A Request: The Painlevé Project

In recent years the Painlevé equations, particularly the six Painlevé transcendents PI,…,PVI, have emerged as the core of modern special function theory. In the eighteenth and nineteenth centuries, the classical special functions such as the Bessel functions, the Airy function, the Legendre functions, the hypergeometric functions, and so on, were recognized and developed in response to the problems of the day in electromagnetism, acoustics, hydrodynamics, elasticity, and many other areas. In the same way, around the middle of the twentieth century, as science and engineering continued to expand in new directions, a new class of functions, the Painlevé functions, started to appear in applications. The list of problems now known to be described by the Painlevé equations is large, varied, and expanding rapidly. The list includes, at one end, the scattering of neutrons off heavy nuclei, and at the other, the statistics of the zeros of the Riemann-zeta function on the critical line Re $z = 1/2$. And in between, among many others, there is random matrix theory, the asymptotic theory of orthogonal polynomials, self-similar solutions of integrable equations, combinatorial problems such as Ulam's longest increasing subsequence problem, tiling problems, multivariate statistics in the important asymptotic regime where the number of variables and the number of samples are comparable and large, and also random particle systems.

Random matrix theory is a common portal for the Painlevé equations into science and engineering. A striking recent example is the discovery that the statistics of the bus delivery system in Cuernavaca, Mexico, is described with remarkable accuracy by random matrix theory: The bus delivery system in Cuernavaca has particular distinguishing features and is a prototype for the bus transportation system in many cities in Latin America. Equally striking is the recent experimental verification of random matrix behavior in turbulent liquid crystal growth.

Over the years, the properties of the classical special functions—algebraic, analytical, asymptotic, and numerical—have been organized and tabulated in various handbooks such as the Bateman Project or the National Bureau of Standards *Handbook of Mathematical Functions*, edited by Abramowitz and Stegun. What is needed now is a comparable organization and tabulation of the properties—algebraic, analytical, asymptotic, and numerical—of the Painlevé functions. This letter is an appeal to interested parties in the scientific community at large for help in developing such a "Painlevé Project". What we have in mind will be described below.

Although the Painlevé equations are nonlinear, much is already known about their solutions, particularly their algebraic, analytical, and asymptotic properties. This is because the equations are integrable in the sense that they have a Lax-Pair and also a Riemann-Hilbert representation from which the asymptotic behavior of the solutions can be inferred using the nonlinear steepest-descent method. The numerical analysis of the equations is less developed and presents novel challenges; in particular, in contrast to the classical special functions, where the linearity of the equations greatly simplifies the situation, each problem for the nonlinear Painlevé equations arises essentially anew.

As a first step in the Painlevé Project, we have established an e-site, maintained at the National Institute of Standards and Technology (NIST). We ask interested readers to send to the site

(1) pointers to new work on the theory of the Painlevé equations, algebraic, analytical, asymptotic, or numerical;

(2) pointers to new applications of the Painlevé equations;

(3) suggestions for possible new applications of the Painlevé equations;

(4) requests for specific information about the Painlevé equations.

The e-site will work as follows:

(1) You must be a "subscriber" to post messages to the e-site. To become a subscriber, send email to `daniel.lozier@nist.gov`.

(2) To post a message after becoming a subscriber, send email to `PainleveProject@nist.gov`. The message will be forwarded to every subscriber.

(3) See `http://cio.nist.gov/esd/emaildir/lists/painleveproject/threads.html` for the complete archive of posted messages. This archive is visible to anyone, not just subscribers.

(4) See `http://cio.nist.gov/esd/emaildir/lists/painleveproject/subscribers.html` for the complete list of subscribers. This list is visible to anyone, not just subscribers.

Depending on the response to our appeal, we plan to set up a Wiki page for the Painlevé equations, and then ultimately a comprehensive handbook in a style befitting our digital age, along the lines of the hyperlinked version (`http://dlmf.nist.gov`) of the new *NIST Handbook of Mathematical Functions*, edited by Olver, Lozier, Boisvert, and Clark and published by Cambridge University Press. Incidentally, this work contains, for the first time, a chapter on the Painlevé equations.

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