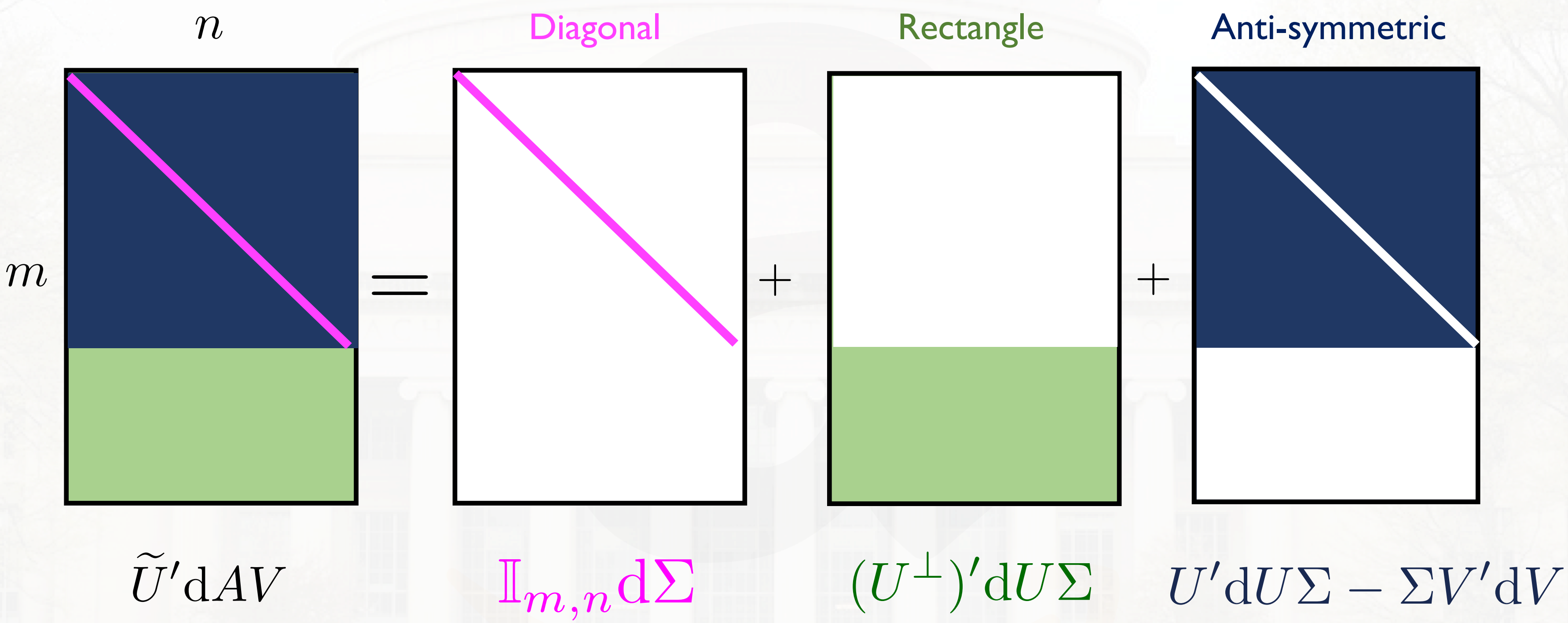
The Differential

$$\mathrm{d}A = U\mathrm{d}\Sigma V' + \mathrm{d}U\Sigma V' + U\Sigma \mathrm{d}V'$$

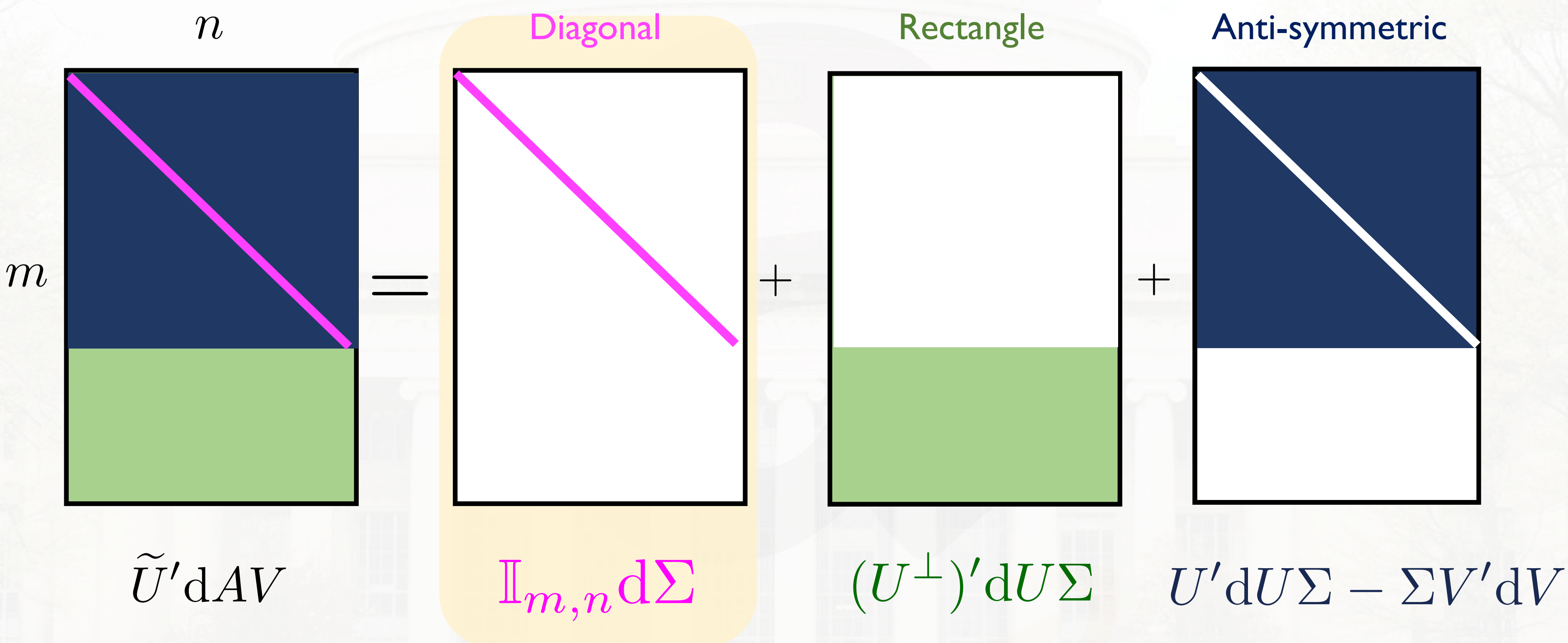
$$(\widetilde{U}'\mathrm{d}AV)^{\wedge} = (\mathrm{d}A)^{\wedge}$$

$$\widetilde{U}'\mathrm{d}AV = \mathbb{I}_{m,n}\mathrm{d}\Sigma + \widetilde{U}'\mathrm{d}U\Sigma - \mathbb{I}_{m,n}\Sigma V'\mathrm{d}V$$

$$\widetilde{U}'\mathrm{d}AV = \mathbb{I}_{m,n}\mathrm{d}\Sigma + \widetilde{U}'\mathrm{d}U\Sigma - \mathbb{I}_{m,n}\Sigma V'\mathrm{d}V$$

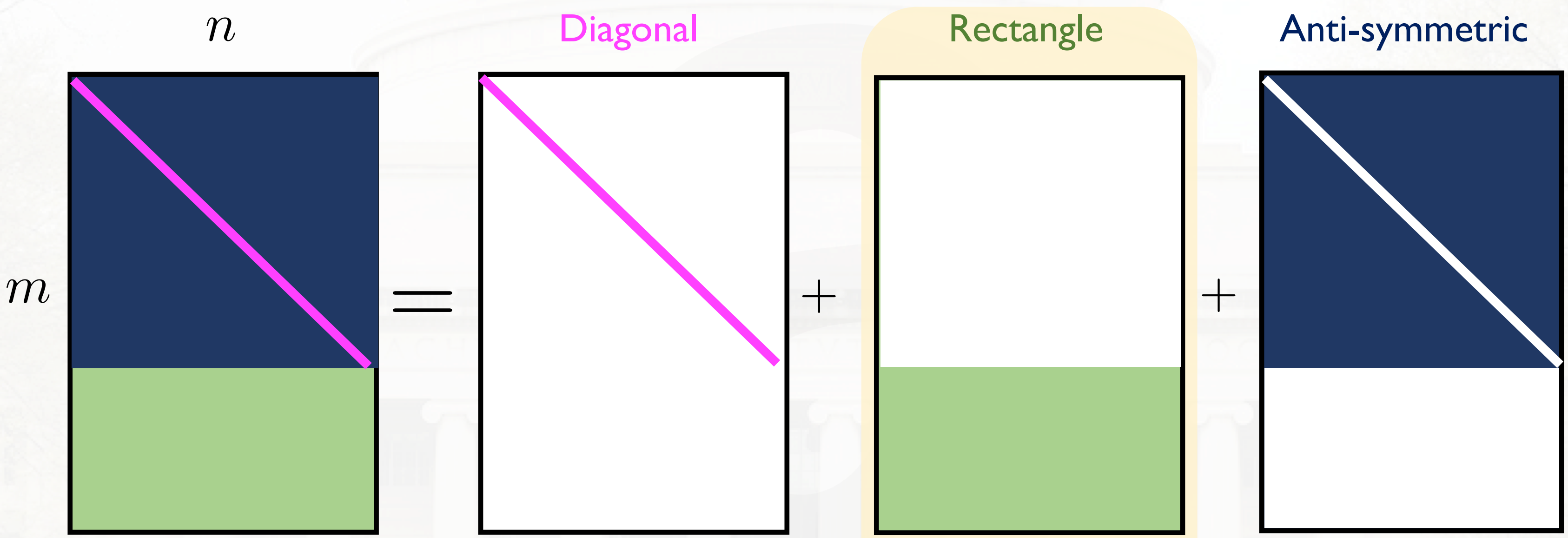


$$\widetilde{U}'\mathrm{d}AV = \mathbb{I}_{m,n}\mathrm{d}\Sigma + \widetilde{U}'\mathrm{d}U\Sigma - \mathbb{I}_{m,n}\Sigma V'\mathrm{d}V$$




$$(\mathbb{I}_{m,n}\mathrm{d}\Sigma)^{\wedge} = \mathrm{d}\sigma_1 \wedge \cdots \wedge \mathrm{d}\sigma_n = (\mathrm{d}\Sigma)^{\wedge}$$

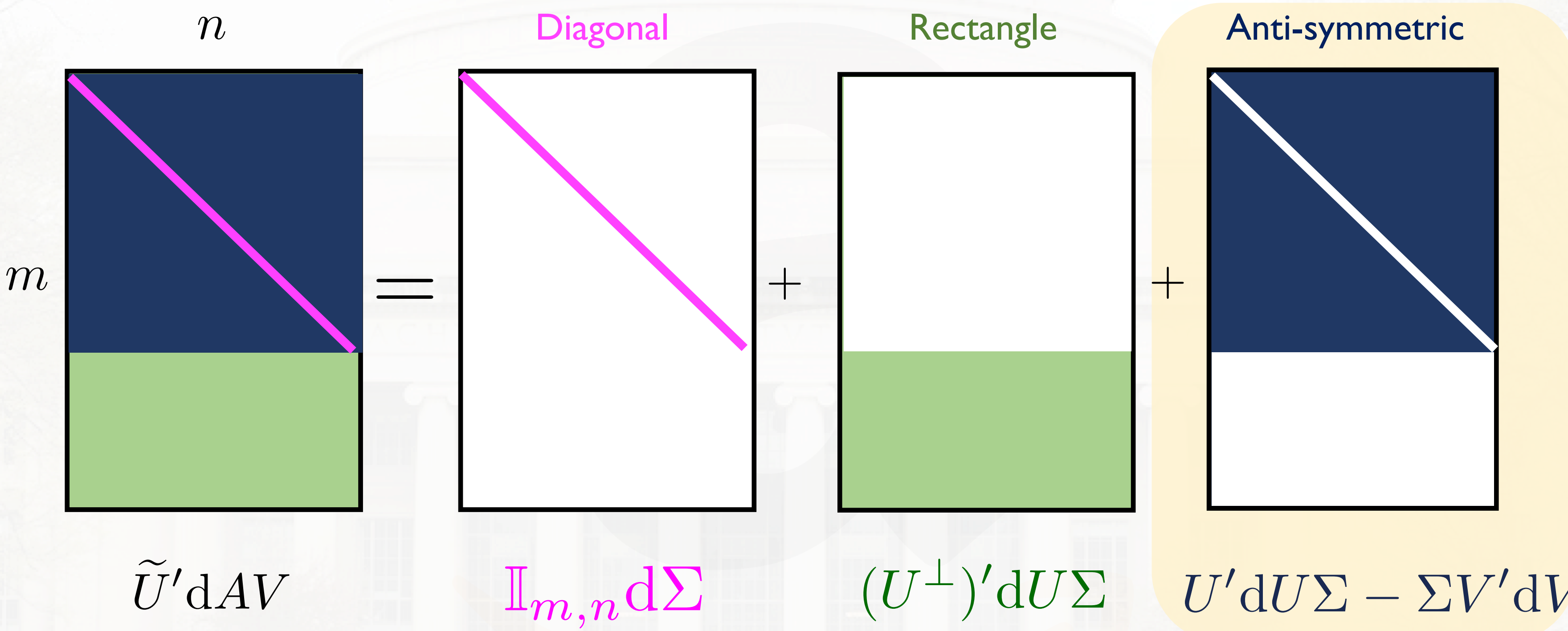
$$\widetilde{U}'\mathrm{d}AV = \mathbb{I}_{m,n}\mathrm{d}\Sigma + \widetilde{U}'\mathrm{d}U\Sigma - \mathbb{I}_{m,n}\Sigma V'\mathrm{d}V$$




$\widetilde{U}'\mathrm{d}AV$ | $\mathbb{I}_{m,n}\mathrm{d}\Sigma$ | $(U^{\perp})'\mathrm{d}U\Sigma$ | $U'\mathrm{d}U\Sigma - \Sigma V'\mathrm{d}V$

$$((U^{\perp})'\mathrm{d}U\Sigma)^{\wedge} = \prod_i \sigma_i^{m-n}((U^{\perp})'\mathrm{d}U)^{\wedge}$$

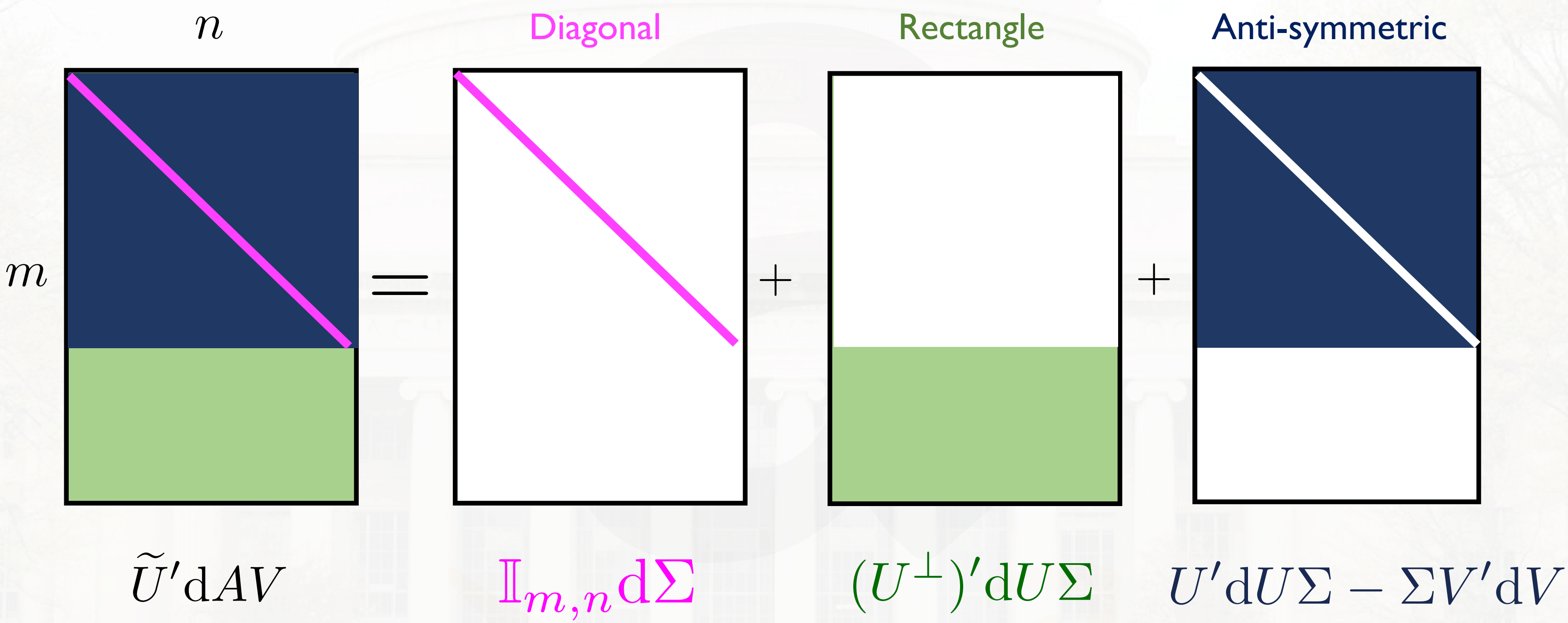
$$\widetilde{U}'\mathrm{d}AV = \mathbb{I}_{m,n}\mathrm{d}\Sigma + \widetilde{U}'\mathrm{d}U\Sigma - \mathbb{I}_{m,n}\Sigma V'\mathrm{d}V$$




$$(U'\mathrm{d}U\Sigma - \Sigma V'\mathrm{d}V)^{\wedge} = \prod_{i<j}(\sigma_i^2 - \sigma_j^2)(U'\mathrm{d}U)^{\wedge}(V'\mathrm{d}V)^{\wedge}$$

Diagonal 0 when $\beta = 1$

$$\widetilde{U}'\mathrm{d}AV = \mathbb{I}_{m,n}\mathrm{d}\Sigma + \widetilde{U}'\mathrm{d}U\Sigma - \mathbb{I}_{m,n}\Sigma V'\mathrm{d}V$$



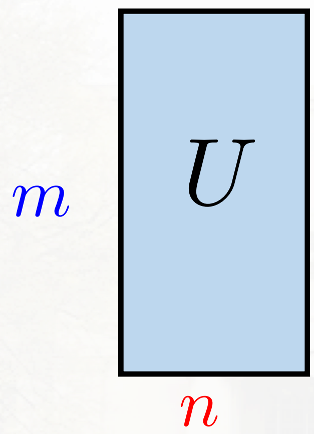

**Laguerre** $$\mathrm{d}A = \prod_{i<j}(\sigma_i^2 - \sigma_j^2)\prod_{i=1}^{n}\sigma_i^{m-n}(\widetilde{U}'\mathrm{d}U)(V'\mathrm{d}V)\mathrm{d}\Sigma$$

# Integration over Stiefel Manifold

$$V_{\text{Stiefel}}(m, n) \cdot V_{\text{Orthogonal}}(n) / V_{\text{phases}}(n)$$

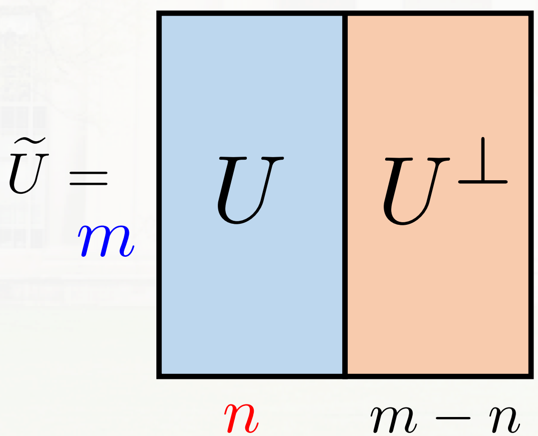
# *Stiefel Manifold : Encodings*

**set of n-dim (orthonormal) frames in $\mathbb{R}^m$ $n \leqslant m$**




Tall Skinny

$$U'U = \mathbb{I}_n$$



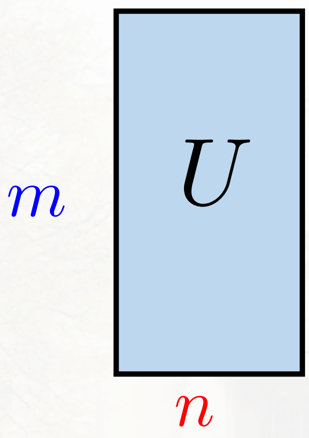

Full orthogonal

Equivalence Class = $\left\{ \widetilde{U} \begin{bmatrix} \mathbb{I}_n & \\ & W_2 \end{bmatrix} \right\}$

$$W_2 \in \mathcal{O}(m-n)$$

# *Grassmann Manifold : Encodings*

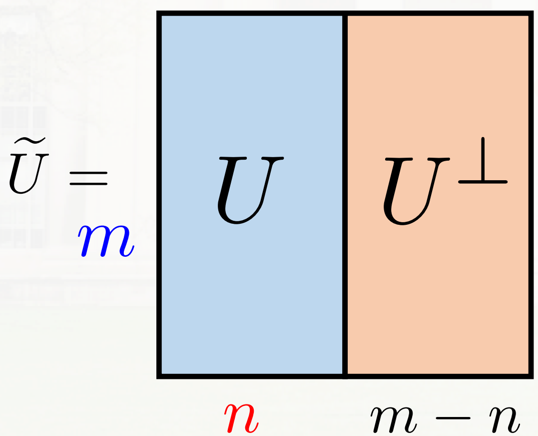
**set of n-dim subspaces in** $\mathbb{R}^m$ $n \leqslant m$




$$U'U = \mathbb{I}_n$$

Equivalence Class = $\{UW : W \in \mathcal{O}(n)\}$

Tall Skinny




Equivalence Class = $\left\{ \widetilde{U} \begin{bmatrix} W_1 & \\ & W_2 \end{bmatrix} \right\}$

$$W_1 \in \mathcal{O}(n), \quad W_2 \in \mathcal{O}(m-n)$$

Full orthogonal

# Volume Cheat Sheet ($\beta = 1$)

$$V_{\text{sphere}}(n) = 2\pi^{n/2}/\Gamma(n/2)$$

$$V_{\text{phases}}(n) = V_{\text{sphere}}(1)^n \;\; (= 2^n)$$

$$V_{\text{Stiefel}}(m,n) = \prod_{i=1}^{n} V_{\text{sphere}}(m-i+1)$$

$$V_{\text{orthogonal}}(n) = V_{\text{Stiefel}}(n,n)$$

$$V_{\text{Grassmann}}(m,n) = V_{\text{Stiefel}}(m,n)/V_{\text{orthogonal}}(n)$$

# Volume Cheat Sheet

$$V_{\text{sphere}}(n;\boldsymbol{\beta}) = 2\pi^{\boldsymbol{\beta}\cdot n/2}/\Gamma(\boldsymbol{\beta}\cdot n/2)$$

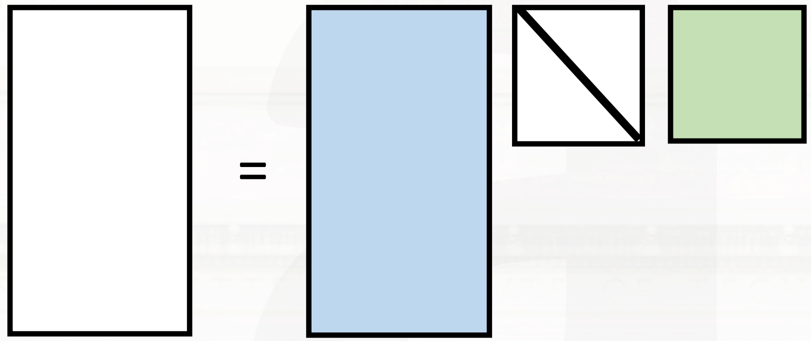
$$V_{\text{phases}}(n;\boldsymbol{\beta}) = V_{\text{sphere}}(1;\boldsymbol{\beta})^n$$

$$V_{\text{Stiefel}}(m,n;\boldsymbol{\beta}) = \prod_{i=1}^{n} V_{\text{sphere}}(m-i+1;\boldsymbol{\beta})$$

$$V_{\text{orthogonal}}(n;\boldsymbol{\beta}) = V_{\text{Stiefel}}(n,n;\boldsymbol{\beta})$$

$$V_{\text{Grassmann}}(m,n;\boldsymbol{\beta}) = V_{\text{Stiefel}}(m,n;\boldsymbol{\beta})/V_{\text{orthogonal}}(n;\boldsymbol{\beta})$$

# Constant in Integration for Laguerre Ensemble for general **beta**

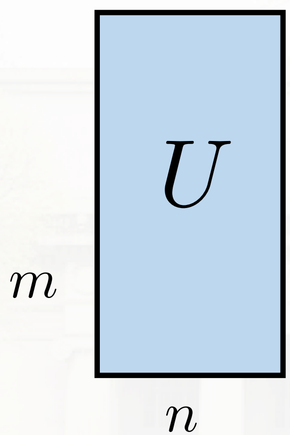



$\text{Stiefel}(m, n; \beta)$ $\mathcal{O}(n; \beta)$

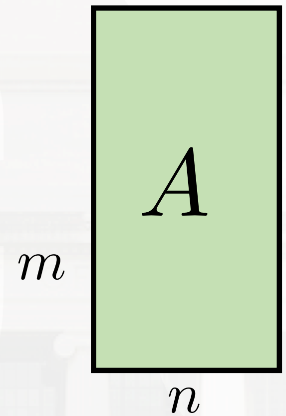
$$V_{\text{Stiefel}}(m, n; \beta) \times V_{\text{orthogonal}}(n; \beta) / V_{\text{phases}}(n; \beta)$$

only tricky part is the $\beta - 1$ symmetric matrices in the differential and only the diagonals

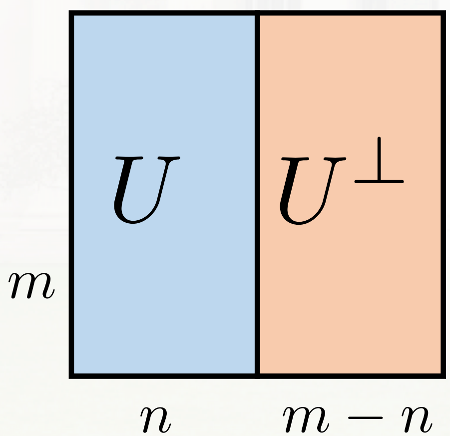
# Grassmann with Matrices




$$U'U = \mathbb{I}_n$$

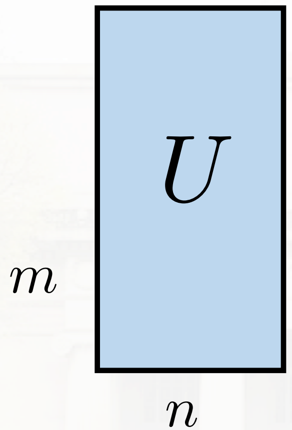



$$\mathrm{rank}(A) = n$$

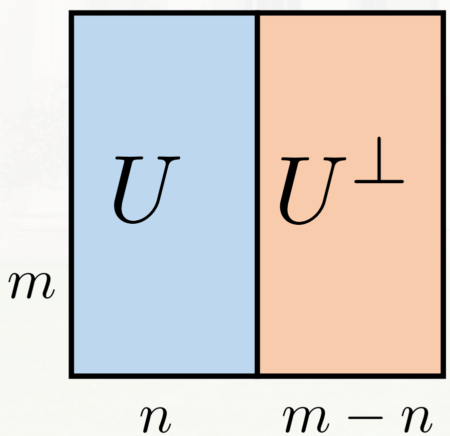


# Grassmann with Matrices

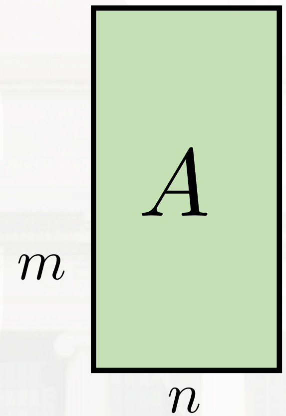
## Orthogonal Representations




$$U'U = \mathbb{I}_n$$



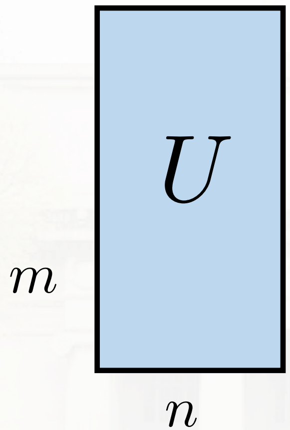

## General Representations



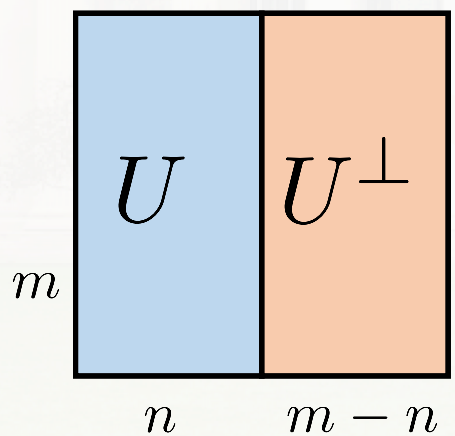

$$\text{rank}(A) = n$$

# Grassmann with Matrices

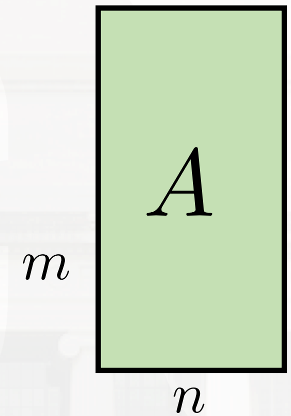
## Orthogonal Representations




$$U'U = \mathbb{I}_n$$




## General Representations




$$\text{rank}(A) = n$$

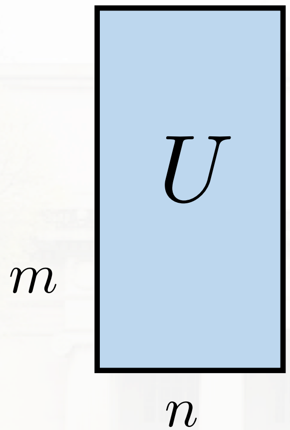
**Uniform Random Subspace**

$A = \texttt{randn(m,n)}$

$$\text{span}\left\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\right\}$$

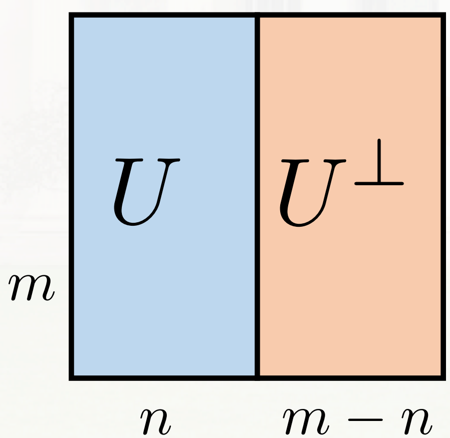
# Grassmann with Matrices

## Orthogonal Representations



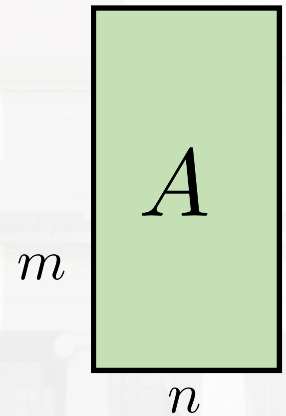

$$U'U = \mathbb{I}_n$$

Partial CS
Or Partial GSVD




CS

## General Representations




$$\text{rank}(A) = n$$

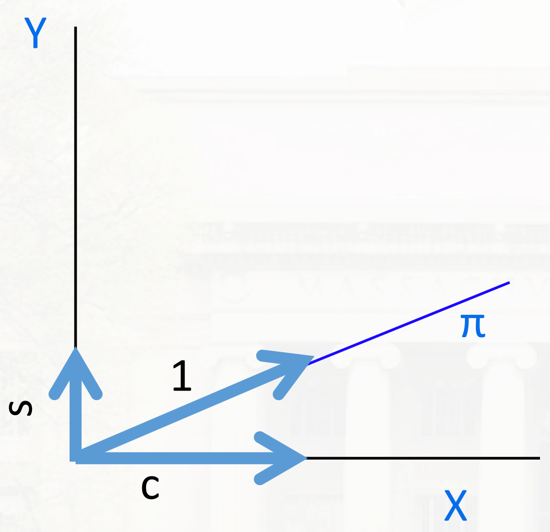
**Uniform**
**Random Subspace**

$$A = \texttt{randn(m,n)}$$

$$\text{span}\,\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$$

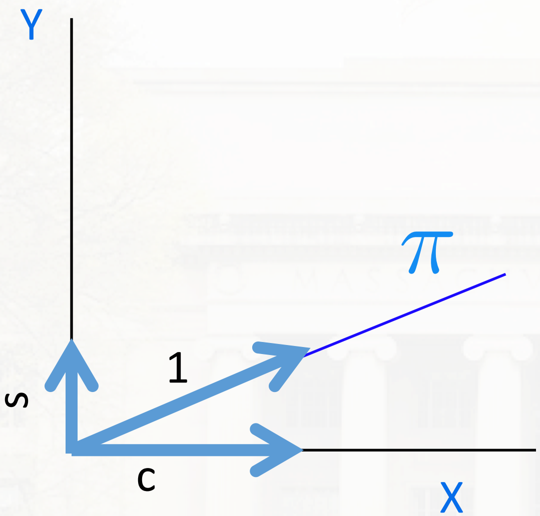
GSVD

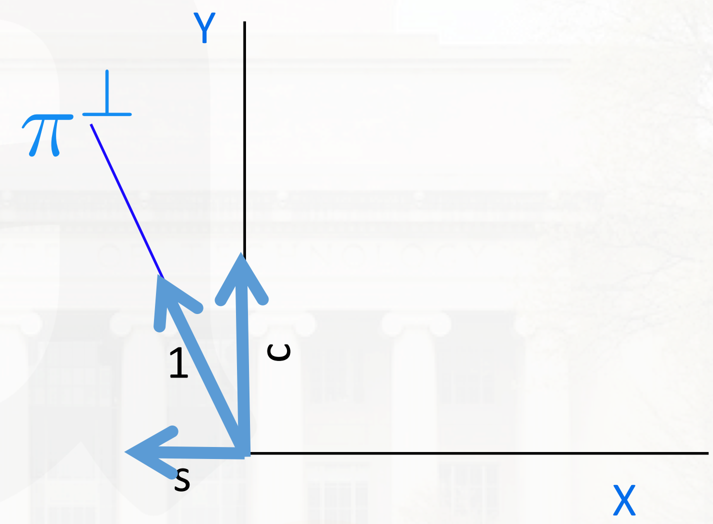
# In $\mathbb{R}^2$ a subspace is given by its angle with the $x$ and $y$ axes




Or equivalently by a (c,s) "cosine-sin" pair with c,s non-negative, and + or - directions on the axes.

# In $\mathbb{R}^2$ a subspace is given by its angle with the $x$ and $y$ axes




Or equivalently by a (c,s) "cosine-sin" pair with c,s non-negative, and + or - directions on the axes. The orthogonal subspace yields the reverse pair (s,c).

# In $\mathbb{R}^m$ we can write $m = m_1 + m_2$ and obtain principal angles between *X* and *Y*

A point $\boldsymbol{\pi}$ on the Grassmann manifold is

$$W_\pi = \begin{bmatrix} U & \\ & V \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} = \begin{bmatrix} UC \\ VS \end{bmatrix}$$

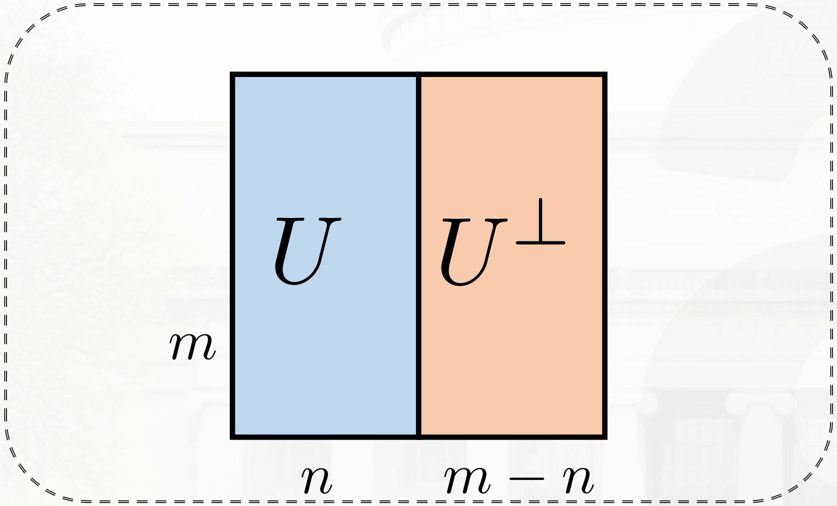
Unique up to column signs/phases

$$U \in \text{Stiefel}(m_1, n), \quad V \in \text{Stiefel}(m_2, n)$$

$$C = \text{diag}(c_1, \cdots, c_n)$$
$$S = \text{diag}(s_1, \cdots, s_n)$$

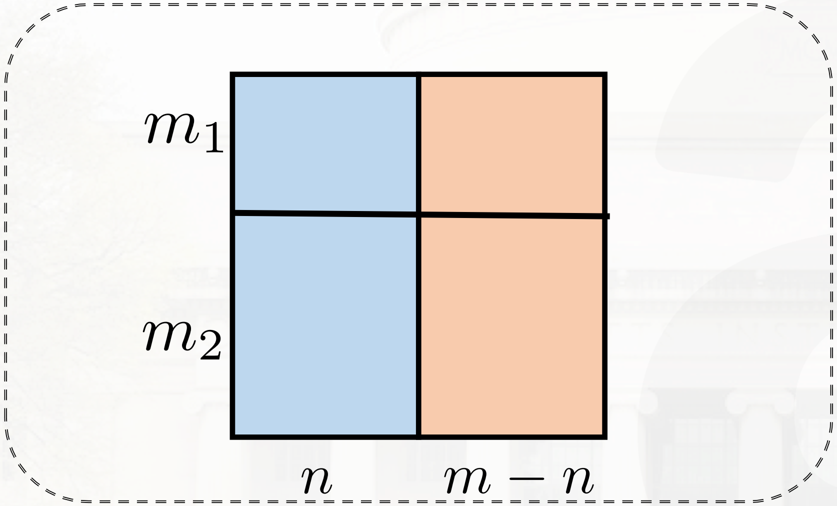
$$c_i^2 + s_i^2 = 1$$

# The CS Decomposition



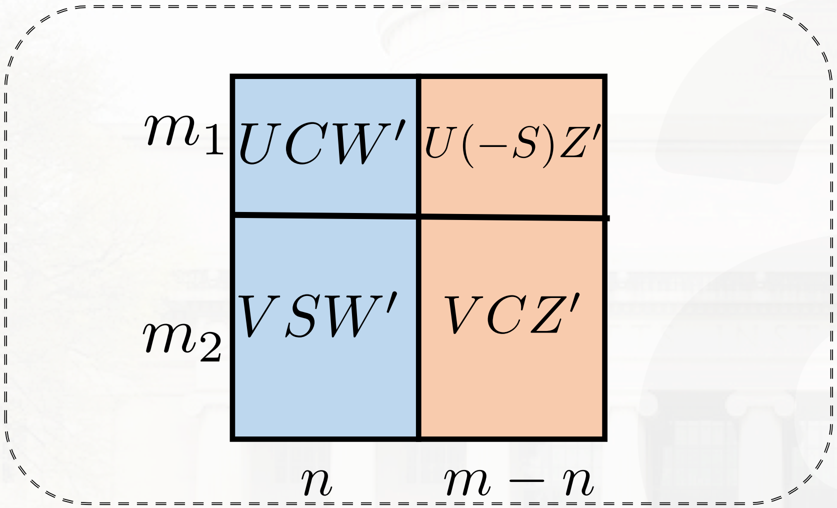

n dimensional subspace of $\mathbb{R}^m$

# The CS Decomposition




n dimensional subspace of $\mathbb{R}^m$
X=first $m_1$ coordinate dimensions
Y=last $m_2$ coordinate dimensions

# The CS Decomposition




n dimensional subspace of $\mathbb{R}^m$
X=first $m_1$ coordinate dimensions
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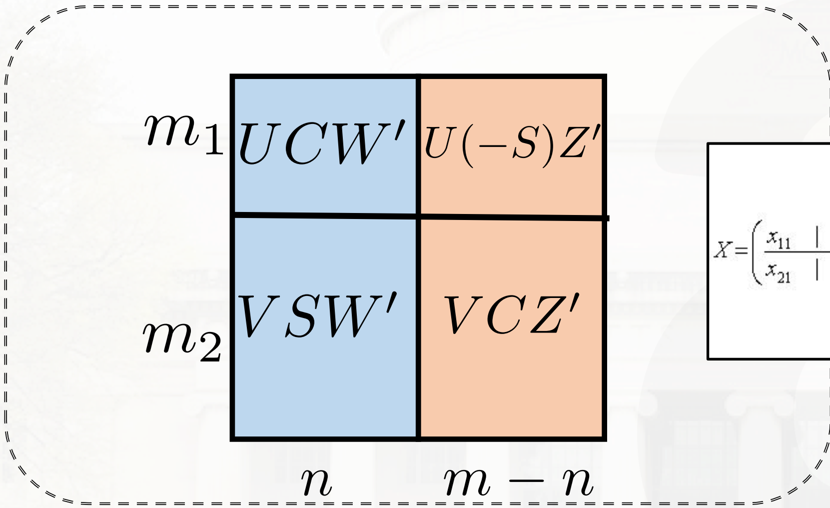
# The CS Decomposition




$$X=\begin{pmatrix} x_{11} & \vert & x_{12} \\ \hline x_{21} & \vert & x_{22} \end{pmatrix}=\begin{pmatrix} u_1 & \vert & \\ \hline & \vert & u_2 \end{pmatrix}\left(\begin{array}{ccc|ccc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & -S & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \\ \hline 0 & 0 & 0 & I & 0 & 0 \\ 0 & S & 0 & 0 & C & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{array}\right)\begin{pmatrix} v_1 & \vert & \\ \hline & \vert & v_2 \end{pmatrix}^T$$

n dimensional subspace of $\mathbb{R}^m$

X=first $m_1$ coordinate dimensions

Y=last $m_2$ coordinate dimensions

# GSVD Formulation (non-orthogonal)

**Description**

```
[U,V,X,C,S] = gsvd(A,B)
```

that

```
A = U*C*X'
B = V*S*X'
C'*C + S'*S = I
```

- e=gsvd(A,B)

$$\left[\begin{array}{c} A \\ \hline B \end{array}\right] = \left[\begin{array}{c|c} U & \\ \hline & V \end{array}\right] \left[\begin{array}{c} C \\ \hline S \end{array}\right] X'$$

$$U^T U = \mathbb{I}$$

$$V^T V = \mathbb{I}$$

$$C^2 + S^2 = \mathbb{I}$$

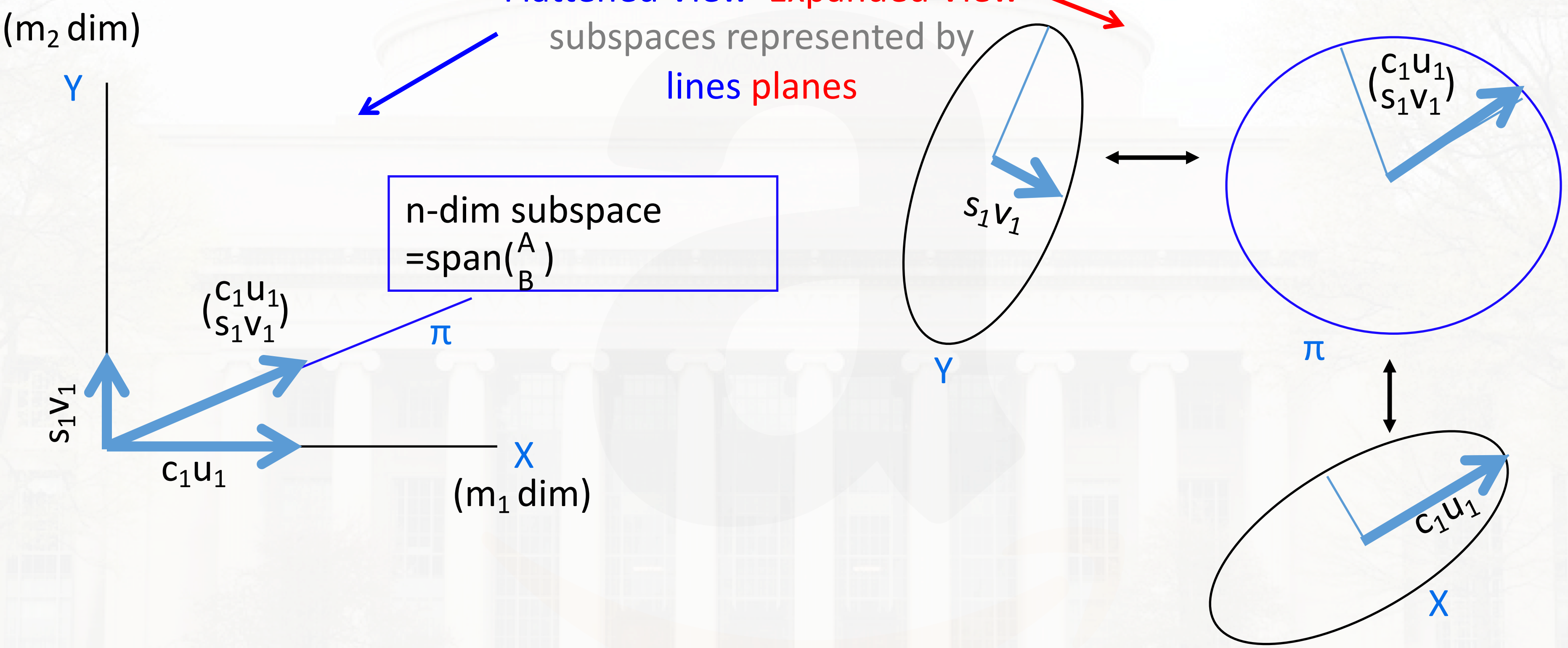
$$(A'A + B'B)^{-1} A'A \sim C^2$$

# GSVD(A,B)

A,B have n columns

$m = m_1 + m_2$ dimensions
$n \le m$

Flattened View Expanded View
subspaces represented by
lines planes




Ex 1: Random line in $R^2$ through 0:
On the x axis: c
On the y axis: s

Ex 2: Random plane in $R^4$ through 0:
On xy plane: c1,c2
On zw plane: s1,s2

# GSVD(A,B)

A,B have n columns

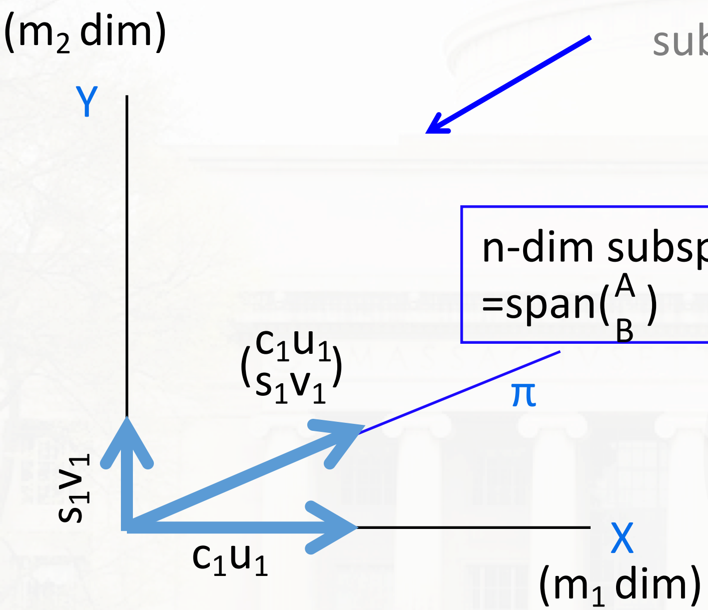
$m = m_1 + m_2$ dimensions
$n \le m$

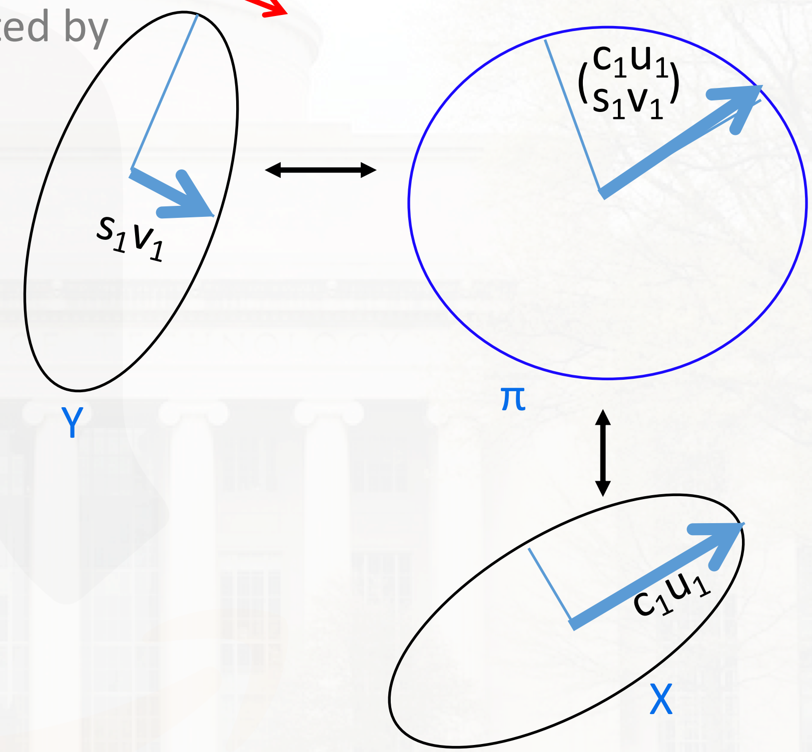
Flattened View Expanded View

subspaces represented by

lines planes

n-dim subspace
$=\text{span}\begin{pmatrix} A \\ B \end{pmatrix}$




Ex 3: Random line in $R^3$ through 0:
On the xy plane: c and 0
On the z axis: s

Ex 4:
Random plane in $R^3$ through 0:
On the xy plane: c and 1 (one axis in the xy plane)
On the z axis: s

# GSVD Jacobian

The Players

$$W_\pi = \begin{bmatrix} UC \\ VS \end{bmatrix} \qquad W_\pi^\perp = \begin{bmatrix} -US & U^\perp & \\ VC & & V^\perp \end{bmatrix}$$

# GSVD Jacobian

The Players

$$W_\pi = \begin{bmatrix} UC \\ VS \end{bmatrix} \qquad W_\pi^\perp = \begin{bmatrix} -US & U^\perp & \\ VC & & V^\perp \end{bmatrix}$$

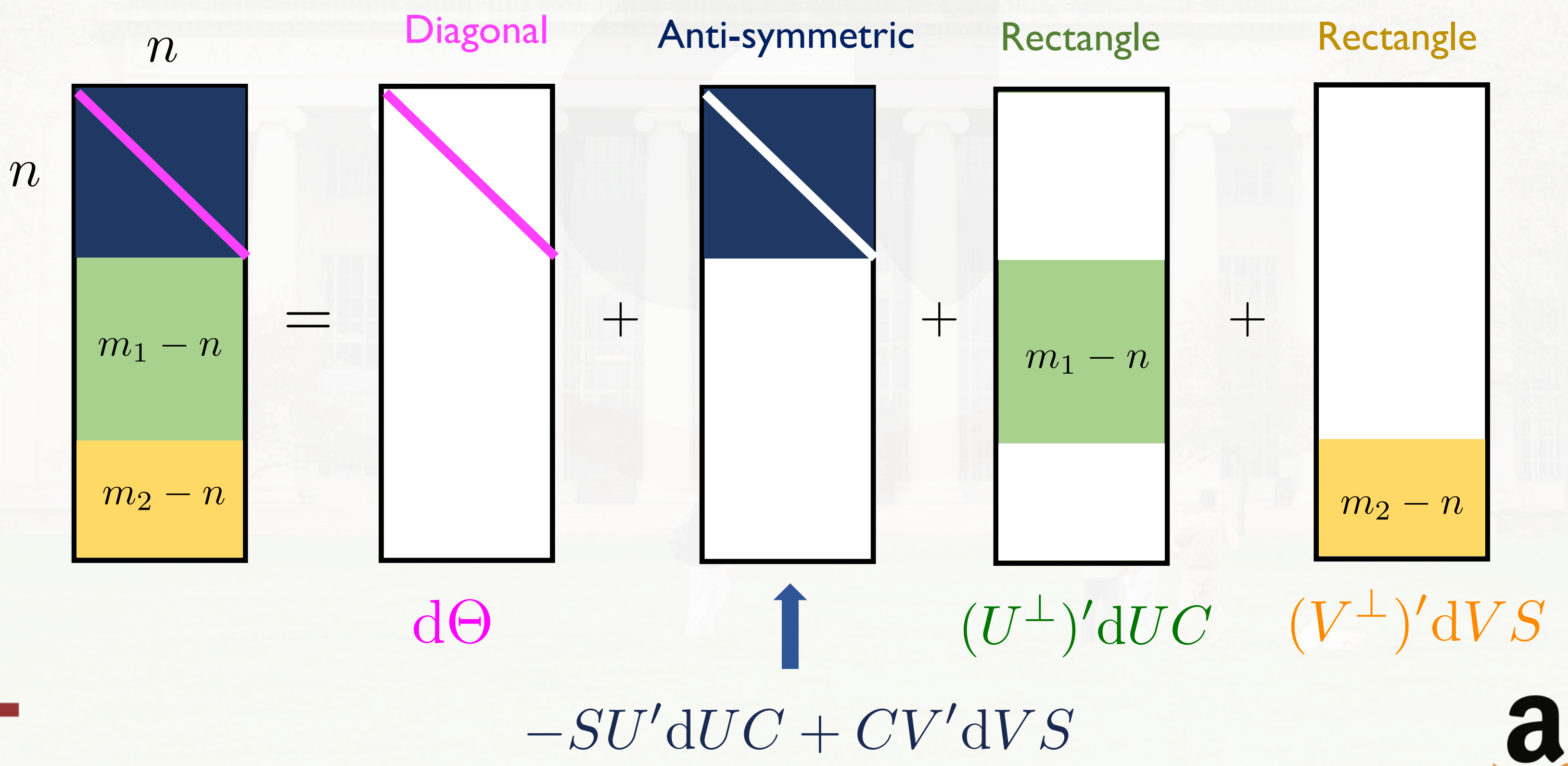
The Differential

$$(W_\pi^\perp)' \mathrm{d}W_\pi = \begin{bmatrix} -SU'\mathrm{d}UC + CV'\mathrm{d}VS + \mathrm{d}\Theta \\ (U^\perp)'\mathrm{d}UC \\ (V^\perp)'\mathrm{d}VS \end{bmatrix}$$

# GSVD Jacobian

**The Differential**

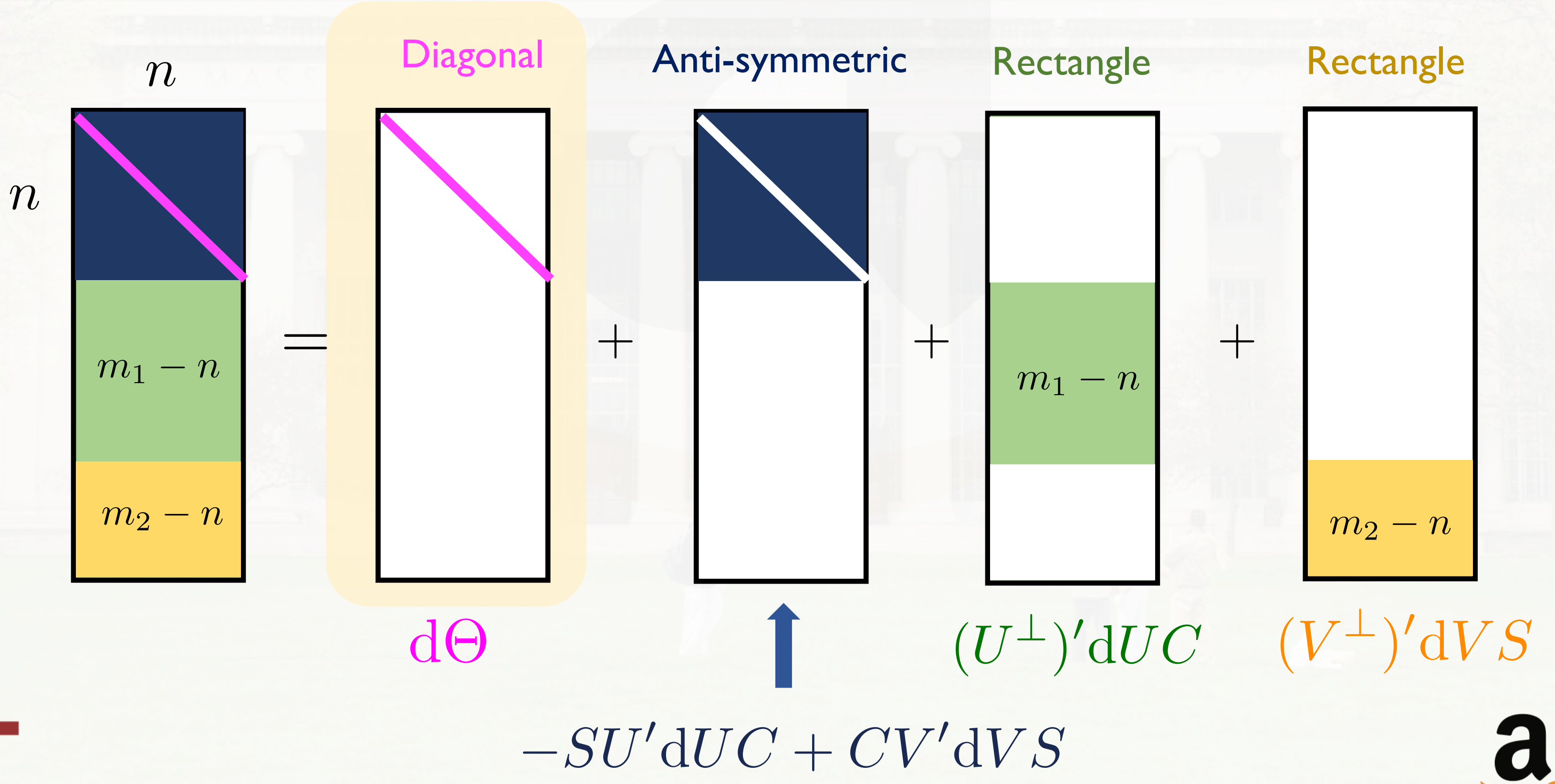
$$(W_\pi^\perp)' \mathrm{d}W_\pi = \begin{bmatrix} -SU'\mathrm{d}UC + CV'\mathrm{d}VS + \mathrm{d}\Theta \\ (U^\perp)'\mathrm{d}UC \\ (V^\perp)'\mathrm{d}VS \end{bmatrix}$$



# GSVD Jacobian

**Diagonal**

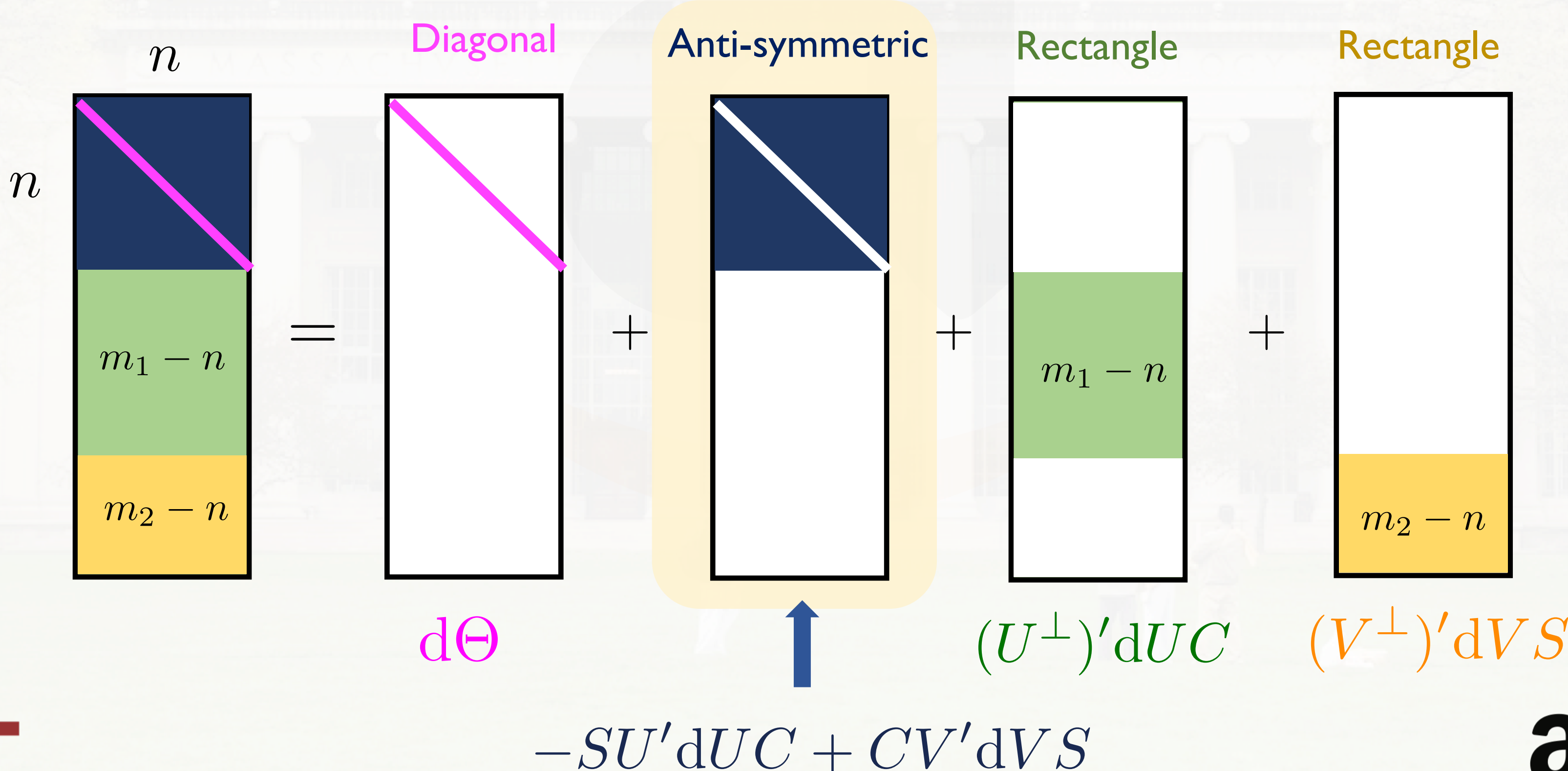
$$(\mathrm{d}\Theta)^\wedge = \mathrm{d}\theta_1 \wedge \cdots \wedge \mathrm{d}\theta_n$$



# GSVD Jacobian

**Anti-symmetric (Only for $\beta = 1$)**

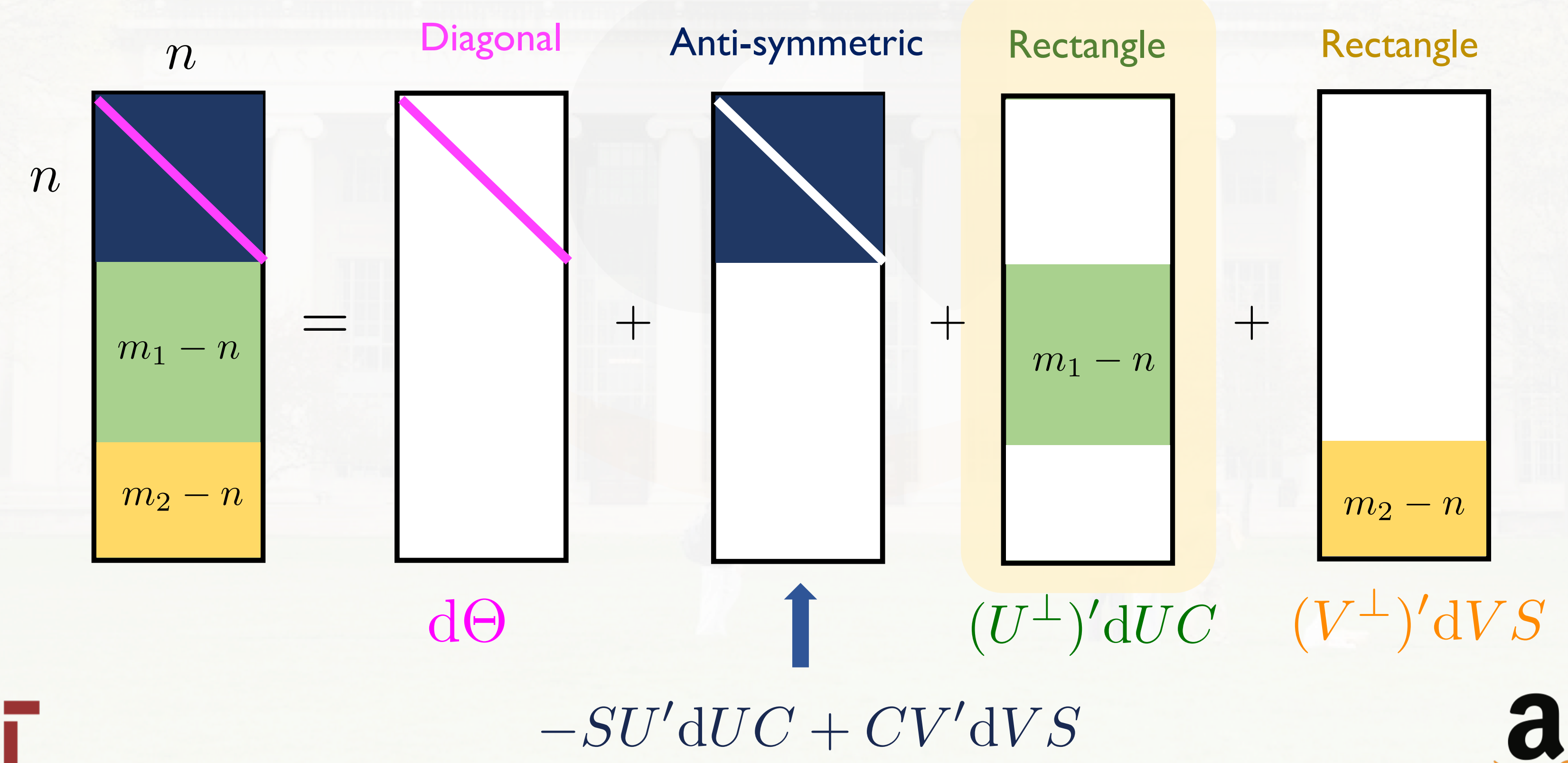
$$(-SU'\mathrm{d}UC + CV'\mathrm{d}VS)^\wedge = \prod_{i<j}(c_i^2 - c_j^2)(U'\mathrm{d}U)^\wedge(V'\mathrm{d}V)^\wedge$$



# GSVD Jacobian

**Rectangle**

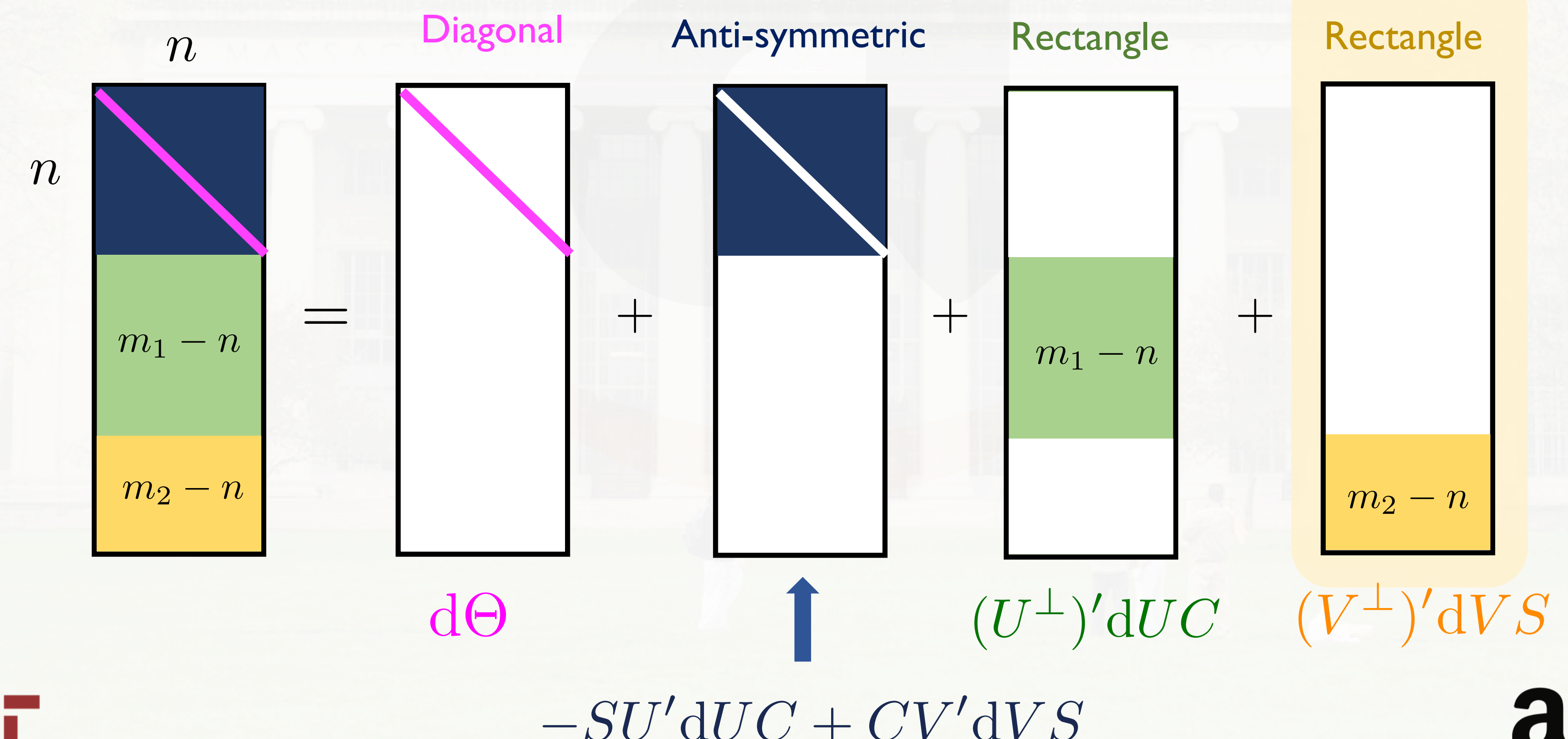
$$((U^{\perp})'\mathrm{d}UC)^{\wedge} = \prod_{i=1}^{n} c_i^{m_1-n}((U^{\perp})'\mathrm{d}U)^{\wedge}$$



# GSVD Jacobian

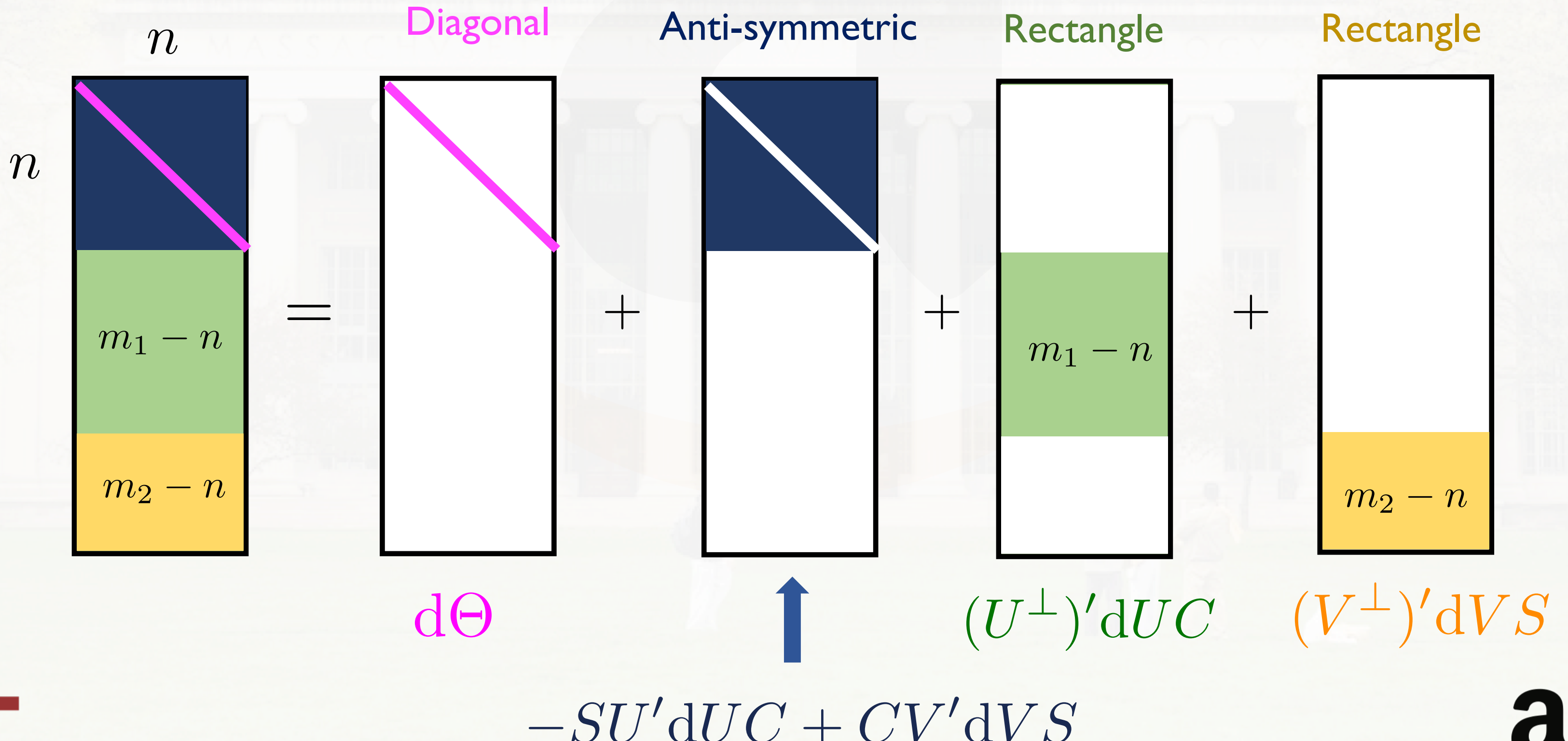
**Rectangle**

$$((V^{\perp})'\mathrm{d}VS)^{\wedge} = \prod_{i=1}^{n} s_i^{m_2-n}\,((V^{\perp})'\mathrm{d}V)^{\wedge}$$



# GSVD Jacobian

$$((W_\pi)'\mathrm{d}W_\pi)^\wedge = \prod_{i<j}(c_i^2 - c_j^2)\prod_i c_i^{m_1-n} s_i^{m_2-n}(\widetilde{U}\mathrm{d}U)^\wedge(\widetilde{V}\mathrm{d}V)^\wedge(\mathrm{d}\Theta)^\wedge$$



# What's left

- Integration Over the Stiefel Manifolds

$$\frac{V_{Stiefel}(m_1,n)V_{Stiefel}(m_2,n)}{V_{Grassmann}(m,n)V_{phases}(n)}\prod_{i<j}(c_i^2-c_j^2)\prod_i c_i^{(m_1-n)}\prod_i s_i^{(m_1-n)}\mathrm{d}\theta_i$$

- A change of variable gives the eigenvalue density in MANOVA format

# Ghost Geometry

$$\frac{V_{Stiefel}(m_1, n; \beta) V_{Stiefel}(m_2, n; \beta)}{V_{Grassmann}(m, n; \beta) V_{phases}(n; \beta)}$$

$$\times \prod_{i<j} (c_i^2 - c_j^2)^\beta \prod_i c_i^{\beta(m_1 - n) + (\beta - 1)} \prod_i s_i^{\beta(m_1 - n) + (\beta - 1)} \mathrm{d}\Theta$$

# Ghost Laguerre and Ghost Jacobi

$$\frac{V_{Stiefel}(m_1,n;\beta)V_{Stiefel}(m_2,n;\beta)}{V_{Grassmann}(m,n;\beta)V_{phases}(n;\beta)}$$
$$\times \prod_{i<j}(c_i^2 - c_j^2)^\beta \prod_i c_i^{\beta(m_1-n)+(\beta-1)} \prod_i s_i^{\beta(m_1-n)+(\beta-1)} \mathrm{d}\Theta$$

$$\frac{V_{Stiefel}(m,n;\beta)V_{orthogonal}(n,n;\beta)}{V_{phases}(n;\beta)}$$
$$\times \prod_{i<j}(\sigma_i - \sigma_j)^\beta \prod_i \sigma_i^{\beta(m-n)+(\beta-1)} \mathrm{d}\Sigma$$

Given a set of $n$ principal angles from the references hyperplane $\mathbb{R}^{m_1}$, what's the volume of the $n$-dim subspaces in $\mathbb{R}^m$ that are at that set of angles?

Given a set of $n$ principal angles from the references hyperplane $\mathbb{R}^{m_1}$, what's the volume of the $n$-dim subspaces in $\mathbb{R}^m$ that are at that set of angles?

$$\frac{V_{\text{Stiefel}}(m_1, n) V_{\text{Stiefel}}(m - m_1, n)}{V_{\text{Grassmann}}(m, n) V_{\text{phases}}(n)}$$

# Selberg Integral

$$S_n(\alpha,\beta,\gamma) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2\gamma}\, dt_1 \cdots dt_n$$

$$= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(1+\gamma)}$$

Upon variable change gives exactly the constant

$$\frac{V_{\text{Stiefel}}(m_1, n; \beta) V_{\text{Stiefel}}(m-m_1, n; \beta)}{V_{\text{Grassmann}}(m, n; \beta) V_{\text{phases}}(n; \beta)}$$

# Summing it all up

- GSVD more natural than MANOVA formulation
- One can write Jacobians for Hermite, Laguerre, Jacobi for all beta as if full models exist
- The ghost Stiefel and Grassmann volumes are exactly the right constants in the traditional pdf’s
- All terms in integrand have a natural ghost interpretation
- Selberg Integrals, Jacobians, Differential Geometry all love ghosts

# Summing it all up

- GSVD … mulation
- One … Laguerre, Jacob … st
- The g … mes are exact … tional pdf's


- All terms in integrand have a natural ghost interpretation
- Selberg Integrals, Jacobians, Differential Geometry all love ghosts